Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–54, *2024*
DOI:10.1017/prm.2024.109

Numerical invariants on twisted noncommutative polyballs: curvature and multiplicity

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(Received 19 January 2024; revised 5 September 2024; accepted 5 September 2024)

The goal of this paper is to show that the theory of curvature invariant, as introduced by Arveson, admits a natural extension to the framework of $\mathcal{U}$-twisted polyballs $B^{\mathcal{U}}(\mathcal{H})$ which consist of k-tuples $(A_1,\ldots,A_k)$ of row contractions $A_i=(A_{i,1},\ldots,A_{i,n_i})$ satisfying certain $\mathcal{U}$-commutation relations with respect to a set $\mathcal{U}$ of unitary commuting operators on a Hilbert space $\mathcal{H}$. Throughout this paper, we will be concerned with the curvature of the elements $A\in B^{\mathcal{U}}(\mathcal{H})$ with positive trace class defect operator $\Delta_A(I)$. We prove the existence of the curvature invariant and present some of its basic properties. A distinguished role as a universal model among the pure elements in $\mathcal{U}$-twisted polyballs is played by the standard $I\otimes\mathcal{U}$-twisted multi-shift $\mathbf{S}$ acting on $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$. The curvature invariant $\operatorname{curv}(A)$ can be any non-negative real number and measures the amount by which A deviates from the universal model $\mathbf{S}$. Special attention is given to the $I\otimes\mathcal{U}$-twisted multi-shift $\mathbf{S}$ and the invariant subspaces (co-invariant) under $\mathbf{S}$ and $I\otimes\mathcal{U}$, due to the fact that any pure element $A\in B^{\mathcal{U}}(\mathcal{H})$ with $\Delta_A(I)\geq 0$ is the compression of $\mathbf{S}$ to such a co-invariant subspace.

Keywords: $\mathscr{U}$-twisted polyball; curvature invariant; multiplicity invariant; multivariable operator theory

2010 *Mathematics Subject Classification:* Primary: 47A65; 47A13; 47A15
Secondary: 47A20; 46L65

1. Introduction

In 2000, Arveson [2] introduced and studied the curvature and Euler characteristic for finite rank contractive Hilbert modules over $\mathbb{C}[z_1,\ldots,z_n]$, which are in fact numerical invariants for the commuting n-tuples of operators $X:=(X_1,\ldots,X_n)\in B(\mathcal{H})^n$, with $\Delta_X:=I-X_1X_1^*-\cdots-X_nX_n^*\geq 0$ and $\operatorname{rank}\Delta_X<\infty$, where $B(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Shortly after, the author [20] and, independently, Kribs [14] defined and studied a notion of curvature for the elements in the noncommutative unit ball

$$[B(\mathcal{H})^n]_1^-:=\{(X_1,\ldots,X_n)\in B(\mathcal{H})^n:\ I-X_1X_1^*-\cdots-X_nX_n^*\geq 0\}$$

and, as a consequence, for the invariant subspaces under the left creation operators $S_1, \ldots, S_n$ on the full Fock space $F^2(H_n)$ with n generators. Some of these results were extended by Muhly and Solel [16] to a class of completely positive maps on semifinite factors. The theory of Arveson's curvature on the symmetric Fock space $F_s^2(H_n)$ with n generators was significantly expanded due to the work by Greene, Richter, and Sundberg [11]; Fang [7]; and Gleason, Richter, and Sundberg [10]. Engliš remarked in [6] that using Arveson's ideas, one can extend the notion of curvature to complete Nevanlinna–Pick kernels. The extension of Arveson's theory to holomorphic spaces with non-Nevanlinna–Pick kernels was first considered by Fang [9] who was able to show that the main results about the curvature invariant on the symmetric Fock space carry over to the Hardy space $H^2(\mathbb{D}^k)$ over the polydisc. He also extended the theory to the invariant subspaces of the Dirichlet shift [8]. In the noncommutative setting, a notion of curvature invariant for noncommutative domains generated by positive regular free polynomials was considered in [21].

In [22], we developed a theory of curvature invariant for the *regular noncommutative polyball* and formulated a theory of curvature and multiplicity invariants for the tensor product of full Fock spaces $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ and also for the tensor product of symmetric Fock spaces. These results were used in [24] to study the Euler characteristic associated with the elements of the regular noncommutative polyball and obtain an analogue of Arveson's version of the Gauss–Bonnet–Chern theorem from Riemannian geometry, which connects the curvature to the Euler characteristic of some associated algebraic modules.

We say that $V := (V_1, \ldots, V_k)$, $k \geq 2$, is a k-tuple of *doubly $\mathcal{U}$-commuting row isometries* $V_i := [V_{i,1} \cdots V_{i,n_i}]$ with $V_{i,s} \in B(\mathcal{H})$ if

$$V_{i,s} \in \mathcal{U}' \quad \text{and} \quad V_{i,s}^* V_{j,t} = U_{i,j}(s,t)^* V_{j,t} V_{i,s}^* \text{ if } \ i \neq j,$$

where $\mathcal{U} := \{U_{i,j}(s,t)\} \subset B(\mathcal{H})$ is a set of commuting unitary operators such that $U_{j,i}(t,s) = U_{i,j}(s,t)^*$ if $i \neq j$ and $\mathcal{U}'$ is the commutant of $\mathcal{U}$. We note that, in the particular case when $n_i = 1$, $V_1, \ldots, V_k$ are unitary operators, and $U_{i,j} := \lambda_{i,j} I_{\mathcal{H}}$, $\lambda_{i,j} \in \mathbb{T}$, the corresponding universal C^*-algebras generated by $V_1, \ldots, V_k$ are the higher-dimensional noncommutative tori which are studied in noncommutative differential geometry (see [4, 30]). In the same setting, but assuming that $V_1, \ldots, V_k$ are isometries, De Jeu and Pinto [5] obtained Wold decompositions for doubly $\mathcal{U}$-commuting isometries. Inspired by their work, we studied in [25, 26] the structure of the k-tuples of doubly Λ-commuting row isometries, which corresponds to the particular case $U_{i,j}(s,t) = \lambda_{i,j}(s,t) I_{\mathcal{H}}$, where $\lambda_{i,j}(s,t) \in \mathbb{T}$.

The rotation algebras, noncommutative tori, the Heisenberg group C^*-algebras, as well as C^*-algebras generated by isometries with twisted commutation relations have been studied in the literature in various particular cases (see [1, 5, 12, 13, 15, 17–19, 25, 29, 31] and [26]). More recently, we studied in [27] the structure of the k-tuples of doubly $\mathcal{U}$-commuting *row isometries*, obtained Wold type decompositions [32], and used them to classify the k-tuples of doubly $\mathcal{U}$-commuting row isometries up to a unitary equivalence. Furthermore, in [28], we showed that many of the classical results concerning the dilation theory of contractions on Hilbert spaces have analogues for regular $\mathcal{U}$-twisted polyballs.

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Due to the Wold decomposition from [27], each k-tuple of doubly $\mathcal{U}$-commuting pure row isometries is unitarily equivalent to a *standard multi-shift* $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ with $\mathbf{S}_i := [\mathbf{S}_{i,1} \cdots \mathbf{S}_{i,n_i}]$, which is a k-tuple of doubly $I \otimes \mathcal{U}$-commuting pure row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$, where $\mathbb{F}_{n_i}^+$ is the unital free semigroup with n_i generators (see § 3 for the definition). It was proved in [28] that the standard multi-shift plays the role of a universal model for the pure elements in the $\mathcal{U}$-twisted polyball. To present this result, we introduce the *$\mathcal{U}$-twisted polyball* $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ as the set of all $\mathcal{U}$-commuting k-tuples $A := (A_1, \ldots, A_k)$ of row contractions $A_i := (A_{i,1}, \ldots, A_{i,n_i}) \in B(\mathcal{H})^{n_i}$, i.e.

$$A_{i,s} \in \mathcal{U}' \quad \text{and} \quad A_{i,s}A_{j,t} = U_{i,j}(s,t)A_{j,t}A_{i,s} \text{ if } \ i \neq j.$$

We proved in [28] that $A \in \mathbf{B}^{\mathcal{U}}(\mathcal{H})$ is a pure element, i.e. $\Phi_{A_i}^m(I) \to 0$ strongly as $m \to \infty$, where $\Phi_{A_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by $\Phi_{A_i}(X) := \sum_{s=1}^{n_i} A_{i,s}XA_{i,s}^*$, and the defect operator

$$\Delta_A(I) := (id - \Phi_{A_1}) \circ \cdots \circ (id - \Phi_{A_k})(I) \geq 0,$$

if and only if there are a Hilbert space $\mathcal{D} \subset \mathcal{H}$ and a multi-shift $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ with $\mathbf{S}_i := [\mathbf{S}_{i,1} \cdots \mathbf{S}_{i,n_i}]$ of doubly $I \otimes \mathcal{U}$-commuting pure isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}$ such that $\mathcal{H}$ is co-invariant under all operators $\mathbf{S}_{i,s}$ and $I \otimes U_{i,j}(s,t)$ and $A_{i,s}^* = \mathbf{S}_{i,s}^*|_{\mathcal{H}}$. Due to this reason, to understand the structure of pure elements A in the $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ with positive defect operators $\Delta_A(I)$, one should focus on the $I \otimes \mathcal{U}$-twisted multi-shifts $\mathbf{S}$ and the closed invariant (resp. co-invariant subspaces) under $\mathbf{S}$ and $I \otimes \mathcal{U}$.

The goal of the present paper is to show that the notion of curvature admits a natural extension to the framework of $\mathcal{U}$-twisted polyballs and to present its basic properties followed by several consequences. We remark that if $\mathcal{U} \neq \{I\}$, the row contractions $A_1, \ldots, A_k$ are not pairwise commuting. Due to this reason, the 'tensor product' techniques used in the theory of the regular polyballs and the curvature [22, 23] need to be replaced with new ones appropriate for the $\mathcal{U}$-twisted polyballs [27, 28].

In § 2, after a few preliminary results, we introduce (see Definition (2.7)) and prove the existence of the M-curvature, where $M = (M_1, \ldots, M_k) \in \mathbb{R}_+^k$, $M_i \geq \|\Phi_{A_i}^*(I)\| > 0$, and $\Phi_{A_i}^*(X) := \sum_{j=1}^{n_i} A_i^*XA_i$, associated with any k-tuple $A := (A_1, \ldots, A_k)$ of $\mathcal{U}$-commuting operators with positive trace class defect $\Delta_A(I)$ and show that

$$0 \leq \operatorname{curv}_M(A) \leq \operatorname{trace}\,[\Delta_A(I)]\,.$$

We also show that if A and $A^{'}$ are k-tuples of $\mathcal{U}$-commuting and $\mathcal{U}'$-commuting operators, respectively, with positive trace class defect operators, then $A \oplus A'$ is a k-tuple of $\mathcal{U} \oplus \mathcal{U}'$-commuting operators with a positive trace class defect operator and

$$\operatorname{curv}_M(A \oplus A') = \operatorname{curv}_M(A) + \operatorname{curv}_M(A').$$

If, in addition, $\dim \mathcal{H}' < \infty$, then $\operatorname{curv}_M(A \oplus A') = \operatorname{curv}_M(A)$.

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In § 3, under the assumption that A is in the $\mathcal{U}$-twisted polyball $B^{\mathcal{U}}(\mathcal{H})$ and has positive trace class defect operator $\Delta_A(I)$, we established several asymptotic formulas for the M-curvature invariant $\operatorname{curv}_M(A)$ in terms of the noncommutative Berezin kernel K_A associated with A.

Throughout this paper, special attention is given to the case $M = (n_1, \ldots, n_k)$, when the corresponding curvature, denote by $\operatorname{curv}(A)$, satisfies the asymptotic formula

$$\operatorname{curv}(A) = \lim_{m\to\infty} \frac{1}{\binom{m+k}{k}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})K_A\right]}{\operatorname{trace} P_{(q_1,\ldots,q_k)}},$$

where $P_{(q_1,\ldots,q_k)}$ is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$, with orthonormal basis $\{\chi_{(\alpha_1,\ldots,\alpha_k)}\}$, onto the subspace span $\{\chi_{(\alpha_1,\ldots,\alpha_k)} :\ \alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i| = q_i\}$. We provide several asymptotic formulas including

$$\operatorname{curv}(A) = \lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty} \frac{\operatorname{trace}\left[(id-\Phi_{A_1}^{q_1+1})\circ\cdots\circ(id-\Phi_{A_k}^{q_k+1})(I)\right]}{\prod_{i=1}^k(1+n_i+\cdots+n_i^{q_i})},$$

where Φ_{A_i} is the completely positive linear map associated with the row contraction A_i. We also show that the curvature invariant is upper semi-continuous.

The standard multi-shift $\mathbf{S}_{\mathbf{z}}$ associated with the scalar weights $\mathbf{z} = (z_{i,j}(s,t))$, where $z_{i,j}(s,t)\in\mathbb{T}$ and $z_{j,i}(t,s) = \overline{z_{i,j}(s,t)}$, plays an important role in this paper. This is due to the fact that any pure element A in the $\mathcal{U}$-twisted polyball with the property that $\Delta_A(I)\geq 0$ and $\operatorname{rank}\Delta_A(I) = m\in\mathbb{N}$ is the compression of a direct sum $\oplus_{p=1}^m \mathbf{S}_{\mathbf{z}^{(p)}}$ of scalar multi-shifts to a co-invariant subspace $\mathcal{M}^\perp$, i.e. $A_{i,s} = P_{\mathcal{M}^\perp}(\mathbf{S}_{\mathbf{z}^{(p)}})_{i,s}|_{\mathcal{M}^\perp}$.

Unlike the non-twisted case ($\mathcal{U} = \{I\}$) where we have, up to a unitary equivalence, just one standard shift $\mathbf{S}$ with $\operatorname{rank}\Delta_{\mathbf{S}}(I) = 1$, in the twisted case, all the multi-shifts $\mathbf{S}_{\mathbf{z}}$ satisfy the relation

$$\operatorname{curv}(\mathbf{S}_{\mathbf{z}}) = \operatorname{rank}\Delta_{\mathbf{S}_{\mathbf{z}}}(I) = 1,$$

and these are the only $\mathcal{U}$-twisted multi-shifts $\mathbf{S}$ with $\operatorname{rank}\Delta_{\mathbf{S}}(I) = 1$. Moreover, we prove, in § 4, that if $\mathbf{S}_{\mathcal{U}}$ and $\mathbf{S}_{\mathcal{U}'}$ are the standard multi-shifts associated with $\mathcal{U}\subset B(\mathcal{H})$ and $\mathcal{U}'\subset B(\mathcal{H}')$, respectively, then $\mathbf{S}_{\mathcal{U}}$ is jointly similar to $\mathbf{S}_{\mathcal{U}'}$ if and only if there is an invertible operator $W\in B(\mathcal{H},\mathcal{H}')$ such that $U_{i,j}(s,t) = W^{-1}U'_{i,j}(s,t)W$. Consequently, if $\mathbf{S}_{\mathbf{z}}$ and $\mathbf{S}_{\mathbf{z}'}$ are standard multi-shifts associated with the scalar weights $\mathbf{z}$ and $\mathbf{z}'$, respectively, then they are unitarily equivalent if and only if $\mathbf{z} = \mathbf{z}'$.

In § 5, we prove that if A is an element in the $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ such that it admits characteristic function Θ_A and the defect $\Delta_A(I)$ is a positive finite rank operator, then the *curvature operator* $\Delta_{\mathbf{S}}(K_AK_A^*)(N\otimes I_{\mathcal{H}})$ is a trace class

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and

$$\operatorname{curv}(A) = \operatorname{trace}\left[\Delta_{\mathbf{S}}(K_A K_A^*)(N \otimes I_{\mathcal{H}})\right],$$

where

$$N := \sum_{(s_1,\ldots,s_k)\in\mathbb{Z}_+^k} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} P_{(s_1,\ldots,s_k)}.$$

This leads to the index type formula

$$\operatorname{curv}(A) = \operatorname{rank}\Delta_A(I) - \operatorname{trace}\left[\Theta_A(P_{\mathbb{C}}\otimes I)\Theta_A^*(N\otimes I_{\mathcal{H}})\right]$$

which is used to show that the curvature invariant detects the elements in $B(\mathcal{H})^{n_1+\cdots+n_k}$ which are unitarily equivalent to an $I\otimes\mathcal{U}$-twisted multi-shift $\mathbf{S}$ of finite rank defect operator, i.e. acting on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$ with $\dim\mathcal{K}<\infty$ (see theorem 5.5).

In § 6, under the assumption that $\mathbf{S}$ is a $I\otimes\mathcal{U}$-twisted multi-shift on $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$ with $\dim\mathcal{H}<\infty$ and $\mathcal{M}$ is any invariant subspace under $\mathbf{S}$ and $I\otimes\mathcal{U}$, we introduce the multiplicity of $\mathcal{M}$ by setting

$$m(\mathcal{M}) := \lim_{m\to\infty}\frac{1}{\binom{m+k}{k}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}}\frac{\operatorname{trace}\left[P_{\mathcal{M}}(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]}{\operatorname{trace}P_{(q_1,\ldots,q_k)}}.$$

We prove its existence, provide several asymptotic formulas, and connect it to the curvature invariant by showing that

$$m(\mathcal{M}) = \dim\mathcal{H} - \operatorname{curv}(P_{\mathcal{M}^\perp}\mathbf{S}|_{\mathcal{M}^\perp}).$$

In particular, if $\mathbf{S_z}$ is the scalar $\mathbf{z}$-twisted multi-shift on $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$ and $\mathcal{M}$ is an invariant subspace under $\mathbf{S_z}\otimes I_{\mathcal{E}}$ with $\dim\mathcal{E}<\infty$, then its multiplicity exists. We remark that if $n_1=\cdots=n_k=1$, then $\mathcal{M}$ is in the vector-valued Hardy space $H^2(\mathbb{D}^k)\otimes\mathcal{E}$ and

$$m(\mathcal{M}) = \lim_{m\to\infty}\frac{\operatorname{trace}\left[P_{\mathcal{M}}(P_{\leq m}\otimes I_{\mathcal{E}})\right]}{\operatorname{trace}\left[P_{\leq m}\right]},$$

where $P_{\leq m}$ is the orthogonal projection on the polynomials of degree $\leq m$. This is a twisted version of Fang's [9] commutative result for $H^2(\mathbb{D}^k)\otimes\mathcal{E}$ when $\mathbf{z}=\{1\}$.

In § 6, we also obtain some results concerning the semi-continuity for the curvature and the multiplicity invariants. More precisely, we prove that if $\mathbf{S}$ is a $\mathcal{U}$-twisted multi-shift with $\mathcal{U}\subset B(\mathcal{H})$ and $\dim\mathcal{H}<\infty$, acting on $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$, and $\mathcal{M}$ and $\mathcal{M}_p$ are invariant subspaces of $\mathbf{S}$ and $\mathcal{U}$ such that $P_{\mathcal{M}_p}\to P_{\mathcal{M}}$ in the weak operator topology, then

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$$\limsup_{p\to\infty} \operatorname{curv}(P_{\mathcal{M}_p^\perp}\mathbf{S}|_{\mathcal{M}_p^\perp}) \le \operatorname{curv}(P_{\mathcal{M}^\perp}\mathbf{S}|_{\mathcal{M}^\perp})$$

and

$$\liminf_{p\to\infty} m(\mathcal{M}_p) \ge m(\mathcal{M}).$$

If $\mathcal{M}$ is a Beurling type invariant subspace under $\mathbf{S}$ and $I\otimes\mathcal{U}$, which does not contain nontrivial reducing subspace under $\mathbf{S}$, then we show that the *multiplicity operator* $\Delta_{\mathbf{S}}(P_{\mathcal{M}})(N\otimes I)$ is a trace class and

$$m(\mathcal{M}) = \operatorname{trace}\left[\Delta_{\mathbf{S}}(P_{\mathcal{M}})(N\otimes I)\right].$$

In particular, this relation holds for the Beurling type invariant subspace under $\mathbf{S_z}\otimes I_{\mathcal{E}}$ with $\dim\mathcal{E}<\infty$.

