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The determination of Green's Function by means of Cylindrical or Spherical Harmonics.

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INTRODUCTION.

In the following paper, the determination of Green's function, for spaces bounded by surfaces of the cylindrical and spherical polar systems, is effected by what is believed to be a novel process, in which are utilised the properties of cylindrical and spherical harmonics, regarded as functions of their parameters.

The functions considered are the cylindrical harmonic

$$(Ae^{\kappa z} + Be^{-\kappa z})B_m(\kappa\rho) \quad (C\cos m\phi + D\sin m\phi)$$

and the spherical harmonic

$$(Ar^n + Br^{-n-1})S_n^m(\cos\theta)(C\cos m\phi + D\sin m\phi);$$

ρ, z, ϕ and r, θ, ϕ being the cylindrical and spherical coordinates of a point in space. Here, as is known, $B_m(\kappa\rho)$ or $S_n^m(\cos\theta)$ can be so chosen that the function in which it occurs as a factor is a potential function, that is, satisfies Laplace's equation, $\nabla^2 V = 0$.

Taking $S_n^m(\cos\theta)$ for example, S must satisfy a certain well-known differential equation of the second order, in which $\mu \equiv \cos\theta$ is the independent variable. In this order of procedure the parameters n, m, the *degree* and *rank* of the harmonic, occupy the place of constants, and the nature of the functionality of S, as depending on n, m, does not come into question. It is, however, possible to define two solutions, in general independent, of the equation for S, which are, for a given μ, continuous and indeed holomorphic functions of the complex variables m and n. For most applications, it is sufficient to regard one only of the parameters as variable; the other, along with the geometrical variable μ, is for the moment constant. Then, supposing for example that μ, m are given, the properties of holomorphic function of n we have to consider are chiefly those relating to the distribution of the zeros, and the nature

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of the singularity at infinity. The importance of these questions has in special cases long been recognised. Thus, for instance, Laplace's investigation of an approximate formula for the Legendre's coefficient of high order, clearly essentially belongs to the theory of the zonal harmonic as a function of its degree.

Of the Bessel functions, a similar view may be taken. We define two solutions of Bessel's equation, $J_m(\kappa\rho)$, $G_m(\kappa\rho)$; as functions of the *factor* κ, the properties of these solutions are well known; as functions of the *rank* m, they have received little consideration, although they possess many simple and elegant properties, which may be usefully applied in various problems of physical mathematics.

The first part of the paper is occupied with the definitions of the functions used, and, in the case particularly of the spherical harmonics, with a short sketch of their leading properties. The methods here are perhaps to some extent novel, but for general conceptions I am greatly indebted to a valuable memoir by Dr Hobson in the *Phil. Trans.*, 1896.

The second part of the paper is devoted to the determination of Green's function. The problem is, to determine a potential function which shall be zero at the boundary of a given space, and discontinuous at only one point within the space, at which point, or pole, it becomes infinite as $1/r$, in the usual mode of expression; that is, the difference between Green's function and the reciprocal of the distance from the pole must tend to a definite limit as the variable point approaches the pole. The method of solving the problem may be described as *direct;* the function $1/r$ is taken as basis, and Green's function found by adding to this a function, continuous throughout the space, and neutralising, or balancing, $1/r$ at the boundary. For the application of this direct method we require first of all a representation of the function $1/r$ in terms of the appropriate harmonic functions. For the Bessels and spherical harmonics, two such representations are already well known, and are immediately available for the purpose of neutralising $1/r$ at cylindrical, or parallel plane boundaries, in the case of the Bessels; at conical, or spherical boundaries in the case of the spherical harmonics. A third representation, the natural complement of the other two, is here given; this involves harmonics of pure imaginary rank, and serves the purpose of neutralising $1/r$ at a boundary consisting of two axial planes.

When in this way the Green's function has been found for, say, a space bounded by two spheres, the difficulty of this method hitherto has been to neutralise this Green's function at additional boundaries, without disturbing the balance already attained at the first boundary. This difficulty is here overcome very simply by means of transformations depending on the use of Cauchy's fundamental integration theorem. Thus, just as we find three forms for the fundamental function $1/r$, so also three forms are found for the Green's function for a space bounded by, say, two spheres. One of these forms can be immediately applied to neutralise the Green's function at one or two additional conical boundaries; the other of the two forms may be similarly used when the additional boundary consists of two axial planes. For each of these new Green's functions, three forms are likewise obtained, and so the process can be continued. In all cases the function is expressed by means of a double series of harmonics, or the integral of a series, or a double integral.

The advantage of possessing alternative forms for the functions does not cease when the analytical transformations are completed; each form has, in fact, its own particular region of very rapid convergence, and hence, from the purely arithmetical point of view, it is useful to have all.

A totally different and, from many points of view, extremely beautiful method of dealing with the problem, has been given by Stokes, who takes the case of a finite space bounded by the six faces of a rectangular solid. This method consists essentially in assuming for the function required an expansion in the form of a triple series of functions, which are not potentials, but each of which vanishes at every part of the boundary. The coefficients are determined by differentiation and integration. The triple series can be reduced to a double series in three different ways. These correspond to the three forms alluded to above.

There is nothing to prevent the application of this method to all the cases given below. (For an example, see a paper by Mr H. M. Macdonald, L.M.S., vol. 26, "The electrical distribution on a conductor, etc.") The advantage of the method of this paper is that it is independent of any theorem for the expansion of an arbitrary function.

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THE BESSEL FUNCTIONS.

1. The definition of the Bessel Function of the first kind is firmly established, having come down, indeed, from Bessel himself, viz. :—

$$J_m(x) = \sum_s (-)^s \frac{x^{m+2s}}{2^{m+2s}\Pi(s)\Pi(m+s)}.$$

(Throughout the paper $\sum_s f(s)$ means, in the absence of a statement to the contrary, $f(0)+f(1)+f(2)+\ldots\ldots$, and $\sum_s' f(s)$ means, $\frac{1}{2}f(0)+f(1)+f(2)+\ldots\ldots$.)

For the Bessel Function of the second kind, various definitions and symbols are, unfortunately, prevalent. The suggestion of Gray and Matthews is here adopted, so that we define

$$G_m(x) = \frac{\pi}{2\sin m\pi}\left\{J_{-m}(x) - e^{-im\pi}J_m(x)\right\}.$$

If m be a given arbitrary constant, J and G are functions of x, in general multiform. They become perfectly defined when the phase of x is restricted to a range of 2π; this range we usually take to be from $-\frac{\pi}{2}$ to $+\frac{3\pi}{2}$. The form of the functions in the vicinity of the singular point $x=0$ is at once evident from the definitions, but the case of m an integer is exceptional. In this case $J_m(x)$ is uniform and $J_{-m}x = (-)^m J_m x$; the definition of $G_m x$ becomes illusory. For this special case, we define G_m as the limit of the function which defines it in the general case. The singularity at $x=0$ is then logarithmic and $G_m(x) = -J_m(x)\log x +$ a uniform function of x.

For both J and G, $x=\infty$ is an essential singularity. All the information necessary as to the form of the functions in the vicinity of $x=\infty$, is obtainable from the very important semi-convergent expansion of G. We have when $x>0$

$$G_m(x) = e^{-\frac{m\pi i}{2}} e^{i(x+\frac{\pi}{4})} \sqrt{\frac{\pi}{2x}} \sum_{p=0}^{p=r-1} \frac{\Pi(p-m-\frac{1}{2})\Pi(p+m-\frac{1}{2})}{\Pi(-m-\frac{1}{2})\Pi(m-\frac{1}{2})\Pi p}(2ix)^{-p}$$
$$+ R,$$

where if $r>m-\frac{1}{2}$, and m be real, R is less in absolute value than the next, or $(r+1)^{\text{th}}$ term of the series.

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The formula holds if $-\frac{\pi}{2} <$ phase of $x < \frac{3\pi}{2}$.

The corresponding formula for $J_m(x)$ is obtained from the equation $\pi i J_m(x) = G_m(x) - e^{im\pi} G_m(xe^{i\pi})$, which follows at once from the definition of G. If the real part of x is negative, we replace this by

$$\pi i J_m(x) = e^{im\pi} G_m(xe^{-i\pi}) - e^{2im\pi} G_m(x).$$

But whether m be real or not, we have for the limiting forms of the functions for x infinite,

$$G_m(x) = e^{-\frac{m\pi i}{2}} e^{i(x+\frac{\pi}{4})} \sqrt{\frac{\pi}{2x}};$$

$$J_m(x) = \sqrt{\frac{2}{\pi x}} \cos\left\{x - (m+\tfrac{1}{2})\frac{\pi}{2}\right\}, \text{ if real part of } x \text{ is positive};$$

$$= e^{m\pi i} J_m(xe^{-i\pi}), \text{ if real part of } x \text{ is negative.}$$

At infinity in the upper part of the plane, therefore,

$J_m(x)$ is infinite, $G_m(x)$ vanishes, and $G_m(ax)J_m(bx)$,

where a, b are real positives, vanishes if $a > b$.

2. As regards the zeros of J and G as functions of x, it is sufficient to state here the following well-known theorems:—

(*a*) When m is real and positive, the zeros of $J_m(x)$ are all real and simple, but
$G_m(x)$ has no zeros for which the imaginary part of x is positive or zero.

(*b*) $J_m(ax)G_m(bx) - J_m(bx)G_m(ax)$, where a, b are real and positive, is a uniform, even function of x, whose zeros are all real and simple.

3. Passing to the consideration of J, G as functions of their rank m, we note first that the functions are holomorphic, and therefore, for instance, expansible in series of ascending powers of m convergent over the whole plane; the x in $J_m(x)$, $G_m(x)$ is supposed to have any constant value, but the particular value $x=0$ is excluded from consideration.

The forms of the functions for an m of very large modulus are given at once by the defining expansions.

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Thus, for a very large m, we have approximately

$$J_m(x) = \frac{x^m}{2^m \Pi m}.$$

Also $$\Pi m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m = \sqrt{2\pi m}\, e^{m\log m - m};$$

this formula holds if the phase of m lie between $-\pi$, $+\pi$. For the succeeding applications the only case of importance is that in which x is a positive pure imaginary, ia say, where a is real and positive.

We then have modulus of

$$e^{-\frac{m\pi i}{2}} J_m(ia) = \frac{1}{\sqrt{2\pi M}}\left(\frac{ae}{2M}\right)^{M\cos a} e^{M a \sin a},$$

$$\text{where } m = Me^{ia}; \quad -\pi < a < \pi;$$

$$\text{or modulus } J_m(ia) = \frac{1}{\sqrt{2\pi M}}\left(\frac{ae}{2M}\right)^{M\cos a} e^{-M\left(\frac{\pi}{2} - a\right)\sin a}$$

Again, we have

$$e^{\frac{m\pi i}{2}} G_m(ib) = \frac{\pi}{2\sin m\pi}\left\{e^{\frac{m\pi i}{2}} J_{-m}(ib) - e^{-\frac{m\pi i}{2}} J_m(ib)\right\}$$

$\therefore$ $e^{\frac{m\pi i}{2}} G_m(ib)$ is an *even* function of m. Taking b to be real and positive, this function is real for m real, and therefore for m a pure imaginary; further, for the four values of m, $\pm p \pm iq$, where p, q are real, the function has the same modulus.

Consider then an m in the first quadrant.

$$\text{We have} \quad e^{\frac{m\pi i}{2}} G_m(ib) = \frac{\pi}{2\sin m\pi} e^{\frac{im\pi}{2}} J_{-m} ib$$

$$= \frac{\Pi m}{2m}\left(\frac{b}{2}\right)^{-m} \text{ approximately.}$$

This holds except for m a pure imaginary, and therefore, except for such values of m, $G_m(ib)$ is infinite with m. For a pure imaginary m, say $m = is$, s real and positive, we have

$$e^{\frac{m\pi i}{2}} G_m(ib) = \frac{1}{2m}\left\{\Pi(m)\left(\frac{b}{2}\right)^{-m} - \Pi(-m)\left(\frac{b}{2}\right)^m\right\}$$

and therefore

$$G_u(ib) = \sqrt{\frac{2\pi}{s}} \sin\left\{\frac{\pi}{4} + s\log s - s - \log\frac{b}{2}\right\}.$$

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Lastly, the function $J_m(\lambda a)G_m(\lambda b) - J_m(\lambda b)G_m(\lambda a)$,

being equal to $\frac{\pi}{2\sin m\pi}\{J_m(\lambda a)J_{-m}(\lambda b) - J_m(\lambda b)J_{-m}(\lambda a)\}$,

is a uniform even function of m, just as it is of λ.

When m is large, this function

$$= \frac{1}{2m}\left\{\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m\right\}$$

$$= \frac{1}{m}\sinh(m\log\overline{a/b}).$$

It is interesting to compare this formula with its analogue for m fixed, λ very great, namely,

$$\frac{1}{\lambda\sqrt{ab}}\sin\lambda(a-b).$$

4. We require next a few theorems relative to the zeros of the functions of m, for the proof of which the following definite integrals are convenient.

Let u, v be any Bessel Functions of ranks m, n, and factors κ, λ respectively;

so that
$$\rho\frac{d^2u}{d\rho^2} + \frac{du}{d\rho} + \left(\kappa^2\rho - \frac{m^2}{\rho}\right)u = 0$$

$$\rho\frac{d^2v}{d\rho^2} + \frac{dv}{d\rho} + \left(\lambda^2\rho - \frac{n^2}{\rho}\right)v = 0.$$

Multiply these equations by v, u, respectively, subtract, and integrate from a to b.

Thus
$$\int_a^b \left\{(\kappa^2 - \lambda^2)\rho + \frac{n^2 - m^2}{\rho}\right\}uv d\rho = \left|\rho\left(u\frac{dv}{d\rho} - v\frac{du}{d\rho}\right)\right|_a^b.$$

The consequences which follow from taking $m^2 = n^2$ are familiar; we obtain results of precisely analogous significance for the theory of the functions of m, by taking $\kappa^2 = \lambda^2$.

(i) Take $\left.\begin{matrix} u = G_m\lambda\rho \\ v = G_n\lambda\rho \end{matrix}\right\}$ λ a positive pure imaginary.

