BULL. AUSTRAL. MATH. SOC. 47D25, 46L40
VOL. 53 (1996) [391–400]

TWO CHARACTERISATIONS OF ADDITIVE *-AUTOMORPHISMS OF $\mathcal{B}(H)$

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Dedicated to the memory of my best friend Mazsó

Let H be a complex Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on H. In this paper we give two necessary and sufficient conditions for an additive bijection of $\mathcal{B}(H)$ to be a *-automorphism. Both of the results in the paper are related to the so-called preserver problems.

INTRODUCTION

Linear preserver problems concern the characterisation of linear operators on matrix algebras that leave certain functions, subsets or relations invariant. This subject has attracted the attention of many mathematicians during this century (see the survey paper [**7**]). In fact, it represents one of the most active areas in matrix theory. In the last decade interest in similar questions on operator algebras over infinite dimensional spaces has also been growing (for example, [**9**, **10**, **11**, **12**] and the references therein). This research includes the study of preserver problems for mappings which are not linear but are merely additive (for example, [**9**, **10**]). This follows an approach that goes back to the classical work [**5**] on homomorphisms.

In our paper we intend to give the solutions of two additive preserver problems which turn out to characterise *-automorphisms. By a *-automorphism we just mean a bijective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ which preserves the ring structure (rather than the algebra structure) and for which $\Phi(A^*) = \Phi(A)^*$. In Theorem 1 we deal with additive mappings on $\mathcal{B}(H)$ which commute with the function $|\,.\,|^k : \mathcal{B}(H) \to \mathcal{B}(H)$ for some fixed natural number k where $|A| = (A^*A)^{1/2}$ denotes the absolute value of an operator A (see [**7**, Problem IV]). A similar question for linear mappings on matrix algebras was treated in [**2**, Theorem 1]. In Theorem 2 additive mappings satisfying a certain operator-valued orthogonality preserving property are considered (see [**7**, Problem III]). A similar linear preserver problem was studied in [**12**, Theorem 4].

Received 13th July, 1995.
Research partially supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA 1652 and K&H Bank Ltd., Universitas Foundation.

Results

Besides giving a characterisation of additive *-automorphisms of $\mathcal{B}(H)$, Theorem 1 below also contains an automatic bijectivity result. Namely, for an additive mapping $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ with the commuting property described above we assume only the inclusion of the smallest operator ideal $\mathcal{F}(H)$ of finite rank operators in the range $\operatorname{rng}\Phi$ of Φ and then we obtain that in this case Φ is automatically surjective and injective.

Theorem 1. *Let H be a complex Hilbert space, $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ an additive function and $k \geqslant 1$ a natural number. Suppose that $\mathcal{F}(H) \subset \operatorname{rng}\Phi$ and*

$$|\Phi(A)|^k = \Phi\left(|A|^k\right) \qquad (A \in \mathcal{B}(H)).$$

*If $k = 1$, then there exists a positive real number c and an additive *-automorphism $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ such that $\Phi = c\Phi$. If $k > 1$, then Φ is a *-automorphism.*

Proof: We divide the proof into several steps.

Step 1. Φ sends the sets of all positive and all self-adjoint operators into themselves. It is also order-preserving, real-linear and continuous.

Since every positive operator has a unique positive kth root, the first part of the assertion is trivial. Moreover, we also have

$$|\Phi(A)|^k = \Phi\left(|A|^k\right) = \Phi(|A|)^k$$

which implies $|\Phi(A)| = \Phi(|A|)$ $(A \in \mathcal{B}(H))$. Since every self-adjoint operator is the difference of two positive ones, thus Φ preserves self-adjointness. This implies that Φ is order-preserving. Let $A \geqslant 0$ and $\lambda \in \mathbb{R}$ be fixed for the moment and consider arbitrary rational numbers r, s with $r < \lambda < s$. Since Φ is additive, it is $\mathbb{Q}$-linear. Consequently, we have

$$r\langle\Phi(A)x, x\rangle = \langle\Phi(rA)x, x\rangle \leqslant \langle\Phi(\lambda A)x, x\rangle \leqslant \langle\Phi(sA)x, x\rangle = s\langle\Phi(A)x, x\rangle \qquad (x \in H).$$