We remark that if A is an element in $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ and $\mathcal{M}$ is an invariant subspace under A and $\mathcal{U}$, then $A|_{\mathcal{M}}$ is not necessarily in the $\mathcal{U}|_{\mathcal{M}}$-twisted polyball in general. However, we will provide necessary and sufficient conditions when $A|_{\mathcal{M}}$ is in $\mathbf{B}^{\mathcal{U}|_{\mathcal{M}}}(\mathcal{H})$ and prove a stability result for the curvature invariant.

In § 7, we present some results concerning the range of the curvature and the multiplicity invariants. More precisely, we show that if $(n_1,\ldots,n_k)\in\mathbb{N}^k$ with $n_j\ge 2$ for some j and $t\in[0,m]$, then there exists a pure element A in the $\mathcal{U}$-twisted polyball such that $\operatorname{rank} A = m$ and $\operatorname{curv}(A)=t$. Consequently, the range of the curvature on the $\mathcal{U}$-twisted polyballs is $[0,\infty)$. This also implies that the range of the multiplicity invariant is $[0,\infty)$.

On the other hand, we show that the range of the curvature restricted to the class of doubly $\mathcal{U}$-commuting row isometries with trace class defect operators is $\mathbb{Z}_+$. In addition, if $V:=(V_1,\ldots,V_k)$ with $V_i:=[V_{i,1}\cdots V_{i,n_i}]$ and $V_{i,s}\in B(\mathcal{H})$ is a k-tuple of doubly $\mathcal{U}$-commuting row isometries with the trace class defect operator, we prove that

$$\operatorname{curv}(V) = \operatorname{trace}[\Delta_V(I)] = \operatorname{rank}\Delta_V(I)$$

and

$$\operatorname{curv}(V) = m\in\mathbb{Z}_+ \quad \text{if and only if} \quad \dim \bigcap_{\substack{i\in\{1,\ldots,k\}\\ s\in\{1,\ldots,n_i\}}} \ker V_{i,s}^* = m.$$

Moreover, if $\operatorname{curv}(V)\ne 0$ and $n_j\ge 2$ for some $j\in\{1,\ldots,k\}$, then we prove that, for any $t\in[0,\operatorname{curv}(V)]$, there is an invariant subspace $\mathcal{M}\subset\mathcal{H}$ under V and $\mathcal{U}$ such that $\operatorname{curv}(P_{\mathcal{M}^\perp}V|_{\mathcal{M}^\perp})=t$.

In the sequel to this paper, we study the Euler characteristic associated with the elements of the $\mathcal{U}$-twisted polyballs and obtain an analogue of Arveson's version of the Gauss–Bonnet–Chern theorem from Riemannian geometry.

2. Curvature invariant for $\mathcal{U}$-commuting operators

In this section, we consider a few preliminary results which are needed throughout the paper, introduce and prove the existence of the M-curvature, and present several asymptotic formulas and basic properties for the curvature invariant.

Let $k \in \mathbb{N} := \{1, 2, \ldots\}$ with $k \geq 2$, and consider the set

$$\Gamma := \{(i, j, s, t) : \ i, j \in \{1, \ldots, k\}, i \neq j, s \in \{1, \ldots, n_i\}, t \in \{1, \ldots, n_j\}\},$$

where $n_i, n_j \in \mathbb{N}$. Throughout this paper, $\mathcal{U} := \{U_{i,j}(s,t)\}_{(i,j,s,t)\in\Gamma}$ is a set of commuting unitary operators on a Hilbert space such that

$$U_{j,i}(t,s) = U_{i,j}(s,t)^*, \qquad (i,j,s,t) \in \Gamma.$$

Definition. *A k-tuple* $A := (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ *with* $A_i := (A_{i,1}, \ldots, A_{i,n_i})$ *is called* $\mathcal{U}$*-commuting if*

$$A_{i,s} \in \mathcal{U}' \quad \text{and} \quad A_{i,s}A_{j,t} = U_{i,j}(s,t)A_{j,t}A_{i,s}, \qquad (i,j,s,t) \in \Gamma,$$

where $\mathcal{U}'$ *is the commutant of* $\mathcal{U} \subset B(\mathcal{H})$*. If, in addition,*

$$A_{i,s}^*A_{j,t} = U_{i,j}(s,t)^*A_{j,t}A_{i,s}^*, \qquad (i,j,s,t) \in \Gamma,$$

we say that A *is a* k*-tuple of doubly* $\mathcal{U}$*-commuting operators.*

Definition. *The* $\mathcal{U}$*-twisted polyball* $\boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ *is the set of all* $\mathcal{U}$*-commuting* k*-tuples* $A := (A_1, \ldots, A_k)$ *of row contractions* $A_i := (A_{i,1}, \ldots, A_{i,n_i}) \in B(\mathcal{H})^{n_i}$*, i.e.*

$$A_{i,1}A_{i,1}^* + \cdots + A_{i,n_i}A_{i,n_i}^* \leq I.$$

We say that A *has a positive defect operator if* $\Delta_A(I) \geq 0$*, where*

$$\Delta_A(X) := (id - \Phi_{A_1}) \circ \cdots \circ (id - \Phi_{A_k})(X), \qquad X \in B(\mathcal{H}),$$

and $\Phi_{A_i} : B(\mathcal{H}) \to B(\mathcal{H})$ *is the completely positive linear map defined by* $\Phi_{A_i}(X) := \sum_{s=1}^{n_i} A_{i.s}XA_{i,s}^*$.

The regular $\mathcal{U}$*-twisted polyball* $\boldsymbol{B}^{\mathcal{U}}_{reg}(\mathcal{H})$ *is the set of all* k*-tuples* A *of* $\mathcal{U}$*-commuting row contractions such that* $\Delta_{rA}(I) \geq 0$ *for any* $r \in [0, 1)$*.*

According to Proposition 1.2 from [28], a k-tuple A of $\mathcal{U}$-commuting row contractions is in $\mathbf{B}^{\mathcal{U}}_{reg}(\mathcal{H})$ if and only if $(id - \Phi_{A_1})^{s_1} \circ \cdots \circ (id - \Phi_{A_k})^{s_k}(I) \geq 0$ for any $s_1, \ldots, s_k \in \{0, 1\}$. We note that $\mathbf{B}^{\mathcal{U}}_{reg}(\mathcal{H})$ is included in $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ and the inclusion is strict in general. On the other hand, it is easy to see that if $A \in \mathbf{B}^{\mathcal{U}}_{reg}(\mathcal{H})$, then $(z_1A_1, \ldots, z_kA_k) \in \mathbf{B}^{\mathcal{U}}_{reg}(\mathcal{H})$ for any $z_i \in \mathbb{D} := \{z \in \mathbb{C} : \ |z| \leq 1\}$.

Proposition 2.3. *If* $A := (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ *is a* k*-tuple of* $\mathcal{U}$*-commuting row operators and* $\mathcal{U}'$ *is the commutant* $\mathcal{U}$*, then*

$$\Phi_{A_{i_1}} \circ \Phi_{A_{i_2}} \circ \cdots \circ \Phi_{A_{i_p}}(Y) = \Phi_{A_{i_{\sigma(1)}}} \circ \Phi_{A_{i_{\sigma(2)}}} \circ \cdots \circ \Phi_{A_{i_{\sigma(p)}}}(Y), \qquad Y \in \mathcal{U}',$$

for any $i_1, \ldots, i_p \in \{1, \ldots, k\}$ *and any permutation* σ *of the set* $\{1, \ldots, p\}$ *and*

$$\Delta_A(Y) = (id - \Phi_{\omega(1)}) \circ \cdots \circ (id - \Phi_{\omega(k)})(Y)$$

for any permutation ω *of the set* $\{1, \ldots, k\}$*.*

If, in addition, $Y = Y^* \in \mathcal{U}'$ is such that $\Delta_A(Y) \geq 0$, then the following statements hold:

(i) For any $q_1, \ldots, q_k \in \mathbb{N}$,

$$0 \leq \sum_{p_k=0}^{q_k-1} \cdots \sum_{p_1=0}^{q_1-1} \Phi_{A_k}^{p_k} \circ \cdots \circ \Phi_{A_1}^{p_1}(\Delta_A(Y)) = (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_1}^{q_1})(Y).$$

(ii) If each A_i is pure, i.e. $\Phi_{A_i}^m(I) \to 0$ weakly as $m \to \infty$, then $Y \geq 0$,

$$Y = \sum_{p_k=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \Phi_{A_k}^{p_k} \circ \cdots \circ \Phi_{A_1}^{p_1}(\Delta_A(Y))$$

and

$$(id - \Phi_{A_1})^{s_1} \circ \cdots \circ (id - \Phi_{A_k})^{s_k}(Y) \geq 0 \quad \textit{for any} \quad s_1, \ldots, s_k \in \{0, 1\}$$

Proof. Since $A_{i,s}$ is in the commutant of $\mathcal{U}$ which is a set consisting of unitary operators, it is clear that $A_{i,s}$ is in the commutant of $\mathcal{U}^*$. Consequently, using the fact that $A_{i,s}A_{j,t} = U_{i,j}(s,t)A_{j,t}A_{i,s}$ for $(i, j, s, t) \in \Gamma$, we deduce that

$$\begin{aligned}
\Phi_{A_i} \circ \Phi_{A_j}(Y) &= \sum_{s=1}^{n_i} \sum_{t=1}^{n_j} A_{i,s}A_{j,t}YA_{j,t}^*A_{i,s}^* \\
&= \sum_{t=1}^{n_j} \sum_{s=1}^{n_i} U_{i,j}(s,t)A_{j,t}A_{i,s}YA_{i,s}^*A_{j,t}^*U_{i,j}(s,t)^* \\
&= \Phi_{A_j} \circ \Phi_{A_i}(Y)
\end{aligned}$$

for any $Y \in \mathcal{U}'$ and $i, j \in \{1, \ldots, k\}$. Using this argument repeatedly, one can easily prove the first part of the proposition.

Now, assume that $Y = Y^* \in \mathcal{U}'$ and $\Delta_A(Y) \geq 0$. Since $A_{i,s} \in \mathcal{U}'$, we also have $\Delta_A(Y) \in \mathcal{U}'$. Due to the first part of this proposition and the fact that $\Delta_A(Y) := (id - \Phi_{A_1}) \circ \cdots \circ (id - \Phi_{A_k})(Y) \geq 0$, item (i) follows immediately. To prove item (ii), assume that each A_i is pure, i.e. $\Phi_{A_i}^m(I) \to 0$ weakly as $m \to \infty$. Note that if $X = X^* \in B(\mathcal{H})$, then

$$-\|X\|\Phi_{A_i}^m(I) \leq \Phi_{A_i}^m(X) \leq \|X\|\Phi_{A_i}^m(I).$$

Hence, $\Phi_{A_i}^m(X) \to 0$ weakly as $m \to \infty$. Using this fact repeatedly in item (i), we conclude that

$$0 \leq Y = \sum_{p_k=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \Phi_{A_k}^{p_k} \circ \cdots \circ \Phi_{A_1}^{p_1}(\Delta_A(Y)),$$

where the convergence is in the weak operator topology. Since $\Delta_A(Y) \geq 0$ and $Y \geq 0$, we deduce that $\Phi_{A_1}(\Delta_{(A_2,\ldots,A_k)}(Y)) \leq \Delta_{(A_2,\ldots,A_k)}(Y)$, where

$\Delta_{(A_2,\ldots,A_k)}(Y) := (id - \Phi_{A_2}) \circ \cdots \circ (id - \Phi_{A_k})(Y)$ is a self-adjoint operator. Hence, $\Phi_{A_1}^m(\Delta_{(A_2,\ldots,A_k)}(Y)) \leq \Delta_{(A_2,\ldots,A_k)}(Y)$ for any $m \in \mathbb{N}$ and

$$-\|\Delta_{(A_2,\ldots,A_k)}(Y)\|\Phi_{A_1}^m(I) \leq \Phi_{A_1}^m(\Delta_{(A_2,\ldots,A_k)}(Y)) \leq \|\Delta_{(A_2,\ldots,A_k)}(Y)\|\Phi_{A_1}^m(I)$$

for any $m \in \mathbb{N}$. Taking $m \to \infty$ and using the fact that $\Phi_{A_1}^m(I) \to 0$ weakly as $m \to \infty$, we conclude that $\Delta_{(A_2,\ldots,A_k)}(Y) \geq 0$. Using similar arguments and the first part of the proposition, we can deduce that $(id - \Phi_{A_1})^{p_1} \circ \cdots \circ (id - \Phi_{A_k})^{p_k}(Y) \geq 0$ for any $p_i \in \{0, 1\}$. $\square$

Let $A := (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ with $A_i := [A_{i,1} \cdots A_{i,n_i}]$. We say that A_i is *row power bounded* if there is $M > 0$ such that $\|\Phi_{A_i}^m(I)\| \leq M$ for any $m \in \mathbb{N}$.

COROLLARY. Let $A := (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ be a k-tuple of $\mathcal{U}$-commuting row operators.

(i) If $\Delta_A(I) \geq 0$ and each A_i is a pure tuple, then A is in the regular $\mathcal{U}$-twisted polyball $\mathbf{B}_{reg}^{\mathcal{U}}(\mathcal{H})$ and

$$\sum_{p_k=0}^{\infty} \cdots \sum_{p_1=0}^{\infty} \Phi_{A_k}^{p_k} \circ \cdots \circ \Phi_{A_1}^{p_1}(\Delta_A(I)) = I.$$

(ii) If $\Delta_{rA}(I) \geq 0$ for any $r \in [0, 1)$ and each A_i is a row power bounded tuple, then A is in the regular $\mathcal{U}$-twisted polyball.

Proof. Part (i) follows from proposition 2.3, part (ii), in the particular case when $Y = I$. To prove item (ii), we note that since A_i is power bounded, rA_i is pure for any $r \in [0, 1)$. Applying proposition 2.3 part (ii) to rA, we deduce that

$$(id - \Phi_{rA_1})^{s_1} \circ \cdots \circ (id - \Phi_{rA_k})^{s_k}(I) \geq 0 \quad \text{for any} \quad s_1, \ldots, s_k \in \{0, 1\}, r \in [0, 1).$$

Hence, each A_i is a row contraction and A is in the regular $\mathcal{U}$-twisted polyball. This completes the proof. $\square$

Given two k-tuples $\mathbf{q} = (q_1, \ldots, q_k)$ and $\mathbf{p} = (p_1, \ldots, p_k)$ in $\mathbb{Z}_+^k$, where $\mathbb{Z}_+ := \{0, 1, \ldots\}$, we set $\mathbf{q} \leq \mathbf{p}$ if $q_i \leq p_i$ for any $i \in \{1, \ldots, k\}$. We consider $\mathbb{Z}_+^k$ as a directed set with respect to this partial order.

THEOREM 2.5 *Let $A := (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ with $A_i := (A_{i,1}, \ldots, A_{i,n_i})$ be a k-tuple of $\mathcal{U}$-commuting operators, and let Φ_{A_i} and $\Phi_{A_i}^*$ be the completely positive linear maps on $B(\mathcal{H})$ defined by*

$$\Phi_{A_i}(X) := \sum_{j=1}^{n_i} A_i X A_i^* \quad \textit{and} \quad \Phi_{A_i}^*(X) := \sum_{j=1}^{n_i} A_i^* X A_i.$$

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If $\Delta_A(I)$ *is a positive trace class operator and* $M_i \geq \|\Phi^*_{A_i}(I)\| > 0$*, then the limit*

$$\lim_{m\to\infty} \frac{1}{\binom{m+k-1}{k-1}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k=m}} \frac{1}{M_1^{q_1}\cdots M_k^{q_k}} \mathrm{trace}\left[\Phi^{q_1}_{A_1}\circ\cdots\circ\Phi^{q_k}_{A_k}(\Delta_A(I))\right]$$

exists and is bounded above by $\mathrm{trace}\,[\Delta_A(I)]$.

Proof. If $Y \geq 0$ is a trace class operator, then

$$\begin{aligned}\mathrm{trace}\,[\Phi_{A_i}(Y)] &= \sum_{j=1}^{n_i} \mathrm{trace}\,(A_{i,j} Y A^*_{i,j}) = \mathrm{trace}\left[\left(\sum_{j=1}^{n_i} A^*_{i,j}A_{i,j}\right) Y\right]\\ &\leq \|\Phi^*_{A_i}(I)\|\mathrm{trace}\, Y \leq M_i \mathrm{trace}\, Y\end{aligned}$$

for any $i\in\{1,\ldots,k\}$. Hence, $\frac{1}{M_i}\mathrm{trace}\,[\Phi_{A_i}(Y)] \leq \mathrm{trace}\, Y$ for any $i\in\{1,\ldots,k\}$. Consequently, since $\Delta_A(I)$ is a positive trace class operator and A is a $\mathcal{U}$-commuting tuple of operators, $\Delta_A(I)\in\mathcal{U}'$ and we can use proposition 2.3 to deduce that the multi-sequence

$$\begin{aligned}s_{q_1,\ldots,q_k} &:= \frac{1}{M_1^{q_1}\cdots M_k^{q_k}}\mathrm{trace}\left[\Phi^{q_1}_{A_1}\circ\cdots\circ\Phi^{q_k}_{A_k}(\Delta_A(I))\right]\\ &= \frac{1}{M_{\sigma(1)}^{q_1}\cdots M_{\sigma(k)}^{q_k}}\mathrm{trace}\left[\Phi^{q_1}_{A_{\sigma(1)}}\circ\cdots\circ\Phi^{q_k}_{A_{\sigma(k)}}(\Delta_A(I))\right]\end{aligned}$$

is decreasing with respect to each of the indices $q_1,\ldots,q_k$ and $s_{q_1,\ldots,q_k} \leq \mathrm{trace}\,(\Delta_A(I))$. Now, it is easy to see that

$$\lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty} s_{q_1,\ldots,q_k} = \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} s_{q_1,\ldots,q_k} = \inf_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} s_{q_1,\ldots,q_k} \leq \mathrm{trace}\,(\Delta_A(I)).$$

For each $\ell\in\mathbb{Z}_+$, denote

$$\Omega_\ell := \{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k:\ q_1+\cdots+q_k=m \text{ and } q_1\geq\ell,\ldots,q_k\geq\ell\}.$$

It is easy to see that $\mathrm{card}\,\Omega_0 = \binom{m+k-1}{k-1}$ and $\mathrm{card}\,(\Omega\backslash\Omega_\ell) \leq k\ell\binom{m+k-2}{k-1}$ for $\ell\in\mathbb{Z}_+$. Set $y := \inf_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} s_{q_1,\ldots,q_k}$, and let $\epsilon>0$. Then, there exists $p\in\mathbb{N}$, $p\geq 1$, such that $|s_{q_1,\ldots,q_k} - y| < \epsilon$ for any $q_1\geq p,\ldots,q_k\geq p$. Consequently, we deduce that

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$$\left|\frac{1}{\binom{m+k-1}{k-1}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k=m}} s_{q_1,\ldots,q_k}-y\right| = \left|\frac{1}{\binom{m+k-1}{k-1}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k=m}} (s_{q_1,\ldots,q_k}-y)\right|$$
$$\leq \frac{1}{\binom{m+k-1}{k-1}}\sum_{(q_1,\ldots,q_k)\in\Omega_0\setminus\Omega_p} |s_{q_1,\ldots,q_k}-y| + \frac{1}{\binom{m+k-1}{k-1}}\sum_{(q_1,\ldots,q_k)\in\Omega_p} |s_{q_1,\ldots,q_k}-y|$$
$$\leq \frac{1}{\binom{m+k-1}{k-1}} kp\binom{m+k-2}{k-2}[\mathrm{trace}\,(\Delta_A(I)+y]+\epsilon.$$

Taking $m\to\infty$, we deduce that

$$\lim_{m\to\infty}\frac{1}{\binom{m+k-1}{k-1}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k=m}} s_{q_1,\ldots,q_k}=y.$$

This completes the proof. □

Corollary 2.6. The limit in theorem 2.5 is equal to

$$\lim_{m\to\infty}\frac{1}{\binom{m+k}{k}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}} \frac{1}{M_1^{q_1}\cdots M_k^{q_k}}\mathrm{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right]$$
$$=\lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty}\frac{1}{M_{\sigma(1)}^{q_1}\cdots M_{\sigma(k)}^{q_k}}\mathrm{trace}\left[\Phi_{A_{\sigma(1)}}^{q_1}\circ\cdots\circ\Phi_{A_{\sigma(k)}}^{q_k}(\Delta_A(I))\right]$$
$$=\lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty}\frac{1}{q_1}\sum_{s_1=0}^{q_1}\cdots\frac{1}{q_k}\sum_{s_k=0}^{q_k}\frac{1}{M_{\sigma(1)}^{s_1}\cdots M_{\sigma(k)}^{s_k}}\mathrm{trace}$$
$$\left[\Phi_{A_{\sigma(1)}}^{s_1}\circ\cdots\circ\Phi_{A_{\sigma(k)}}^{s_k}(\Delta_A(I))\right]$$

for any permutation σ of the set $\{1,\ldots,k\}$.