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Then $\int_1^\infty G_m(\lambda\rho)G_n(\lambda\rho)\frac{d\rho}{\rho}=\frac{\lambda}{n^2-m^2}\left\{G_n\lambda\,.\,\frac{d}{d\lambda}G_m\lambda-G_m\lambda\frac{d}{d\lambda}G_n\lambda\right\}$

$$\int_1^\infty (G_m\,\lambda\rho)^2\frac{d\rho}{\rho}=\frac{\lambda}{2m}\left\{\frac{dG\lambda}{dm}\,.\,\frac{dG\lambda}{d\lambda}-G\lambda\frac{d^2G\lambda}{dmd\lambda}\right\}.$$

so that if $G_m(\lambda)=0$,

$$\int_1^\infty G_m(\lambda\rho)^2\frac{d\rho}{\rho}=\frac{\lambda}{2m}\,\frac{dG\lambda}{dm}\,.\,\frac{dG\lambda}{d\lambda}.$$

(ii) Similarly if the real part of $m+n$ be positive,

$$\int_0^1 J_m(\lambda\rho)J_n(\lambda\rho)\frac{d\rho}{\rho}=\frac{\lambda}{m^2-n^2}\left\{J_n(\lambda)\frac{d}{d\lambda}J_m(\lambda)-J_m(\lambda)\frac{d}{d\lambda}J_n(\lambda)\right\}.$$

$$\int_0^1 (J_m\lambda\rho)^2\frac{d\rho}{\rho}=\frac{\lambda}{2m}\left\{J\lambda\frac{d^2J\lambda}{dmd\lambda}-\frac{dJ\lambda}{d\lambda}\,\frac{dJ\lambda}{dm}\right\}.$$

(iii) Take $u=J_m\lambda\rho\,.\,G_m\lambda b-J_m\lambda b\,.\,G_m\lambda\rho$

$$v=J_n\lambda\rho\,.\,G_n\lambda b-J_n\lambda b\,.\,G_n\lambda\rho$$

Then $\int_a^b uv\frac{d\rho}{\rho}=\frac{a}{n^2-m^2}\left(v\frac{du}{d\rho}-u\frac{dv}{d\rho}\right)_{\rho=a}.$

$$\int_a^b u^2\frac{d\rho}{\rho}=\frac{a}{2m}\left\{\frac{du}{dm}\,\frac{du}{d\rho}-u\frac{d^2u}{dmd\rho}\right\}_{\rho=a}.$$

5. We now prove the few theorems we require relative to the zeros of the functions of m.

In the following λ is a positive pure imaginary; a and b real and positive.

(a) $J_m(\lambda)$ has no zeros with real part positive or nil.

For $i^{-m}J_m(\lambda)$ is real for m real; therefore for $m=p\pm iq$, the values of $i^{-m}J_m\lambda\rho$ are conjugate complexes;

therefore $\int_0^1 J_{m1}(\lambda\rho)J_{m2}(\lambda\rho)\frac{d\rho}{\rho}$ cannot vanish,

where $m_1=p+iq$, $m_2=p-iq$; therefore by the first result of 4 (ii) $J_{m1}(\lambda)$, $J_{m2}(\lambda)$ cannot both vanish, and hence neither can vanish; that is, $J_m\lambda$ has no *complex* zeros.

It has no *real* zeros; otherwise we should have a potential

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function, say sinh $\lambda z . J_m\lambda\rho . \sin m\phi$ vanishing at the whole of the external boundary of the space within the cylinder $\rho = 1$, the parallel planes $z = 0$, $z = \frac{\pi i}{\lambda}$, and the axial planes $\phi = 0$, $\phi = \frac{\pi}{m}$. But this is impossible, the case $\frac{\pi}{m} > 2\pi$ not excepted. Thirdly, $J_m\lambda$ has no pure imaginary zeros; otherwise we should have, as in the first part of the proof $J_m\lambda$, $J_{-m}\lambda$ vanishing simultaneously; that is J, G vanishing simultaneously, a supposition inconsistent with the fundamental relation

$$G_m\lambda . \frac{d}{d\lambda}J_m\lambda - J_m\lambda\frac{d}{d\lambda}G_m\lambda = \frac{1}{\lambda}.$$

(*b*) That the functions $G_m(\lambda)$ and $J_m a\lambda . G_m b\lambda - J_m b\lambda G_m a\lambda$ have no *complex* zeros is proved just as above; that they have no *real* zeros follows by considering the potentials

$$\sinh\lambda z . G_m\lambda\rho . \sin m\phi, \text{ with } \rho \gtreqless 1;$$

and $\sinh \lambda z(J_m\lambda\rho G_m\lambda b - G_m\lambda\rho J_m\lambda b)\sin m\phi$, with ρ between a and b.

Each function has, however, an infinite number of purely imaginary zeros; this almost follows from the approximate formulas for m large which have already been given, or may be formally proved by the following considerations.

Take $G_m\lambda$ for example: this, as has been shown, is infinite for an infinite, not pure imaginary, m; but $m^2G_m\lambda$ is infinite for every infinite m. Now it follows easily from the second formula of 4 (i) that any zero must be simple, as G, $\frac{dG}{dm}$ cannot vanish together. If, therefore, G have only a finite number of zeros, the uniform function $\frac{1}{m^2G}$ would have only a finite number of accidental singularities; vanishing at infinity as it does, it would therefore be a rational function, a supposition excluded by the form of G at infinity.

Hence $G_m\lambda$ and similarly $J_m\lambda a G_m\lambda b - J_m\lambda b G_m\lambda a$ have an infinite number of pure imaginary zeros.

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The relation we have made use of in (*a*) above, namely,

$$G_m\lambda\frac{d}{d\lambda}J_m\lambda - J_m\lambda\frac{d}{d\lambda}G_m\lambda = \frac{1}{\lambda},$$

is usually proved by the common method applicable to two solutions of a linear differential equation of the second order. It may also be proved geometrically, by a simple application of Green's Theorem, a method essentially the same as the former. But it is interesting to observe that it follows easily from the character of the left hand expression as a function of m; it is, namely, a holomorphic function whose form at infinity is $\frac{1}{\lambda}$; the function is therefore $\frac{1}{\lambda}$ for every m.

Spherical Harmonics.

6. If the potential function $(x+iy)^m$ be inverted from the point $(0, 0, -\xi)$ we obtain another potential function

$$(x+iy)^m/\{x^2+y^2+(z+\xi)^2\}^{m+\frac{1}{2}}.$$

Hence also
$$(x+iy)^m\int_0^\infty\frac{\xi^{m+n}d\xi}{\{\rho^2+(z+\xi)^2\}^{m+\frac{1}{2}}}$$

is a potential function, if the integral be convergent.

Change to polar coordinates r, θ, ϕ and put $\xi = r\lambda$, where λ is the new variable of integration.

The potential becomes

$$r^n\sin^m\theta e^{im\phi}\int_0^\infty\frac{\lambda^{m+n}d\lambda}{1+2\lambda\cos\theta+\lambda^2)^{m+\frac{1}{2}}},$$

and is therefore, by definition, a spherical harmonic of degree n, and rank m.

The function of θ represented by the integral may be expanded as follows.

We have $\quad 1+2\lambda\cos\theta+\lambda^2 = (1+\lambda)^2 - 2\lambda(1-\cos\theta)$

$$= (1+\lambda)^2\left\{1-\frac{4\lambda}{(1+\lambda)^2}a\right\},$$

writing a for $\frac{1}{2}(1-\cos\theta)$ or $\frac{1}{2}(1-\mu)$

$$\therefore \int_0^\infty\frac{\lambda^{m+n}d\lambda}{(1+2\lambda\cos\theta+\lambda^2)^{m+\frac{1}{2}}} = \int_0^\infty\frac{\lambda^{m+n}}{(1+\lambda)^{2m+1}}\left\{1-\frac{4\lambda}{(1+\lambda)^2}a\right\}^{-m-\frac{1}{2}}d\lambda.$$

If modulus $a<1$, the last factor can be expanded by the Binomial Theorem, and the series integrated term by term.

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Coefficient of a^p will be

$$\frac{\Pi(m-\frac{1}{2}+p)}{\Pi(m-\frac{1}{2})\Pi p}\,4^p\int_0^\infty \lambda^{m+n+p}(1+\lambda)^{-2m-2p-1}d\lambda$$
$$=\frac{\Pi(m-\frac{1}{2}+p)}{\Pi(m-\frac{1}{2})\Pi p}4^p\frac{\Pi(m+n+p)\Pi(m-n-1+p)}{\Pi(2m+2p)}.$$

Coefficient of a^{p+1} is obtained from this by multiplying by

$$\frac{m+p+\frac{1}{2}}{p+1}\cdot 4\cdot\frac{(m+n+p+1)(m-n+p)}{(2m+2p+1)(2m+2p+2)}$$
$$=\frac{(m+n+1+p)(m-n+p)}{(p+1)(m+p+1)}.$$

Hence the integral is

$$\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi 2m}\left\{1+\frac{(m-n)(m+n+1)}{1\,.\,(m+1)}a+\dots\right\}$$
$$=\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi 2m}\mathrm{F}\left(m-n,\ m+n+1,\ m+1,\ \frac{1-\mu}{2}\right)$$
$$=\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi 2m}\left(\frac{1+\mu}{2}\right)^{-m}\mathrm{F}\left(n+1,\ -n,\ m+1,\ \frac{1-\mu}{2}\right),$$

where we use Euler's Theorem for the hypergeometric function,

$$\mathrm{F}(a,\ \beta,\ \gamma,\ x)=(1-x)^{\gamma-a-\beta}\mathrm{F}(\gamma-a,\ \gamma-\beta,\ \gamma,\ x).$$

The function of θ

$$\frac{1}{\Pi m}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}\mathrm{F}\left(n+1,\ -n,\ m+1,\ \frac{1-\mu}{2}\right)$$

will be denoted by $\mathrm{P}_n^m(\mu)$.

When $m=0$, the above solid harmonic

$$\text{(i)}\quad \int_0^\infty\frac{\xi^n d\xi}{\{\rho^2+(z+\xi)^2\}^{\frac{1}{2}}}=\Pi n\,.\,\Pi(-n-1)r^n\mathrm{P}_n(\mu)$$
$$=\ -\ \frac{\pi}{\sin n\pi}\ r^n\mathrm{P}_n(\mu).$$

For the convergence of the definite integral we have used, the real parts of $m+n+1$, $m-n$, must be positive; this restriction still allows a continuous range of values for the variables m, n. But the definition of $\mathrm{P}_n^m(\mu)$ clearly defines a function of m, n as well as of μ; the solid harmonic is also a function of m, n, as is likewise $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ of the solid harmonic; this being zero for a continuous range of values of m, n, is zero for all values. P_n^m is clearly a holomorphic function both of m and of n.

As a function of μ, P_n^m will here be considered only for real values of μ from -1 to $+1$, this being all that is needed for the physical applications that follow. The definition exhibits sufficiently the behaviour of the function near $\mu=1$; a formula showing its character near $\mu=-1$ will be found presently.

We note here, as easily derived from the definite integral, the two formulæ

$$(m+n+1)P_{n+1}^m-\mu(2n+1)P_n^m+(n-m)P_{n+1}^m=0$$

and

$$P_n^m(0)=\frac{\sqrt{\pi}}{2^m}\cdot\frac{1}{\Pi\frac{m+n}{2}\cdot\Pi\frac{m-n-1}{2}}.$$

7. $P_n^m(\mu)$ is a solution of the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP}{d\mu}\right\}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}P=0.$$

Since the equation is unaltered by changing μ into $-\mu$, or m into $-m$, we have three other solutions

$$P_n^{-m}(\mu),\quad P_n^m(-\mu),\quad P_n^{-m}(-\mu)$$

The two linear relations connecting the four solutions are found below. Change of n into $-n-1$ does not affect the equation, but this does not lead to any new solution, since the definition gives at once $P_n^m(\mu)=P_{-n-1}^m(\mu)$, an important relation.

Expression of the Harmonics by Bessel Functions.

8.

$$\int_0^\infty e^{-\lambda z}J_m(\lambda\rho)\lambda^n d\lambda\,.\cos m\phi \text{ is a potential function, } (n+m>-1)$$

Putting $z=r\cos\theta$, $\rho=r\sin\theta$, $\lambda r=\kappa$, we change the integral to

$$r^{-n-1}\cos m\phi\int_0^\infty e^{-\kappa\cos\theta}J_m(\kappa\sin\theta)\kappa^n d\kappa.$$

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The function is, then, a solid spherical harmonic of degree $-n-1$, rank m.

Hence $\int_0^\infty e^{-\kappa\cos\theta} J_m(\kappa\sin\theta)\kappa^n d\kappa = AP_n^m(\mu) + BP_n^{-m}(\mu)$.

To find the constants, suppose $m > 0$; then $B = 0$, since P_n^{-m} is infinite for $\theta = 0$; further dividing by $\sin^m\theta$ and putting $\theta = 0$,

$$\int_0^\infty \frac{e^{-\kappa}\kappa^{m+n}}{2^m \Pi m} d\kappa = A \cdot \frac{1}{2^m \Pi m}.$$

$$\therefore \quad A = \Pi(m+n)$$

and
$$\int_0^\infty e^{-\lambda z} J_m \lambda\rho \cdot \lambda^n d\lambda = \frac{\Pi(m+n)}{r^{n+1}} P_n^m(\cos\theta),$$

for all values of m, n, θ for which both sides of the equation retain a meaning.

Also
$$\int_0^\infty e^{-\lambda\cos\theta} J_m(\lambda\sin\theta)\lambda^n d\lambda = \Pi(m+n)P_n^m(\cos\theta)$$
$$= p_n^m(\cos\theta), \text{ say.}$$

We have thus a representation of P by means of J, provided

$$\theta < \frac{\pi}{2}, \quad m+n > -1, \quad (\textit{i.e.}, \text{ real part of } m+n > -1).$$

The restriction upon m, n may at once be removed by taking a complex path of integration. Thus, supposing the λ plane cut from 0 to $-\infty$,

$$\int e^{\lambda z} J_m \lambda\rho \cdot \lambda^n d\lambda = \frac{2\pi i}{\Pi(-m-n-1)} \frac{1}{r^{n+1}} P_n^m\left(\frac{z}{r}\right),$$

and
$$\int e^{\lambda z} J_m \lambda\rho \cdot \lambda^{-n-1} d\lambda = \frac{2\pi i}{\Pi(n-m)} \cdot r^n P_n^m\left(\frac{z}{r}\right),$$

the path being from $-\infty$, under 0, round 0, back to $-\infty$, and the only restriction $z > 0$.

We proceed to obtain a representation of P in which there is no restriction upon θ.

9. We may conveniently begin by explaining here a notation which will frequently be used in dealing with complex integrals. In the plane of any variable λ, let E, W be points on the positive and negative sides respectively of the real axis; N, S points on the positive and negative sides of the axis of imaginaries;

E, N, W, S thus corresponding to the points of the compass. These points will be supposed at an indefinitely great distance; and the symbol $\int f(\lambda)d\lambda \Big|$ ON, for example, will be used to denote an integration over the whole of the upper half of the imaginary axis.

$\int f(\lambda)d\lambda \Big|$ OE is therefore the same as $\int_0^\infty f(\lambda)d\lambda$.

Consider now the integral

$$\int_0^\infty e^{i\lambda\cos\theta} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda \,.\, (i^m \equiv e^{\frac{m\pi i}{2}}).$$

This may be written

$$\int e^{\lambda\cos\theta} i^m \mathrm{G}_m(\lambda\sin\theta)\,.\,\lambda^n d\lambda\,.\,i^{-n-1} \Big| \text{ON}$$

and the path may be deformed into OW, provided $\cos\theta > 0$, since $\mathrm{G}_m(\lambda\sin\theta)$ vanishes at infinity in the upper part of the plane.