This gives that $\langle\Phi(\lambda A)x, x\rangle = \langle\lambda\Phi(A)x, x\rangle$ holds for every $x \in H$, that is,

$$\Phi(\lambda A) = \lambda\Phi(A).$$

Let us turn to the continuity of Φ. Let $A \in \mathcal{B}(H)$. By the inequality $|A| \leqslant \||A|\| I = \|A\| I$ we can compute

$$|\Phi(A)| = \Phi(|A|) \leqslant \|A\| \, \Phi(I).$$

Since the norm of a positive operator is equal to its numerical radius, we arrive at

$$\|\Phi(A)\| = \||\Phi(A)|\| \leqslant \|A\| \, \|\Phi(I)\|$$

which yields the continuity of Φ.

STEP 2. For every rank-one projection P there is a positive number c and a projection E such that $\Phi(cE) = P$.

Since $\mathcal{F}(H) \subset \operatorname{rng}\Phi$ and P is positive, it follows that there exists a positive operator A such that $\Phi(A) = P$. Let

$$A = \int_{[0,\|A\|]} \lambda d E(\lambda)$$

be the spectral resolution of A and let

$$N_n = \int_{[1/n,\|A\|]} \lambda d E(\lambda)$$

and

$$K_n = \int_{[0,1/n[} \lambda d E(\lambda) = A - N_n \qquad (n \in \mathbb{N}).$$

We assert that there is an $n \in \mathbb{N}$ such that for $B_n = [1/n, \|A\|]$ we have $\Phi(E(B_n)) \neq 0$. Indeed, in the opposite case, using the continuity and real-linearity of Φ, we obtain $\Phi(N_n) = 0$ for every $n \in \mathbb{N}$. Since $K_n \longrightarrow 0$, we also have $\Phi(K_n) \longrightarrow 0$. These imply $\Phi(A) = 0$ which is a contradiction. Now, let $n \in \mathbb{N}$ be such that $\Phi(E(B_n)) \neq 0$. Since $(1/n)E(B_n) \leqslant A$, we infer $\Phi((1/n)E(B_n)) \leqslant \Phi(A) = P$. The minimality of the projection P and the positivity of $\Phi((1/n)E(B_n))$ imply that there is a positive constant c such that $\Phi(cE(B_n)) = P$.

STEP 3. For every self-adjoint operator $S \in \mathcal{B}(H)$ we have

$$\Phi(iS)^*\Phi(I) + \Phi(I)\Phi(iS) = 0$$
$$\Phi(S)^2 - 1/2\Phi(S^2)\Phi(I) - 1/2\Phi(I)\Phi(S^2) = 0.$$

Observe that e^{itS} is a unitary operator for every $t \in \mathbb{R}$. Hence

$$\Phi(e^{itS})^*\Phi(e^{itS}) = |\Phi(e^{itS})|^2 = \Phi(|e^{itS}|)^2 = \Phi(I)^2 = \Phi(I)^*\Phi(I).$$

Now, using the continuity and real-linearity of Φ as well as the power series expansion $e^{itS} = I + itS + (itS)^2/2 + \ldots$ and the uniqueness theorem on holomorphic functions, we infer from the equality above that

$$\Phi(iS)^*\Phi(I) + \Phi(I)^*\Phi(iS) = 0$$
$$\Phi(iS)^*\Phi(iS) + 1/2\Phi\Big((iS)^2\Big)^*\Phi(I) + 1/2\Phi(I)^*\Phi\Big((iS)^2\Big) = 0.$$

Since $\Phi(I), \Phi(S), \Phi(S^2)$ are self-adjoint and $\Phi(iS)^*\Phi(iS) = |\Phi(iS)|^2 = \Phi(|iS|)^2 = \Phi(|S|)^2 = |\Phi(S)|^2 = \Phi(S)^*\Phi(S)$, thus we obtain the result.