Proof. Setting $a_m:=\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k=m}}\frac{1}{M_1^{q_1}\cdots M_k^{q_k}}\mathrm{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right]$ and $b_m=\binom{m+k-1}{k-1}$, and taking into account that

$$\binom{k-1}{k-1}+\binom{k}{k-1}+\cdots+\binom{m+k-1}{k-1}=\binom{m+k}{k},$$

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we have

$$\frac{a_0+a_1+\cdots+a_m}{b_0+b_1+\cdots+b_m}=\frac{1}{\binom{m+k}{k}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}}\frac{1}{M_1^{q_1}\cdots M_k^{q_k}}$$
$$\operatorname{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right].$$

Due to theorem 2.5, $\lim_{m\to\infty}\frac{a_m}{b_m}$ exists. Consequently, denoting by y the limit in theorem 2.5 and using Stolz–Cesáro convergence theorem, we deduce that

$$\lim_{m\to\infty}\frac{1}{\binom{m+k}{k}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}}\frac{1}{M_1^{q_1}\cdots M_k^{q_k}}\operatorname{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right]=y.$$

According to the proof of theorem 2.5, we also have

$$y=\lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty}\frac{1}{M_{\sigma(1)}^{q_1}\cdots M_{\sigma(k)}^{q_k}}\operatorname{trace}\left[\Phi_{A_{\sigma(1)}}^{q_1}\circ\cdots\circ\Phi_{A_{\sigma(k)}}^{q_k}(\Delta_A(I))\right].$$

The last equality in the corollary follows after a repeated application of the Stolz-Cesásro convergence theorem. We leave it to the reader. □

Let $A:=(A_1,\ldots,A_k)\in B(\mathcal{H})^{n_1+\cdots+n_k}$ with $A_i:=(A_{i,1},\ldots,A_{i,n_i})$ be a k-tuple of $\mathcal{U}$-commuting operators and let Φ_{A_i} and $\Phi_{A_i}^*$ be the completely positive linear maps on $B(\mathcal{H})$ defined by

$$\Phi_{A_i}(X):=\sum_{j=1}^{n_i}A_iXA_i^* \quad \text{and} \quad \Phi_{A_i}^*(X):=\sum_{j=1}^{n_i}A_i^*XA_i.$$

Throughout this paper, we assume that the defect operator $\Delta_A(I)$ is a positive trace class operator and $M:=(M_1,\ldots,M_k)$ is such that $M_i\geq\|\Phi_{A_i}^*(I)\|>0$.

Definition 2.7. *The M-curvature of A is defined by the relation*

$$\operatorname{curv}_M(A):=\lim_{m\to\infty}\frac{1}{\binom{m+k}{k}}\sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}}\frac{1}{M_1^{q_1}\cdots M_k^{q_k}}\operatorname{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right].$$

If $M_i=\|\Phi_{A_i}^*(I)\|$ for any $i\in\{1,\ldots,k\}$, the corresponding M-curvature is called $*$-*curvature* and we use the notation $\operatorname{curv}_*(A)$. We remark that if A belongs to the $\mathcal{U}$-twisted polyball, then we can take $M_i=n_i$, in which case the corresponding M-curvature is called *curvature* and we denote it by $\operatorname{curv}(A)$.

Corollary 2.8. Let $A:=(A_1,\ldots,A_k)$ be a k-tuple of $\mathcal{U}$-commuting operators such that $\Delta_A(I)$ is a positive trace class operator. Then the following statements hold:

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(i) For any $M := (M_1, \dots, M_k)$ such that $M_i \geq \|\Phi^*_{A_i}(I)\| > 0$,

$$0 \leq \operatorname{curv}_M(A) \leq \operatorname{curv}_*(A) \leq \operatorname{trace}\left[\Delta_A(I)\right].$$

(ii) If $\operatorname{curv}_*(A) > 0$ and there is j such that $M_j > \|\Phi^*_{A_j}(I)\|$, then

$$\operatorname{curv}_M(A) = 0.$$

(iii) If A is in the $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ and $\operatorname{curv}(A) > 0$, then

$$\operatorname{curv}_*(A) = \operatorname{curv}(A) \quad \text{and} \quad \|\Phi^*_{A_i}(I)\| = n_i$$

for any $i \in \{1, \dots, k\}$.

(iv) If A is in the regular $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}_{reg}(\mathcal{H})$, then

$$\operatorname{curv}_*(A) \leq \operatorname{trace}\left[\Delta_A(I)\right] \leq \operatorname{rank}\left[\Delta_A(I)\right].$$

Proof. Using theorem 2.5, we obtain the inequalities in item (i). To prove (ii), assume that $\operatorname{curv}_*(A) > 0$ and there is j such that $M_j > \|\Phi^*_{A_j}(I)\|$. Suppose that $\operatorname{curv}_M(A) > 0$. Since $\operatorname{curv}_*(A) < \infty$, we have $\frac{\operatorname{curv}_*(A)}{\operatorname{curv}_M(A)} < \infty$. On the other hand, due to theorem 2.5 and corollary 2.6,

$$\operatorname{curv}_M(A) = \lim_{q_1 \to \infty} \cdots \lim_{q_k \to \infty} \frac{1}{M_1^{q_1} \cdots M_k^{q_k}} \operatorname{trace}\left[\Phi^{q_1}_{A_1} \circ \cdots \circ \Phi^{q_k}_{A_k}(\Delta_A(I))\right].$$

Consequently,

$$\frac{\operatorname{curv}_*(A)}{\operatorname{curv}_M(A)} = \lim_{q_1 \to \infty} \cdots \lim_{q_k \to \infty} \left(\frac{M_1}{\|\Phi^*_{A_1}(I)\|}\right)^{q_1} \cdots \left(\frac{M_k}{\|\Phi^*_{A_k}(I)\|}\right)^{q_k} = \infty,$$

a contradiction. Therefore, $\operatorname{curv}_M(A) = 0$.

Now, we prove item (iii). If $A \in \mathbf{B}^{\mathcal{U}}(\mathcal{H})$, then each A_i is a row contraction and, consequently, $\|\Phi^*_{A_i}(I)\| \leq n_i$ for any $i \in \{1, \dots, k\}$. Due to item (i) when $M_i := n_i$, we have

$$0 \leq \operatorname{curv}(A) \leq \operatorname{curv}_*(A) \leq \operatorname{trace}\left[\Delta_A(I)\right].$$

Since $\operatorname{curv}(A) > 0$, we also have $\operatorname{curv}_*(A) > 0$. Applying item (ii) when $M_i := n_i$, we conclude that $\|\Phi^*_{A_i}(I)\| = n_i$ for any $i \in \{1, \dots, k\}$, which implies $\operatorname{curv}_*(A) = \operatorname{curv}(A)$. This proves item (iii).

If $A \in \mathbf{B}^{\mathcal{U}}_{reg}(\mathcal{H})$, then the defect $\Delta_A(I)$ is a positive contraction. Consequently, we have $\operatorname{trace}\left[\Delta_A(I)\right] \leq \operatorname{rank}\left[\Delta_A(I)\right]$, which proves item (iv). The proof is complete. □

Proposition 2.9. *Let $A \in B(\mathcal{H})^{n_1 + \cdots + n_k}$ and $A' \in B(\mathcal{H}')^{n_1 + \cdots + n_k}$ be k-tuples of $\mathcal{U}$-commuting and $\mathcal{U}'$-commuting operators, respectively, with positive trace class*

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defect operators. Then, $A \oplus A'$ is a k-tuple of $\mathcal{U} \oplus \mathcal{U}'$-commuting operators with a positive trace class defect operator and

$$\operatorname{curv}_M(A \oplus A') = \operatorname{curv}_M(A) + \operatorname{curv}_M(A').$$

If, in addition, $\dim \mathcal{H}' < \infty$, then $\operatorname{curv}_M(A \oplus A') = \operatorname{curv}_M(A)$.

Proof. It is straightforward due to either theorem 2.5 or corollary 2.6. □

3. The curvature and the noncommutative Berezin kernel on $\mathcal{U}$-twisted polyballs

Under the assumption that A is in the $\mathcal{U}$-twisted polyball $B^{\mathcal{U}}(\mathcal{H})$ and has a positive trace class defect operator $\Delta_A(I)$, we established several asymptotic formulas for the curvature invariant in terms of the noncommutative Berezin kernel associated with A. We also show that the curvature is upper semi-continuous.

Definition. *We say that $V := (V_1, \ldots, V_k)$ is a k-tuple of doubly $\mathcal{U}$-commuting row isometries $V_i := [V_{i,1} \cdots V_{i,n_i}]$ with $V_{i,s} \in B(\mathcal{H})$ if*

$$V_{i,s} \in \mathcal{U}' \quad \textit{and} \quad V_{i,s}^* V_{j,t} = U_{i,j}(s,t)^* V_{j,t} V_{i,s}^*, \qquad (i,j,s,t) \in \Gamma.$$

The above doubly $\mathcal{U}$-commutation relation implies the $\mathcal{U}$-commutation relation $V_{i,s}V_{j,t} = U_{i,j}(s,t)V_{j,t}V_{i,s}$ for any $(i,j,s,t) \in \Gamma$ (see [27]). Therefore, V is also a $\mathcal{U}$-commuting k-tuple.

In what follows, we fix a set $\mathcal{U} := \{U_{i,j}(s,t)\}_{(i,j,s,t)\in\Gamma} \subset B(\mathcal{H})$ of commuting unitary operators such that $U_{j,i}(t,s) = U_{i,j}(s,t)^*$ for $(i,j,s,t) \in \Gamma$. We recall the definition of the *standard $I \otimes \mathcal{U}$-twisted multi-shift* $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ acting on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$, where $\mathbb{F}_{n_i}^+$ is the unital free semigroup with generators $g_1^i, \ldots g_{n_i}^i$ and neutral element $g^i{}_0$. Let $\{\chi_{(\alpha_1,\ldots,\alpha_k)}\}$, $\alpha_i \in \mathbb{F}_{n_i}^+$, be the orthonormal basis for $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$. For each $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$, let $\mathbf{S}_i := [\mathbf{S}_{i,1} \cdots \mathbf{S}_{i,n_i}]$ be the row operator defined by setting

$$\mathbf{S}_{i,s}\left(\chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h\right) := \begin{cases} \chi_{(g_s^i\alpha_1,\alpha_2,\ldots,\alpha_k)} \otimes h, & \text{if } i = 1 \\ \chi_{(\alpha_1,\ldots,\alpha_{i-1},g_s^i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)} \otimes \boldsymbol{U}_{i,1}(s,\alpha_1)\cdots \\ \qquad \boldsymbol{U}_{i,i-1}(s,\alpha_{i-1})h, & \text{if } i \in \{2,\ldots,k\} \end{cases}$$

for any $h \in \mathcal{H}$, $\alpha_1 \in \mathbb{F}_{n_1}^+, \ldots, \alpha_k \in \mathbb{F}_{n_k}^+$, where

$$\boldsymbol{U}_{i,j}(s,\beta) := \begin{cases} \prod_{b=1}^q U_{i,j}(s,j_b) & \text{if } \beta = g_{j_1}^j \cdots g_{j_q}^j \in \mathbb{F}_{n_j}^+ \\ I & \text{if } \beta = g_0^j \end{cases}$$

for any $j \in \{1, \ldots, k\}$. Due to [27], $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ is a k-tuple of doubly $I \otimes \mathcal{U}$-commuting pure row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$. If we need to emphasize the dependence of $\mathbf{S}$ on the set $\mathcal{U}$, we use the notation $\mathbf{S}_{\mathcal{U}}$.

Let $A := (A_1, \dots, A_k)$ be a k-tuple such that its defect operator

$$\Delta_A(I) := (id - \Phi_{A_1}) \circ \cdots \circ (id - \Phi_{A_k})(I) \geq 0$$

and each $A_i := [A_{i,1} \cdots A_{i,n_i}]$ is *row power bounded*, i.e. there is a constant $C > 0$ such that $\|\Phi_{A_i}^m(I)\| \leq C$ for any $m \in \mathbb{N}$. Following [28], the *noncommutative Berezin kernel* $K_A : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$ associated with A is defined by setting

$$K_A h := \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+} \chi_{(\beta_1, \dots, \beta_k)} \otimes \Delta_A(I)^{1/2} A_{k,\beta_k}^* \cdots A_{1,\beta_1}^* h, \qquad h \in \mathcal{H},$$

where $A_{i,\beta_i} := A_{i,p_1} \cdots A_{i,pm}$ if $\beta_i = g_{p_1}^i \cdots g_{pm}^i \in \mathbb{F}_{n_i}^+$ and $A_{i,g_0^i} = I$. A simple extension of theorem 1.5 from [28] is the following. We denote by $vN(\mathcal{U})$ the von Neumann algebra generated by $\mathcal{U}$.

THEOREM 3.2 *Let $A := (A_1, \dots, A_k) \in B(\mathcal{H})^{n_1 + \cdots + n_k}$ be a $\mathcal{U}$-commuting k-tuple such that $\Delta_A(I) \geq 0$ and each $A_i := [A_{i,1} \cdots A_{i,n_i}]$ is row power bounded. If $\boldsymbol{S} := (\boldsymbol{S}_1, \dots, \boldsymbol{S}_k)$ with $\boldsymbol{S}_i := [\boldsymbol{S}_{i,1} \cdots \boldsymbol{S}_{i,n_i}]$ is the standard k-tuple of doubly $I \otimes \mathcal{U}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$, then the following statements hold:*

(i) The noncommutative Berezin kernel K_A is a bounded linear operator.
(ii) $K_A^* K_A = \lim_{q_k \to \infty} \dots \lim_{q_1 \to \infty} (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_1}^{q_1})(I)$.
(iii) For any $i \in \{1, \dots, k\}$ and $s \in \{1, \dots, n_i\}$,

$$K_A A_{i,s}^* = \boldsymbol{S}_{i,s}^* K_A.$$

(iv) For any $(i, j, s, t) \in \Gamma$,

$$K_A U = (I \otimes U) K_A, \qquad U \in vN(\mathcal{U}).$$

Proof. Note that

$$\begin{aligned}\|K_A h\|^2 &= \left\langle \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+} A_{1,\beta_1} \cdots A_{k,\beta_k} \Delta_A(I) A_{k,\beta_k}^* \cdots A_{1,\beta_1}^* h, h \right\rangle \\ &= \left\langle \sum_{p_1, \dots, p_k = 0}^{\infty} \Phi_{A_1}^{p_1} \circ \cdots \circ \Phi_{A_k}^{p_k} [\Delta_A(I)] h, h \right\rangle\end{aligned}$$

for any $h \in \mathcal{H}$. Due to proposition 2.3, for any $q_1, \dots, q_k \in \mathbb{N}$,

$$\sum_{p_k=0}^{q_k-1} \cdots \sum_{p_1=0}^{q_1-1} \Phi_{A_k}^{p_k} \circ \cdots \circ \Phi_{A_1}^{p_1}(\Delta_A(I)) = (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_1}^{q_1})(I).$$

Since the latter product is a sum of 2^k terms of the form $\pm \Phi_{A_{i_1}}^{q_{i_1}} \circ \cdots \circ \Phi_{A_{i_p}}^{q_{i_p}}(I)$ and there is $C > 0$ such that $\|\Phi_{A_i}^m(I)\| \leq C$ for any $m \in \mathbb{N}$ and $i \in \{1, \dots, k\}$, we have

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$\|(id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_1}^{q_1})(I)\| \leq C^k 2^k$ for any $q_1, \ldots, q_k \in \mathbb{Z}_+$. Consequently, taking into account that $\sum_{p_k=0}^{q_k-1} \cdots \sum_{p_1=0}^{q_1-1} \Phi_{A_k}^{p_k} \circ \cdots \circ \Phi_{A_1}^{p_1}(\Delta_A(I))$ is an increasing multi-sequence of positive operators with respect to the indices $q_1, \ldots, q_k$, we conclude that K_A is a bounded linear operator and item (ii) holds. The proof of items (iii) and (iv) is similar to the one corresponding to the particular case when A is in the regular $\mathcal{U}$-twisted polyball (see theorem 1.5 from [28]). We shall omit it. □

Under the assumption that A is in the $\mathcal{U}$-twisted polyball $B^{\mathcal{U}}(\mathcal{H})$ and has positive trace class defect operator $\Delta_A(I)$, we established several asymptotic formulas for the curvature invariant in terms of the noncommutative Berezin kernel associated with A.

For each $(q_1, \ldots, q_k) \in \mathbb{Z}_+^k$, let $P_{(q_1,\ldots,q_k)}$ be the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ onto the subspace

$$\operatorname{span}\{\chi_{(\alpha_1,\ldots,\alpha_k)} : \ \alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| = q_i\}.$$

We also denote by $P_{\leq(q_1,\ldots,q_k)}$ the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ onto the subspace

$$\operatorname{span}\{\chi_{(\alpha_1,\ldots,\alpha_k)} : \ \alpha_i \in \mathbb{F}_{n_i}^+, |\alpha_i| \leq q_i\}.$$

Theorem 3.3 *Let A be in the $\mathcal{U}$-twisted polyball $\boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ such that $\Delta_A(I)$ is a positive trace class operator. Then, the M-curvature of A satisfies the asymptotic formulas*

$$\begin{aligned}
\operatorname{curv}_M(A) &= \lim_{m\to\infty} \frac{1}{\binom{m+k}{k}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})K_A\right]}{M_1^{q_1}\cdots M_k^{q_k}}\\
&= \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})K_A\right]}{M_1^{q_1}\cdots M_k^{q_k}}\\
&= \lim_{m\to\infty} \frac{1}{\binom{m+k-1}{k-1}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k= m}} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})K_A\right]}{M_1^{q_1}\cdots M_k^{q_k}}.
\end{aligned}$$

If, in addition, $M_i \geq 1$ for any $i \in \{1, \ldots, k\}$, then

$$\operatorname{curv}_M(A) = \lim_{q_1\to\infty} \cdots \lim_{q_k\to\infty} \frac{\operatorname{trace}\left[K_A^*(P_{\leq(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})K_A\right]}{\prod_{i=1}^k (1 + M_i + \cdots + M_i^{q_i})}.$$

Proof. Let $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$, with $\mathbf{S}_i := (\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i})$ being the standard k-tuple of doubly $I \otimes \mathcal{U}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$. Note that