Hence $\int_0^\infty e^{i\lambda\cos\theta} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda$

$$= \int_0^\infty e^{-\lambda\cos\theta} i^m \mathrm{G}_m(-\lambda\sin\theta)\lambda^n d\lambda\,.\,i^{n+1}$$

$$= \frac{\pi}{2\sin m\pi}\,.\,i^{n+1}\int_0^\infty e^{-\lambda\cos\theta}(i^{-m}\mathrm{J}_{-m}\lambda\sin\theta - i^m \mathrm{J}_m\lambda\sin\theta)\lambda^n d\lambda$$

(i) $$= \frac{\pi}{2\sin m\pi} i^{n+1}\{i^{-m}p_n^{-m}(\mu) - i^m p_n^m(\mu)\}$$

Similarly $\int_0^\infty e^{-i\lambda\cos\theta} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda$

$$= \int e^{-\lambda\cos\theta} i^m \mathrm{G}_m(\lambda\sin\theta)\lambda^n d\lambda\,.\,i^{-n-1} \Big| \text{ON}$$

$$= \int \Big| \text{OE, if } \cos\theta > 0$$

(ii) $$= \frac{\pi}{2\sin m\pi} i^{-n-1}\{i^m p_n^{-m}(\mu) - i^{-m} p_n^m(\mu)\}.$$

From these two relations

(iii) $$\int_0^\infty \sin\left\{\lambda\cos\theta + (m-n-1)\frac{\pi}{2}\right\} i^m \mathrm{G}_m(i\lambda\sin\theta)\lambda^n d\lambda$$

$$= -\frac{\pi}{2}p_n^m(\mu).$$

This formula extends the representation of the function to values of θ greater than $\frac{\pi}{2}$, provided the real parts of both $n+m$ and $n-m$ be greater than -1.

In (iii) we may change θ into $\pi-\theta$, or m into $-m$; neither of these changes, it may be observed, affects $i^m G_m(i\lambda\sin\theta)$.

In (i) change θ into $\pi-\theta$; the integral becomes the integral of (ii). Hence

(iv) $$i^{n+1-m}p_n^{-m}(-\mu) - i^{n+1+m}p_n^{m}(-\mu) = i^{m-n-1}p_n^{-m}(\mu) - i^{-m-n-1}p_n^{m}(\mu).$$

Similarly, by changing θ into $\pi-\theta$ in (ii),

(v) $$i^{m-n-1}p_n^{-m}(-\mu) - i^{-n-1-m}p_n^{m}(-\mu) = i^{-m+n+1}p_n^{-m}(\mu) - i^{m+n+1}p_n^{m}(\mu).$$

By means of these two equations, any two of the four functions

$$p_n^m(\mu),\quad p_n^m(-\mu),\quad p_n^{-m}(\mu),\quad p_n^{-m}(-\mu)$$

can be expressed in terms of the remaining two.

10. Eliminating $p_n^{-m}(\mu)$ from these equations, we find

$$\sin m\pi p_n^m(\mu) = \sin n\pi p_n^m(-\mu) + \sin(m-n)\pi p_n^{-m}(-\mu),$$

or returning to the P functions,

(i) $$\Pi(m+n)\sin m\pi \mathrm{P}_n^m(\mu)$$
$$= \Pi(m+n)\sin n\pi \mathrm{P}_n^m(-\mu) + \Pi(n-m)\sin(m-n)\pi \mathrm{P}_n^{-m}(-\mu).$$

This equation, and that obtained from it by changing μ into $-\mu$, may be regarded as the linear relations connecting the four solutions referred to in §7.

Since $$\Pi(n-m)\sin(m-n)\pi = \pi/\Pi(m-n-1),$$

we may write (i) in the form

(ii) $$\pi \mathrm{P}_n^{-m}(\mu) = \Pi(m+n)\Pi(m-n-1)\{\sin m\pi \mathrm{P}_n^m(-\mu) - \sin n\pi \mathrm{P}_n^m(\mu)\}.$$

The equation (i) or (ii) has many important applications, of which one or two will be pointed out.

(*a*) The equation gives the form of $\mathrm{P}_n^m(\mu)$ in the neighbourhood of $\mu=-1$, or of $\mathrm{P}_n^m(-\mu)$ near $\mu=+1$.

When m is an integer, the method of limits has to be applied, as in the case of the Bessel function $G_m(x)$ and the expansion will involve a logarithm, unless n also be an integer.

(*b*) When $n-m$ is a positive integer p, we have

$$\Pi(m+n)\sin m\pi P_n^m(\mu)=\Pi(m+n)(-)^p\sin m\pi P_n^m(-\mu)$$

so that if m be positive and not integral,

$$P_{m+p}^m(\mu)=(-)^p P_{m+p}^m(-\mu).$$

But the two sides of this equation are, for a given p, continuous functions of m, so that this holds for every m.

The function $P_{m+p}^m(\mu)$ is important; it is the product of $(1-\mu^2)^{\frac{m}{2}}$ by a rational integral function of μ, and for m positive vanishes both at $\mu=1$ and $\mu=-1$

We have, in fact, by definition,

$$P_n^m(\mu)=\frac{1}{2^m\Pi m}(1-\mu^2)^{\frac{m}{2}}F\left(m-n,\ m+n+1,\ m+1,\ \frac{1-\mu}{2}\right)$$

and the series terminates when $m-n$ is zero or a negative integer.

(*c*) When m is an integer, we similarly obtain

$$\Pi(n+m)P_n^m(\mu)=(-)^m\Pi(n-m)P_n^{-m}(\mu).$$

If now n also is taken integral, then if $n-m<0$, $P_n^{-m}(\mu)=0$.

Some Fundamental Integrals.

11. Let u, v be two harmonics of degrees, n, p and ranks m, l, so that

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}u=0$$

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dv}{d\mu}\right\}+\left\{p(p+1)-\frac{l^2}{1-\mu^2}\right\}v=0.$$

Multiply these by v, u; subtract, and integrate from h to k.

$$\therefore\quad\int_h^k\left\{(n-p)(n+p+1)+\frac{l^2-m^2}{1-\mu^2}\right\}uvd\mu$$
$$=\left[(1-\mu^2)\left(u\frac{dv}{d\mu}-v\frac{du}{d\mu}\right)\right]_h^k$$

This equation has applications of precisely the same sort as the

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corresponding equation for the Bessel. Into the details of these it is not proposed to enter, but one or two results will be given.

(i) Take $u = P_n^m(\mu)$, $v = P_n^{-m}(\mu)$.

$$\text{Then} \quad \left| (1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right) \right|_h^k = 0$$

i.e., $(1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right)$ is independent of μ; to find its value, we may take the limit for $\mu = 1$.

Now near $\mu = 1$, we have $u = \dfrac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}}$

$$v = \frac{1}{2^{-m}\Pi - m}(1-\mu^2)^{-\frac{m}{2}}.$$

Hence we find $(1-\mu^2)\left\{P_n^m(\mu)\frac{d}{d\mu}P_n^{-m}(\mu) - P_n^{-m}(\mu)\frac{d}{d\mu}P_n^m(\mu)\right\}$

$$= \frac{2}{\pi}\sin m\pi.$$

From this again, by means of 10 (ii) we readily deduce $(1-\mu^2)\left\{P_n^m(\mu)\frac{d}{d\mu}P_n^m(-\mu) - P_n^m(-\mu)\frac{d}{d\mu}P_n^m(\mu)\right\}$

$$= \frac{2}{\Pi(m+n)\Pi(m-n-1)}.$$

(ii) Take $u = P_n^m(\mu)$, $v = P_p^m(-\mu)$; also suppose real part of m positive.

Then $\displaystyle\int_{-1}^{+1} P_n^m(\mu)P_p^m(-\mu)d\mu$

$$= \frac{1}{(n-p)(n+p+1)}\left[(1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right)\right]_{-1}^{+1}$$

From 10 (ii), near $\mu = 1$, we have

$$P_p^m(-\mu) = \frac{\pi}{\sin m\pi}\frac{1}{\Pi(m+p)\Pi(m-p-1)}\cdot\frac{2^m}{\Pi(-m)}(1-\mu^2)^{-\frac{m}{2}}$$

and therefore

$$\left| (1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right) \right|_{\mu=1} = \frac{2}{\Pi(m+p)\Pi(m-p-1)}$$

Similarly

$$\left| (1-\mu^2)\left(u\frac{dv}{d\mu} - v\frac{du}{d\mu}\right) \right|_{\mu=-1} = \frac{2}{\Pi(m+n)\Pi(m-n-1)}.$$

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Hence $\int_{-1}^{+1} P_n^m(\mu) P_p^m(-\mu) d\mu$

$$= \frac{2}{(n-p)(n+p+1)}\left[\frac{1}{\Pi(m+p)\Pi(m-p-1)} - \frac{1}{\Pi(m+n)\Pi(m-n-1)}\right]$$

a result interesting as the generalisation of various known theorems.

In special cases, the value of the dexter has to be found by limits.

By far the most important case is that where $n-m$, $p-m$ are zero or positive integers. Since $P_n = P_{-n-1}$ we may suppose $n+\frac{1}{2}$ and $p+\frac{1}{2}$ not less than 0.

Then the equation shows that if n, p are different, the integral vanishes, since $\Pi(m-p-1)$, $\Pi(m-n-1)$ are infinite.

In order to find the value of the integral when $n=p$, put $p=m+q$, q zero or a positive integer;
and find the limit as the single variable n approaches the fixed $m+q$.

We have, the integral

$$= \frac{1}{\pi}\frac{\sin(n-m)\pi}{n-q-m} \cdot \frac{2}{n+q+m+1} \cdot \frac{\Pi(n-m)}{\Pi(n+m)}.$$

$$= (-)^q \frac{2}{2m+2q+1} \cdot \frac{\Pi q}{\Pi(m+2q)}, \quad \text{in the limit.}$$

But, q being an integer,

$$P_{m+q}^m(-\mu) = (-)^q P_{m+q}^m(\mu), \quad [10\ (b)].$$

Hence we may state the result: when $n-m$ is a positive integer or zero,

$$\int_{-1}^{+1} \{P_n^m(\mu)\}^2 d\mu = \frac{2}{2n+1} \cdot \frac{\Pi(n-m)}{\Pi(n+m)}.$$

12. The harmonic $P_n^m(\mu)$ as a function of m, its rank.

The function is holomorphic; its form for m infinite is given by the definition $P_n^m(\mu) = \frac{1}{\Pi m}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}} F\left(n+1,\ -n,\ m+1,\ \frac{1-\mu}{2}\right)$.

For m very large, this gives $P^m = \frac{1}{\Pi m}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{m}{2}}$.

As to the zeros of the function, the important case is that in which the given n is of the form $-\frac{1}{2}+\lambda i$, λ real.

By methods altogether similar to those used for the Bessels, it may be proved, $(n = -\frac{1}{2} + \lambda i)$.

(i) $P_n^m(\mu)$ has no zeros for which the real part of m is positive or zero.

(ii) The function

$$\Pi(m+n)\Pi(m-n-1)\{P_n^m(h)P_n^m(-k) - P_n^m(k)P_n^m(-h)\},$$

which, by 10 (ii), is the same as

$$\frac{\pi}{\sin m\pi}\{P_n^m(h)P_n^{-m}(k) - P_n^m(k)P_n^{-m}(h)\},$$

and is therefore even and holomorphic, has an infinite number of purely imaginary simple zeros.

When m is very large, this function has the form

$$\frac{2}{m}\sinh\{m\log(\tan\alpha\cot\beta)\}; \quad h = \cos\alpha, \quad k = \cos\beta.$$

13. The harmonic $P_n^m(\mu)$ as a function of n, its degree.

As to the form of the function for n infinite, we have room here for only a few statements; for various investigations bearing on the question, reference may be made to the memoir of Dr Hobson, *l.c.*, and to Heine, I. p. 178, II. p. 223.

In the first place, if $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$, we have

$$\text{(i)}\quad P_n^m(\cos\theta) = \frac{1}{\sqrt{2\pi\sin\theta}}\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}\left[\begin{array}{l} i^{m+\frac{1}{2}}e^{-(n+\frac{1}{2})i\theta}F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ n+\frac{3}{2},\ \frac{ie^{-i\theta}}{2\sin\theta}\right) \\ +i^{-(m+\frac{1}{2})}e^{+(n+\frac{1}{2})i\theta}F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ n+\frac{3}{2},\ \frac{-ie^{i\theta}}{2\sin\theta}\right)\end{array}\right]$$

but the series diverge for values of θ outside the limits specified. For any value of θ, not 0 or π, however, it is possible to show that the limiting form of the function is given by the first terms of these series, so that for any n with large modulus, and phase not equal to π

$$\text{(ii)}\quad P_n^m(\cos\theta) = \frac{1}{\sqrt{2\pi\sin\theta}}\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}\left\{\begin{array}{l} i^{m+\frac{1}{2}}e^{-(n+\frac{1}{2})i\theta} \\ +i^{-(m+\frac{1}{2})}e^{+(n+\frac{1}{2})i\theta}\end{array}\right\}.$$

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It is very curious that if in the equation,

$$-\frac{\pi}{2}\Pi(m+n)P_n^m(\cos\theta)$$
$$=\int_0^\infty \sin\left\{\lambda\cos\theta+(m-n-1)\frac{\pi}{2}\right\}i^m G_m(i\lambda\sin\theta)\lambda^n d\lambda$$

we substitute for G its semi-convergent expansion and integrate term by term (a process not, of course, defensible) we obtain

$$\text{(iii)}\quad P_n^m(\cos\theta)=\frac{1}{\sqrt{2\pi\sin\theta}}\frac{\Pi(n-\frac{1}{2})}{\Pi(n+m)}$$
$$\left[\begin{array}{l} i^{m+\frac{1}{2}}e^{-(n+\frac{1}{2})i\theta}F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ \frac{1}{2}-n,\ -\frac{ie^{i\theta}}{2\sin\theta}\right) \\ +i^{-(m+\frac{1}{2})}e^{+(n+\frac{1}{2})i\theta}F\left(\frac{1}{2}-m,\ \frac{1}{2}+m,\ \frac{1}{2}-n,\ \frac{ie^{-i\theta}}{2\sin\theta}\right)\end{array}\right]$$

If n be replaced by $-n-1$, the hypergeometic series are those of (i), and the interesting question suggests itself, does the right hand of (iii) actually represent the function P for the range of values of θ for which it has a meaning?

For a large n, (i) and (iii) give the same limiting form for P, but on examination it appears that the dexter members are not really identical, unless m is half an odd integer, in which case, it may be noticed, the series terminate, like the series for J_m in the like case.

14. With regard to the zeros of P as a function of n, it is only necessary to consider the case of m real and positive, or zero. Then

(i) $P_n^m(\cos\alpha)$ has an infinite number of real, simple zeros, and no others.

For a large n, we have from (ii) of last paragraph

$$P_n^m(\cos\theta)=\frac{\sqrt{2}}{\sqrt{\pi\sin\theta}}\frac{\Pi(n-m)}{\Pi(n+\frac{1}{2})}\left\{\cos\left[(n+\tfrac{1}{2})\theta-(m+\tfrac{1}{2})\frac{\pi}{2}\right]\right\}.$$

(ii) The function

$$\Pi(m+n)\Pi(m-n-1)\{P_n^m(\cos\alpha)P_n^m(-\cos\beta)-P_n^m(\cos\beta)P_n^m(-\cos\alpha)\},$$

or the same thing,

$$\frac{\pi}{\sin m\pi}\{P_n^m(\cos\alpha)P_n^{-m}(\cos\beta)-P_n^m(\cos\beta)P_n^{-m}(\cos\alpha)\}$$

has also an infinite number of real, simple zeros, and no others.