STEP 4. $\Phi(I)$ is a positive scalar multiple of the identity. If $k > 1$, then $\Phi(I) = I$.

Let $x \in H$ be a unit vector and let $P = x \otimes x$ be the corresponding rank-one projection. If $c > 0$ and E is a projection with $P = \Phi(cE)$ (see Step 2), then by Step 3 we have

$$\begin{aligned} P = P^2 = \Phi(cE)^2 &= 1/2\big(\Phi(c^2E)\Phi(I) + \Phi(I)\Phi(c^2E)\big) \\ &= c/2(\Phi(cE)\Phi(I) + \Phi(I)\Phi(cE)) = c/2(P\Phi(I) + \Phi(I)P). \end{aligned}$$

Evaluating the operators on both sides at x we arrive at

$$x = c/2(\langle \Phi(I)x, x\rangle x + \Phi(I)x).$$

This gives that for every $x \in H$ there is a scalar $\lambda_x \in \mathbb{C}$ such that $\Phi(I)x = \lambda_x x$. One can check rather easily that λ_x can be chosen to be independent of x and hence there is a non-negative real number λ such that $\Phi(I) = \lambda I$. Let us show that $\lambda \neq 0$. Indeed, if $\Phi(I) = 0$, then for every unitary operator $U \in \mathcal{B}(H)$ we have

$$|\Phi(\mu U)| = \Phi(|\mu U|) = |\mu|\,\Phi(I) = 0$$

and thus $\Phi(\mu U) = 0$ $(\mu \in \mathbb{C})$. Since every element of $\mathcal{B}(H)$ is a finite linear combination of unitary operators, it follows that $\Phi = 0$ which is a contradiction. Therefore, $\lambda > 0$ and we may and do assume that in the case $k = 1$ the relation $\Phi(I) = I$ holds.

If $k > 1$, then $\Phi(I)$ is a projection. Indeed, we have $\Phi(I)^k = \Phi(I^k) = \Phi(I)$ and using, for example, the continuous function calculus we get $\sigma(\Phi(I)) \subset \{0,1\}$. This gives that $\Phi(I)$ is a projection and we infer $\Phi(I) = I$.

STEP 5. For every self-adjoint operator S we have $\Phi(iS) = \Phi(iI)\Phi(S)$ and $\Phi(iI) \in \{iI, -iI\}$. Consequently, Φ is either linear or conjugate-linear.

We first show that $W = \Phi(iI)$ is unitary. From Step 3 we know that the set of all skew-symmetric elements of $\mathcal{B}(H)$ is invariant under Φ. It follows that W is skew-symmetric. Since $W^*W = |\Phi(iI)|^2 = \Phi(I)^2 = I$, thus $W^*W = I$ and $W^2 = -I$. These imply that W is a surjective isometry, that is, W is unitary. Let S be an arbitrary self-adjoint operator. From the proof of Step 3 we obtain that $|\Phi(iS)| = |\Phi(S)|$. Since the partial isometry in the polar decomposition of a normal operator can be chosen to be unitary, we infer that for an arbitrary projection P there are unitary operators U, V such that

$$\Phi(iP) = U\,|\Phi(iP)| = U\Phi(P) \quad \text{and} \quad \Phi(i(I+P)) = V\Phi(I+P).$$

Consequently, we obtain the equality

$$V + V\Phi(P) = W + U\Phi(P).$$

From Step 3 it follows that $\Phi(P)$ is idempotent. If $x \in \operatorname{rng}\Phi(P)$ is a unit vector, then we have

$$2Vx = Vx + Vx = Wx + Ux.$$

Since Vx, Wx, Ux are unit vectors and the sum of two unit vectors has norm 2 if and only if they coincide, we infer $Wx = Ux$. Consequently, we have $W\Phi(P) = U\Phi(P)$ and this gives us that $\Phi(iP) = \Phi(iI)\Phi(P)$. Moreover, we compute