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$$\mathbf{S}_{i,s}^*\left(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h\right) = \begin{cases} \chi_{(\alpha_1,\ldots,\alpha_{i-1},\beta_i,\alpha_{i+1},\ldots,\alpha_k)}\otimes \boldsymbol{U}_{i,1}(s,\alpha_1)^*\cdots \\ \qquad \boldsymbol{U}_{i,i-1}(s,\alpha_{i-1})^*h, & \text{if } \alpha_i = g_s^i\beta_i \\ 0, & \text{otherwise} \end{cases}$$

for any $h\in\mathcal{H}$, $\alpha_1\in\mathbb{F}_{n_1}^+,\ldots,\alpha_k\in\mathbb{F}_{n_k}^+$, and $\beta_i\in\mathbb{F}_{n_i}^+$, and, consequently,

$$\begin{aligned}\sum_{s=1}^{n_i}\mathbf{S}_{i,s}\mathbf{S}_{i,s}^*\left(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h\right) &= \begin{cases} \chi_{(\alpha_1,\ldots,\alpha_{i-1},\alpha_i,\alpha_{i+1},\ldots,\alpha_k)}\otimes \boldsymbol{U}_{i,1}(s,\alpha_1)\boldsymbol{U}_{i,1}(s,\alpha_1)^*\cdots \\ \qquad \boldsymbol{U}_{i,i-1}(s,\alpha_{i-1})\boldsymbol{U}_{i,i-1}(s,\alpha_{i-1})^*h, & \text{if } |\alpha_i|\geq 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h, & \text{if } |\alpha_i|\geq 1 \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Since $\mathbf{S}:=(\mathbf{S}_1,\ldots,\mathbf{S}_k)$ with $\mathbf{S}_i:=(\mathbf{S}_{i,1},\ldots,\mathbf{S}_{i,n_i})$ is k-tuple of doubly $I\otimes\mathcal{U}$-commuting row isometries and $\mathcal{U}:=\{U_{i,j}(s,t)\}_{(i,j,s,t)\in\Gamma}$ is a set of commuting unitary operators on a Hilbert space $\mathcal{H}$ such that $U_{j,i}(t,s)=U_{i,j}(s,t)^*$ for $(i,j,s,t)\in\Gamma$, one can use the $I\otimes\mathcal{U}$-commutation relations for $\mathbf{S}$ to check that

$$(\mathbf{S}_{i,s}\mathbf{S}_{i,s}^*)(\mathbf{S}_{j,t}\mathbf{S}_{j,t}^*) = (\mathbf{S}_{j,t}\mathbf{S}_{j,t}^*)(\mathbf{S}_{i,s}\mathbf{S}_{i,s}^*)$$

for any $i,j\in\{1,\ldots,k\}$, $i\neq j$, $s\in\{1,\ldots,n_i\}$, $t\in\{1,\ldots,n_j\}$. Moreover, for any $i_1,\ldots,i_p$ distinct elements in $\{1,\ldots,k\}$ and $\alpha_1\in\mathbb{F}_{n_{i_1}}^+,\ldots,\ \alpha_p\in\mathbb{F}_{n_{i_p}}^+$, one can prove that

$$\left(\mathbf{S}_{i_1,\alpha_1}\mathbf{S}_{i_1,\alpha_1}^*\right)\cdots\left(\mathbf{S}_{i_p,\alpha_p}\mathbf{S}_{i_p,\alpha_p}^*\right) = \mathbf{S}_{i_1,\alpha_1}\cdots\mathbf{S}_{i_p,\alpha_p}\mathbf{S}_{i_p,\alpha_p}^*\cdots\mathbf{S}_{i_1,\alpha_1}^*. \tag{3.1}$$

Consequently, it is easy to see that

$$(id-\Phi_{\mathbf{S}_1})\circ\cdots\circ(id-\Phi_{\mathbf{S}_k})(I) = \prod_{i=1}^k\left(I-\sum_{s=1}^{n_i}\mathbf{S}_{i,s}\mathbf{S}_{i,s}^*\right) = P_{\mathbb{C}}\otimes I_{\mathcal{H}},$$

where $P_{\mathbb{C}}$ is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$ onto $\mathbb{C}\subset\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$, where $\mathbb{C}$ is identified with $\chi_{(g_0^1,\ldots,g_0^k)}\mathbb{C}$. On the other hand, one can check that

$$\left(\sum_{\alpha\in\mathbb{F}_{n_i}^+,|\alpha|=q_i}\mathbf{S}_{i,\alpha}\mathbf{S}_{i,\alpha}^*\right)\left(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h\right) = \begin{cases} \chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h, & \text{if } |\alpha_i|\geq q_i \\ 0, & \text{otherwise} \end{cases}$$

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for any $q_i \in \mathbb{N}$. Hence, we deduce that

$$
\begin{aligned}
&(\Phi_{\mathbf{S}_i}^{q_i} - \Phi_{\mathbf{S}_i}^{q_i+1})(I)\left(\chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h\right) \\
&\qquad = \left(\sum_{\alpha\in\mathbb{F}_{n_i}^+, |\alpha|=q_i} \mathbf{S}_{i,\alpha}\mathbf{S}_{i,\alpha}^* - \sum_{\alpha\in\mathbb{F}_{n_i}^+, |\alpha|=q_i+1} \mathbf{S}_{i,\alpha}\mathbf{S}_{i,\alpha}^*\right)\left(\chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h\right) \\
&\qquad = \begin{cases} \chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h, & \text{if } |\alpha_i| = q_i \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Consequently, using relation (3.1), we obtain

$$
\begin{aligned}
&(\Phi_{\mathbf{S}_1}^{q_1} - \Phi_{\mathbf{S}_1}^{q_1+1}) \circ \cdots \circ (\Phi_{\mathbf{S}_k}^{q_k} - \Phi_{\mathbf{S}_k}^{q_k+1})(I) \\
&\qquad = \prod_{i=1}^k \left(\sum_{\alpha\in\mathbb{F}_{n_i}^+, |\alpha|=q} \mathbf{S}_{i,\alpha}\mathbf{S}_{i,\alpha}^* - \sum_{\alpha\in\mathbb{F}_{n_i}^+, |\alpha|=q_i+1} \mathbf{S}_{i,\alpha}\mathbf{S}_{i,\alpha}^*\right) \\
&\qquad = P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}}.
\end{aligned} \tag{3.2}
$$

On the other hand, due to theorem 3.2, we have

$$
K_A^* K_A = \lim_{q_k\to\infty} \cdots \lim_{q_1\to\infty} (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_1}^{q_1})(I),
$$

where the limits are in the weak operator topology. Since

$$
0 \le (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_1}^{q_1})(I) = \sum_{s_k=0}^{q_k-1} \Phi_{A_k}^{s_k} \circ \cdots \sum_{s_1=0}^{q_1-1} \Phi_{A_1}^{s_1} \circ (id - \Phi_{A_k}) \circ \cdots \circ (id - \Phi_{A_1})(I)
$$

and $\Delta_A(I) := (id - \Phi_{A_k}) \circ \cdots \circ (id - \Phi_{A_1})(I) \ge 0$, the sequence $\{(id - \Phi_{T_k}^{q_k}) \circ \cdots \circ (id - \Phi_{T_1}^{q_1})(I)\}_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k}$ is increasing with respect to each index q_i. Using proposition 2.3, the fact that $\Phi_{A_1}, \ldots, \Phi_{A_k}$ are WOT-continuous completely positive and contractive linear maps, and WOT-$\lim_{q_i\to\infty} \Phi_{A_i}^{q_i}(I)$ exists for each $i \in \{1, \ldots, k\}$, we deduce that

$$
\begin{aligned}
(id - \Phi_{A_1})(K_A^* K_A) &= \lim_{q_k\to\infty} \cdots \lim_{q_1\to\infty} (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_1}^{q_1}) \circ (id - \Phi_{A_1})(I) \\
&= \lim_{q_k\to\infty} \cdots \lim_{q_2\to\infty} (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_2}^{q_2}) \\
&\qquad \left[\lim_{q_1\to\infty} (id - \Phi_{A_1}^{q_1}) \circ (id - \Phi_{A_1})(I)\right] \\
&= \lim_{q_k\to\infty} \cdots \lim_{q_2\to\infty} (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_2}^{q_2}) \circ (id - \Phi_{A_1})(I).
\end{aligned}
$$

Consequently, composing to the left by $id - \Phi_{A_2}$, a similar reasoning leads to

$$
(id - \Phi_{A_2}) \circ (id - \Phi_{A_1})(K_A^* K_A) = \lim_{q_k\to\infty} \cdots \lim_{q_3\to\infty} (id - \Phi_{A_k}^{q_k}) \circ \cdots \circ (id - \Phi_{A_3}^{q_3}) \circ
$$

$$(id - \Phi_{A_1}) \circ (id - \Phi_{A_2})(I).$$

Continuing this process, we obtain

$$\Delta_A(K_A^* K_A) = \Delta_A(I). \tag{3.3}$$

On the other hand, due to theorem 3.2, we have $K_A A_{i,j}^* = \mathbf{S}_{i,j}^* K_A$ for any $i \in \{1, \ldots, k\}$ and any $j \in \{1, \ldots, n_i\}$. Consequently, we deduce that

$$\begin{aligned} K_A^* &\left[\Phi_{\mathbf{S}_1}^{q_1} \circ \cdots \circ \Phi_{\mathbf{S}_k}^{q_k} \circ (id - \Phi_{\mathbf{S}_k}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_1})(I)\right] K_A \\ &= \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k} \circ (id - \Phi_{A_1}) \circ \cdots \circ (id - \Phi_{A_k})(K_A^* K_A). \end{aligned}$$

Hence and using relations (3.2) and (3.3), we deduce that

$$\begin{aligned} K_A^*(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})K_A &= K_A^* \left[(\Phi_{\mathbf{S}_1}^{q_1} - \Phi_{\mathbf{S}_1}^{q_1+1}) \circ \cdots \circ (\Phi_{\mathbf{S}_k}^{q_k} - \Phi_{\mathbf{S}_k}^{q_k+1})(I)\right] K_A \\ &= K_A^* \left[\Phi_{\mathbf{S}_1}^{q_1} \circ \cdots \circ \Phi_{\mathbf{S}_k}^{q_k} \circ (id - \Phi_{\mathbf{S}_k}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_1})(I)\right] K_A \\ &= \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k} \circ (id - \Phi_{A_1}) \circ \cdots \circ (id - \Phi_{A_k})(K_A^* K_A) \\ &= \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I)). \end{aligned}$$

Therefore,

$$K_A^*(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})K_A = \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I)) \tag{3.4}$$

for any $q_1, \ldots, q_k \in \mathbb{Z}_+$. Using this relation, theorem 2.5, and corollary 2.6, we deduce the first three equalities in the theorem.

Assume now that $M_i \geq 1$ for any $i \in \{1, \ldots, k\}$. Using again theorem 2.5 and corollary 2.6, we have

$$\operatorname{curv}_M(A) = \lim_{q_1\to\infty} \cdots \lim_{q_k\to\infty} \frac{1}{M_1^{q_1} \cdots M_k^{q_k}} \operatorname{trace}\left[\Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I))\right].$$

A repeated application of Stolz–Cesáro convergence theorem with respect to each limit leads to

$$\operatorname{curv}_M(A) = \lim_{q_1\to\infty} \cdots \lim_{q_k\to\infty} \frac{\operatorname{trace}\left[\sum_{s_1=0}^{q_1} \cdots \sum_{s_k=0}^{q_k} \Phi_{A_1}^{s_1} \circ \cdots \circ \Phi_{A_k}^{s_k}(\Delta_A(I))\right]}{\prod_{i=1}^k (1 + M_i + \cdots + M_i^{q_i})}.$$

Consequently, due to relation (3.4), we deduce that

$$\operatorname{curv}_M(A) = \lim_{q_1\to\infty} \cdots \lim_{q_k\to\infty} \frac{\operatorname{trace}\left[K_A^*(P_{\leq(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})K_A\right]}{\prod_{i=1}^k (1 + M_i + \cdots + M_i^{q_i})}.$$

The proof is complete. □

In the particular case when $M = (M_1, \ldots, M_k) = (n_1, \ldots, n_k)$, we obtain the following asymptotic formulas for the curvature which will be used later on.

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Corollary 3.4. Let $A := (A_1, \ldots, A_k)$ be an element in the $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ such that $\Delta_A(I)$ is a positive trace class operator. Then, the curvature of A satisfies the asymptotic formulas

$$\begin{aligned}
\operatorname{curv}(A) &= \lim_{m\to\infty} \frac{1}{\binom{m+k}{k}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})K_A\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]}\\
&= \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})K_A\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]}\\
&= \lim_{m\to\infty} \frac{1}{\binom{m+k-1}{k-1}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k= m}} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})K_A\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]}\\
&= \lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty} \frac{\operatorname{trace}\left[K_A^*(P_{\leq(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})K_A\right]}{\operatorname{trace}\left[P_{\leq(q_1,\ldots,q_k)}\right]}\\
&= \lim_{m\to\infty} \frac{1}{\binom{m+k}{k}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}} \frac{1}{n_1^{q_1}\cdots n_k^{q_k}}\operatorname{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right]\\
&= \lim_{m\to\infty} \frac{1}{\binom{m+k-1}{k-1}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k= m}} \frac{1}{n_1^{q_1}\cdots n_k^{q_k}}\operatorname{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right]\\
&= \lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty} \frac{\operatorname{trace}\left[(id-\Phi_{A_1}^{q_1+1})\circ\cdots\circ(id-\Phi_{A_k}^{q_k+1})(I)\right]}{\prod_{i=1}^k(1+n_i+\cdots+n_i^{q_i})}\\
&= \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{1}{n_1^{q_1}\cdots n_k^{q_k}}\operatorname{trace}\left[\Phi_{A_1}^{q_1}\circ\cdots\circ\Phi_{A_k}^{q_k}(\Delta_A(I))\right].
\end{aligned}$$

We remark that in the particular case of the *$\mathcal{U}$-twisted polydisk*, i.e. $n_1 = \cdots = n_k = 1$, we have $\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right] = 1$, and therefore, the first of the assymptotic formulas for the curvature (see corollary 3.4) becomes

$$\operatorname{curv}(A) = \lim_{m\to\infty} \frac{\operatorname{trace}\left[K_A^*(P_{\leq m}\otimes I_{\mathcal{H}})K_A\right]}{\operatorname{trace} P_{\leq m}},$$

where $P_{\leq m}$ stands for the orthogonal projection of $\ell^2(\mathbb{Z}_+^k)$ onto

$$\operatorname{span}\{\chi_{(m_1,\ldots,m_k)} :\ m_i\in\mathbb{Z}_+, m_1+\cdots+m_k\leq m\}.$$

Definition. *Let $A := (A_1, \ldots, A_k)$ with $A_i := (A_{i,1}, \ldots, A_{i,n_i}) \in B(\mathcal{H})^{n_i}$ and $C := (C_1, \ldots, C_k)$ with $C_i := (C_{i,1}, \ldots, C_{i,n_i}) \in B(\mathcal{H}')^{n_i}$.*

(i) *We say that A is unitarily equivalent to C if there is a unitary operator $\Psi : \mathcal{H} \to \mathcal{H}'$ such that $\Psi A_{i,s} = C_{i,s}\Psi$ for any $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$.*
(ii) *If A and C are $\mathcal{U}$-commuting and $\mathcal{W}$-commuting, respectively, we say that $(A, \mathcal{U})$ is unitarily equivalent to $(C, \mathcal{W})$ if there is a unitary operator Ψ :*

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$\mathcal{H} \to \mathcal{H}'$ such that

$$\Psi A_{i,s} = C_{i,s}\Psi \quad \textit{and} \quad \Psi U_{i,j}(s,t) = W_{i,j}(s,t)\Psi \textit{ for any } (i,j,s,t) \in \Gamma.$$

We note that if $(A, \mathcal{U})$ is unitarily equivalent to $(C, \mathcal{W})$, then $A \in \mathbf{B}^{\mathcal{U}}(\mathcal{H})$ has a positive trace class defect operator $\Delta_A(I_{\mathcal{H}})$ if and only if $C \in \mathbf{B}^{\mathcal{W}}(\mathcal{H}')$ has a positive trace class defect operator $\Delta_C(I_{\mathcal{H}'})$. Using one of the asymptotic expressions for the curvature, it is easy to see that $\operatorname{curv}(A) = \operatorname{curv}(C)$. Therefore, the curvature is invariant under unitary equivalence of the pairs $(A, \mathcal{U})$ and $(C, \mathcal{W})$. We remark that if A and C are just unitarily equivalent, then $(A, \mathcal{U})$ is not necessarily unitarily equivalent to $(C, \mathcal{W})$ in general.

THEOREM 3.6 *Let A and $\{A^{(p)}\}_{p\in\mathbb{N}}$ be elements in the $\mathcal{U}$-twisted polyball $\boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ with positive defect operators and defect spaces included in a finite dimensional subspace of $\mathcal{H}$. If $K_{A^{(p)}}K^*_{A^{(p)}} \to K_A K_A^*$ in the weak operator topology as $p \to \infty$, then*

$$\limsup_{p\to\infty} \operatorname{curv}(A^{(p)}) \leq \operatorname{curv}(A).$$

Proof. Let $\mathcal{K} \subset \mathcal{H}$ be a subspace with $\dim \mathcal{K} < \infty$ such that $\mathcal{D}_A \subset \mathcal{K}$ and $\mathcal{D}_{A^{(p)}} \subset \mathcal{K}$ for any $p \in \mathbb{N}$. Note that

$$\lim_{p\to\infty} \operatorname{trace}\left[(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{K}})K_{A^{(p)}}K^*_{A^{(p)}}\right] = \operatorname{trace}\left[(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{K}})K_A K_A^*\right]$$

for any $(q_1, \ldots, q_k) \in \mathbb{Z}_+^k$. A close look at the proof of theorem 3.3 reveals that

$$\begin{aligned}\lim_{p\to\infty} \operatorname{trace}\left[\Phi^{q_1}_{A_1^{(p)}} \circ \cdots \circ \Phi^{q_k}_{A_k^{(p)}}(\Delta_{A^{(p)}}(I))\right] &= \lim_{m\to\infty} \operatorname{trace}\left[K^*_{A^{(p)}}(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{K}})K_{A^{(p)}}\right] \\ &= \operatorname{trace}\left[(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{K}})K_A K_A^*\right] \\ &= \operatorname{trace}\left[\Phi^{q_1}_{A_1} \circ \cdots \circ \Phi^{q_k}_{A_k}(\Delta_A(I))\right]\end{aligned}$$

for any $(q_1, \ldots, q_k) \in \mathbb{Z}_+^k$. For each $p \in \mathbb{N}$ and $\mathbf{q} := (q_1, \ldots, q_k) \in \mathbb{Z}_+^k$, let

$$a_{\mathbf{q}} := \frac{\operatorname{trace}\left[\Phi^{q_1}_{A_1} \circ \cdots \circ \Phi^{q_k}_{A_k}(\Delta_A(I))\right]}{n_1^{q_1} \cdots n_k^{q_k}} \quad \text{and}$$

$$a_{\mathbf{q}}^{(p)} := \frac{\operatorname{trace}\left[\Phi^{q_1}_{A_1^{(p)}} \circ \cdots \circ \Phi^{q_k}_{A_k^{(p)}}(\Delta_{A^{(p)}}(I))\right]}{n_1^{q_1} \cdots n_k^{q_k}}.$$

As we saw in the proof of theorem 2.5, $\{a_{\mathbf{q}}\}_{\mathbf{q}\in\mathbb{Z}_+^k}$ and $\{a_{\mathbf{q}}^{(p)}\}_{\mathbf{q}\in\mathbb{Z}_+^k}$ are decreasing multi-sequences with respect to the indices $q_1, \ldots, q_k$. Setting $a := \lim_{\mathbf{q}\in\mathbb{Z}_+^k} a_{\mathbf{q}}$ and $a^{(p)} := \lim_{\mathbf{q}\in\mathbb{Z}_+^k} a_{\mathbf{q}}^{(p)}$, we prove that $\limsup_{p\to\infty} a^{(p)} \leq a$, by contradiction. Passing to a subsequence, we may assume that there is $\epsilon > 0$ such that $a^{(p)} - a \geq 2\epsilon > 0$ for

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any $p \in \mathbb{N}$. Taking into account that $a := \lim_{\mathbf{q}\in\mathbb{Z}_+^k} a_{\mathbf{q}}$, we can find $N \geq 1$ such that $|a_{\mathbf{q}} - a| < \epsilon$ for any $\mathbf{q} = (q_1, \ldots, q_k) \in \mathbb{Z}_+^k$ with $q_i \geq N$. Consequently, $a_{\mathbf{q}}^{(p)} \geq a^{(p)}$ and

$$|a_{\mathbf{q}}^{(p)} - a_{\mathbf{q}}| \geq |a_{\mathbf{q}}^{(p)} - a| - |a_{\mathbf{q}} - a| \geq 2\epsilon - \epsilon = \epsilon,$$

which contradicts that $\lim_{p\to\infty} a_{\mathbf{q}}^{(p)} = a_{\mathbf{q}}$. Consequently, using corollary 3.4, we deduce that

$$\limsup_{p\to\infty} \operatorname{curv}(A^{(p)}) \leq \operatorname{curv}(A).$$

The proof is complete. □

The next result shows that the curvature invariant is upper semi-continuous.