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For a large n, this function

$$=\frac{2}{n\sqrt{\sin\alpha\sin\beta}}\sin\{(n+\tfrac{1}{2})(\alpha-\beta)\}.$$

15. We have now considered the limiting forms of $P_n^m(\mu)$ in three cases when

(i) μ approaches -1, with n, m fixed
(ii) m approaches ∞, with n, μ fixed
(iii) n approaches ∞, with m, μ fixed.

Two important cases remain, in which one only of the variables is held fast, while the other two vary, subject to a constant relation.

First, let n tend to infinity, θ to zero, while the product $n\theta$ remains constant $=\lambda$, suppose.

Then from the definition of P, we obtain, in a well-known way,

$$\text{Limit } n^m P_n^m(\cos\theta) = J_m(\lambda).$$

Supposing now, that not merely modulus n, but the *imaginary* part of n tends to infinity (remaining positive), we have from 10 (ii)

$$\Pi(m+n)\Pi(m-n-1)\sin m\pi P_n^m(-\mu)$$
$$=\pi P_n^{-m}(\mu)+\sin n\pi\,\Pi(m+n)\Pi(m-n-1)P_n^m(\mu)$$
$$=\pi P_n^{-m}(\mu)-\frac{\pi\sin n\pi}{\sin(n-m)\pi}\;\frac{\Pi(n+m)}{\Pi(n-m)}\;P_n^m(\mu)$$

or since $\Pi(n+m)=n^m\Pi n$ approximately

$$n^{-m}\Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)$$
$$=\frac{\pi}{\sin m\pi}\{n^{-m}P_n^{-m}(\mu)-e^{-im\pi}n^m P_n^m(\mu)\}$$

or Limit $n^{-m}\Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)$

$$=\frac{\pi}{\sin m\pi}\{J_{-m}(\lambda)-e^{-im\pi}J_m(\lambda)\}$$
$$=2G_m(\lambda),$$

where the imaginary part of λ is positive.

Second, while θ remains fixed, let m and n tend to infinity, so that $n-m=p$, a given positive integer.

Since $P_n^m(\mu)=\dfrac{1}{2^m\Pi m}(1-\mu^2)^{\frac{m}{2}}F\left(m-n,\ m+n+1,\ m+1,\ \dfrac{1-\mu}{2}\right)$

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we have, in this case

$$P_n^m(\mu) = \frac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}}\left\{1 - \frac{p}{1}(1-\mu) + \frac{p(p-1)}{1.2}(1-\mu)^2, \text{ etc.}\right\}$$

approximately.

So that $\underset{m=\infty}{\text{Limit}} \quad P_{m+p}^m(\mu) = \frac{1}{2^m \Pi m}(1-\mu^2)^{\frac{m}{2}}\mu^p.$

Addition Theorem.

16. The distance D between two points (r, α, ϕ), (r', β, ϕ') is given by $D^2 = r^2 - 2rr'\cos\gamma + r'^2$, where

$$\cos\gamma = \cos\alpha \cos\beta + \sin\alpha \sin\beta \cos(\phi - \phi').$$

$P_n(-\cos\gamma)$ is, if $\gamma > 0$, a continuous periodic function of $\phi - \phi'$, which by Fourier's Theorem can be expanded in the form

$$P_n(-\cos\gamma) = A_0 + A_1\cos(\phi - \phi') + \ldots + A_m\cos m(\phi - \phi') + \ldots$$

with $\qquad A_m \cos m\phi' = \frac{1}{\pi}\int_0^{2\pi} P_n(-\cos\gamma)\cos m\phi d\phi.$

The definite integral may be determined as follows.

Consider the potential V, given by

$$V = r^n \cos m\phi \, P_n^m(\cos\theta)\, P_n^m(-\cos\alpha)\,; \quad 0 < \theta < \alpha$$

$$V = r^n \cos m\phi \, P_n^m(\cos\alpha)\, P_n^m(-\cos\theta)\,; \quad \alpha < \theta < \pi$$

where m is a positive integer, and $n = -\frac{1}{2} + \lambda i$, λ real.

This potential can be produced by a distribution of matter on the cone $\theta = \alpha$, whose density at r, α, ϕ is σ,

given by $4\pi\sigma = \frac{1}{r}\left|\frac{dV}{d\theta}\right| \begin{matrix}\theta = \alpha - 0 \\ \theta = \alpha + 0\end{matrix}$

$$= r^{n-1}\cos m\phi \begin{Bmatrix} P_n^m(-\cos\alpha)\frac{d}{d\theta}P_n^m(\cos\theta) \\ -P_n^m(\cos\alpha)\frac{d}{d\theta}P_n^m(-\cos\theta) \end{Bmatrix}\theta = \alpha.$$

$$= r^{n-1}\cos m\phi \,.\, \frac{2}{\sin\alpha} \,.\, \frac{1}{\Pi(m+n)\Pi(m-n-1)}$$

from 11 (i).

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The potential of this distribution at (r', β, ϕ') is

$$\iint \frac{\sigma dS}{D}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \cos m\phi d\phi \int_0^\infty \frac{2}{\Pi(m+n)\Pi(m-n-1)} \frac{r^n dr}{\sqrt{(r^2 - 2rr'\cos\gamma + r'^2)}}$$

$$= \frac{1}{2\pi} \frac{\Pi(n)\Pi(-n-1)(r')^n}{\Pi(m+n)\Pi(m-n-1)} \int_0^{2\pi} \cos m\phi \,.\, P_n(-\cos\gamma) d\phi,$$

from 6 (i).

But the potential is, if $a > \beta$, $r'^n \cos m\phi' P_n^m(\cos\beta) P_n^m(-\cos a)$

$$\therefore \quad A_m = \frac{1}{\pi}\int_0^{2\pi} \cos m\phi \, P_n(-\cos\gamma) d\phi / \cos m\phi'$$

$$= 2\frac{\Pi(m+n)\Pi(m-n-1)}{\Pi n \,.\, \Pi(-n-1)} P_n^m(\cos\beta) P_n^m(-\cos a),$$

or half of this, if $m = 0$.

Hence

(1) $P_n(-\cos\gamma)$

$$= P_n(\cos\beta)P_n(-\cos a) + 2\sum_{m=1} c_m P_n^m(\cos\beta) P_n^m(-\cos a)\cos m(\phi - \phi').$$

where $a > \beta$; $c_m = \dfrac{\Pi(m+n)\Pi(m-n-1)}{\Pi n \,.\, \Pi(-n-1)}$

$$= (-)^m (n+m)(n+m-1)\ldots(n-m+1).$$

For m very large,

$$c_m P_n^m(\cos\beta) P_n^m(-\cos a) = \frac{1}{m}\left(-\frac{\sin n\pi}{\pi}\right)\tan^m\frac{\beta}{2}\cot^m\frac{a}{2}.$$

The series therefore converges absolutely if $a > \beta$; it still converges if $a = \beta$, provided $\phi - \phi'$ is not zero.

The series represents a function of n, which for a continuous range of values of n, has been proved equal to the function $P_n(-\cos\gamma)$; the series therefore represents this function for all values of n.

(2) Now change a into $\pi - a$, $\phi - \phi'$ into $\pi - (\phi - \phi')$; we obtain

$$P_n\{\cos a \cos\beta + \sin a \sin\beta \cos\overline{\phi - \phi'}\}$$

$$= 2\sum_m{}' (-)^m c_m P_n^m(\cos a) P_n^m(\cos\beta)\cos m(\phi - \phi'),$$

the restriction becoming $\pi > a + \beta$.

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(3) Again change β into $\pi-\beta$, $\phi-\phi'$ into $\pi-(\phi-\phi')$;

hence $$P_n\{\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\overline{\phi-\phi'}\}$$
$$=2\sum_m{}'(-)^m c_m P_n^m(-\cos\alpha)P_n^m(-\cos\beta)\cos m(\phi-\phi');$$

with the restriction $\pi<\alpha+\beta$.

Of these three forms, the one originally obtained is the most important.

It is to be observed that when n is a positive integer,

$$\text{and} \quad m>n, \quad c_m=0.$$

Many formulæ are shortened by the introduction of an additional function S, defined by the equation

$$S_n^m(\mu)\equiv\Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)$$

The equation (1), for instance, may be written

(4) $$S_n(\cos\gamma)=2\sum_m{}' S_n^m(\cos\alpha)\,.\,P_n^m(\cos\beta)\cos m(\phi-\phi').$$
$$\alpha>\beta.$$

17. Addition Theorems involving harmonics of pure imaginary rank.

The well-known addition theorem for the Bessel function G is

(1) $$G_0(\lambda R)=G_0\lambda a\,.\,J_0\lambda b+2\Sigma G_m\lambda a\,.\,J_m\lambda b\,.\,\cos m(\phi-\phi')$$
$$=2\sum_m{}' G_m\lambda a\,.\,J_m\lambda b\,.\,\cos m(\phi-\phi');$$

where $R=\sqrt{\{a^2+b^2-2ab\cos(\phi-\phi')\}}$; and a, b are real and positive with $a>b$.

The series converges absolutely, and the theorem holds, whatever be the phase of λ. Hence taking the increment of the two members of the equation when the phase of λ increases by 2π, we deduce the addition theorem for J,

(2) $$J_0(\lambda R)=2\sum_m{}' J_m\lambda a J_m\lambda b\cos m(\phi-\phi').$$

The series for $G_0(i\lambda R)$ may be transformed into an integral as follows.

Consider the function of m,

$$\frac{\cos m(\pi-\phi+\phi')}{\sin m\pi}G_m i\lambda a\,.\,J_m i\lambda b,$$

in which we suppose a, b, λ real and positive, and

$$0<\phi-\phi'<2\pi.$$

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Then $\cos m(\pi - \phi + \phi')/\sin m\pi$ vanishes for m infinite, unless m is real. Also (§ 3) for a large m,

$$i^{-m}J_m i\lambda b = \frac{(\lambda b)^m}{2^m \Pi m}$$

$$i^m \; G_m i\lambda a = \frac{\Pi m}{2m}\left(\frac{2}{\lambda a}\right)^m, \text{ if real part of } m \text{ be positive,}$$

and therefore $\quad G_m i\lambda a \,.\, J_m i\lambda b = \dfrac{1}{2m}\left(\dfrac{b}{a}\right)^m$

For a pure imaginary m, modulus $G_m i\lambda a \,.\, J_m i\lambda b = 1/\text{modulus } m$. Hence $G_m i\lambda a \,.\, J_m i\lambda b \,.\, \cos m(\pi - \phi + \phi')/\sin m\pi$ vanishes for m infinite with real part positive or zero. (The vanishing is *effective*, by which we mean that, even when multiplied by m, the function tends to zero; a condition sufficient to ensure the evanescence of the integral of the function taken over any part of the circle at an infinite distance. But in cases like this, we often, for the sake of brevity, omit to call attention to this point explicitly, as it cannot fail to be observed.)

Take now a path of integration from S to N in the m plane. This path must not pass through O, which is a pole of the function; we therefore describe a small circle about O as centre, and take a path from S to N along the axis of imaginaries, but passing O by the semicircle, first, on the eastern side, second, on the western side.

The first integral

$$= (-2\pi i) \text{ (sum of residues at poles to the right of O).}$$

The second integral

$$= (-2\pi i) \text{ (sum of the same residues + residue at O).}$$

Since $i^m G_m i\lambda a$ is an even function of m, we may write the sum of the integrals over the straight portions in the form

$$2\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi}(i^m G_m i\lambda a)(i^{-m}J_m i\lambda b - i^m J_{-m} i\lambda b)dm,$$

path from the north point of the small circle to N. We now diminish the small circle indefinitely, so that its contributions to the integrals neutralize each other, and obtain

$$2\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi} G_m i\lambda a(-e^{im\pi})\frac{2\sin m\pi}{\pi} G_m i\lambda b dm \,\Big|\, \text{ON}$$

$$= -2\pi i \,.\, \frac{2}{\pi} \,.\, \sum_m{}' \cos m(\phi - \phi') G_m i\lambda a \, J_m i\lambda b$$

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or

(3) $$G_0(i\lambda R) = \frac{2}{\pi}\int_0^\infty \cosh s(\pi - \phi + \phi')e^{-s\pi}G_{is}i\lambda a \,.\, G_{is}i\lambda b \,.\, ds.$$

$o < \phi - \phi' < 2\pi$; λ, a, b real and positive.

Similarly may be treated the expansion of $S_n(\cos\gamma)$, 16 (4). For, when m is large

$$S_n^m\cos\alpha \,.\, P_n^m\cos\beta = \frac{1}{m}\left(\tan\frac{\beta}{2}\Big/\tan\frac{\alpha}{2}\right)^m,$$

which vanishes when the real part of m is positive, and $\alpha > \beta$.

We find, just as above,

(4) $S_n(\cos\gamma)$

$$= i\int_0^\infty \frac{\cosh s(\pi - \phi + \phi')}{\sinh s\pi}(S_n^{is}\cos\alpha \,.\, P_n^{is}\cos\beta - S_n^{-is}\cos\alpha \,.\, P_n^{-is}\cos\beta)ds$$

$o < \phi - \phi' < 2\pi$.

The factor of the integrand involving the S and P functions does not appear to be expressible in the form of the product of a function of α by the same function of β ; but it is important to observe that this factor is really symmetrical in α and β.

For (§ 12) $S_n^m k \,.\, P_n^m h - S_n^m h \,.\, P_n^m k$ is even in m,

that is, equal to $S_n^{-m}k\, P_n^{-m}h - S_n^{-m}h\, P_n^{-m}k$.

Hence, transposing, we have

$$S_n^m k\, P_n^m h - S_n^{-m}k\, P_n^{-m}h = S_n^m h\, P_n^m k - S_n^{-m}h\, P_n^{-m}k.$$

Green's Function.

We proceed now to obtain Green's function in terms of harmonic potentials, for spaces bounded by surfaces of the cylindrical and spherical polar systems.

The bounding surfaces are complete, or, at any rate, if they be supposed completed, they do not intersect the space considered. It has not been thought necessary to give the solution for every variety of space satisfying this condition, but the cases omitted can easily be treated after the same manner as the others.

The coordinates of the pole are expressed by accented, and of the variable point by unaccented letters; these two points may, it is well known, be interchanged without altering the value of the function.

The reciprocal of the distance between the points is denoted by T, and the Green's function by V.

Cylindrical Coordinates.

$$z,\ \rho,\ \phi;\quad \mathrm{R} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi')}.$$

18. *Whole of space.* The Green's function is simply T.