$$-\Phi(iP) = \Phi(iP)^* = \Phi(P)^*\Phi(iI)^* = -\Phi(P)\Phi(iI).$$

Since the projection P was arbitrary, by the spectral theorem and the continuity of Φ we have $\Phi(iS) = \Phi(iI)\Phi(S) = \Phi(S)\Phi(iI)$ for every self-adjoint $S \in \mathcal{B}(H)$. This further implies

$$\Phi(iI)\Phi(iS) = \Phi(iI)\Phi(S)\Phi(iI) = \Phi(iS)\Phi(iI)$$

and hence we infer that $\Phi(iI)$ commutes with the range of Φ. In particular $\Phi(iI)$ commutes with every finite rank operator. It is an elementary computation to show that in this case $\Phi(iI)$ is of the form λI. Now, one can see that $\lambda \in \{i, -i\}$. Having this in mind, the final assertion is easy to get.

STEP 6. Φ is an additive *-homomorphism or *-antihomomorphism.

The *-preserving property of Φ is trivial to check. From Step 3 we have

$$\Phi(S)^2 = \Phi(S^2)$$

for every self-adjoint $S \in \mathcal{B}(H)$. Linearising this latter equality, that is, replacing S by $S + T$, we obtain that

$$\Phi(ST + TS) = \Phi(S)\Phi(T) + \Phi(T)\Phi(S)$$

holds for every self-adjoint $S, T \in \mathcal{B}(H)$. Hence, we can compute

$$\begin{aligned}\Phi(S + iT)^2 &= (\Phi(S) + \Phi(iI)\Phi(T))^2 = \Phi(S)^2 - \Phi(T)^2 + \Phi(iI)(\Phi(S)\Phi(T) + \Phi(T)\Phi(S)) \\ &= \Phi(S^2) - \Phi(T^2) + \Phi(iI)\Phi(ST + TS) = \Phi(S^2 - T^2 + i(ST + TS)) = \Phi\left((S + iT)^2\right).\end{aligned}$$

This means that $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a so-called Jordan homomorphism. By a well-known algebraic argument of Herstein [4] (see [**8**, 6.3.2 Lemma, 6.3.6 Lemma and 6.3.7 Theorem]), if ϕ is an additive function from a ring $\mathcal{R}$ into $\mathcal{B}(H)$ such that $\mathcal{F}(H) \subset \operatorname{rng}\phi$ and

$$\phi(a)^2 = \phi(a^2) \qquad (a \in \mathcal{R}),$$

then ϕ is *either a homomorphism or an antihomomorphism*. Using this, we have the result.

STEP 7. Φ is injective.

The kernel of Φ is an ideal. Let P be a rank-one projection. Suppose that $E \in \mathcal{B}(H)$ is a projection with $\Phi(E) = P$ (see Step 2). We assert that E is finite dimensional. Indeed, if E is infinite dimensional, then it is the sum of two mutually orthogonal infinite dimensional projections $E' = UE$, $E'' = VE$ where U, V are isometries. Since $P = \Phi(E') + \Phi(E'')$, by the minimality of P it follows that either $\Phi(E') = 0$ or $\Phi(E'') = 0$. However, in both cases we have $\Phi(E) = 0$ because of the relations $U^*E' = E$, $V^*E'' = E$. So, E is finite dimensional. Since every non-zero ideal of $\mathcal{B}(H)$ contains $\mathcal{F}(H)$, thus we have the injectivity of Φ.

STEP 8. Φ is a homomorphism.

Suppose that $\dim H > 1$ and Φ is an antihomomorphism. If $A \in \mathcal{B}(H)$, then we have

$$\begin{aligned}\Phi(AA^*) &= \Phi(A^*)\Phi(A) = \Phi(A)^*\Phi(A)\\ &= |\Phi(A)|^2 = \Phi(|A|)^2 = \Phi\left(|A|^2\right) = \Phi(A^*A).\end{aligned}$$

By the injectivity of Φ it follows that $AA^* = A^*A$. Since this obviously does not hold for every $A \in \mathcal{B}(H)$, the proof is complete. (If $\dim H = 1$, then every antihomomorphism Φ is also a homomorphism.)

STEP 9. Φ is surjective.