THEOREM *Let A and $\{A^{(p)}\}_{p\in\mathbb{N}}$ be elements in the $\mathcal{U}$-twisted polyball $\boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ such that they have positive defect operators and* $\operatorname{rank} \Delta_{A^{(p)}}(I) \leq C$ *for any $p \in \mathbb{N}$. If $A^{(p)} \to A$ in the norm topology as $p \to \infty$, then*

$$\limsup_{p\to\infty} \operatorname{curv}(A^{(p)}) \leq \operatorname{curv}(A).$$

Proof. According to the proof of theorem 3.6, it suffices to show that, for any $(q_1, \ldots, q_k) \in \mathbb{Z}_+^k$,

$$\lim_{p\to\infty} \operatorname{trace}\left[\Phi_{A_1^{(p)}}^{q_1} \circ \cdots \circ \Phi_{A_k^{(p)}}^{q_k}(\Delta_{A^{(p)}}(I))\right] = \operatorname{trace}\left[\Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I))\right].$$

Consequently, using the fact that $\operatorname{rank} \Delta_{A^{(p)}}(I)) \leq C$, $p \in \mathbb{N}$, for some $C > 0$, we have

$$\begin{aligned}
&\left|\operatorname{trace}\left[\Phi_{A_1^{(p)}}^{q_1} \circ \cdots \circ \Phi_{A_k^{(p)}}^{q_k}(\Delta_{A^{(p)}}(I)) - \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I))\right]\right| \\
&\leq \|\Phi_{A_1^{(p)}}^{q_1} \circ \cdots \circ \Phi_{A_k^{(p)}}^{q_k}(\Delta_{A^{(p)}}(I)) - \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I))\| \\
&\qquad \times \operatorname{rank}\left[\Phi_{A_1^{(p)}}^{q_1} \circ \cdots \circ \Phi_{A_k^{(p)}}^{q_k}(\Delta_{A^{(p)}}(I)) - \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I))\right] \\
&\leq n_1^{q_1} \cdots n_k^{q_k} \left(\operatorname{rank}[\Delta_{A^{(p)}}(I)] + \operatorname{rank}[\Delta_A(I)]\right) \|\Phi_{A_1^{(p)}}^{q_1} \circ \cdots \circ \Phi_{A_k^{(p)}}^{q_k}(\Delta_{A^{(p)}}(I)) \\
&\qquad - \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I))\| \\
&\leq 2C n_1^{q_1} \cdots n_k^{q_k} \|\Phi_{A_1^{(p)}}^{q_1} \circ \cdots \circ \Phi_{A_k^{(p)}}^{q_k}(\Delta_{A^{(p)}}(I)) - \Phi_{A_1}^{q_1} \circ \cdots \circ \Phi_{A_k}^{q_k}(\Delta_A(I))\|.
\end{aligned}$$

Since $A^{(p)} \to A$ in the norm topology as $p \to \infty$, it is clear that the later expression converges to 0. The rest of the proof is similar to that of theorem 3.6. The proof is complete. □

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4. The $\mathcal{U}$-twisted multi-shifts as universal operator models

In [28], we proved the following classification result for the pure elements in the regular $\mathcal{U}$-twisted polyball having defect operators of finite rank. Let $A := (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ be a k-tuple with $A_i := [A_{i,1} \cdots A_{i,n_i}]$. Then, A is a pure k-tuple in the regular $\mathcal{U}$-twisted polyball, with $\operatorname{rank} \Delta_A(I) = m$, where $m \in \mathbb{N}$, if and only if there is a set $\mathcal{W} := \{W_{i,j}(s,t)\}_{(i,j,s,t)\in\Gamma}$ of commuting unitary operators on $\mathbb{C}^m$ with

$$W_{j,i}(t,s) = W_{i,j}(s,t)^*, \qquad (i,j,s,t) \in \Gamma,$$

and there is a jointly invariant subspace $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathbb{C}^m$ under the $I \otimes \mathcal{W}$-twisted multi-shift $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$ with $\mathbf{S}_i := [\mathbf{S}_{i,1} \cdots \mathbf{S}_{i,n_i}]$ and under $I \otimes W_{i,j}(s,t)$ such that $\dim[(P_{\mathbb{C}} \otimes I_{\mathbb{C}^m})\mathcal{M}^\perp] = m$ and, up to a unitary equivalence,

$$A_{i,s} = P_{\mathcal{M}^\perp} \mathbf{S}_{i,s}|_{\mathcal{M}^\perp}.$$

Due to the spectral theorem, since $W_{i,j}(s,t)$ are commuting unitary operators on $\mathbb{C}^m$, there is an orthonormal basis $\{v_1, \ldots, v_m\}$ of $\mathbb{C}^m$ that simultaneously diagonalizes each $W_{i,j}(s,t)$, i.e.

$$W_{i,j}(s,t)v_p = z_{i,j}^{(p)}(s,t)v_p, \qquad z_{i,j}^{(p)}(s,t) \in \mathbb{T},$$

for any $(i,j,s,t) \in \Gamma$ and $p \in \{1, \ldots, m\}$. Setting $\mathbf{z}^{(p)} := (z_{i,j}^{(p)}(s,t))_{(i,j,s,t)\in\Gamma}$, it is clear that each $\mathbb{C}v_p$ is a reducing subspace for all $W_{i,j}(s,t)$ and $W_{i,j}(s,t)|_{\mathbb{C}v_p} = z_{i,j}^{(p)}(s,t)I_{\mathbb{C}v_p}$. Consequently, due to the definition of the multi-shift, we have $\mathbf{S} = \bigoplus_{p=1}^m \mathbf{S}_{\mathbf{z}^{(p)}}$, i.e $\mathbf{S}_{i,s} = \bigoplus_{p=1}^m (\mathbf{S}_{\mathbf{z}^{(p)}})_{i,s}$. Therefore, the $I \otimes \mathcal{W}$-twisted multi-shift $\mathbf{S}$ is a direct sum of multi-shifts $\mathbf{S}_{\mathbf{z}} \in B(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+))$ with scalar weights $\mathbf{z} = (z_{i,j}(s,t))_{(i,j,s,t)\in\Gamma}$, where $z_{i,j}(s,t) \in \mathbb{T}$ and $z_{j,i}(t,s) = \overline{z_{i,j}(s,t)}$ for $(i,j,s,t) \in \Gamma$.

In particular, according to Corollary 3.5 from [28], A is a pure k-tuple in the regular $\mathcal{U}$-twisted polyball, with $\operatorname{rank} \Delta_A(I) = 1$ if and only if there is a set $\mathbf{z} = (z_{i,j}(s,t))_{(i,j,s,t)\in\Gamma}$ of complex numbers in the torus $\mathbb{T}$ with

$$z_{j,i}(t,s) = \overline{z_{i,j}(s,t)}, \qquad (i,j,s,t) \in \Gamma,$$

and a jointly invariant subspace $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ under the standard multi-shift $\mathbf{S}_{\mathbf{z}}$ such that $\dim[P_{\mathbb{C}}\mathcal{M}^\perp] = 1$ and, up to a unitary equivalence,

$$A_{i,s} = P_{\mathcal{M}^\perp} (\mathbf{S}_{\mathbf{z}})_{i,s}|_{\mathcal{M}^\perp}.$$

If $\mathcal{M}'$ is another jointly co-invariant subspace under $(\mathbf{S}_{\mathbf{z}})_{i,s}$, then $P_{\mathcal{M}^\perp} \mathbf{S}_{\mathbf{z}}|_{\mathcal{M}^\perp}$ and $P_{\mathcal{M}'^\perp} \mathbf{S}_{\mathbf{z}}|_{\mathcal{M}'^\perp}$ are unitarily equivalent if and only if $\mathcal{M}^\perp = \mathcal{M}'^\perp$. As a consequence of the next result, we deduce that if $\mathbf{S}_{\mathbf{z}}$ and $\mathbf{S}_{\mathbf{z}'}$ are the standard multi-shifts associated with the scalar weights $\mathbf{z}$ and $\mathbf{z}'$, respectively, $\mathbf{S}_{\mathbf{z}}$ is unitarily equivalent $\mathbf{S}_{\mathbf{z}'}$ if and only if $\mathbf{z} = \mathbf{z}'$.

THEOREM 4.1 *Let $\boldsymbol{S}_{\mathcal{U}}$ and $\boldsymbol{S}_{\mathcal{U}'}$ be the standard multi-shifts associated with $\mathcal{U} \subset B(\mathcal{H})$ and $\mathcal{U}' \subset B(\mathcal{H}')$, respectively. Then, $\boldsymbol{S}_{\mathcal{U}}$ is jointly similar to $\boldsymbol{S}_{\mathcal{U}'}$ if and only if*

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there is an invertible operator $W \in B(\mathcal{H}, \mathcal{H}')$ *such that* $U_{i,j}(s,t) = W^{-1}U'_{i,j}(s,t)W$ *for any* $(i,j,s,t) \in \Gamma$.

Moreover, $\boldsymbol{S}_{\mathcal{U}}$ *is unitarily equivalent* $\boldsymbol{S}_{\mathcal{U}'}$ *if and only if there is a unitary operator* $W \in B(\mathcal{H}, \mathcal{H}')$ *such that* $U_{i,j}(s,t) = W^*U'_{i,j}(s,t)W$.

Proof. Let $\{\chi_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ be the orthonormal basis for the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$. Recall that $\mathbf{S}_{\mathcal{U}} = ((\mathbf{S}_{\mathcal{U}})_1, \ldots, (\mathbf{S}_{\mathcal{U}})_k)$ with $(\mathbf{S}_{\mathcal{U}})_i := [(\mathbf{S}_{\mathcal{U}})_{i,1} \cdots (\mathbf{S}_{\mathcal{U}})_{i,n_i}]$, where

$$(\mathbf{S}_{\mathcal{U}})_{i,s}\left(\chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h\right) := \begin{cases} \chi_{(g_s^i\alpha_1,\alpha_2,\ldots,\alpha_k)} \otimes h, & \text{if } i = 1 \\ \chi_{(\alpha_1,\ldots,\alpha_{i-1},g_s^i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)} \otimes \boldsymbol{U}_{i,1}(s,\alpha_1)\cdots \\ \qquad \boldsymbol{U}_{i,i-1}(s,\alpha_{i-1})h, & \text{if } i \in \{2,\ldots,k\} \end{cases}$$

for any $h \in \mathcal{H}$, $\alpha_1 \in \mathbb{F}_{n_1}^+, \ldots, \alpha_k \in \mathbb{F}_{n_k}^+$, where

$$\boldsymbol{U}_{i,j}(s,\beta) := \begin{cases} \prod_{b=1}^q U_{i,j}(s,j_b) & \text{if } \beta = g_{j_1}^j \cdots g_{j_q}^j \in \mathbb{F}_{n_j}^+ \\ I & \text{if } \beta = g_0^j \end{cases}$$

for any $j \in \{1, \ldots, k\}$.

Assume that there is an invertible operator $X \in \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}'$ such that

$$X(\mathbf{S}_{\mathcal{U}})_{i,s} = (\mathbf{S}_{\mathcal{U}'})_{i,s}X \tag{4.1}$$

for any $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. Let $[X_{\boldsymbol{\alpha},\boldsymbol{\beta}}]_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+}$ be the operator block matrix of X, where $\langle X_{\boldsymbol{\alpha},\boldsymbol{\beta}}h, h'\rangle = \langle X(\chi_{\boldsymbol{\beta}} \otimes h, \chi_{\boldsymbol{\alpha}} \otimes h'\rangle$ for any $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$. Note that

$$\begin{aligned} \langle X(\mathbf{S}_{\mathcal{U}})_{1,s}(\chi_{\boldsymbol{\alpha}} \otimes h), (\chi_{\boldsymbol{\beta}} \otimes h')\rangle &= \left\langle X(\chi_{(g_s^1\alpha_1,\ldots,\alpha_k)} \otimes h), (\chi_{(\beta_1,\ldots,\beta_k)} \otimes h')\right\rangle \\ &= \left\langle X_{(\beta_1,\ldots,\beta_k),(g_s^1\alpha_1,\ldots,\alpha_k)}h, h'\right\rangle \end{aligned}$$

and

$$\begin{aligned} &\langle (\mathbf{S}_{\mathcal{U}'})_{1,s}X(\chi_{\boldsymbol{\alpha}} \otimes h), (\chi_{\boldsymbol{\beta}} \otimes h')\rangle \\ &= \left\langle X(\chi_{\boldsymbol{\alpha}} \otimes h), (\mathbf{S}_{\mathcal{U}'})_{1,s}^*(\chi_{\boldsymbol{\beta}} \otimes h')\right\rangle \\ &= \begin{cases} \left\langle X(\chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h), (\chi_{(\gamma_1,\beta_2,\ldots,\beta_k)} \otimes h')\right\rangle & \text{if } \beta_1 = g_s^1\gamma_1 \text{ for some } \gamma_1 \in \mathbb{F}_{n_1}^+ \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} X_{(\gamma_1,\beta_2,\ldots,\beta_k),(\alpha_1,\ldots,\alpha_k)} & \text{if } \beta_1 = g_s^1\gamma_1 \text{ for some } \gamma_1 \in \mathbb{F}_{n_1}^+ \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consequently, due to relation (4.1), we must have

$$X_{(g_s^1\gamma_1,\beta_2,\ldots,\beta_k),(g_s^1\sigma_1,\alpha_2,\ldots,\alpha_k)} = X_{(\gamma_1,\beta_2,\ldots,\beta_k),(\sigma_1,\alpha_2,\ldots,\alpha_k)}, \quad \gamma_1,\sigma_1 \in \mathbb{F}_{n_1}^+, \alpha_i,\beta_i \in \mathbb{F}_{n_i}^+,$$

and

$$X_{(\beta_1,\ldots,\beta_k),(\alpha_1,\ldots,\alpha_k)} = 0 \qquad \text{if} \ \ \beta_1 \neq \alpha_1, \alpha_i,\beta_i \in \mathbb{F}_{n_i}^+.$$

Consequently, if $\beta_1 = \alpha_1$, then

$$X_{(\beta_1,\beta_2,\ldots,\beta_k),(\alpha_1,\alpha_2,\ldots,\alpha_k)} = X_{(g_0^1,\beta_2,\ldots,\beta_k),(g_0^1,\alpha_2,\ldots,\alpha_k)} \tag{4.2}$$

for any $\alpha_2,\beta_2 \in \mathbb{F}_{n_2}^+,\ldots,\alpha_k,\beta_k \in \mathbb{F}_{n_k}^+$.

Now, fix $i \in \{2,\ldots,k\}$ and $t \in \{1,\ldots,n_i\}$. As above, using the definition of the standard multi-shifts, we deduce that

$$\begin{aligned}
&\left\langle X(\mathbf{S}_{\mathcal{U}})_{i,t}(\chi_{\boldsymbol{\alpha}}\otimes h),(\chi_{\boldsymbol{\beta}}\otimes h')\right\rangle \\
&= \left\langle X(\chi_{(\alpha_1,\ldots,\alpha_{i-1},g_t^i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)}\otimes \boldsymbol{U}_{i,1}(t,\alpha_1)\cdots\boldsymbol{U}_{i,i-1}(t,\alpha_{i-1})h),(\chi_{(\beta_1,\ldots,\beta_k)}\otimes h')\right\rangle \\
&= \left\langle X_{(\beta_1,\ldots,\beta_k),(\alpha_1,\ldots,\alpha_{i-1},g_t^i\alpha_i,\alpha_{i+1},\ldots,\alpha_k)}\boldsymbol{U}_{i,1}(t,\alpha_1)\cdots\boldsymbol{U}_{i,i-1}(t,\alpha_{i-1})h,h'\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle(\mathbf{S}_{\mathcal{U}'})_{i,t}X(\chi_{\boldsymbol{\alpha}}\otimes h),(\chi_{\boldsymbol{\beta}}\otimes h')\rangle \\
&= \left\langle X(\chi_{\boldsymbol{\alpha}}\otimes h),(\mathbf{S}_{\mathcal{U}'})_{i,t}^*(\chi_{\boldsymbol{\beta}}\otimes h')\right\rangle \\
&= \begin{cases} \Big\langle X(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h),(\chi_{(\beta_1,\ldots,\beta_{i-1},\gamma_i,\beta_{i+1},\ldots,\beta_k)}\otimes \\ \qquad \boldsymbol{U}'_{i,1}(t,\beta_1)^*\cdots\boldsymbol{U}'_{i,i-1}(t,\beta_{i-1})^*h')\Big\rangle & \text{if } \beta_i = g_t^i\gamma_i, \gamma_i \in \mathbb{F}_{n_i}^+ \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \Big\langle \boldsymbol{U}'_{i,1}(t,\beta_1)\cdots\boldsymbol{U}'_{i,i-1}(t,\beta_{i-1}) \\ \qquad X_{(\beta_1,\ldots,\beta_{i-1},\gamma_i,\beta_{i+1},\ldots,\beta_k),(\alpha_1,\ldots,\alpha_k)},h'\Big\rangle & \text{if } \beta_i = g_t^i\gamma_i \text{with } \gamma_i \in \mathbb{F}_{n_i}^+ \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Now, using relation (4.1), we deduce that

$$X_{(\beta_1,\ldots,\beta_i,\ldots,\beta_k),(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_k)} = 0 \qquad \text{if} \ \ \beta_i \neq \alpha_i, \alpha_j,\beta_j \in \mathbb{F}_{n_j}^+$$

and

$$\begin{aligned}
X_{(\beta_1,\ldots,g_t^i\gamma_i,\ldots,\beta_k),(\alpha_1,\ldots,g_t^i\sigma_i,\ldots,\alpha_k)}&\boldsymbol{U}_{i,1}(t,\alpha_1)\cdots\boldsymbol{U}_{i,i-1}(t,\alpha_{i-1}) \\
&= \boldsymbol{U}'_{i,1}(t,\beta_1)\cdots\boldsymbol{U}'_{i,i-1}(t,\beta_{i-1})X_{(\beta_1,\ldots,\gamma_i,\ldots,\beta_k),(\alpha_1,\ldots,\sigma_i,\ldots,\alpha_k)}.
\end{aligned}$$

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Consequently, if $\beta_i = \alpha_i$, then

$$\begin{aligned} &X_{(\beta_1,\ldots,\beta_i,\ldots,\beta_k),(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_k)}\boldsymbol{U}_{i,1}(\beta_i,\alpha_1)\cdots\boldsymbol{U}_{i,i-1}(\beta_i,\alpha_{i-1}) \\ &\qquad = \boldsymbol{U}'_{i,1}(\beta_i,\beta_1)\cdots\boldsymbol{U}'_{i,i-1}(\beta_i,\beta_{i-1})X_{(\beta_1,\ldots,g_0^i,\ldots,\beta_k),(\alpha_1,\ldots,g_0^i,\ldots,\alpha_k)} \end{aligned} \tag{4.3}$$

where

$$\boldsymbol{U}_{i,j}(\beta_i,\alpha_j) = \prod_{b=1}^{q}\boldsymbol{U}_{i,j}(j_b,\alpha_j) \qquad \text{if } \ \beta_i = g_{j_1}^i\cdots g_{j_q}^i \in \mathbb{F}_{n_i}^+, \alpha_j \in \mathbb{F}_{n_j}^+,$$

and $\boldsymbol{U}_{i,j}(\beta_i,\alpha_j) = I$ if $\beta_i = g_0^i$.

Due to the above results, if there is $i \in \{1,\ldots,k\}$ such that $\alpha_i \neq \beta_i$, then $X_{(\beta_1,\ldots,\beta_k),(\alpha_1,\ldots,\alpha_k)} = 0$. Therefore, the non-zero entries of the operator block matrix $[X_{\boldsymbol{\alpha},\boldsymbol{\beta}}]_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}$ are those on the diagonal, that is, $X_{(\alpha_1,\ldots,\alpha_k),(\alpha_1,\ldots,\alpha_k)}$. Using relations (4.2) and (4.3), we deduce that

$$\begin{aligned} X_{(\alpha_1,\ldots,\alpha_k),(\alpha_1,\ldots,\alpha_k)} &= X_{(g_0^1,\alpha_2,\ldots,\alpha_k),(g_0^1,\alpha_2,\ldots,\alpha_k)} = X_{(g_0^1,g_0^2,\alpha_3,\ldots,\alpha_k),(g_0^1,g_0^2,\alpha_3,\ldots,\alpha_k)} \\ &= \cdots = X_{(g_0^1,g_0^2,\ldots,g_0^k),(g_0^1,g_0^2,\ldots,g_0^k)} \end{aligned} \tag{4.4}$$

for any $\alpha_i \in \mathbb{F}_{n_i}^+$. Since X is a an invertible operator, $X_{(g_0^1,\ldots,g_0^k),(g_0^1,\ldots,g_0^k)} \in B(\mathcal{H},\mathcal{H}')$ must be invertible as well.