(1) From 8

$$\mathrm{T} = \int_0^\infty e^{-\lambda(z-z')} \mathrm{J}_0 \lambda \mathrm{R} \, . \, d\lambda$$

$$= 2\int_0^\infty e^{-\lambda(z-z')} d\lambda \sum_m{}' \mathrm{J}_m \lambda\rho \, . \, \mathrm{J}_m \lambda\rho' . \cos m(\phi - \phi').$$

$$z > z'$$

(2) From 9 (iii)

$$\mathrm{T} = \frac{2}{\pi}\int_0^\infty \cos\lambda(z - z') \mathrm{G}_0 i\lambda \mathrm{R} \, . \, d\lambda$$

$$= \frac{4}{\pi}\int_0^\infty \cos\lambda(z - z') d\lambda \sum_m{}' \mathrm{G}_m i\lambda\rho \, . \, \mathrm{J}_m i\lambda\rho' . \cos m(\phi - \phi').$$

$$\rho > \rho'.$$

(3) The above forms for T are familiar; the first, which we may call the z form, is formally discontinuous at the z surface through the pole, the second, or ρ form, at the ρ surface through the pole. The third, or ϕ form, does not seem to have been given before. It is, from 18 (3)

$$\mathrm{T} = \frac{2}{\pi}\int_0^\infty \cos\lambda(z - z') \mathrm{G}_0 i\lambda \mathrm{R} \, . \, d\lambda$$

$$= \frac{4}{\pi^2}\int_0^\infty \cos\lambda(z - z') d\lambda \int_0^\infty \cosh s(\pi - \phi + \phi') e^{-s\pi} \mathrm{G}_{is} i\lambda\rho \, . \, \mathrm{G}_{is} i\lambda\rho' \, ds.$$

$$0 < \phi - \phi' < 2\pi.$$

19. *Space bounded by two parallel planes*

$$z = 0, \quad z = c > 0.$$

(1) The Green's function is obtained by adding to T a potential non-singular throughout the space, and equal to $(-\mathrm{T})$ on the boundary.

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Using the z form for T, we see that this complementary potential must take the values

$$-\int_0^\infty e^{-\lambda(c-z')} J_0 \lambda R d\lambda \text{ on } z=c, \text{ and}$$

$$-\int_0^\infty e^{-\lambda z'} \quad J_0 \lambda R d\lambda \text{ on } z=0.$$

Such a potential is

$$-\int_0^\infty \left\{ \frac{\sinh \lambda z}{\sinh \lambda c} e^{-\lambda(c-z')} + \frac{\sinh \lambda(c-z)}{\sinh \lambda c} e^{-\lambda z'} \right\} J_0 \lambda R d\lambda,$$

for the integral converges absolutely so long as

$z+z'-2c<0$, $z+z'>0$, conditions necessarily fulfilled.

To obtain V, add T, which when $z>z'$, is

$$\int_0^\infty e^{-\lambda(z-z')} J_0 \lambda R d\lambda.$$

Thus $$V = 2\int_0^\infty \frac{\sinh \lambda(c-z) \sinh \lambda z'}{\sinh \lambda c} J_0 \lambda R \,.\, d\lambda\,;\ z>z'.$$

For $z<z'$, interchange z and z'.

There may seem to be a certain loss of simplicity in uniting the part of the Green's function which contains the singularity, namely, T, to the other part of the function (often, by itself, with sign changed, called the Green's function) which is non-singular. This, however, is an essential step in the transformations we propose to make. Moreover, the form given for V has the advantage of showing at a glance that V vanishes for $z=c$, and (since for $z<z'$, we interchange z and z') for $z=0$.

(2) Since $J_0(\lambda R) = \frac{1}{\pi i}\{G_0(\lambda R) - G_0(-\lambda R)\}$, $R>0$,

we may write

$$V = \frac{2}{\pi i}\int \frac{\sinh \lambda(c-z) \,.\, \sinh \lambda z'}{\sinh \lambda c} G_0 \lambda R \,.\, d\lambda \,\Big|\, \text{WOE},$$

the function multiplying $G_0 \lambda R$ being *odd* in λ.

The integrand effectively vanishes in the upper part of the λ plane, and therefore the integral equals $2\pi i$ into sum of residues at the poles on ON.

Hence $$V = \frac{4}{c} \sum_p \sin p\pi z/c \,.\, \sin p\pi z'/c \,.\, G_0 i p\pi R/c.$$

$$R>0.$$

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This series converges with striking rapidity, except very near the pole, the rapidity increasing with R.

Since $G_0 ix = \sqrt{\pi/2x}\, e^{-x}$ for a large real x we have, for the limiting form

$$V = \frac{4}{c}\sin\frac{\pi z}{c} \cdot \sin\frac{\pi z'}{c} \cdot \sqrt{\frac{c}{2R}}\, e^{-\pi R/c},$$

a sufficient approximation if R is not less than, say, four or five times c.

Using the ρ form for $G_0 i\lambda R$, 17 (2),

$$V = 8/c \sum_p \sin\lambda z \sin\lambda z' \sum_m{}' G_m i\lambda\rho \,.\, J_m i\lambda\rho' \,.\, \cos m(\phi - \phi').$$

$$\lambda = p\pi/c, \quad \rho > \rho'.$$

(3) Using the ϕ form for $G_0 i\lambda R$, 17 (3),

$$V = 8/\pi c \,.\, \sum_p \sin\lambda z \sin\lambda z' \int_0^\infty e^{-s\pi} \cosh s(\pi - \phi + \phi') G_{iu} i\lambda\rho \,.\, G_{iu} i\lambda\rho' ds.$$

20. *Space bounded externally by a cylinder*

$$\rho = a.$$

(1) The ρ form for T is

$$T = \frac{4}{\pi}\int_0^\infty \cos\lambda(z - z')d\lambda \sum_m{}' G_m i\lambda\rho \,.\, J_m i\lambda\rho' \cos m(\phi - \phi').$$

$$\rho > \rho'$$

Hence

$$V = \frac{4}{\pi}\int_0^\infty \cos\lambda(z - z')d\lambda \,.\, \sum_m{}' \frac{J_m i\lambda\rho'}{J_m i\lambda a} \,.$$

$$(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a)\cos m(\phi - \phi').$$

$$\rho > \rho'.$$

(2) In the ρ form for T we may interchange the order summation, integration; or, in other words, integrate term by term. The proof is in this case easy, for

$$G_m i\lambda\rho\, J_m i\lambda\rho' = (i^m G_m i\lambda\rho)(i^{-m} J_m i\lambda\rho'),$$

the product of two factors constantly positive, so that the modulus of the elementary term

$$\cos\lambda(z - z') G_m i\lambda\rho\, J_m i\lambda\rho' \cos m(\phi - \phi')$$

is not greater than $\quad G_m i\lambda\rho\, J_m i\lambda\rho'$.

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But $\int_0^\infty d\lambda \sum_m' G_m i\lambda\rho\, J_m i\lambda\rho'$

$= \int_0^\infty d\lambda \,.\, G_0 i\lambda(\rho - \rho')$, which is finite, so that the sum-integral converges absolutely,

The same applies to the sum-integral that has been added to T to obtain V.

Hence $V = 4/\pi \sum_m' \cos m(\phi - \phi')$.

$$\int_0^\infty \cos\lambda(z - z')\frac{J_m i\lambda\rho'}{J_m i\lambda a}(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a)d\lambda.$$

The integrand here is a *uniform*, *even*, function of λ, so that the integral may be written

$$\frac{1}{2i}\int e^{-\lambda(z-z')}\frac{J_m\lambda\rho'}{J_m\lambda a}(J_m\lambda a\, G_m\lambda\rho - J_m\lambda\rho\, G_m\lambda a)d\lambda \Big|\ \mathrm{SN}.$$

If $z > z'$ and $\rho > \rho'$, the integrand vanishes at infinity in the eastern half of the λ plane. The infinities are simple poles at the zeros of $J_m\lambda a$. The integral therefore equals $-2\pi i$ (residues to right of O), and

$$V = 4/a \sum_m' \cos m(\phi - \phi') \sum_\lambda e^{-\lambda(z-z')} J_m\lambda\rho\, J_m\lambda\rho'\, G_m\lambda a / J'_m\lambda a$$

$$= 4/a^2 \sum_m' \cos m(\phi - \phi') \sum_\lambda e^{-\lambda(z-z')} J_m\lambda\rho\, J_m\lambda\rho' / \overline{\lambda(J'_m\lambda a)}^2,$$

using the relation $G_m\lambda a\, J'_m\lambda a - J_m\lambda a\, G'_m\lambda a = 1/\lambda a$.

Here the λ's are the positive zeros, in ascending order, of

$J_m\lambda a$, and $z > z'$.

(3) The ϕ form for V is deduced from the form (1) by a process similar to that used in proving 17 (3).

Take the complex integral

$$\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi}\frac{J_m i\lambda\rho'}{J_m i\lambda a}(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a)dm$$

along the two paths from S to N used in 17.

For a very large m, the factors of the integrand involving the Bessels approximate to

$$\left(\frac{\rho'}{a}\right)^m \frac{\pi}{2\sin m\pi}\left\{\left(\frac{a}{\rho}\right)^m - \left(\frac{\rho}{a}\right)^m\right\}\frac{1}{\Pi m \,.\, \Pi(-m)}$$

$$= \frac{1}{2m}\left\{\left(\frac{\rho'}{\rho}\right)^m - \left(\frac{\rho\rho'}{a^2}\right)^m\right\},$$

which vanishes on the eastern side of the plane if $\rho > \rho'$.
The other factor vanishes if $0 < \phi - \phi' < 2\pi$.
Under these conditions, the sum of the two integrals is

$$\frac{2}{\pi}(-2\pi i)\sum_m{}' \frac{J_m i\lambda\rho'}{J_m i\lambda a}(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a)\cos m(\phi - \phi').$$

But that sum is

$$2\int \frac{\cos m(\pi - \phi + \phi')}{\sin m\pi}\left(\frac{J_m i\lambda\rho'}{J_m i\lambda a} - \frac{J_{-m} i\lambda\rho'}{J_{-m} i\lambda a}\right)$$

$$[J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a]dm \;\Big|\; \text{ON},$$

the function in square brackets being *even* in m.
Hence, from (1) and using the definition of G,

$$V = \frac{4}{\pi^2}\int_0^\infty \cos\lambda(z - z')d\lambda$$

$$\int_0^\infty \left\{\begin{aligned}&\cosh s(\pi - \phi + \phi')(J_{is} i\lambda a\, G_{is} i\lambda\rho' - J_{is} i\lambda\rho'\, G_{is} i\lambda a).\\ &\quad (J_{is} i\lambda a\, G_{is} i\lambda\rho - J_{is} i\lambda\rho\, G_{is} i\lambda a)ds/(J_{is} i\lambda a\, J_{-is} i\lambda a)\end{aligned}\right\}$$

$$0 < \phi - \phi' < 2\pi.$$

21. *Space bounded by two axial planes*

$$\phi = 0, \;\; \phi = \alpha > 0.$$

(1) Starting with the ϕ form for T, 18 (3),
and observing that the factor of the integrand involving ϕ is

$$\cosh s(\pi - \phi + \phi') \quad \text{when} \quad \phi > \phi',$$

$$\text{but} \quad \cosh s(\pi - \phi' + \phi) \quad \text{when} \quad \phi < \phi',$$

we proceed as in 19 (1), and obtain

$$V = \frac{8}{\pi^2}\int_0^\infty \cos\lambda(z - z')d\lambda.$$

$$\int_0^\infty e^{-s\pi}(\sinh s\pi/\sinh s\alpha)\sinh s(\alpha - \phi)\sinh s\phi'\, G_{is} i\lambda\rho\, G_{is} i\lambda\rho' ds.$$

$$\phi > \phi'.$$

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(2) Bearing in mind that $i^s G_s$ is an even function of s, we write the second integral

$$\int (\sin s\pi/\sin s\alpha)\sin s(\alpha-\phi)\sin s\phi'(i^s G_s i\lambda\rho)(i^s G_s i\lambda\rho')\, i\, ds \,\Big|\, \mathrm{ON}$$

$$= -\frac{\pi i}{2}\int \frac{\sin s(\alpha-\phi)\sin s\phi'}{\sin s\alpha} G_s i\lambda\rho\, J_s i\lambda\rho'\, ds \,\Big|\, \mathrm{SN}$$

$$= (-2\pi i)\ (\text{residues to right of O}),\ \text{if } \rho>\rho'.$$

Hence

$$V = \frac{8}{\alpha}\int_0^\infty \cos\lambda(z-z')\,d\lambda\, \sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha} G_{\frac{m\pi}{\alpha}} i\lambda\rho\, J_{\frac{m\pi}{\alpha}} i\lambda\rho'.$$

$$\rho>\rho'.$$

(3) The sum-integral just written converges absolutely, and we may put (writing s for $m\pi/\alpha$)

$$V = \frac{8}{\alpha}\sum_m \sin s\phi \,.\, \sin s\phi' \int_0^\infty \cos\lambda(z-z') G_s i\lambda\rho\, J_s i\lambda\rho'\, d\lambda.$$

If $z>z'$, $\rho>\rho'$, the integral is

$$\frac{1}{2i}\int \{e^{\lambda(z-z')} + e^{-\lambda(z-z')}\} G_s\lambda\rho \,.\, J_s\lambda\rho'\, d\lambda \,\Big|\, \mathrm{ON}$$

$$= \frac{1}{2i}\int e^{\lambda(z-z')} G_s\lambda\rho\, J_s\lambda\rho'\, d\lambda \,\Big|\, \mathrm{OW}$$

$$+ \frac{1}{2i}\int e^{-\lambda(z-z')} G_s\lambda\rho\, J_s\lambda\rho'\, d\lambda \,\Big|\, \mathrm{OE}$$

$$= \frac{1}{2i}\int e^{-\lambda(z-z')} J_s\lambda\rho' \{G_s\lambda\rho - e^{is\pi} G_s(-\lambda\rho)\}\, d\lambda \,\Big|\, \mathrm{OE}$$

$$= \frac{\pi}{2}\int_0^\infty e^{-\lambda(z-z')} J_s\lambda\rho\, J_s\lambda\rho'\, d\lambda. \quad (\S 1.)$$

Substitution in the form (2) gives

$$V = \frac{4\pi}{\alpha}\sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\int_0^\infty e^{-\lambda(z-z')} J_{\frac{m\pi}{\alpha}}\lambda\rho \,.\, J_{\frac{m\pi}{\alpha}}\lambda\rho' \,.\, d\lambda.$$

$$z>z'.$$

22. *Space bounded externally by two parallel planes, and a cylinder,* $z=0, \quad z=c, \quad \rho=a.$

(1) From 19 (2)

$$V=\frac{8}{c}\sum_p \sin\lambda z \,.\, \sin\lambda z'.$$
$$\sum_m{}' \frac{J_m i\lambda\rho'}{J_m i\lambda a}(J_m i\lambda a\, G_m i\lambda\rho - J_m i\lambda\rho\, G_m i\lambda a)\cos m(\phi-\phi').$$
$$\lambda=p\pi/c, \quad \rho>\rho'.$$

(2) From 20 (2) by the same work as in 19 (1),

$$V=\frac{8}{a^2}\sum_m{}'\cos m(\phi-\phi')\sum_\lambda\frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c}J_m\lambda\rho \,.\, J_m\lambda\rho'/\overline{\lambda(J'_m\lambda a)^2}.$$

The λ's are the positive zeros of $J_m\lambda a$, $z>z'$.