Without loss of generality we may suppose that Φ is conjugate-linear. Let $x_0, y, z \in H$ be vectors such that $\Phi(x_0 \otimes y)z \neq 0$. Then for the function $U : H \to H$ defined by $Ux = \Phi(x \otimes y)z$ $(x \in H)$ we have

$$UAx = \Phi(Ax \otimes y)z = \Phi(A)\Phi(x \otimes y)z = \Phi(A)Ux \qquad (x \in H)$$

for every $A \in \mathcal{B}(H)$. Plainly, U is a conjugate-linear operator. To see that it is injective, let $x \neq 0$ be such that $Ux = 0$. The equality above implies that $UAx = 0$ for every $A \in \mathcal{B}(H)$ which gives $U = 0$. Since $Ux_0 \neq 0$, we arrive at a contradiction. Using that equality as well as the fact that the range of Φ contains $\mathcal{F}(H)$, one can readily verify that U is surjective. Hence we get that U is an invertible bounded conjugate-linear operator. In fact, it is a positive scalar multiple of an antiunitary operator. This follows from the computation

$$\begin{aligned}\|Ux\|^2 &= \langle \Phi(x \otimes y)z, \Phi(x \otimes y)z\rangle = \langle \Phi(x \otimes y)^*\Phi(x \otimes y)z, z\rangle = \langle \Phi(y \otimes x \cdot x \otimes y)z, z\rangle\\ &= \langle \Phi\left(\|x\|^2\, y \otimes y\right)z, z\rangle = \|x\|^2 \langle \Phi(y \otimes y)z, z\rangle \quad (x \in H).\end{aligned}$$

Since $\Phi(A) = UAU^{-1}$ $(A \in \mathcal{B}(H))$, we have the surjectivity of Φ.

This completes the proof the theorem. $\square$

https://doi.org/10.1017/S0004972700017147 Published online by Cambridge University Press

REMARK. It is well-known that every additive *-automorphism of $B(H)$ is of the form

$$A \longmapsto UAU^*,$$

where U is a fixed unitary or antiunitary operator. In fact, this is a part of the proof above.

In contrast with our Theorem 1, the argument given in the proof of the next result works only in the infinite dimensional case.

THEOREM 2. *Let H be a complex infinite dimensional separable Hilbert space and let $\Phi : B(H) \to B(H)$ be an additive bijection such that $\Phi(I) = I$. Suppose that*

(i) *$A^*B = 0$ if and only if $\Phi(A)^*\Phi(B) = 0$ and*

(ii) *$AB^* = 0$ if and only if $\Phi(A)\Phi(B)^* = 0$.*

*Then Φ is a *-automorphism.*

PROOF: Just as above, we divide the proof into steps.

STEP 1. Φ maps projections into projections and rank-one projections into rank-one projections. If $(P_\alpha)_\alpha$ is a maximal family of pairwise orthogonal rank-one projections, then so is $(\Phi(P_\alpha))_\alpha$.

If P is a projection, then $P^*(I-P) = 0$. This implies $\Phi(P)^*(I - \Phi(P)) = 0$, that is, $\Phi(P)^* = \Phi(P)^*\Phi(P)$. Consequently, $\Phi(P)$ is self-adjoint and idempotent. Suppose that P is rank-one while $\Phi(P)$ is not rank-one. Then $\Phi(P)$ can be written as a sum of two nonzero projections. Since Φ^{-1} satisfies the same hypotheses as Φ, what we have just proved shows that P can also be written as a sum of two nonzero projections. This is a contradiction. The remaining part is very easy to see.