Now, let $i \in \{2,\ldots,k\}$ and let $j \in \{1,\ldots,k-1\}$ be such that $i > j$. In what follows, we show that $\boldsymbol{U}_{i,j}(s,t) = \boldsymbol{U}'_{i,j}(s,t)$ for any $s \in \{1,\ldots,n_i\}$ and $t \in \{1,\ldots,n_j\}$. Indeed, take $\alpha_i = g_s^i$, $\alpha_j = g_t^j$, and $\alpha_p = g_0^p$ for any $p \in \{1,\ldots,k\}$ with $p \neq i$ and $p \neq j$. Applying relations (4.3) and (4.4), we obtain

$$\begin{aligned} &X_{(\alpha_1,\ldots,\alpha_k),(\alpha_1,\ldots,\alpha_k)}\boldsymbol{U}_{i,1}(g_s^i,g_0^1)\cdots\boldsymbol{U}_{i,j}(g_s^i,g_t^j)\cdots\boldsymbol{U}_{i,i-1}(g_s^i,g_0^{i-1}) \\ &= \boldsymbol{U}'_{i,1}(g_s^i,g_0^1)\cdots\boldsymbol{U}'_{i,j}(g_s^i,g_t^j)\cdots\boldsymbol{U}'_{i,i-1}(g_s^i,g_0^{i-1}) \\ &\quad X_{(\alpha_1,\ldots,\alpha_{i-1},g_0^i,\alpha_{i+1}\cdots\alpha_k),(\alpha_1,\ldots,\alpha_{i-1},g_0^i,\alpha_{i+1},\cdots\alpha_k)}, \end{aligned}$$

which implies

$$X_{(\alpha_1,\ldots,\alpha_k),(\alpha_1,\ldots,\alpha_k)}\boldsymbol{U}_{i,j}(s,t) = \boldsymbol{U}'_{i,j}(s,t)X_{(g_0^1,\ldots,g_0^k),(g_0^1,\ldots,g_0^k)}.$$

Consequently, again using relation (4.4), i.e. $X_{(\alpha_1,\ldots,\alpha_k),(\alpha_1,\ldots,\alpha_k)} = X_{(g_0^1,\ldots,g_0^k),(g_0^1,\ldots,g_0^k)} := W$, we deduce that $W\boldsymbol{U}_{i,j}(s,t) = \boldsymbol{U}'_{i,j}(s,t)W$, which proves our assertion. The converse is obviously true. In a similar manner, one can show that $\mathbf{S}_{\mathcal{U}}$ is unitarily equivalent $\mathbf{S}_{\mathcal{U}'}$ if and only if there is a unitary operator $W \in B(\mathcal{H},\mathcal{H}')$ such that $U_{i,j}(s,t) = W^*U'_{i,j}(s,t)W$ for any $(i,j,s,t) \in \Gamma$. The proof is complete. □

Corollary 4.2. Let $\mathbf{S}_{\mathbf{z}}$ and $\mathbf{S}_{\mathbf{z}'}$ be the standard multi-shifts associated with the scalar weights $\mathbf{z}$ and $\mathbf{z}'$, respectively. Then, $\mathbf{S}_{\mathbf{z}}$ is unitarily equivalent $\mathbf{S}_{\mathbf{z}'}$ if and only if $\mathbf{z} = \mathbf{z}'$.

5. Curvature operator on the $\mathcal{U}$-twisted polyball and some classification results

In this section, we prove that if A is an element in the $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ which admits characteristic function Θ_A and the defect $\Delta_A(I)$ is a positive finite rank operator, then we can introduce a trace class *curvature operator* whose trace is exactly the curvature of A. This result is used to show that the curvature invariant detects the elements in $B(\mathcal{H})^{n_1+\cdots+n_k}$ which are unitarily equivalent to a $I\otimes\mathcal{U}$-twisted multi-shift $\mathbf{S}$ of finite rank defect operator and completely classify them. We also show that the curvature invariant completely classifies the finite rank Beurling type jointly invariant subspace under $\mathbf{S}$ and $I\otimes\mathcal{U}$.

We recall that

$$N_{\leq(q_1,\ldots,q_k)} := \sum_{s_i\in\mathbb{Z}_+, s_i\leq q_i} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} P_{(s_1,\ldots,s_k)},$$

where $P_{(s_1,\ldots,s_k)}$ is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$ onto the subspace

$$\text{span}\,\{\chi_{(\alpha_1,\ldots,\alpha_k)}:\ \alpha_i\in\mathbb{F}_{n_i}^+, |\alpha_i|=s_i\}.$$

Theorem 5.1 *Let A be an element in the $\mathcal{U}$-twisted polyball $\boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ such that $\Delta_A(I)$ is a positive trace class operator. Then,*

$$\text{curv}(A) = \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \text{trace}\left[\Delta_{\boldsymbol{S}}(K_AK_A^*)(N_{\leq(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right],$$

where K_A is the noncommutative Berezin kernel of A.

Proof. Fix $j\in\{1,\ldots,k\}$, and let $\alpha_j\in\mathbb{F}_{n_j}^+$ with $|\alpha_j|=s_j\leq q_j\in\mathbb{Z}_+$. Denote by $P_{q_j}^{(j)}$ the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$ onto

$$\overline{\text{span}}\left\{\chi_{(\beta_1,\ldots,\beta_k)}:\ \beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+ \text{and } |\beta_j|=q_j\right\}.$$

Using the definition of the $I\otimes\mathcal{U}$-twisted multi-shift $\mathbf{S}$ on $B(\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$, we have

$$\begin{aligned}
&\mathbf{S}_{j,\alpha_j}^*(P_{q_j}^{(j)}\otimes I_{\mathcal{H}})(\chi_{(\beta_1,\ldots,\beta_k)}\otimes h)\\
&=\begin{cases}\mathbf{S}_{j,\alpha_j}^*(\chi_{(\beta_1,\ldots,\beta_k)}\otimes h) & \text{if } |\beta_j|=q_j\\ 0, & \text{otherwise}\end{cases}\\
&=\begin{cases}\chi_{(\beta_1,\ldots,\beta_{j-1},\gamma_j,\beta_{j+1},\ldots,\beta_k)}\otimes \boldsymbol{U}_{j,1}(\alpha_j,\beta_1)^*\cdots\boldsymbol{U}_{j,j-1}(\alpha_j,\beta_{j-1})^*h, & \text{if } \beta_j=\alpha_j\gamma_j\\ 0, & \text{otherwise}\end{cases}
\end{aligned}$$

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if $j \geq 2$, where

$$\mathbf{U}_{j,p}(\alpha_j, \beta_p) := \prod_{b=1}^{s_j} \mathbf{U}_{j,p}(j_b, \beta_p)$$

for each $p \in \{1, \ldots, j-1\}$, $\alpha_j = g_{j_1}^j \cdots g_{j s_j}^j \in \mathbb{F}_{n_j}^+$ and $\beta_p \in \mathbb{F}_{np}^+$. If $j = 1$, we have

$$\mathbf{S}_{1,\alpha_1}^*(P_{q_1}^{(1)} \otimes I_{\mathcal{H}})(\chi_{(\beta_1,\ldots,\beta_k)} \otimes h) = \begin{cases} \chi_{(\gamma_1,\beta_2,\ldots,\beta_k)} \otimes h, & \text{if } \beta_1 = \alpha_1\gamma_1 \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, if $j \geq 2$, we deduce that

$$\begin{aligned} &(P_{q_j-s_j}^{(j)} \otimes I_{\mathcal{H}})\mathbf{S}_{j,\alpha_j}^*(\chi_{(\beta_1,\ldots,\beta_k)} \otimes h) \\ &= \begin{cases} (P_{q_j-s_j}^{(j)} \otimes I_{\mathcal{H}})\Big(\chi_{(\beta_1,\ldots,\beta_{j-1},\gamma_j,\beta_{j+1},\ldots,\beta_k)} \otimes \boldsymbol{U}_{j,1}(\alpha_j,\beta_1)^* \cdots \\ \qquad \boldsymbol{U}_{j,j-1}(\alpha_j,\beta_{j-1})^* h\Big), & \text{if } \beta_j = \alpha_j\gamma_j \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \chi_{(\beta_1,\ldots,\beta_{j-1},\gamma_j,\beta_{j+1},\ldots,\beta_k)} \otimes \boldsymbol{U}_{j,1}(\alpha_j,\beta_1)^* \cdots \boldsymbol{U}_{j,j-1}(\alpha_j,\beta_{j-1})^* h, & \text{if } \beta_j = \alpha_j\gamma_j \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, if $j = 1$, we have

$$(P_{q_1-s_1}^{(1)} \otimes I_{\mathcal{H}})\mathbf{S}_{1,\alpha_1}^*(\chi_{(\beta_1,\ldots,\beta_k)} \otimes h) = \begin{cases} \chi_{(\gamma_1,\beta_2,\ldots,\beta_k)} \otimes h, & \text{if } \beta_1 = \alpha_1\gamma_1 \\ 0, & \text{otherwise.} \end{cases}$$

The above relations imply

$$\mathbf{S}_{j,\alpha_j}^*(P_{q_j}^{(j)} \otimes I_{\mathcal{H}}) = (P_{q_j-s_j}^{(j)} \otimes I_{\mathcal{H}})\mathbf{S}_{j,\alpha_j}^*,$$

for any $\alpha_j \in \mathbb{F}_{n_j}^+$ with $|\alpha_j| = s_j \leq q_j \in \mathbb{Z}_+$. Since $P_{(s_1,\ldots,s_k)} = P_{s_1}^{(1)} \cdots P_{s_k}^{(k)}$ and taking into account that $\Phi_{\mathbf{S}_i}^{*s_i}(I) = n_i^{s_i} I$, we obtain

$$\begin{aligned} \Phi_{\mathbf{S}_1}^{*s_1} \circ \cdots \circ \Phi_{\mathbf{S}_k}^{*s_k}\left(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}}\right) &= \left(P_{(q_1-s_1,\ldots,q_k-s_k)} \otimes I_{\mathcal{H}}\right)\Phi_{\mathbf{S}_1}^{*s_1} \circ \cdots \circ \Phi_{\mathbf{S}_k}^{*s_k}(I) \\ &= n_1^{s_1} \cdots n_k^{s_k}\left(P_{(q_1-s_1,\ldots,q_k-s_k)} \otimes I_{\mathcal{H}}\right) \end{aligned} \tag{5.1}$$

for any $s_1 \leq q_1, \ldots, s_k \leq q_k$. Using proposition 2.3, we have

$$X = \sum_{s_1=0}^{\infty} \cdots \sum_{s_k=0}^{\infty} \Phi_{\mathbf{S}_1}^{s_1} \circ \cdots \circ \Phi_{\mathbf{S}_k}^{s_k}(\Delta_{\mathbf{S}}(X))$$

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for any $X \in (I \otimes \mathcal{U})'$. Hence,

$$(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})X = (P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}}) \sum_{s_1=0}^{q_1} \cdots \sum_{s_k=0}^{q_k} \Phi_{\mathbf{S}_1}^{s_1} \circ \cdots \circ \Phi_{\mathbf{S}_k}^{s_k}(\Delta_{\mathbf{S}}(X)).$$

Now, using this relation and (5.1), we deduce that

$$\begin{aligned}
&\operatorname{trace}\left[(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})X\right]\\
&= \sum_{s_1=0}^{q_1} \cdots \sum_{s_k=0}^{q_k} \operatorname{trace}\\
&\qquad \left[(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}}) \sum_{|\alpha_1|=s_1,\ldots,|\alpha_k|=s_k} \mathbf{S}_{1,\alpha_1} \cdots \mathbf{S}_{k,\alpha_k} \Delta_{\mathbf{S}}(X) \mathbf{S}_{k,\alpha_k}^* \cdots \mathbf{S}_{1,\alpha_1}^*\right]\\
&= \sum_{s_1=0}^{q_1} \cdots \sum_{s_k=0}^{q_k} \operatorname{trace}\\
&\qquad \left[\sum_{|\alpha_1|=s_1,\ldots,|\alpha_k|=s_k} \mathbf{S}_{k,\alpha_k}^* \cdots \mathbf{S}_{1,\alpha_1}^* (P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}}) \mathbf{S}_{1,\alpha_1} \cdots \mathbf{S}_{k,\alpha_k} \Delta_{\mathbf{S}}(X)\right]\\
&= \sum_{s_1=0}^{q_1} \cdots \sum_{s_k=0}^{q_k} \operatorname{trace}\left[\Delta_{\mathbf{S}}(X) \Phi_{\mathbf{S}_k}^{*s_k} \circ \cdots \circ \Phi_{\mathbf{S}_1}^{*s_1}\left(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}}\right)\right]\\
&= \operatorname{trace}\left[\Delta_{\mathbf{S}}(X) \sum_{s_1=0}^{q_1} \cdots \sum_{s_k=0}^{q_k} n_1^{s_1} \cdots n_k^{s_k} \left(P_{(q_1-s_1,\ldots,q_k-s_k)} \otimes I_{\mathcal{H}}\right)\right]\\
&= \operatorname{trace}\left\{\Delta_{\mathbf{S}}(X) \left[\left(\sum_{s_1=0}^{q_1} n_1^{s_1} P_{q_1-s_1}^{(1)}\right) \cdots \left(\sum_{s_k=0}^{q_k} n_k^{s_k} P_{q_k-s_k}^{(k)}\right) \otimes I_{\mathcal{H}}\right]\right\}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\frac{\operatorname{trace}\left[(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})X\right]}{n_1^{q_1} \cdots n_k^{q_k}}\\
&= \operatorname{trace}\left\{\Delta_{\mathbf{S}}(X) \left[\left(\sum_{s_1=0}^{q_1} \frac{1}{n_1^{s_1}} P_{s_1}^{(1)}\right) \cdots \left(\sum_{s_k=0}^{q_k} \frac{1}{n_k^{s_k}} P_{s_k}^{(k)}\right) \otimes I_{\mathcal{H}}\right]\right\}\\
&= \operatorname{trace}\left[\Delta_{\mathbf{S}}(X) \sum_{s_i \in \mathbb{Z}_+, s_i \le q_i} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} P_{(s_1,\ldots,s_k)} \otimes I_{\mathcal{H}}\right]\\
&= \operatorname{trace}\left[\Delta_{\mathbf{S}}(X)(N_{\le(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})\right]
\end{aligned} \tag{5.2}$$

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for any $X \in (I \otimes \mathcal{U})'$. Since $K_A U_{i,j}(s,t) = (I \otimes U_{i,j}(s,t))K_A$ for any $(i,j,s,t) \in \Gamma$, it is clear that $K_A K_A^* \in (I \otimes \mathcal{U})'$. Using corollary 3.4 and relation (5.2) when $X = K_A K_A^*$, we obtain

$$\begin{aligned} \operatorname{curv}(A) &= \lim_{(q_1,\dots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[K_A^*(P_{(q_1,\dots,q_k)} \otimes I_{\mathcal{H}})K_A\right]}{\operatorname{trace}\left[P_{(q_1,\dots,q_k)}\right]} \\ &= \lim_{(q_1,\dots,q_k)\in\mathbb{Z}_+^k} \operatorname{trace}\left[\Delta_{\mathbf{S}}(K_A K_A^*)(N_{\leq(q_1,\dots,q_k)} \otimes I_{\mathcal{H}})\right]. \end{aligned} \tag{5.3}$$

This completes the proof. □

THEOREM 5.2 *Let* $A := (A_1, \dots, A_k)$ *be an element in the* $\mathcal{U}$*-twisted polyball* $\boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ *such that* $\Delta_A(I)$ *is a positive finite rank operator and* $\Delta_{\boldsymbol{S}}(I - K_A K_A^*) \geq 0$. *Then, the curvature operator* $\Delta_{\boldsymbol{S}}(K_A K_A^*)(N \otimes I_{\mathcal{H}})$ *is trace class and*

$$\operatorname{curv}(A) = \operatorname{trace}\left[\Delta_{\boldsymbol{S}}(K_A K_A^*)(N \otimes I_{\mathcal{H}})\right],$$

where

$$N := \sum_{(s_1,\dots,s_k)\in\mathbb{Z}_+^k} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}} P_{(s_1,\dots,s_k)}.$$

Proof. Let $\mathcal{D}_A := \Delta_A(I)\mathcal{H}$, and note that

$$\begin{aligned} \frac{\operatorname{trace}\left[(P_{(q_1,\dots,q_k)} \otimes I_{\mathcal{D}_A})(I - K_A K_A^*)\right]}{n^{q_1}\cdots n_k^{q_k}} &\leq \|I - K_A K_A^*\| \frac{\operatorname{trace}\left[(P_{(q_1,\dots,q_k)} \otimes I_{\mathcal{D}_A})\right]}{n^{q_1}\cdots n_k^{q_k}} \\ &\leq \|I - K_A K_A^*\| \dim \mathcal{D}_A. \end{aligned}$$

Hence, and using relation (5.2) when $X := I - K_A K_A^* \in (I \otimes \mathcal{U})'$, we obtain

$$\operatorname{trace}\left[\Delta_{\mathbf{S}}(I - K_A K_A^*)(N_{\leq(q_1,\dots,q_k)} \otimes I_{\mathcal{H}})\right] \leq \|I - K_A K_A^*\| \dim \mathcal{D}_A.$$

Since $\{N_{\leq(q_1,\dots,q_k)}\}$ is an increasing multi-sequence of positive operators converging to N, we deduce that

$$\begin{aligned} \operatorname{trace}\left[\Delta_{\mathbf{S}}(I - K_A K_A^*)(N \otimes I_{\mathcal{H}})\right] &= \lim_{(q_1,\dots,q_k)\in\mathbb{Z}_+^k} \operatorname{trace}\left[\Delta_{\mathbf{S}}(I - K_A K_A^*)(N_{\leq(q_1,\dots,q_k)} \otimes I_{\mathcal{H}})\right] \\ &\leq \|I - K_A K_A^*\| \dim \mathcal{D}_A. \end{aligned} \tag{5.4}$$

Consequently, $\Delta_{\mathbf{S}}(I - K_A K_A^*)(N \otimes I_{\mathcal{H}})$ is a trace class operator. On the other hand, we have

$$\begin{aligned} \operatorname{trace}\left[\Delta_{\mathbf{S}}(K_A K_A^*)(N \otimes I_{\mathcal{H}})\right] &= \operatorname{trace}\left[\Delta_{\mathbf{S}}(I)(N \otimes I_{\mathcal{D}_A})\right] \\ &\quad - \operatorname{trace}\left[\Delta_{\mathbf{S}}(I - K_A K_A^*)(N \otimes I_{\mathcal{D}_A})\right] \\ &= \operatorname{trace}\left[P_{\mathbb{C}} \otimes I_{\mathcal{D}_A}\right] - \operatorname{trace}\left[\Delta_{\mathbf{S}}(I - K_A K_A^*)(N \otimes I_{\mathcal{D}_A})\right] \\ &= \operatorname{rank} \Delta_A(I) - \operatorname{trace}\left[\Delta_{\mathbf{S}}(I - K_A K_A^*)(N \otimes I_{\mathcal{D}_A})\right]. \end{aligned} \tag{5.5}$$

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This shows that $\Delta_{\mathbf{S}}(K_AK_A^*)(N\otimes I_{\mathcal{H}})$ is a trace class operator. Similarly, we obtain

$$\begin{aligned}&\text{trace}\left[\Delta_{\mathbf{S}}(K_AK_A^*)(N_{\leq(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]\\&\quad=\text{rank}\,\Delta_A(I)-\text{trace}\left[\Delta_{\mathbf{S}}(I-K_AK_A^*)(N_{\leq(q_1,\ldots,q_k)}\otimes I_{\mathcal{D}_A})\right].\end{aligned}\tag{5.6}$$

Using relations (5.3), (5.6), (5.4), and (5.5), we obtain

$$\begin{aligned}\text{curv}(A)&=\lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k}\text{trace}\left[\Delta_{\mathbf{S}}(K_AK_A^*)(N_{\leq(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]\\&=\text{rank}\,\Delta_A(I)-\lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k}\text{trace}\left[\Delta_{\mathbf{S}}(I-K_AK_A^*)(N_{\leq(q_1,\ldots,q_k)}\otimes I_{\mathcal{D}_A})\right]\\&=\text{rank}\,\Delta_A(I)-\text{trace}\left[\Delta_{\mathbf{S}}(I-K_AK_A^*)(N\otimes I_{\mathcal{D}_A})\right]\\&=\text{trace}\left[\Delta_{\mathbf{S}}(K_AK_A^*)(N\otimes I_{\mathcal{H}})\right].\end{aligned}$$

The proof is complete. □

Let $\mathbf{S}_{\mathcal{V}}$ be the standard k-tuple of doubly $I\otimes\mathcal{V}$-commuting pure isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}$, and let $\mathbf{S}_{\mathcal{W}}$ be the standard k-tuple of doubly $I\otimes\mathcal{W}$-commuting pure isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$. A bounded linear operator $M:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$ is called *multi-analytic* with respect to the multi-shifts $\mathbf{S}_{\mathcal{V}}$ and $\mathbf{S}_{\mathcal{W}}$ if

$$M(\mathbf{S}_{\mathcal{V}})_{i,s}=(\mathbf{S}_{\mathcal{W}})_{i,s}M,\qquad i\in\{1,\ldots,k\},s\in\{1,\ldots,n_i\},$$

and

$$M(I\otimes V_{i,j}(s,t))=(I\otimes W_{i,j}(s,t))M,\qquad (i,j,s,t)\in\Gamma.$$

If, in addition, M is a partial isometry, we call it an *inner multi-analytic* operator.

