A REMARK ON FREE TOPOLOGICAL GROUPS WITH NO SMALL SUBGROUPS

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1. Introduction

For a completely regular space X let $G(X)$ be the Graev free topological group on X. While proving $G(X)$ exists for completely regular spaces X, Graev showed that every pseudo-metric on X can be extended to a two-sided invariant pseudo-metric on the abstract group $G(X)$. The free group topology on $G(X)$ is usually strictly finer than this pseudo-metric topology. In particular this is the case when X is not totally disconnected (see Morris and Thompson [7]). It is of interest to know when $G(X)$ has no small subgroups (see Morris [5]). Morris and Thompson [6] showed that this is the case if and only if X admits a continuous metric. The proof relied on properties of the free group topology and it is natural to ask if $G(X)$ with its pseudo-metric topology has no small subgroups when and only when X admits a continuous metric. We show that this is the case. Topological properties of $G(X)$ associated with the pseudo-metric topology have recently been studied by Joiner [3] and Abels [1].

2. Notation and preliminaries

Let X be a completely regular topological space with a distinguished point e.

The space X is said to *admit a continuous metric* if there is a continuous one-to-one mapping of X onto a metric space.

A topological group is said to have *no small subgroups* if there exists a neighbourhood of the identity which contains no non-trivial subgroups.

As the proof of our result depends on Graev's extension of pseudo-metrics on X to $G(X)$ we describe the essential features of this process.

The group $G(X)$ is said to be the *Graev topological group on X* [2] if it has the properties:

(a) X is a subspace of $G(X)$,

(b) X generates $G(X)$ algebraically, and e is the identity element of $G(X)$,

(c) for any continuous mapping ϕ of X into any topological group H such that $\phi(e)$ is the identity element of H, there exists a continuous homomorphism Φ of $G(X)$ into H such that $\Phi|X = \phi$.

Graev showed that $G(X)$ is the free abstract group on the set $X - e$ having the finest group topology which induces the given topology on X. Let $X' = X \cup \{x^{-1}: x \in X - e\}$ and $N = \{1, 2, \cdots\}$. For w in $G(X)$ with reduced form $w = x_1 \cdots x_n$ let $S(w) = \{x_1, \cdots, x_n, e, x_1^{-1}, \cdots, x_n^{-1}\}$. The topology on X is defined by a family of continuous pseudo-metrics. Let ρ be a pseudo-metric on X. Graev extended ρ to a two-sided invariant pseudo-metric on $G(X)$ as follows. Extend ρ to X' by setting $\rho(x^{-1}, y^{-1}) = \rho(x,y)$ and $\rho(x^{-1}, y) = \rho(x,e) + \rho(e,y)$ for x and y in X. For u and v in $G(X)$ we have an infinity of representations $u = x_1 \cdots x_n$ and $v = y_1 \cdots y_n$ where the x_i and y_i are in X'. Extend ρ to $G(X)$ by setting

$$\rho(u,v) = \inf\left\{\sum_{i=1}^{n} \rho(x_i, y_i): u = x_1 \cdots x_n \text{ and } v = y_1 \cdots y_n\right\}$$

We are interested in the case $v = e$ and Graev's results restricted to this case are that the infimum is attained when u has its reduced representation $x_1 \cdots x_n$ and the y_i are suitably chosen from $S(u)$.

We need the following result (see Kurosh [4], page 127).

LEMMA 1. *For any* $w \in G(X) - e$ *there is* $l \in G(X)$ *and* $c \in G(X) - e$ *such that* $w = lcl^{-1}$ *where* c *has the reduced form* $c = x_1 \cdots x_n$ *where* $x_i \in X' - e$ *for* $i = 1, \cdots, n$ *for some* $n \in N$ *and* $x_1 \neq x_n^{-1}$. *Further for any* $t \in N$, $l^{-1} w^t l = c^t$ *and* c^t *has reduced form* $c^t = x_1 \cdots x_n x_1 \cdots x_n \cdots x_1 \cdots x_n$.

Let X admit a continuous metric d. Extend d to a continuous pseudo-metric on $G(X)$ as described above. For any w in $G(X) - e$ set

$$f(w) = \min\{d(p,q): p \neq q;\, p, q \in S(w)\}$$

The following properties of f need no explanation.

LEMMA 2. *The function* f *satisfies*

(i) $f(w) > 0$ *for all* $w \in G(X) - e$,

(ii) *if* w *has reduced form* lcm *for some* l, c, *and* $m \in G(X) - e$, *then* $f(c) \geqq f(w)$, *and*

(iii) *for any* $c \in G(X) - e$ *and any* $t \in N$, $f(c^t) = f(c)$.

3. Results

Let c in $G(X)$ have the reduced form $c = x_1 \cdots x_n$ where the x_i are in X' and $x_1 \neq x_n^{-1}$.

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LEMMA 3. *If $n \geqq 3$, then for any $t \in N$*

$$d(c^t, e) \geqq tf(c).$$

PROOF. In reduced form $c^t = x_1 \cdots x_n x_1 \cdots x_n \cdots x_1 \cdots x_n = s_1 \cdots s_{tn}$ where $s_i = x_j$ for $i = (p-1)n + j$, $1 \leqq p \leqq t$ and $1 \leqq j \leqq n$. From Graev's construction described in the previous section we may write $e = y_1 \cdots y_{tn}$ such that $d(c^t, e) = \sum_{i=1}^{tn} d(s_i, y_i)$ where $y_i \in S(c^t) = S(c)$. It suffices to show that for each p with $1 \leqq p \leqq t$, $\sum_{i=(p-1)n+1}^{(p-1)n+3} d(s_i, y_i) \geqq f(c)$. Consider the case $p = 1$ where the inequality becomes

$$d(x_1, y_1) + d(x_2, y_2) + d(x_3, y_3) \geqq f(c).$$

Now $y_2 \in S(c)$ and if $y_2 \neq x_2$ then $d(x_2, y_2) \geqq f(c)$ and the result follows. If $y_2 = x_2$, since $y_1 \cdots y_{tn} = e$, either $y_1 = x_2^{-1}$ or $y_3 = x_2^{-1}$ so that either $d(x_1, y_1) = d(x_1, x_2^{-1}) \geqq f(c)$ or $d(x_3, y_3) = d(x_3, x_2^{-1}) \geqq f(c)$, and the inequality follows. The cases $2 \leqq p \leqq t$ follow analogously.

THEOREM 4. *If X admits a continuous metric then $G(X)$ with the pseudo-metric topology has no small subgroups.*

REMARK. If $G(X)$ has no small subgroups then X must admit a continuous pseudo-metric, by [6].

PROOF. Let d be the extension of the continuous metric on X to a two-sided invariant pseudo-metric on $G(X)$. The open set $U = \{w \in G(X): d(w, e) < 1\}$ contains no nontrivial subgroups. This can be seen as follows. Let $w \in U - e$. Then by Lemma 1, $w^3 = 1c1^{-1}$ where $1 \in G(X)$ and $c = x_1 \cdots x_n$, $x_1 \neq x_n^{-1}$, and length $c \geqq 3$. Therefore for $t \in N$, $d(w^{3t}, e) = d(1c^t1^{-1}, e) = d(c^t, e)$ by two-sided invariance of d. By Lemmas 2 and 3, $(d(c^t, e) \geqq tf(c) \geqq tf(w)$. Thus $d(w^{3t}, e) \geqq tf(w)$ and for $t > f(w)^{-1}$, $w^{3t} \notin U$. Therefore U cannot contain a nontrivial subgroup.

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