STEP 2. Let $x, y \in H$. There exists an additive function $\tau : \mathbb{C} \to \mathbb{C}$ such that

$$\Phi(\lambda x \otimes y) = \tau(\lambda)\Phi(x \otimes y) \qquad (\lambda \in \mathbb{C}).$$

Let $x, y \neq 0$ and let x', y' denote the corresponding unit vectors. Suppose that u, v are arbitrary unit vectors such that $\langle u, x\rangle = \langle v, y\rangle = 0$. Then by (i) and (ii) we have

$$\Phi(u \otimes u)\Phi(\lambda x \otimes y) = 0, \quad \Phi(\lambda x \otimes y)\Phi(v \otimes v) = 0.$$

Hence, we obtain

$$\Phi(\lambda x \otimes y) = \Phi(x' \otimes x')\Phi(\lambda x \otimes y)\Phi(y' \otimes y') = \tau'(\lambda)x'' \otimes y'',$$

where x'', y'' are unit vectors such that $x'' \otimes x'' = \Phi(x' \otimes x')$ and $y'' \otimes y'' = \Phi(y' \otimes y')$. Considering the equations above for $\lambda = 1$ as well, one can get the result.

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STEP 3. The function τ in Step 2 can be chosen to be independent of x, y.

We show that τ does not depend on x. The remaining part can be proved in the same way. Let x, x', y be non-zero vectors and suppose that $\langle x, x' \rangle = 0$. Let $\tau, \tau', \mu : \mathbb{C} \to \mathbb{C}$ be additive functions such that

$$\Phi(\lambda x \otimes y) = \tau(\lambda)\Phi(x \otimes y), \quad \Phi(\lambda x' \otimes y) = \tau'(\lambda)\Phi(x' \otimes y)$$
$$\Phi(\lambda(x + x') \otimes y) = \mu(\lambda)\Phi((x + x') \otimes y)$$

for every $\lambda \in \mathbb{C}$. Then we have

$$\mu(\lambda)\Phi(x \otimes y) + \mu(\lambda)\Phi(x' \otimes y) = \mu(\lambda)\Phi((x + x') \otimes y) = \Phi(\lambda(x + x') \otimes y) =$$
$$\Phi(\lambda x \otimes y) + \Phi(\lambda x' \otimes y) = \tau(\lambda)\Phi(x \otimes y) + \tau'(\lambda)\Phi(x' \otimes y).$$

If we multiply the equality above by $\Phi(x \otimes x)^*$ from the left, we arrive at

$$\mu(\lambda)\Phi(x \otimes x)^*\Phi(x \otimes y) = \tau(\lambda)\Phi(x \otimes x)^*\Phi(x \otimes y).$$

Similarly, multiplying it by $\Phi(x' \otimes x')^*$ we have

$$\mu(\lambda)\Phi(x' \otimes x')^*\Phi(x' \otimes y) = \tau'(\lambda)\Phi(x' \otimes x')^*\Phi(x' \otimes y).$$

Consequently, we obtain $\tau = \tau'$. If x' is a non-zero vector which is not orthogonal to x, then one can consider a third non-zero vector x'' which is orthogonal to x as well as to x'. This completes the proof of this step.

STEP 4. τ is either the identity or the conjugation on $\mathbb{C}$.

We first show that τ is a ring homomorphism. Indeed, we compute

$$\tau(\lambda)\tau(\mu)\Phi(x \otimes y) = \tau(\lambda)\Phi(\mu x \otimes y) = \Phi(\lambda\mu x \otimes y) = \tau(\lambda\mu)\Phi(x \otimes y)$$

and this implies that τ is multiplicative. We assert that τ is continuous. If τ is not continuous, then by an elementary result from the theory of functional equations, τ is unbounded on some bounded subset of $\mathbb{C}$. Let (λ_n) be a bounded sequence such that $|\tau(\lambda_n)| \longrightarrow \infty$. Let (P_n) be a sequence of mutually orthogonal rank-one projections. Consider the operator $A = \Phi\left(\sum_n \lambda_n P_n\right)$. For an arbitrary $n_0 \in \mathbb{N}$ let $x \in \operatorname{rng} \Phi(P_{n_0})$ be a unit vector. Then we have

$$\begin{aligned}\|A\| \geqslant \|Ax\| &= \left\|\Phi\left(\sum_n \lambda_n P_n\right)\Phi(P_{n_0})x\right\| \\ &= \left\|\Phi(\lambda_{n_0}P_{n_0})\Phi(P_{n_0})x + \Phi\left(\sum_{n \neq n_0} \lambda_n P_n\right)\Phi(P_{n_0})x\right\| = |\tau(\lambda_{n_0})|.\end{aligned}$$

This implies that the operator A is not bounded, which is a contradiction. Therefore, τ is continuous. Since every nontrivial continuous ring endomorphism of $\mathbb{C}$ is either the identity or the conjugation [**6**, Lemma 1, p.356] (see [**1**, Chapter 5]) we have the assertion.