(3) From (1), utilizing the work of 20 (3), we write down at once

$$V=\frac{8}{\pi c}\sum_p \sin\lambda z \,.\, \sin\lambda z'.$$
$$\int_0^\infty \cosh s(\pi-\phi+\phi')f(\rho)\,.\,f(\rho')ds/(J_{is}i\lambda a \,.\, J_{-is}i\lambda a),$$

where $f(\rho)=J_{is}i\lambda a\, G_{is}i\lambda\rho - J_{is}i\lambda\rho\, G_{is}i\lambda a,$

$$\lambda=p\pi/c, \quad 0<\phi-\phi'<2\pi.$$

23. *Space bounded by two parallel planes, and two axial planes* $z=0, \quad z=c, \quad \phi=0, \quad \phi=a.$

(1) Starting from 19 (3), the process of 21 (1) gives

$$V=\frac{16}{\pi c}\sum_p \sin\lambda z \,.\, \sin\lambda z'.$$
$$\int_0^\infty e^{-s\pi}(\sinh s\pi/\sinh sa)\sinh s(a-\phi)\sinh s\phi' \,.\, G_{is}i\lambda\rho \,.\, G_{is}i\lambda\rho' \,.\, ds$$
$$\lambda=p\pi/c, \quad \phi>\phi',$$

(2) From 21 (3),

$$V=\frac{8\pi}{a}\sum_m \sin s\phi \sin s\phi'\int_0^\infty\frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c}J_s\lambda\rho\, J_s\lambda\rho' d\lambda.$$
$$s=m\pi/a, \quad z>z'.$$

(3) From (1) as in 21 (2),

$$V=\frac{16\pi}{ca}\sum_p \sin\lambda z \sin\lambda z'.$$
$$\sum_m \sin s\phi \sin s\phi' \,G_s i\lambda\rho\, J_s i\lambda\rho'.$$
$$\lambda=p\pi/c, \quad s=m\pi/a, \quad \rho>\rho'.$$

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24. *Space bounded by two axial planes and a cylinder*

$$\phi = 0, \quad \phi = a, \quad \rho = a.$$

(1) From 21 (2),

$$V = \frac{8}{a}\int_0^\infty \cos\lambda(z-z')d\lambda.$$

$$\sum_m \sin s\phi \sin s\phi' (J_s i\lambda\rho'/J_s i\lambda a)(J_s i\lambda a\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda a),$$

$$s = m\pi/a, \quad \rho > \rho'.$$

(2) From 20 (3),

$$V = \frac{8}{\pi^2}\int_0^\infty \cos\lambda(z-z')d\lambda.$$

$$\int_0^\infty \frac{\sinh s\pi}{\sinh sa} \sinh s(a-\phi)\sinh s\phi' . f(\rho) f(\rho') . \frac{ds}{J_{is} i\lambda a\, J_{-is} i\lambda a}.$$

$f(\rho)$ as in 22 (3), $\phi > \phi'$.

(3) From (1) as in 20 (2),

$$V = \frac{8\pi}{a^2 a}\sum_m \sin s\phi \sin s\phi'.$$

$$\sum_\lambda e^{-\lambda(z-z')} J_s\lambda\rho . J_s\lambda\rho' / \overline{\lambda(J'_s\lambda a)}^2.$$

$s = m\pi/a$, λ a positive zero of $J_s\lambda a$, $z > z'$.

25. *Space bounded by two axial planes, two parallel planes, and a cylinder*

$$\phi = 0, \quad \phi = a, \quad z = 0, \quad z = c, \quad \rho = a.$$

(1) From 24 (3)

$$V = \frac{16\pi}{a^2 a}\sum_m \sin s\phi \sin s\phi'$$

$$\sum_\lambda \frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c} J_s\lambda\rho . J_s\lambda\rho' / \overline{\lambda(J_s'\lambda a)}^2.$$

$s = m\pi/a$, λ a positive zero of $J_s\lambda a$, $z > z'$.

(2) From 23 (3),

$$V = \frac{16\pi}{ca}\sum_p \sin\lambda z \sin\lambda z'.$$

$$\sum_m \sin s\phi \sin s\phi' (J_s i\lambda\rho'/J_s i\lambda a)(J_s i\lambda a\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda a).$$

$$\lambda = p\pi/c, \quad s = m\pi/a, \quad \rho > \rho'.$$

(3) In 22 (3), instead of $\cosh s(\pi - \phi + \phi')$,

write $\quad 2(\sinh s\pi/\sinh sa)\sinh s(a-\phi)\sinh s\phi'.$

$$\phi > \phi'.$$

26. *Space bounded by two parallel planes, two axial planes, and two cylinders*

$$z=0,\ z=c,\ \phi=0,\ \phi=\alpha,\ \rho=a,\ \rho=b,\ b>a.$$

Since this is the first occurrence of a boundary partly composed of *two* cylinders, we can deduce from preceding results only one of the forms for V. From this the other two are derived directly.

(1) Taking the function of 23 (3) we subtract from it a potential taking the same value on the cylinders and vanishing on the planes.

This potential is clearly obtained by writing for $G_s i\lambda\rho\, J_s i\lambda\rho'$ in 23 (3),

$$G_s i\lambda b\,.\,J_s i\lambda\rho'.\frac{J_s i\lambda a\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda a}{J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a}$$

$$+J_s i\lambda a\quad G_s i\lambda\rho'.\frac{J_s i\lambda\rho\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda\rho}{J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a},$$

for this expression is

$$G_s i\lambda b\, J_s i\lambda\rho' \text{ for } \rho=b, \text{ and } J_s i\lambda a\, G_s i\lambda\rho' \text{ for } \rho=a.$$

A slight reduction gives

$$V=\frac{16\pi}{ca}\sum_p \sin\lambda z\sin\lambda z'\sum_m \sin s\phi\sin s\phi'.$$

$$(J_s i\lambda a\, G_s i\lambda\rho' - J_s i\lambda\rho'\, G_s i\lambda a)(J_s i\lambda b\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda b)$$

$$\div(J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a).$$

$$\lambda=p\pi/c,\quad s=m\pi/\alpha,\quad \rho>\rho'.$$

(2) To deduce the z form from this, suppose $z>z'$, change the order of summation, and consider the function of λ

$$\frac{\sinh\lambda(c-z)\sinh\lambda z'}{\sinh\lambda c}\cdot\frac{(J_s\lambda a G_s\lambda\rho' - J_s\lambda\rho' G_s\lambda a)(J_s\lambda b G_s\lambda\rho - J_s\lambda\rho G_s\lambda b)}{J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a}.$$

This is a uniform, odd function of λ which, if $z>z'$, $\rho>\rho'$, vanishes for every infinite λ. Hence the total sum of the residues of the function vanishes. The infinities of the function are simple poles, namely, the (pure imaginary) zeros of $\sinh\lambda c$, and the (real) zeros of $J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a$.

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Substituting one series of residues for the other in (1) gives

$$V = -\frac{16\pi}{a}\sum_m \sin s\phi \sin s\phi' . \sum_\lambda \frac{\sinh \lambda(c-z)\sinh \lambda z'}{\sinh \lambda c} .$$

$$(J_s\lambda a\, G_s\lambda\rho' - J_s\lambda\rho'\, G_s\lambda a)(J_s\lambda b\, G_s\lambda\rho - J_s\lambda\rho\, G_s\lambda b)$$

$$\div \frac{d}{d\lambda}(J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a).$$

$s = m\pi/a$, the λ's are the positive zeros in order of

$$J_s\lambda a\, G_s\lambda b - J_s\lambda b\, G_s\lambda a, \quad z > z'.$$

(3) By a similar process, (1) yields the ϕ form also.

Consider the function of s

$$\frac{\sin s(a-\phi)\sin s\phi'}{\sin sa}\, \frac{(J_s i\lambda a\, G_s i\lambda\rho' - J_s i\lambda\rho'\, G_s i\lambda a)(J_s i\lambda b\, G_s i\lambda\rho - J_s i\lambda\rho\, G_s i\lambda b)}{J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a}.$$

This is a uniform, odd function of s, vanishing for every infinite s if $\phi > \phi'$, $\rho > \rho'$. The infinities are simple poles at the (real) zeros of $\sin sa$, and the (pure imaginary) zeros of $J_s i\lambda a\, G_s i\lambda b - J_s i\lambda b\, G_s i\lambda a$. Hence, as before,

$$V = -\frac{16\pi}{c}\sum_p \sin\lambda z \sin\lambda z' \sum_s \frac{\sinh s(a-\phi)\sinh s\phi'}{\sinh sa} .$$

$$(J_{is} i\lambda a\, G_{is} i\lambda\rho' - J_{is} i\lambda\rho'\, G_{is} i\lambda a)(J_{is} i\lambda b\, G_{is} i\lambda\rho - J_{is} i\lambda\rho\, G_{is} i\lambda b)$$

$$\div \frac{d}{ds}(J_{is} i\lambda a\, G_{is} i\lambda b - J_{is} i\lambda b\, G_{is} i\lambda a).$$

$\lambda = p\pi/c$, the s's are the positive zeros of

$$J_{is} i\lambda a\, G_{is} i\lambda b - J_{is} i\lambda b\, G_{is} i\lambda a, \quad \phi > \phi'.$$

In the forms (2) and (3) it should be noticed that the expression for V is unaltered when ρ, ρ' are interchanged, on account of the equation defining λ or s.

These forms show very strikingly how the Bessels of *real* factor and rank bear the same sort of relation to the parallel planes as the functions of *pure imaginary* factor and rank bear to the axial planes.

From the solutions of this paragraph all those that precede may, of course, be deduced, but the reduction is not always easy, nor its validity obvious.

SPHERICAL HARMONICS.

The spherical polar coordinates of the variable point are r, θ, ϕ; of the pole r', θ', ϕ'.

We write $r = ae^{\rho}$, $r' = ae^{\rho'}$, so that ρ runs from $-\infty$ to $+\infty$ as r runs from 0 to ∞, and $\rho = 0$ when $r = a$.

Also, as in 16, we write $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$, and, as usual, $\cos\theta = \mu$, $\cos\theta' = \mu'$.

27. *Whole of space.*

$$T = 1/\sqrt{(r^2 - 2rr'\cos\gamma + r'^2)}, \text{ which if } r > r'$$

$$= \frac{1}{r}\left\{P_0(\cos\gamma) + \frac{r'}{r}P_1(\cos\gamma) + \left(\frac{r'}{r}\right)^2 P_2(\cos\gamma) + \text{, etc.}\right\}$$

Multiplying by $\sqrt{rr'}$, this may be written

$$T\sqrt{rr'} = \sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}P_n(\cos\gamma)$$

$$= \sum_n (-)^n e^{-(n+\frac{1}{2})(\rho-\rho')}P_n(-\cos\gamma).$$

The point $n = -\frac{1}{2}$ in the plane of n we shall always denote by C; the path SCN is the straight path through C from

$$-\tfrac{1}{2} - \infty i \text{ to } -\tfrac{1}{2} + \infty i.$$

The approximate form of $P_n(-\cos\gamma)$ for a very large n is

$$\sqrt{\frac{2}{n\pi\sin\theta}}\cdot\cos\left\{(n+\tfrac{1}{2})(\pi-\gamma) - \frac{\pi}{4}\right\}. \quad (\S 14.)$$

Hence

$$\frac{\pi}{2\pi i}\int \frac{e^{-(n+\frac{1}{2})(\rho-\rho')}P_n(-\cos\gamma)}{\cos(n+\frac{1}{2})\pi}dn \,\Big|\, \text{SCN}$$

$$= (-2\pi i)\text{ (residues to right of C), if } \rho > \rho',$$

$$= T\sqrt{rr'}, \text{ or putting } n + \tfrac{1}{2} = i\lambda$$

$$T\sqrt{rr'} = \int_0^\infty \frac{\cos\lambda(\rho-\rho')P_{-\frac{1}{2}+i\lambda}(-\cos\gamma)d\lambda}{\cosh\lambda\pi}$$

(Heine II p. 219.

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')S_{-\frac{1}{2}+i\lambda}(\cos\gamma)d\lambda.$$

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In expanded form

(1) $$T\sqrt{rr'} = 2\sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}.$$

$$\sum_m{}'(-)^m \frac{\Pi(m+n)\Pi(m-n-1)}{\Pi n\,.\,\Pi(-n-1)} P_n^m\mu\,.\,P_n^m\mu' \cos m(\phi-\phi'),$$

the m summation stopping, of course, at $m=n$, $r>r'$.

(2) $$T\sqrt{rr'} = \frac{2}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda \sum_m{}' S_n^m\mu\,.\,P_n^m\mu'\,.\,\cos m(\phi-\phi').$$

$$n = -1/2 + i\lambda, \quad \theta > \theta'.$$

(3) $$T\sqrt{rr'} = \frac{i}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda.$$

$$\int_0^\infty \frac{\cosh s(\pi-\phi+\phi')}{\sinh s\pi}(S_n^{is}\mu\, P_n^{is}\mu' - S_n^{-is}\mu\, P_n^{-is}\mu')ds$$

from 18 (4). $n = -1/2 + i\lambda$, $0 < \phi - \phi' < 2\pi$.

28. *Space bounded by two spheres*

$$\rho = 0, \quad \rho = c.$$

(1) From $T\sqrt{rr'} = \sum_n e^{-(n+\frac{1}{2})(\rho-\rho')} P_n\cos\gamma$, $\rho>\rho'$, precisely as in 19,

$$V\sqrt{rr'} = 2\sum_n \frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c} P_n\cos\gamma.$$

(2) The function of n (odd in $n+\frac{1}{2}$),

$$\frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c\,.\,\cos(n+\frac{1}{2})\pi} P_n(-\cos\gamma),$$

where $\rho>\rho'$, $\gamma>0$, vanishes for every infinite n, and the total sum of its residues is *nil*, that is,

$$-\frac{2}{\pi}\sum_n \frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c}(-)^n P_n(-\cos\gamma)$$

$$+\frac{2}{c}\sum_p \frac{\sin\lambda\rho\sin\lambda\rho'}{\cosh\lambda\pi} P_{-\frac{1}{2}+i\lambda}(-\cos\gamma) = 0,$$

where $\lambda = p\pi/c$.

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Therefore from (1)

$$V\sqrt{rr'}=\frac{2\pi}{c}\sum_p \frac{\sin\lambda\rho\sin\lambda\rho'}{\cosh\lambda\pi}P_{-\frac{1}{2}+i\lambda}(-\cos\gamma),$$

or $$V\sqrt{rr'}=\frac{4}{c}\sum_p \sin\lambda\rho\sin\lambda\rho'\sum_m{}' S_n^m\mu\, P_n^m\mu' \,.\cos m(\phi-\phi').$$

$$\lambda=p\pi/c,\ \ n=-1/2+i\lambda,\ \ \theta>\theta'.$$

If c is a small fraction, that is, if the thickness of the spherical shell is small in comparison with its radius, the first term of the unexpanded form for $V\sqrt{rr'}$ will be a good approximation, γ not being a very small angle.

The limiting form, as c diminishes more and more, is

$$V\sqrt{rr'}=\frac{4}{\sqrt{2c\sin\gamma}}e^{-\pi\gamma/c}\sin\pi\rho/c\,.\sin\pi\rho'/c,$$

an extension of the result of 19.