Definition. *We say that an element A in the $\mathcal{U}$-twisted polyball $\boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ with $\Delta_A(I)\geq 0$ has a characteristic function if there is a multi-analytic operator $\Theta_A:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}_A$ with respect to the multi-shifts $\boldsymbol{S}_{\mathcal{W}}$ and $\boldsymbol{S}_{\mathcal{U}|_{\mathcal{D}_A}}$ such that $K_AK_A^*+\Theta_A\Theta_A^*=I$.*

We remark that, since $K_AK_A^*$ is in the commutant of $I\otimes\mathcal{U}$, we can apply theorem 6.1 from [28], to conclude that an element $A\in\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ with $\Delta_A(I)\geq 0$ has a characteristic function if and only if $\Delta_{\mathbf{S}_{\mathcal{U}}}(I-K_AK_A^*)\geq 0$.

Corollary 5.4. If A is an element in the $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ which has characteristic function Θ_A and finite rank defect operator $\Delta_A(I)\geq 0$, then

$$\text{curv}(A)=\text{rank}\,\Delta_A(I)-\text{trace}\,[\Theta_A(P_{\mathbb{C}}\otimes I)\Theta_A^*(N\otimes I_{\mathcal{H}})].$$

Proof. Since A has a characteristic function, there is a multi-analytic operator $\Theta_A:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}_A$ such that $K_AK_A^*+\Theta_A\Theta_A^*=I$.

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Due to theorem 5.2 and its proof, we have

$$\begin{aligned}\operatorname{curv}(A) &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}\,[\Delta_{\mathbf{S}}(I - K_A K_A^*)(N \otimes I_{\mathcal{D}_A})] \\ &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}\,[\Delta_{\mathbf{S}}(\Theta_A \Theta_A^*)(N \otimes I_{\mathcal{D}_A})] \\ &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}\,[\Theta_A \Delta_{\mathbf{S}}(I)\Theta_A^*(N \otimes I_{\mathcal{D}_A})] \\ &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}\,[\Theta_A(P_{\mathbb{C}} \otimes I)\Theta_A^*(N \otimes I_{\mathcal{H}})].\end{aligned}$$

This completes the proof. □

In what follows, we show that the curvature invariant detects the elements in $B(\mathcal{H})^{n_1+\cdots+n_k}$ which are unitarily equivalent to $I \otimes \mathcal{W}$-twisted multi-shifts $\mathbf{S}_{\mathcal{W}}$ of finite rank defect operator, i.e. acting on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ with $\dim \mathcal{K} < \infty$.

Theorem 5.5 *Let $A \in B(\mathcal{H})^{n_1+\cdots+n_k}$. Then A is unitarily equivalent to an $I \otimes \mathcal{W}$-twisted multi-shift $\boldsymbol{S}_{\mathcal{W}}$ acting on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ with $\dim \mathcal{K} < \infty$ if and only if A is a pure element in a regular $\mathcal{U}$-twisted polyball $\boldsymbol{B}_{reg}^{\mathcal{U}}(\mathcal{H})$ such that A has a characteristic function and*

$$\operatorname{curv}(A) = \operatorname{rank}\Delta_A(I) < \infty.$$

In this case, the noncommutative Berezin kernel $K_A : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}_A$ is a unitary operator and

$$A_{i,s} = K_A^*(\boldsymbol{S}_{\mathcal{U}|_{\mathcal{D}_A}})_{i,s} K_A$$

for any $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$.

Proof. Let $A \in B(\mathcal{H})^{n_1+\cdots+n_k}$, and assume that A is unitarily equivalent to an $I \otimes \mathcal{W}$-twisted multi-shift $\mathbf{S}_{\mathcal{W}}$ acting on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ with $\dim \mathcal{K} < \infty$. Then, there is a unitary operator $\Psi : \mathcal{H} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ such that $A_{i,s} = \Psi^*(\mathbf{S}_{\mathcal{W}})_{i,s}\Psi$ for any $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. Note that $\Delta_A(I) = \Psi^* \Delta_{\mathbf{S}_{\mathcal{W}}}(I)\Psi \geq 0$ and

$$\operatorname{rank}\Delta_A(I) = \operatorname{rank}\Delta_{\mathbf{S}_{\mathcal{W}}}(I) = \operatorname{rank}(P_{\mathbb{C}} \otimes I_{\mathcal{K}}) = \dim \mathcal{K}.$$

Since $\mathbf{S}_{\mathcal{W}}$ is a pure k-tuple of row isometries which are doubly $(I \otimes \mathcal{W})$-commuting, it is easy to see that A is a pure k-tuple of row isometries which are doubly $\mathcal{U}$-commuting, where $\mathcal{U} = \{U_{i,j}(s,t)\}$ is defined by setting $U_{i,j}(s,t) := \Psi^*(I \otimes W_{i,j}(s,t))\Psi$. Therefore, A is a pure element in the regular $\mathcal{U}$-twisted polyball $\mathbf{B}_{reg}^{\mathcal{U}}(\mathcal{H})$. On the other hand, using the definition of the $I \otimes \mathcal{W}$-twisted multi-shift $\mathbf{S} := \mathbf{S}_{\mathcal{W}}$ acting on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$, one can show that the noncommutative Berezin kernel $K_{\mathbf{S}}$ can be identified with the identity on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$. Indeed, we have $K_{\mathbf{S}} : \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K} \to \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{D}_{\mathbf{S}}$. Under the identification of $\mathbb{C}\chi_{(g_0^1,\ldots,g_0^k)}$ with $\mathbb{C}$, we have $\Delta_{\mathbf{S}}(I) = P_{\mathbb{C}} \otimes I_{\mathcal{K}}$ and $\mathcal{D}_{\mathbf{S}} = \mathbb{C} \otimes \mathcal{K} = \mathcal{K}$.

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Since

$$K_{\mathbf{S}}(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h):=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}\chi_{(\beta_1,\ldots,\beta_k)}\otimes(P_{\mathbb{C}}\otimes I_{\mathcal{K}})\mathbf{S}_{k,\beta_k}^*\cdots\mathbf{S}_{1,\beta_1}^*(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h),$$

for $h\in\mathcal{K}$, and

$$(P_{\mathbb{C}}\otimes I_{\mathcal{K}})\mathbf{S}_{k,\beta_k}^*\cdots\mathbf{S}_{1,\beta_1}^*(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h)=\begin{cases}\chi_{(g_0^1,\ldots,g_0^k)}\otimes h, & \text{if } \alpha_1=\beta_1,\ldots,\alpha_k=\beta_k\\ 0, & \text{otherwise,}\end{cases}$$

using the above-mentioned identification, we obtain $K_{\mathbf{S}}(\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h)=\chi_{(\alpha_1,\ldots,\alpha_k)}\otimes h$, which proves our assertion. Now, taking into account that A is unitarily equivalent to $\mathbf{S}$, Corollary 3.4 implies

$$\begin{aligned}\operatorname{curv}(A)=&\operatorname{curv}(\mathbf{S})=\lim_{q_1\to\infty}\cdots\lim_{q_k\to\infty}\frac{\operatorname{trace}\left[K_{\mathbf{S}}^*(P_{\leq(q_1,\ldots,q_k)}\otimes I_{\mathcal{K}})K_{\mathbf{S}}\right]}{\operatorname{trace}\left[P_{\leq(q_1,\ldots,q_k)}\right]}\\ &=\dim\mathcal{K}=\operatorname{rank}\Delta_A(I).\end{aligned}$$

On the other hand, since $K_{\mathbf{S}}K_{\mathbf{S}}^*=I$, we have $\Theta_{\mathbf{S}}=0$ as a characteristic function of $\mathbf{S}$. This completes the proof of the direct implication.

To prove the converse, assume that A is a pure element in a regular $\mathcal{U}$-twisted polyball $\mathbf{B}_{reg}^{\mathcal{U}}(\mathcal{H})$ such that A has a characteristic function and $\operatorname{curv}(A)=\operatorname{rank}\Delta_A(I)<\infty$. Therefore, there is a multi-analytic $\Theta_A:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{E}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}_A$ with respect to the multi-shifts $\mathbf{S}_{\mathcal{W}}$ and $\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}}$, i.e.

$$\Theta_A(\mathbf{S}_{\mathcal{W}})_{i,s}=(\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}})_{i,s}\Theta_A,\qquad i\in\{1,\ldots,k\},s\in\{1,\ldots,n_i\}$$

and

$$\Theta_A(I\otimes W_{i,j}(s,t))=(I\otimes U_{i,j}(s,t)|_{\mathcal{D}_A})\Theta_A,\qquad (i,j,s,t)\in\Gamma,$$

such that $K_AK_A^*+\Theta_A\Theta_A^*=I$. Since A is pure, the noncommutative Berezin kernel is an isometry and, consequently, Θ_A is a partial isometry. We remark that

$$\operatorname{range}\Theta_A^*=\left\{x\in\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{E}:\ \|\Theta_A(x)\|=\|x\|\right\}$$

is the initial space of Θ_A which is invariant under all the isometries $(\mathbf{S}_{\mathcal{W}})_{i,s}$, due to the fact that $\Theta_A(\mathbf{S}_{\mathcal{W}})_{i,s}=(\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}})_{i,s}\Theta_A$. Moreover, since $(\operatorname{range}\Theta_A^*)^\perp=\ker\Theta_A$, it is clear that $(\operatorname{range}\Theta_A^*)^\perp$ is invariant under all isometries $(\mathbf{S}_{\mathcal{W}})_{i,s}$, and consequently, it is jointly reducing for these operators. Due to the fact that

$$I\otimes W_{i,j}(s,t)=(\mathbf{S}_{\mathcal{W}})_{i,s}^*(\mathbf{S}_{\mathcal{W}})_{j,t}^*(\mathbf{S}_{\mathcal{W}})_{i,s}(\mathbf{S}_{\mathcal{W}})_{j,t},$$

the subspace $(\operatorname{range}\Theta_A^*)^\perp$ is also reducing for the operators in $I\otimes\mathcal{W}$.

Since the support of Θ_A is defined as the smallest reducing subspace $\operatorname{supp}(\Theta_A)\subset\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{E}$ under the operators $(\mathbf{S}_{\mathcal{W}})_{i,s}$ containing the co-invariant subspace $\operatorname{range}\Theta_A^*$, we conclude that $\operatorname{supp}(\Theta_A) = \operatorname{range}\Theta_A^*$. Note that $\Phi := \Theta_A|_{\operatorname{supp}(\Theta_A)}$ is an isometric operator and $\Phi\Phi^* = \Theta_A\Theta_A^*$. According to theorem 3.1 from [28], $\operatorname{supp}(\Theta_A) = \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L}$, where $\mathcal{L} := \overline{(P_{\mathbb{C}} \otimes I_{\mathcal{E}})\operatorname{range}\Theta_A^*} \subset \mathcal{E}$.

Since $\Delta_{\mathbf{S}_{\mathcal{W}}}(I) := (id - \Phi_{(\mathbf{S}_{\mathcal{W}})_1}) \circ \cdots \circ (id - \Phi_{(\mathbf{S}_{\mathcal{W}})_k})(I) = P_{\mathbb{C}} \otimes I_{\mathcal{E}}$ and $\operatorname{range}\Theta_A^*$ is a reducing subspace under all unitary operators $I \otimes W_{i,j}(s,t)$, we deduce that

$$\begin{aligned} W_{i,j}(s,t)(P_{\mathbb{C}} \otimes I_{\mathcal{E}})\operatorname{range}\Theta_A^* &= (I \otimes W_{i,j}(s,t))\Delta_{\mathbf{S}_{\mathcal{W}}}(I)\operatorname{range}\Theta_A^* \\ &= \Delta_{\mathbf{S}_{\mathcal{W}}}(I)(I \otimes W_{i,j}(s,t))\operatorname{range}\Theta_A^* \\ &= \Delta_{\mathbf{S}_{\mathcal{W}}}(I)\operatorname{range}\Theta_A^*. \end{aligned}$$

Hence, the subspace $\mathcal{L}$ is reducing for all the unitaries $W_{i,j}(s,t)$. Since

$$\Theta_A(\mathbf{S}_{\mathcal{W}})_{i,s} = (\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}})_{i,s}\Theta_A \text{ and } \Theta_A(I \otimes W_{i,j}(s,t)) = (I \otimes U_{i,j}(s,t)|_{\mathcal{D}_A})\Theta_A,$$

taking the restriction to the support of Θ_A, we obtain

$$\Phi(\mathbf{S}_{\mathcal{W}|_{\mathcal{L}}})_{i,s} = (\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}})_{i,s}\Phi \text{ and } \Phi(I \otimes W_{i,j}(s,t)|_{\mathcal{L}}) = (I \otimes U_{i,j}(s,t)|_{\mathcal{D}_A})\Phi.$$

Consequently, Φ is a multi-analytic operator with respect to the multi-shifts $\mathbf{S}_{\mathcal{W}|_{\mathcal{L}}}$ and $\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}}$. Due to corollary 5.4 and its proof, we have

$$\begin{aligned} \operatorname{curv}(A) &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}[\Delta_{\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}}}(\Theta_A\Theta_A^*)(N \otimes I_{\mathcal{D}_A})] \\ &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}[\Delta_{\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}}}(\Phi\Phi^*)(N \otimes I_{\mathcal{D}_A})] \\ &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}[(\Phi\Delta_{\mathbf{S}_{\mathcal{W}|_{\mathcal{L}}}}(I)\Phi^*)(N \otimes I_{\mathcal{D}_A})] \\ &= \operatorname{rank}\Delta_A(I) - \operatorname{trace}[\Phi(P_{\mathbb{C}} \otimes I_{\mathcal{L}})\Phi^*(N \otimes I_{\mathcal{D}_A})]. \end{aligned}$$

Since $\operatorname{curv}(A) = \operatorname{rank}\Delta_A(I) < \infty$, we must have $\operatorname{trace}[\Phi(P_{\mathbb{C}} \otimes I_{\mathcal{L}})\Phi^*(N \otimes I_{\mathcal{D}_A})] = 0$, which implies $\Phi(P_{\mathbb{C}} \otimes I_{\mathcal{L}})\Phi^*(N \otimes I_{\mathcal{D}_A}) = 0$. Hence, we deduce that $\Phi(P_{\mathbb{C}} \otimes I_{\mathcal{L}})\Phi^*(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{D}_A}) = 0$ for any $(q_1, \ldots, q_k) \in \mathbb{Z}_+^k$. Therefore, $\Phi(P_{\mathbb{C}} \otimes I_{\mathcal{L}})\Phi^* = 0$. Taking into account that Ψ is an isometric multi-analytic operator with respect to the multi-shifts $\mathbf{S}_{\mathcal{W}|_{\mathcal{L}}}$ and $\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}}$, we deduce that $\Phi(P_{\mathbb{C}} \otimes I_{\mathcal{L}}) = 0$ and, consequently, $\Phi = 0$. On the other hand, using the fact that $\Phi\Phi^* = \Theta_A\Theta_A^*$ and $K_AK_A^* + \Theta_A\Theta_A^* = I$, we infer that $K_AK_A^* = I$. Since A is pure, we have $K_A^*K_A = I$, which shows that K_A is a unitary operator. Due to theorem 3.2, we have

$$A_{i,s} = K_A^*(\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_A}})_{i,s}K_A$$

for any $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. This completes the proof. □

Let $\mathbf{S}_{\mathbf{z}}$ be the standard multi-shift associated with the scalar weights $\mathbf{z}$. Then,

$$\operatorname{curv}_*(\mathbf{S}_{\mathbf{z}}) = \operatorname{curv}(\mathbf{S}_{\mathbf{z}}) = \operatorname{rank}\Delta_{\mathbf{S}_{\mathbf{z}}}(I) = 1.$$

Proof. From the proof of theorem 5.5, we have $\operatorname{curv}(\mathbf{S}_{\mathbf{z}}) = \operatorname{rank}\Delta_{\mathbf{S}_{\mathbf{z}}}(I) = 1$. Using corollary 2.8, we complete the proof. □

Let $\mathbf{S}_{\mathcal{W}}$ be the standard k-tuple of doubly $I\otimes\mathcal{W}$-commuting pure isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$. We say that $\mathcal{M}\subset\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$ is a Beurling type [3] jointly invariant subspace under the operators $(\mathbf{S}_{\mathcal{W}})_{i,s}$ and $I\otimes W_{i,j}(s,t)$, where $i\in\{1,\dots,k\}$, $s\in\{1,\dots,n_i\}$, and $(i,j,s,t)\in\Gamma$, if there are a Hilbert space $\mathcal{L}$, a standard k-tuple $\mathbf{S}_{\mathcal{U}}$ of doubly $I\otimes\mathcal{U}$-commuting pure isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}$, and an inner multi-analytic operator $\Psi:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$ with respect to the multi-shifts $\mathbf{S}_{\mathcal{U}}$ and $\mathbf{S}_{\mathcal{W}}$ such that

$$\mathcal{M}=\Psi\left(\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\right).$$

In what follows, we use the notation $\mathbf{S}_{\mathcal{W}}|_{\mathcal{M}}$ for the restriction of $\mathbf{S}_{\mathcal{W}}$ to an invariant subspace $\mathcal{M}$ under all the operators $(\mathbf{S}_{\mathcal{W}})_{i,s}$ and $I\otimes W_{i,j}(s,t)$. In [28], we proved that the following statements are equivalent:

(i) $\mathcal{M}$ is a Beurling type jointly invariant subspace under the operators $(\mathbf{S}_{\mathcal{W}})_{i,s}$ and $I\otimes W_{i,j}(s,t)$.
(ii) $(id-\Phi_{(\mathbf{S}_{\mathcal{W}})_1})\circ\cdots\circ(id-\Phi_{(\mathbf{S}_{\mathcal{W}})_k})(P_{\mathcal{M}})\geq 0$.
(iii) The k-tuple $\mathbf{S}_{\mathcal{W}}|_{\mathcal{M}}$ is doubly $(I\otimes\mathcal{W})|_{\mathcal{M}}$-commuting.
(iv) There is an isometric multi-analytic operator $\Psi:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$ with respect to the standard multi-shifts $\mathbf{S}_{\mathcal{U}}$ and $\mathbf{S}_{\mathcal{W}}$ such that

$$\mathcal{M}=\Psi\left(\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\right).$$

If $\mathcal{M}$ is a Beurling type jointly invariant subspace under $\mathbf{S}_{\mathcal{W}}$ and $I\otimes\mathcal{W}$, we say that it has a finite rank if $\mathbf{S}_{\mathcal{W}}|_{\mathcal{M}}$ has a finite rank defect operator.