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STEP 5. There exists a unitary or antiunitary operator U on H such that

$$\Phi(A) = UAU^* \qquad (A \in \mathcal{F}(H)).$$

Let $\Psi = \Phi\mid_{\mathcal{F}(H)}$. Then Ψ is an either linear or conjugate-linear bijection onto $\mathcal{F}(H)$. This follows from the fact that every finite rank-operator is the finite linear combination of rank-one projections. It is also easy to verify that Ψ is *-preserving. We now show that Ψ is a Jordan homomorphism. Let $S \in \mathcal{F}(H)$ be self-adjoint. Then $S = \sum_{k=1}^{n} \lambda_k P_k$ with some $\lambda_k \in \mathbb{R}$ and pairwise orthogonal rank-one projections P_k. We have

$$\Psi(S)^2 = \left(\sum_{k=1}^{n} \lambda_k \Psi(P_k)\right)^2 = \sum_{k=1}^{n} \lambda_k^2 \Psi(P_k) = \Psi\left(\sum_{k=1}^{n} \lambda_k^2 P_k\right) = \Psi(S^2).$$

Using the linearity-antilinearity of Ψ and the argument in Step 6 of Theorem 1, it is easy to check that Ψ is a Jordan automorphism of $\mathcal{F}(H)$. By Herstein's result again, Ψ is either an automorphism or an antiautomorphism. We claim that Ψ is an automorphism. If it is an antiautomorphism, then for arbitrary $A, B \in \mathcal{F}(H)$ we have

$$A^*B = 0 \Leftrightarrow \Psi(A)^*\Psi(B) = 0 \Leftrightarrow \Psi(A^*)\Psi(B) = 0 \Leftrightarrow \Psi(BA^*) = 0 \Leftrightarrow BA^* = 0$$

which obviously does not hold in general. Applying the same argument as in Step 9 of Theorem 1, one can obtain the result.

STEP 6. Φ is a *-automorphism.

We first show that $\Phi(P) = UPU^*$ holds for every projection P. Indeed, from Step 1 we infer $\Phi(P) = \sum_n \Phi(P_n)$, where $(P_n)_n$ is a pairwise orthogonal sequence of rank-one projections and $P = \sum_n P_n$. It is also obvious that $UPU^* = \sum_n UP_nU^*$. Let us show that $\Phi(\lambda P) = \tau(\lambda)\Phi(P)$ holds for every $\lambda \in \mathbb{C}$. If $x \in \operatorname{rng}\Phi(P_{n_0})$ is arbitrary, then we have

$$\begin{aligned}\Phi(\lambda P)x &= \Phi\left(\lambda P_{n_0} + \lambda \sum_{n \neq n_0} P_n\right)\Phi(P_{n_0})x \\ &= \Phi(\lambda P_{n_0})\Phi(P_{n_0})x + \Phi\left(\lambda \sum_{n \neq n_0} P_n\right)\Phi(P_{n_0})x = \tau(\lambda)\Phi(P_{n_0})\Phi(P_{n_0})x\end{aligned}$$

and similarly

$$\Phi(P)x = \Phi(P_{n_0})\Phi(P_{n_0})x.$$

Now, one can readily verify that $\Phi(\lambda P) = \tau(\lambda)\Phi(P)$. As a consequence, we obtain

$$\Phi(\lambda P) = \tau(\lambda)\Phi(P) = \tau(\lambda)UPU^* = U\lambda PU^*.$$

Since by [**3**, Theorem 2] every element of $\mathcal{B}(H)$ is a finite linear combination of projections, this yields

$$\Phi(A) = UAU^* \qquad (A \in \mathcal{B}(H)),$$

which completes the proof of the theorem. □

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