(3) Substituting the ϕ form of $P(-\cos\gamma)$ in (2), we get

$$V\sqrt{rr'}=2i/c\sum_p \sin\lambda\rho\sin\lambda\rho'$$

$$\int_0^\infty \frac{\cosh s(\pi-\phi+\phi')}{\sinh s\pi}(S_n^{is}\mu\, P_n^{is}\mu' - S_n^{-is}\mu\, P_n^{-is}\mu')ds$$

$$\lambda=p\pi/c,\ \ n=-1/2+i\lambda,\ \ 0<\phi-\phi'<2\pi.$$

29. *Space bounded by a single cone*

$$\cos\theta=\beta.$$

(1) From the θ form of T

$$V\sqrt{rr'}=\frac{2}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda$$

$$\sum_m{}' \frac{P_n^m\mu'}{P_n^m\beta}(P_n^m\beta\, S_n^m\mu - P_n^m\mu S_n^m\beta)\cos m(\phi-\phi').$$

$$n=-1/2+i\lambda,\ \ \theta>\theta'.$$

(2) In (1) we may change the order of summation, integration. (Cf. 20.) Then

$$\frac{2}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')\frac{P\mu'}{P\beta}(P\beta\, S\mu - P\mu S\beta)d\lambda$$

$$=\frac{1}{\pi i}\int e^{-(n+\frac{1}{2})(\rho-\rho')}\frac{P_n^m\mu'}{P_n^m\beta}(P_n^m\beta\, S_n^m\mu - P_n^m\mu\, S_n^m\beta)dn \Big|\ \mathrm{SCN}$$

$=(-2\pi i)$ (residues to right of C), if $\theta>\theta'$, $\rho>\rho'$.

The poles of the integrand are the (real) zeros of $P_n^m\beta$.

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Hence

$$V\sqrt{rr'}=2\sum_m{}'\cos m(\phi-\phi')\sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}P_n^m\mu.P_n^m\mu' S_n^m\beta\Big/\frac{d}{dn}P_n^m\beta.$$

The n's are the zeros $>-\frac{1}{2}$ of $P_n^m\beta$, and $r>r'$.

(3) It has been shown in §12, 2 that

$$P_n^m\mu\, S_n^m\mu' - P_n^m\mu'\, S_n^m\mu \text{ is an even function of } m.$$

Hence from (1), by a process quite similar to that of 20 (3), and using the equation of 12 (2) we find

$$V\sqrt{rr'}$$

$$=\frac{1}{\pi^2}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda\int_0^\infty \cosh s(\pi-\phi+\phi')f(\theta)f(\theta')ds/(P_n^{is}\beta.P_n^{-is}\beta)$$

where $f(\theta)=P_n^{is}\beta\,.\,S_n^{is}\mu-P_n^{is}\mu\, S_n^{is}\beta,$

$$n=-1/2+i\lambda,\ 0<\phi-\phi'<2\pi.$$

30. *Space bounded by two axial planes*

$$\phi=0,\ \phi=a.$$

(1) From 27 (3), as in 21 (1),

$$V\sqrt{rr'}=\frac{2i}{\pi}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda.$$

$$\int_0^\infty \frac{\sinh s(a-\phi)\sinh s\phi'}{\sinh sa}(S_n^{is}\mu\, P_n^{is}\mu' - S_n^{-is}\mu\, P_n^{-is}\mu')ds.$$

$$n=-1/2+i\lambda,\ \phi>\phi'.$$

(2) The second integral can be expressed as

$$-\int\frac{\sin m(a-\phi)\sin m\phi'}{\sin ma}S_n^m\mu\,.\,P_n^m\mu'.\,dm\ \Big|\ \text{SON}$$

which if $\phi>\phi'$, $\theta>\theta'$ equals $2\pi i$ (residues of integrand to right of O). Thus

$$V\sqrt{rr'}=\frac{4}{a}\int_0^\infty \cos\lambda(\rho-\rho')d\lambda\sum_m \sin s\phi\sin s\phi'\, S_n^s\mu\, P_n^s\mu'.$$

$$n=-1/2+i\lambda,\ s=m\pi/a,\ \theta>\theta'.$$

(3) We may write (2)

$$V\sqrt{rr'}=\frac{4}{a}\sum_m \sin s\phi\sin s\phi'\int_0^\infty \cos\lambda(\rho-\rho')S_n^s\mu\, P_n^s\mu' d\lambda.$$

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The integral is equivalent to

$$\frac{1}{2i}\int e^{-(n+\frac{1}{2})(\rho-\rho')}\Pi(s+n)\Pi(s-n-1)P_n^s(-\mu)P_n^s\mu'.\,dn \Big| \text{SCN}.$$

The integrand vanishes at infinity to the right of C, provided

$$\rho > \rho', \ \theta > \theta'.$$

Since s is a real positive, the infinities to the right of C are the poles of $\Pi(s-n-1)$; for these,

$$n = s + p, \ p \text{ zero or a positive integer.}$$

Now $\Pi(-p-1) = -\pi/(\sin p\pi\,.\,\Pi p)$, so that the residue of $\Pi(-p-1)$ when p is an integer is $(-)^{p-1}/\Pi p$.
Also when p is an integer, $P_{s+p}^s(-\mu) = (-)^p P_{s+p}^s\mu$.
Hence the above integral equals

$$\pi \sum_p e^{-(s+p+\frac{1}{2})(\rho-\rho')}(\Pi\overline{2s+p}/\Pi p)P_{s+p}^s\mu\,.\,P_{s+p}^s\mu',$$

and

$$V\sqrt{rr'} = \frac{4\pi}{a}\sum_m \sin s\phi \sin s\phi'.$$

$$\sum_p e^{-(s+p+\frac{1}{2})(\rho-\rho')}(\Pi\overline{2s+p}/\Pi p)P_{s+p}^s\mu\,.\,P_{s+p}^s\mu'.$$

$$s = m\pi/a, \ r > r'.$$

Cf. Lord Kelvin and Tait, *Nat. Phil*, Vol. I. App. B. (m).

31. *Space bounded by two spheres and a cone*

$$\rho = a, \ \rho = c, \ \cos\theta = \beta.$$

(1) From 28 (2),

$$V\sqrt{rr'} = \frac{4}{c}\sum_p \sin\lambda\rho.\sin\lambda\rho' \sum_m{}' \frac{P_n^m\mu'}{P_n^m\beta}(P_n^m\beta\, S_n^m\mu - P_n^m\mu\, S_n^m\beta)\cos m(\phi-\phi').$$

$$\lambda = p\pi/c, \ n = -1/2 + i\lambda, \ \theta > \theta'.$$

(2) From 29 (2),

$$V\sqrt{rr'} = 4\sum_m{}' \cos m(\phi-\phi').$$

$$\sum_n \frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c}(P_n^m\mu\, P_n^m\mu'\, S_n^m\beta \Big/ \frac{d}{dn}P_n^m\beta.$$

The n's are the zeros $> -\frac{1}{2}$ of $P_n^m\beta$; $r > r'$.

(3) As, in 29, (3) is obtained from (1), so here from (1)

$$V\sqrt{rr'} = \frac{2}{\pi c}\sum_p \sin\lambda\rho\sin\lambda\rho'\int_0^\infty \cosh s(\pi - \phi + \phi')f(\theta)f(\theta')ds/(P_n^{is}\beta\, P_n^{-is}\beta)$$

$$\lambda = p\pi/c,\ \ n = -1/2 + i\lambda,\ \ f(\theta) = P_n^{is}\beta\, S_n^{is}\mu - P_n^{is}\mu\, S_n^{is}\beta,$$

$$0 < \phi - \phi' < 2\pi.$$

32. *Space bounded by two spheres and two axial planes*

$$\rho = 0,\ \ \rho = c,\ \ \phi = 0,\ \ \phi = a.$$

(1) From 28 (3),

$$V\sqrt{rr'} = \frac{4i}{c}\sum_p \sin\lambda\rho\,\sin\lambda\rho'$$

$$\int_0^\infty \frac{\sinh s(a-\phi)\sinh s\phi'}{\sinh sa}(S_n^{is}\mu\, P_n^{is}\mu' - S_n^{-is}\mu\, P_n^{-is}\mu')ds$$

$$\lambda = p\pi/c,\ \ n = -1/2 + i\lambda,\ \ 0 < \phi - \phi' < 2\pi.$$

(2) From 30 (3),

$$V\sqrt{rr'} = \frac{8\pi}{a}\sum_m \sin s\phi\,.\,\sin s\phi'.$$

$$\sum_p (\Pi\overline{2s+p}/\Pi p)\frac{\sinh(s+p+\frac{1}{2})(c-\rho)\sinh(s+p+\frac{1}{2})\rho'}{\sinh(s+p+\frac{1}{2})c}P_{s+p}^{s}\mu\,.\,P_{s+p}^{s}\mu'.$$

$$s = m\pi/a,\ \ r > r'.$$

(3) From (1) as in 30 (2),

$$V\sqrt{rr'} = \frac{8\pi}{ca}\sum_p \sin\lambda\rho\,\sin\lambda\rho',$$

$$\sum_m \sin s\phi\,\sin s\phi'\,.\,S_n^{s}\mu\, P_n^{s}\mu'$$

$$s = m\pi/a,\ \ n = -1/2 + i\lambda,\ \ \lambda = p\pi/c,\ \ \theta > \theta'.$$

33. *Space bounded by two axial planes and cone*

$$\phi = 0,\ \ \phi = a,\ \ \cos\theta = \beta.$$

(1) From 30 (2),

$$V\sqrt{rr'} = \frac{4}{a}\int_0^\infty \cos\lambda(\rho - \rho')d\lambda.$$

$$\sum_m \sin s\phi\,.\,\sin s\phi'\,.\,\frac{P_n^m\mu'}{P_n^m\beta}(P_n^m\beta\, S_n^m\mu - P_n^m\mu\, S_n^m\beta).$$

$$s = m\pi/a,\ \ n = -1/2 + i\lambda,\ \ \theta > \theta'.$$

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(2) In 29 (3) replace $\cosh s(\pi-\phi+\phi')$ by

$$2\sinh s(a-\phi)\sinh s\phi' . \sinh s\pi/\sinh sa.$$

$$\phi>\phi'.$$

(3) From (1) as in 29 (2),

$$V\sqrt{rr'}=\frac{4\pi}{a}\sum_m \sin s\phi . \sin s\phi'.$$

$$\sum_n e^{-(n+\frac{1}{2})(\rho-\rho')}P_n^s\mu\, P_n^s\mu' \Big/ \frac{d}{dn}P_n^s\beta.$$

$$s=m\pi/a,\ n \text{ a zero} > -\tfrac{1}{2} \text{ of } P_n^s\beta,\ r>r'.$$

34. *Space bounded by two spheres, two axial planes, and a cone*

$$\rho=0,\ \rho=c,\ \phi=0,\ \phi=a,\ \cos\theta=\beta.$$

(1) In 33 (3) change $e^{-(n+\frac{1}{2})(\rho-\rho')}$ into

$$2\sinh(n+\tfrac{1}{2})(c-\rho)\sinh(n+\tfrac{1}{2})\rho'/\sinh(n+\tfrac{1}{2})c.$$

$$r>r'.$$

(2) In 32 (3) change $S_n^s\mu\, P_n^s\mu'$ into

$$(P_n^s\mu'/P_n^s\beta)(P_n^s\beta\, S_n^s\mu - P_n^s\mu\, S_n^s\beta)$$

$$\theta>\theta'.$$

(3) In 31 (3) change $\cosh s(\pi-\phi+\phi')$ into

$$2\sinh s(a-\phi)\sinh s\phi' \sinh s\pi/\sinh sa.$$

$$\phi>\phi'.$$

35. *Space bounded by two spheres, two axial planes, and two cones*

$$\rho=0,\ \rho=c,\ \phi=0,\ \phi=a\ \ \mu=\beta,\ \mu=\delta,\ \beta>\delta.$$

(1) Start from the θ form of 32. Comparison with the corresponding case of the cylindrical system enables us to write down at once

$$V\sqrt{rr'}=\frac{8\pi}{ca}\sum_p \sin\lambda\rho \sin\lambda\rho' \sum_m \sin s\phi \sin s\phi'.$$

$$(P\beta S\mu' - P\mu' S\beta)(P\delta S\mu - P\mu S\delta)/(P\beta . S\delta - P\delta S\beta).$$

Here the harmonics are of degree $-1/2+i\lambda$ and rank s;

$$\lambda=p\pi/c,\ s=m\pi/a,\ \theta>\theta'.$$

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(2) In order to deduce the r form, change the order of summation, and consider the function of n,

$$\sinh(n+\tfrac{1}{2})(c-\rho)\sinh(n+\tfrac{1}{2})\rho'/\sinh(n+\tfrac{1}{2})c$$
$$\cdot\,(P\beta S\mu'-P\mu' S\beta)(P\delta S\mu-P\mu S\delta)/(P\beta S\delta-P\delta\,.\,S\beta),$$

the harmonics being of degree n and rank s.

If $\rho>\rho'$, $\theta>\theta'$, this function, which is odd in $n+1/2$, vanishes for every infinite n; the total sum of its residues is therefore *nil*. The infinities are simple poles, namely, the (pure imaginary) zeros of $\sinh(n+1/2)c$ and the (real) zeros of $P\beta\, S\delta - P\delta\, S\beta$.

Hence from (1)

$$V\sqrt{rr'} = -\frac{8\pi}{a}\sum_m \sin s\phi \sin s\phi' \sum_n \frac{\sinh(n+\frac{1}{2})(c-\rho)\sinh(n+\frac{1}{2})\rho'}{\sinh(n+\frac{1}{2})c}.$$
$$(P\beta S\mu'-P\mu' S\beta)(P\delta S\mu-P\mu S\delta)\Big/\frac{d}{dn}(P\beta S\delta-P\delta S\beta).$$

The harmonics are of degree n, rank s;

$s=m\pi/a$; n is a zero $>-1/2$ of the function of n,

$$P\beta S\delta-P\delta S\beta\,;\ r>r'.$$

(3) Taking (1) as it stands, consider the function of s,

$$\sin s(a-\phi)\sin s\phi'/\sin sa$$
$$\cdot\,(P\beta S\mu'-P\mu' S\beta)(P\delta S\mu-P\mu S\delta)/(P\beta S\delta-P\delta S\beta),$$

the harmonics being of degree $-1/2+i\lambda$, rank s.

If $\phi>\phi'$, $\theta>\theta'$, this function, which is odd in s, vanishes for every infinite s, and has simple poles at the (pure imaginary) zeros of $P\beta S\delta-P\delta S\beta$, and at the (real) zeros of $\sin sa$.

We thus find, as before,

$$V\sqrt{rr'} = -\frac{8\pi}{c}\sum_p \sin\lambda\rho \sin\lambda\rho' \sum_s \frac{\sinh s(a-\phi)\sinh s\phi'}{\sinh sa}.$$
$$(P\beta S\mu'-P\mu' S\beta)(P\delta S\mu-P\mu S\delta)\Big/\frac{d}{ds}(P\beta S\delta-P\delta S\beta).$$

The harmonics are of degree $-1/2+i\lambda$, rank is;

$\lambda=p\pi/c$; s a positive zero of the function of s

$$P\beta S\delta-P\delta S\beta\,;\ \phi>\phi'$$

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36. *Relation of solutions in the two coordinate systems.*

The reader has doubtless observed the pronounced analogy between the forms of the solutions of allied cases in the two systems of coordinates. This similarity is indeed no more than might have been expected, seeing that the cylindrical is a special limiting case of the spherical system, namely, when the centre of the spheres has gone to infinity.

In the equation $r = ae^{\rho}$, we suppose a to become infinite while $r - a$ remains finite; then

the spherical		ρ	becomes the cylindrical		z/a
,,	,,	θ	,, ,,	,,	ρ/a
,,	,,	ϕ	remains	,,	ϕ.

With the aid of these limits, and those of 15 (1), the reduction of the spherical solutions to the others can easily be made.

(Cf. Lord Kelvin and Tait, *Nat. Phil.*, Vol. II., §783. In this paragraph, which appeared in the first edition of 1867, the relation of cylindrical to spherical harmonics is clearly pointed out, and methods are indicated for arriving at formulae given explicitly, but apparently afterwards, by Neumann, Mehler, and Heine.)

Similarly from the cylindrical solutions it is possible to deduce solutions in ordinary rectangular coordinates, by sending the centre of the cylinders to infinity. The reductions are not quite so easy as in the former case, some rather troublesome limits being required, particularly the limits of $J_m \lambda \rho$ and $G_m \lambda \rho$ when m and ρ become infinite, but in a finite ratio.

For the case of $J_m \lambda \rho$ with m and λ real and positive, see Graf und Gubler, "Theorie der Bessel'schen Funktionen," Bern 1898; s. 96.

Of course the rectangular solutions can easily be found by a direct process.

37. *Semi-convergent symmetrical solutions, or Stokes' forms.*

Reference has been made in the introductory paragraph to the solution given by Stokes for a space bounded by six planes of the rectangular coordinate system.* Now for each of the functions considered in the present paper, it appears that there are *two* distinct forms, analagous to that obtained by Stokes, involving harmonics of real, and of purely imaginary rank respectively. These

* "*On the Critical Values of the Sums of Periodic Series.*" (*Collected Works*, vol. I., p. 301.)

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Stokes' forms, though arithmetically much less simple than the forms already given, are of considerable analytical intertest, for this reason among others, that they are explicitly symmetrical in the accented and unaccented coordinates. They can be derived with little trouble from the expressions already found, the transformations in a large number of instances being most easily effected by means of a theorem of Cauchy's, which it may be useful to state, namely :—If the finite singularities of a uniform function $f(z)$ are all simple poles, and if, for an indefinitely increasing z, $f(z)$ tends to zero (except at the poles, if these extend to infinity, and, it may be, for certain values of the phase of z, for which $f(z)$ remains finite), then the function is equal to the sum of its polar elements, a polar element of the function being $A/(z-a)$, where a is a pole, and A the residue at that pole.

The theorem, which includes many well-known formulae of analysis, is proved at once on applying the fundamental residue theorem to the function of a, $f(a)/(z-a)$.

As examples, take the function of λ,

$$\sinh\lambda(c-z)\sinh\lambda z'/\sinh\lambda c, \quad 0<z'<z<c.$$

The conditions are satisfied, and the function is equivalent to

$$\frac{1}{c}\sum_p \sin p\pi z/c \,.\, \sin p\pi z'/c \,.\left(\frac{1}{\lambda - ip\pi/c}+\frac{1}{\lambda+ip\pi/c}\right)$$

$$=\frac{2}{c}\sum_p \sin p\pi z/c \,.\, \sin p\pi z'/c \,.\, \frac{\lambda}{\lambda^2+p^2\pi^2/c^2}\,.$$

Again, the function of n,

$$S_n^m\mu\, P_n^m\mu' = \Pi(m+n)\Pi(m-n-1)P_n^m(-\mu)P_n^m\mu'$$

satisfies the conditions if $\theta>\theta'$.
Its poles are those of $\Pi(m+n)$ and of $\Pi(m-n-1)$,
that is $n=m+p$, $n=-m-p-1$, with p zero or a positive integer.
The residue of $\Pi(m-n-1)$ when $n=m+p$ is $(-)^{p-1}/\Pi p$, §30, (3).
Hence residue of the function when $n=m+p$ is

$$-(\overline{\Pi p+2m}/\Pi p)P_{m+p}^m\mu \,.\, P_{m+p}^m\mu'.$$

Also since the function is even in $n+\frac{1}{2}$,
the residue at $n=-m-p-1$ is the same as this, except for sign.
Hence

$$S_n^m\mu \,.\, P_n^m\mu' = \sum_p \frac{\Pi(p+2m)}{\Pi p} P_{m+p}^m\mu \,.\, P_{m+p}^m\mu'\left(\frac{1}{n+m+p+1}-\frac{1}{n-m-p}\right).$$

Some special cases of this are interesting, *e.g.*, when $m=0$,

$$P_n(-\mu)P_n\mu' = \frac{\sin n\pi}{\pi}\sum_p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)P_p\mu\,P_p\mu'\,;\ \theta>\theta',$$

and putting $\theta'=0$ in this

$$P_n(-\mu) = \frac{\sin n\pi}{\pi}\sum_p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)P_p\mu$$

so that, changing θ into $\pi-\theta$,

$$P_n(\mu) = \frac{\sin n\pi}{\pi}\sum_p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)(-)^p P_p\mu.$$

The two last formulae evidently give the expansions of $P_n\mu$, $P_n(-\mu)$ in terms of Legendre's functions.

38. As specimens of Stokes' forms, and of the methods by which they may be found, we take the Green's function for a space bounded by two axial planes, §21.

In 21 (1) we replace $\sinh s(\alpha-\phi)\sinh s\phi'/\sinh s\alpha$ by

$$\frac{2s}{\alpha}\sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\cdot\frac{1}{s^2+m^2\pi^2/\alpha^2},\ §37,$$

and find

$$(4)\quad V=\frac{16}{\pi^2 a}\int_0^\infty \cos\lambda(z-z')d\lambda\int_0^\infty e^{-s\pi}\sinh s\pi\, G_{is}i\lambda\rho\, G_{is}i\lambda\rho'.\, sds.$$

$$\sum_m \sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\cdot\frac{1}{s^2+m^2\pi^2/\alpha^2}.$$

Again in 21 (3), for $e^{-\lambda(z-z')}$ write

$$\frac{2}{\pi}\int_0^\infty\frac{\lambda\cos\kappa(z-z')d\kappa}{\kappa^2+\lambda^2},$$

and (for convenience of comparision) interchange the symbols λ, κ. Hence

$$(5)\quad V=\frac{8}{\alpha}\sum_m\sin\frac{m\pi\phi}{\alpha}\sin\frac{m\pi\phi'}{\alpha}\int_0^\infty \kappa J_{\frac{m\pi}{\alpha}}\kappa\rho\, J_{\frac{m\pi}{\alpha}}\kappa\rho'.\,d\kappa.$$

$$\int_0^\infty\frac{\cos\lambda(z-z')d\lambda}{\kappa^2+\lambda^2}.$$

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In (4) we may integrate with respect to s before summing with respect to n, provided, as appears from 21 (2),

$$\frac{2}{\pi^2}\int_0^\infty \frac{s}{s^2+m^2\pi^2/a^2}e^{-s\pi}\sinh s\pi\, G_{is}i\lambda\rho\, G_{is}i\lambda\rho' ds$$
$$= G_{\frac{m\pi}{a}}i\lambda\rho\, J_{\frac{m\pi}{a}}i\lambda\rho', \quad \rho>\rho'.$$

Also in (5) we may first integrate with respect to κ, then sum with respect to m, and lastly integrate with respect to λ, provided

$$\int_0^\infty \frac{\kappa}{\kappa^2+\lambda^2} J_{\frac{m\pi}{a}}\kappa\rho\, J_{\frac{m\pi}{a}}\kappa\rho' d\kappa$$
$$= G_{\frac{m\pi}{a}}i\lambda\rho\, J_{\frac{m\pi}{a}}i\lambda\rho', \quad \rho>\rho'.$$

These two theorems are true and can easily be proved by the method of Cauchy, already exemplified so often. The second theorem is given by Sonine, *Math. Annalen* XVI.

If we were to go through the other functions in like manner, we should find suggested many interesting theorems, easy to prove, and indirectly verifying the legitimacy of altering the order of summations and integrations in the Stokes' forms.

39. *Continuation of Green's function beyond the bounded space containing the pole.*

The close analogy between Newtonian potential theory and the theory of functions of a complex variable has often been remarked. In each theory one of the most important ideas is that of the continuation or extension of a function beyond a limited region for which alone it is, in the first instance, analytically defined.

Now the transformations by which in this paper one form for a Green's function has been deduced from another have all this interesting characteristic that they carry on the function to a region in some way more extensive than that for which the original form is valid. This extension is valuable for various reasons, particularly as an aid to the determination of the singularities of the complete function.

Thus in §19, the form (1) defines a potential function for the space $z'<z<2c-z'$; the form (2) extends the function to the space

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$\rho > \rho'$, for all values of z. Moreover, this second form, with the cognate form for $\rho < \rho'$, shows that the function is odd and periodic in z, with period $2c$. Hence knowing the one singularity of the function between the limits $o < z < c$, we can infer the position and nature of the singularities of the complete function, and obtain the results of the method of images. When the boundary is cylindrical or conical, the singularities are of a more complex character, but they may be investigated from the materials given here, as I hope to show later.

40. *Sommerfeld's Function.*

Not the least interesting cases are those in which the boundary consists in part of two axial planes, the special feature being that the extended function, as a function of ϕ, has the period 2α instead of 2π, so that it does not in general return to its original value when the variable point makes a complete circuit about the axis of z. In order to obtain a quasi-geometrical representation of such a function, Professor Sommerfeld introduces the conception of a Riemann's space winding about the axis of z, a space in which we may consider functions not necessarily periodic in ϕ, whose value is supposed to range from $-\infty$ to $+\infty$.

[*Proceedings of the London Mathematical Society*, Vol. 28, "Über verzweigte Potentiale im Raum." See also in Vol. 30 of the same *Proceedings*, "Some multiform solutions of the Partial Differential Equations of Physical Mathematics," by Dr H. S. Carslaw, and compare Lord Kelvin and Tait, *Nat. Phil.*, Vol. I., App. B (c)].

In such a space the function 1/ distance or T is not the simplest conceivable potential, for it has a singularity of the first order, not for $\phi = \phi'$ only but for $\phi = \phi' + 2n\pi$ where n is any integer.

The fundamental potential, or Sommerfeld's function, as we may call it, will have but one singularity in the whole space, namely, at some point whose $\phi = \phi'$. (The axis of z is supposed excluded from the space by a very fine winding cylinder.)

Expressions for this function in terms of cylindrical and spherical harmonics may be found without difficulty. The easiest way, from our present standpoint, is to take one of the solutions for a space bounded by two axial planes, say $\phi = \pm \alpha/2$, and find its limiting form as a increases indefinitely. Thus in 21 (1), when the plane from which ϕ is measured is turned through an angle $a/2$, so that the factor involving a becomes

$$\sinh s\left(\frac{a}{2}-\phi\right)\sinh s\left(\phi'+\frac{a}{2}\right)\Big/\sinh sa,$$

we may, as it is easy to show, find the limit of V by writing its limit instead of this factor; this limit is $\frac{1}{2}e^{-s(\phi-\phi')}$. Hence

(1) $$V=\frac{4}{\pi^2}\int_0^\infty \cos\lambda(z-z')d\lambda\int_0^\infty e^{-s\pi}\sinh s\pi\,.\,e^{-s(\phi-\phi')}\,G_{is}i\lambda\rho\,G_{is}i\lambda\rho' ds.$$
$$\phi>\phi'.$$

(2) The second integral here is

$$-\int e^{im\pi}\sin m\pi\, e^{im(\phi-\phi')}G_m i\lambda\rho\, G_m i\lambda\rho' dm \,\Big|\, \mathrm{ON}$$
$$=\quad \frac{\pi}{2}\int e^{im(\phi-\phi')}(i^m G_m i\lambda\rho)(i^{-m}J_m i\lambda\rho')dm\,\Big|\,\mathrm{ON}$$
$$-\frac{\pi}{2}\int e^{im(\phi-\phi')}(i^m G_m i\lambda\rho)(i^m J_{-m} i\lambda\rho')dm\,\Big|\,\mathrm{ON}.$$

If $\rho>\rho'$, $\phi>\phi'$, the path of integration in the former integral may be changed to OE, in the latter to OW. Expressing these as real integrals, and collecting, we find

$$V=\frac{4}{\pi}\int_0^\infty \cos\lambda(z-z')d\lambda\int_0^\infty \cos m(\phi-\phi')G_m i\lambda\rho\, J_m i\lambda\rho' dm.$$
$$\rho>\rho'$$

(3) The z form is deduced from this as in 21 (3),

$$V=2\int_0^\infty \cos m(\phi-\phi')dm\int_0^\infty e^{-\lambda(z-z')}J_m\lambda\rho\, J_m\lambda\rho' d\lambda.$$
$$z>z'.$$

The Stokes' forms may now be written down as in 38 (4), and the corresponding expressions in spherical harmonics may easily be obtained in the same way.

41. If we denote the above function by $V(\phi)$, then a function having singularities of the first order at ϕ, $\phi\pm 2a$, $\phi\pm 4a$, etc, is

$$W=V(\phi)+V(\phi+2a)+V(\phi+4a)+\ldots\ldots$$
$$+V(\phi-2a)+V(\phi-4a)+\ldots\ldots$$

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The form 40 (1) is convenient for calculating this series. We obtain

(1) $$W = \frac{4}{\pi^2}\int_0^\infty \cos\lambda(z-z')d\lambda$$

$$\int_0^\infty e^{\quad s\pi}\sinh s\pi\frac{\cosh s(a+\phi'-\phi)}{\sinh sa}G_{is}i\lambda\rho G_{is}i\lambda\rho' ds$$

$$o < \phi - \phi' < 2a.$$

(2) From this we deduce as usual

$$W = \frac{4}{a}\int_0^\infty \cos\lambda(z-z')d\lambda \sum_m{}' \cos\frac{m\pi}{a}(\phi-\phi')G_{\frac{m\pi}{a}}i\lambda\rho\, J_{\frac{m\pi}{a}}i\lambda\rho',$$

$$\rho > \rho'.$$

(3) $$W = \frac{2\pi}{a}\int_0^\infty e^{-\lambda(z-z')}d\lambda \sum_m{}' \cos\frac{m\pi}{a}(\phi-\phi')J_{\frac{m\pi}{a}}\lambda\rho \,.\, J_{\frac{m\pi}{a}}\lambda\rho'.$$

$$z > z'.$$

The functions T and W coincide when $a = \pi$.

42. The relation of the solution of Green's problem to the general problem of determining a potential function taking an arbitrary value at a given boundary is well known. When this paper was commenced, it was intended to consider in some detail, and to illustrate from the foregoing solutions, certain questions arising in this connection, particularly with reference to modes of expansion of an arbitrary function. Since, however, the paper is already sufficiently long, these questions must be left over for the present, but I propose to deal with them in a future paper, which I hope to have the privilege of reading to the Society at one of next session's meetings.

On the Discriminant of the General Homogeneous Quadric.

By Chas. Tweedie, M.A., B.Sc., F.R.S.E.

A note on change of Coordinate Axes.

By Prof. Steggall.

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