DEFINITION 5.7. *Let $\boldsymbol{S}_{\mathcal{W}}$ be the standard k-tuple of doubly $I\otimes\mathcal{W}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$, and let $\mathcal{M}$ and $\mathcal{N}$ be invariant subspaces under $\boldsymbol{S}_{\mathcal{W}}$ and $I\otimes\mathcal{W}$. We say that $\mathcal{M}$ and $\mathcal{N}$ are unitarily equivalent if there is a unitary operator $\Gamma:\mathcal{M}\to\mathcal{N}$ such that*

$$\Gamma(\boldsymbol{S}_{\mathcal{W}})_{i,s}|_{\mathcal{M}}=(\boldsymbol{S}_{\mathcal{W}})_{i,s}|_{\mathcal{N}}\Gamma\quad\textit{and}\quad\Gamma(I\otimes W_{i,j}(s,t))|_{\mathcal{M}}=(I\otimes W_{i,j}(s,t))|_{\mathcal{N}}\Gamma.$$

PROPOSITION 5.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be finite rank Beurling type jointly invariant subspace under $\boldsymbol{S}_{\mathcal{W}}$ and $I\otimes\mathcal{W}$. If $\mathcal{M}$ and $\mathcal{N}$ are unitarily equivalent, then*

$$\operatorname{curv}(\boldsymbol{S}_{\mathcal{W}}|_{\mathcal{M}})=\operatorname{rank}\Delta_{\boldsymbol{S}_{\mathcal{W}}|_{\mathcal{M}}}(I_{\mathcal{M}})=\operatorname{curv}(\boldsymbol{S}_{\mathcal{W}}|_{\mathcal{N}})=\operatorname{rank}\Delta_{\boldsymbol{S}_{\mathcal{W}}|_{\mathcal{N}}}(I_{\mathcal{N}}).$$

Proof. Due to the remarks preceding definition 5.7, $\mathbf{S}_{\mathcal{W}}|_{\mathcal{M}}$ is a doubly $(I\otimes\mathcal{W})|_{\mathcal{M}}$-commuting k-tuple of pure row isometry and there is an isometric multi-analytic operator $\Psi:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{K}$ with respect to the

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standard multi-shifts $\mathbf{S}_\mathcal{U}$ and $\mathbf{S}_\mathcal{W}$ such that

$$\mathcal{M} = \Psi\left(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L}\right).$$

Consequently, $P_\mathcal{M} = \Psi\Psi^*$ and

$$\Delta_{\mathbf{S}_{\mathcal{W}|\mathcal{M}}}(I_\mathcal{M}) = \Delta_{\mathbf{S}_{\mathcal{W}|\mathcal{M}}}(P_\mathcal{M}) = \Psi\Delta_{\mathbf{S}_\mathcal{U}}(I)\Psi^*|_\mathcal{M} = \Psi(P_\mathbb{C} \otimes I_\mathcal{L})\Psi^*|_\mathcal{M}.$$

Using the latter relation and taking into account that Ψ is an isometry, we can prove that $\operatorname{rank}\Delta_{\mathbf{S}_{\mathcal{W}|\mathcal{M}}}(I_\mathcal{M}) = \dim\mathcal{L}$. Indeed, if $\{w_\sigma\}_{\sigma\in\Sigma}$ is an orthonormal basis for the subspace $\mathcal{L}$, then

$$\{\Psi(\chi_{(\alpha_1,\ldots,\alpha_k)} \otimes w_\sigma) : \ \sigma \in \Sigma, (\alpha_1,\ldots,\alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\}$$

is an orthonormal basis for $\mathcal{M}$. Moreover, $\overline{\Psi(P_\mathbb{C} \otimes I_\mathcal{L})\Psi^*|_\mathcal{M}}$ is equal to the closure of the range of the defect operator $\Delta_{\mathbf{S}_{\mathcal{W}|\mathcal{M}}}(I_\mathcal{M})$ and also coincides with the closed linear span of the vectors $\{\Psi(1 \otimes w_\sigma) : \ \sigma \in \Sigma\}$. Therefore,

$$\operatorname{rank}\Delta_{\mathbf{S}_{\mathcal{W}|\mathcal{M}}}(I_\mathcal{M}) = \operatorname{card}\Sigma = \dim\mathcal{L}.$$

Now, we prove that $\operatorname{curv}(\mathbf{S}_\mathcal{W}|_\mathcal{M}) = \dim\mathcal{L}$. As in the proof of theorem 3.3, using the definition of the $I\otimes\mathcal{W}$-twisted multi-shift $\mathbf{S} := \mathbf{S}_\mathcal{U}$ acting on $\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{L}$, one can show that

$$\Phi_{\mathbf{S}_i}^{q_i+1}(I)\left(\chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h\right) = \begin{cases} \chi_{(\alpha_1,\ldots,\alpha_k)} \otimes h, & \text{if } |\alpha_i| \geq q_i + 1 \\ 0, & \text{otherwise} \end{cases}$$

for any $q_i \in \mathbb{N}$. Using this relation, it is easy to see that $(id - \Phi_{\mathbf{S}_1}^{q_1+1}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k}^{q_k+1})(I)$ is the orthogonal projection of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L}$ onto

$$\operatorname{span}\{\chi_{(\alpha_1,\ldots,\alpha_k)} : \ |\alpha_1| \leq q_1,\ldots,|\alpha_k| \leq q_k\} \otimes \mathcal{L}.$$

Due to corollary 3.4, we have

$$\operatorname{curv}(\mathbf{S}_\mathcal{U}) = \lim_{q_1\to\infty} \cdots \lim_{q_k\to\infty} \frac{\operatorname{trace}\left[(id - \Phi_{\mathbf{S}_1}^{q_1+1}) \circ \cdots \circ (id - \Phi_{\mathbf{S}_k}^{q_k+1})(I)\right]}{\prod_{i=1}^k (1 + n_i + \cdots + n_i^{q_i})} = \dim\mathcal{L}.$$

Now, note that

$$(id - \Phi_{(\mathbf{S}_\mathcal{W})_1|\mathcal{M}}^{q_1+1}) \circ \cdots \circ (id - \Phi_{(\mathbf{S}_\mathcal{W})_k|\mathcal{M}}^{q_k+1})(I_\mathcal{M}) = \Psi(id - \Phi_{(\mathbf{S}_\mathcal{U})_1}^{q_1+1}) \circ \cdots \circ$$
$$(id - \Phi_{(\mathbf{S}_\mathcal{U})_k|}^{q_k+1})(I)\Psi^*|\mathcal{M}.$$

Using again corollary 3.4, as above, we deduce that

$$\operatorname{curv}(\mathbf{S}_\mathcal{W}|_\mathcal{M}) = \operatorname{curv}(\mathbf{S}_\mathcal{U}) = \dim\mathcal{L}.$$

Consequently, we have $\operatorname{curv}(\mathbf{S}_\mathcal{W}|_\mathcal{M}) = \operatorname{rank}\Delta_{\mathbf{S}_{\mathcal{W}|\mathcal{M}}}(I_\mathcal{M})$. Similarly, we have that $\mathbf{S}_\mathcal{W}|_\mathcal{N}$ is a doubly $(I \otimes \mathcal{W})|_\mathcal{N}$-commuting k-tuple and $\operatorname{curv}(\mathbf{S}_\mathcal{W}|_\mathcal{N}) =$

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rank $\Delta_{\mathbf{S}_{\mathcal{W}}|_{\mathcal{N}}}(I_{\mathcal{N}})$. Now, since the curvature is invariant up to unitary equivalence, if $\mathcal{M}$ and $\mathcal{N}$ are unitarily equivalent, then $\operatorname{curv}(\mathbf{S}_{\mathcal{W}}|_{\mathcal{M}}) = \operatorname{curv}(\mathbf{S}_{\mathcal{W}}|_{\mathcal{N}})$. The proof is complete. □

6. Invariant subspaces under $\mathcal{U}$-twisted multi-shifts and their multiplicity

In this section, we introduce the notion of multiplicity for the invariant subspaces under the multi-shifts with a finite rank defect operator, prove the existence, provide several asymptotic formulas, and connect it to the curvature invariant. Under appropriate conditions, we show that there is a trace class *multiplicity operator* whose trace coincides with the multiplicity. We also obtain results concerning the semi-continuity for the curvature and the multiplicity invariants. Finally, we provide necessary and sufficient conditions when $A|_{\mathcal{M}}$ is in $\mathbf{B}^{\mathcal{U}|_{\mathcal{M}}}(\mathcal{H})$ and consider some consequences.

In what follows, $\mathbf{S}$ is the standard k-tuple of doubly $I \otimes \mathcal{U}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$ and assume that $\dim \mathcal{H} < \infty$. If $\mathcal{M}$ is any invariant subspace under $\mathbf{S}$ and $I \otimes \mathcal{U}$, we introduce the multiplicity of $\mathcal{M}$ by setting

$$m(\mathcal{M}) := \lim_{m\to\infty} \frac{1}{\binom{m+k}{k}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k\leq m}} \frac{\operatorname{trace}\left[P_{\mathcal{M}}(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]}.$$

Theorem 6.1 *Let $\boldsymbol{S}$ be the standard k-tuple of doubly $I \otimes \mathcal{U}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$, where $\dim \mathcal{H} < \infty$. If $\mathcal{M}$ is any invariant subspace under $\boldsymbol{S}$ and $I \otimes \mathcal{U}$, the multiplicity of $\mathcal{M}$ exists and*

$$\begin{aligned} m(\mathcal{M}) &= \lim_{m\to\infty} \frac{1}{\binom{m+k-1}{k-1}} \sum_{\substack{q_1\geq 0,\ldots,q_k\geq 0\\ q_1+\cdots+q_k= m}} \frac{\operatorname{trace}\left[P_{\mathcal{M}}(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]} \\ &= \lim_{q_1\to\infty} \cdots \lim_{q_k\to\infty} \frac{\operatorname{trace}\left[P_{\mathcal{M}}(P_{\leq(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{\leq(q_1,\ldots,q_k)}\right]} \\ &= \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[P_{\mathcal{M}}(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]}. \end{aligned}$$

Proof. Set $T := (T_1, \ldots, T_n)$ with $T_i := (T_{i,1}, \ldots, T_{i,n_i})$ with $T_{i,s} := P_{\mathcal{M}^\perp}\mathbf{S}_{i,s}|_{\mathcal{M}^\perp}$ for any $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. In this case, we also use the notation $T = P_{\mathcal{M}^\perp}\mathbf{S}|_{\mathcal{M}^\perp}$. Since $T_{i,s}^* = \mathbf{S}_{i,s}^*|_{\mathcal{M}^\perp}$ and each $\mathbf{S}_i := [\mathbf{S}_{i,1} \cdots \mathbf{S}_{i,n_i}]$ is a row

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contraction, T_i is also a row contraction. On the other hand, since

$$\sum_{\alpha\in\mathbb{F}_{n_i}^+,|\alpha|=q} T_{i,\alpha}T_{i,\alpha}^* = P_{\mathcal{M}^\perp}\left(\sum_{\alpha\in\mathbb{F}_{n_i}^+,|\alpha|=q} \mathbf{S}_{i,\alpha}\mathbf{S}_{i,\alpha}^*\right)|_{\mathcal{M}^\perp}$$

and $[\mathbf{S}_{i,1}\cdots\mathbf{S}_{i,n_i}]$ is a pure row isometry, we deduce that T_i is a pure row contraction. Also note that

$$(id-\Phi_{rT_1})\circ\cdots\circ(id-\Phi_{rT_k})(I) = P_{\mathcal{M}^\perp}(id-\Phi_{r\mathbf{S}_1})\circ\cdots\circ(id-\Phi_{r\mathbf{S}_k})(I)|_{\mathcal{M}^\perp}\geq 0,\quad r\in[0,1).$$

On the other hand, since $\mathcal{M}$ is reducing under each $I\otimes U_{i,j}(s,t)$, we have

$$\begin{aligned} T_{j,t}^*T_{i,s}^* &= \mathbf{S}_{j,t}^*\mathbf{S}_{i,s}^*|_{\mathcal{M}^\perp} = \mathbf{S}_{i,s}^*\mathbf{S}_{j,t}^*(I\otimes U_{i,j}(s,t)^*)|_{\mathcal{M}^\perp}\\ &= T_{i,s}^*T_{j,t}^*W_{i,j}(s,t)^* \end{aligned}$$

where $W_{i,j}(s,t) := (I\otimes U_{i,j}(s,t))|_{\mathcal{M}^\perp}$. Hence, $T_{i,s}T_{j,t} = W_{ij}(s,t)T_{j,t}T_{i,s}$ for any $(i,j,s,t)\in\Gamma$. Also note that, since

$$(I\otimes U_{i,j}(s,t)^*)\mathbf{S}_{p,q}^* = \mathbf{S}_{p,q}^*(I\otimes U_{i,j}(s,t)^*)$$

for any $p\in\{1,\ldots,k\}$, $q\in\{1,\ldots,n_p\}$, and $(i,j,s,t)\in\Gamma$, we have that $W_{i,j}(s,t)^*T_{p,q}^* = T_{p,q}^*W_{i,j}(s,t)^*$. Hence, $T_{p,q}W_{i,j}(s,t) = W_{i,j}(s,t)T_{p,q}$. Therefore, $T\in\mathbf{B}_{reg}^{\mathcal{W}}(\mathcal{H})$. Taking into account that $\Delta_T(I_{\mathcal{M}^\perp}) = P_{\mathcal{M}^\perp}\Delta_{\mathbf{S}}(I)|_{\mathcal{M}^\perp}$, we deduce that

$$\operatorname{rank}\Delta_T(I_{\mathcal{M}^\perp})\leq\operatorname{rank}\Delta_{\mathbf{S}}(I) = \dim\mathcal{H}.$$

Now, we need to show that $\Phi_{\mathbf{S}_1}^{q_1}\circ\cdots\circ\Phi_{\mathbf{S}_k}^{q_k}(\Delta_{\mathbf{S}}(I)) = P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}}$. Indeed, as we saw in the proof of theorem 3.3,

$$(\Phi_{\mathbf{S}_1}^{q_1}-\Phi_{\mathbf{S}_1}^{q_1+1})\circ\cdots\circ(\Phi_{\mathbf{S}_k}^{q_k}-\Phi_{\mathbf{S}_k}^{q_k+1})(I) = P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}}.$$

Since, due to proposition 2.3,

$$\Phi_{\mathbf{S}_1}^{q_1}\circ\cdots\circ\Phi_{\mathbf{S}_k}^{q_k}(\Delta_{\mathbf{S}}(I)) = (\Phi_{\mathbf{S}_1}^{q_1}-\Phi_{\mathbf{S}_1}^{q_1+1})\circ\cdots\circ(\Phi_{\mathbf{S}_k}^{q_k}-\Phi_{\mathbf{S}_k}^{q_k+1})(I),$$

our assertion follows. Now, using again that $\mathcal{M}$ is an invariant subspace under $\mathbf{S}$, we have

$$\begin{aligned} \operatorname{trace}\left[\Phi_{T_1}^{q_1}\circ\cdots\circ\Phi_{T_k}^{q_k}(\Delta_T(I_{\mathcal{M}^\perp}))\right] &= \operatorname{trace}\left[P_{\mathcal{M}^\perp}\Phi_{\mathbf{S}_1}^{q_1}\circ\cdots\circ\Phi_{\mathbf{S}_k}^{q_k}(\Delta_{\mathbf{S}}(I))|_{\mathcal{M}^\perp}\right]\\ &= \operatorname{trace}\left[P_{\mathcal{M}^\perp}(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]. \end{aligned}$$

Hence, and due to theorem 3.3 and corollary 3.4,

$$\begin{aligned}\operatorname{curv}(T) &= \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[\Phi_{T_1}^{q_1}\circ\cdots\circ\Phi_{T_k}^{q_k}(\Delta_T(I_{\mathcal{M}^\perp}))\right]}{n_1^{q_1}\cdots n_k^{q_k}}\\ &= \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[P_{\mathcal{M}^\perp}(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]}{n_1^{q_1}\cdots n_k^{q_k}}\end{aligned}$$

exists. This implies

$$\begin{aligned}&\lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[P_{\mathcal{M}}(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]}\\ &= \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]} - \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[P_{\mathcal{M}^\perp}(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]}\\ &= \dim\mathcal{H} - \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[P_{\mathcal{M}^\perp}(P_{(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right]}{\operatorname{trace}\left[P_{(q_1,\ldots,q_k)}\right]} = \dim\mathcal{H} - \operatorname{curv}(T).\end{aligned}$$

Now, using again theorem 3.3 and corollary 3.4, one can easily complete the proof. □

The multiplicity invariant measures the size of the invariant subspaces under $\mathbf{S}$ and $I\otimes\mathcal{U}$ in the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$. Note that $m(\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}) = \dim\mathcal{H}$, and if $\mathcal{M}_1$ and $\mathcal{M}_2$ are orthogonal invariant subspaces, then $m(\mathcal{M}_1\oplus\mathcal{M}_2) = m(\mathcal{M}_1) + m(\mathcal{M}_2)$.

Corollary 6.2. Under the hypothesis of theorem 6.1, the following statements hold:

(i) $m(\mathcal{M}) = \dim\mathcal{H} - \operatorname{curv}(P_{\mathcal{M}^\perp}\mathbf{S}_{\mathcal{U}}|_{\mathcal{M}^\perp})$

(ii) If $\mathcal{M}_1,\ldots,\mathcal{M}_n$ are orthogonal invariant subspaces under $\mathbf{S}$ and $I\otimes\mathcal{U}$, then

$$\begin{aligned}&\operatorname{curv}(P_{(\mathcal{M}_1\oplus\cdots\oplus\mathcal{M}_n)^\perp}\mathbf{S}_{\mathcal{U}}|_{(\mathcal{M}_1\oplus\cdots\oplus\mathcal{M}_n)^\perp})\\ &\qquad= \sum_{i=1}^n \operatorname{curv}(P_{\mathcal{M}_i^\perp}\mathbf{S}_{\mathcal{U}}|_{\mathcal{M}_i^\perp}) - (n-1)\dim\mathcal{H}.\end{aligned}$$

(iii) If $T := P_{\mathcal{M}^\perp}\mathbf{S}_{\mathcal{U}}|_{\mathcal{M}^\perp}$ and K_T is the noncommutative Berezin kernel associated with T, then

$$m(\mathcal{M}) = \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \operatorname{trace}\left[\Delta_{\mathbf{S}_{\mathcal{U}}}(I - K_T K_T^*)(N_{\le(q_1,\ldots,q_k)}\otimes I_{\mathcal{H}})\right].$$

Proof. The first relation follows from the proof of theorem 6.1. Item (ii) follows from item (i) and the fact that $m(\mathcal{M}_1\oplus\cdots\oplus\mathcal{M}_n) = \sum_{i=1}^n m(\mathcal{M}_i)$. To prove item

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(iii), we use theorem 5.2 which shows that

$$\operatorname{curv}(T) = \lim_{(q_1,\dots,q_k)\in\mathbb{Z}_+^k} \operatorname{trace}\left[\Delta_{\mathbf{S}_{\mathcal{U}}}(K_T K_T^*)(N_{\le(q_1,\dots,q_k)}\otimes I_{\mathcal{H}})\right].$$

Now, using item (i), we complete the proof. □

Corollary 6.3. Let $\mathbf{S}_{\mathbf{z}}$ be the standard $\mathbf{z}$-twisted multi-shift on $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)$. If $\mathcal{M}$ is an invariant subspace under $\mathbf{S}_{\mathbf{z}}\otimes I_{\mathcal{E}}$ with $\dim\mathcal{E}<\infty$, then its multiplicity exists. In particular, if $n_1=\cdots=n_k=1$, then $\mathcal{M}$ is in the vector-valued Hardy space $H^2(\mathbb{D}^k)\otimes\mathcal{E}$ and

$$m(\mathcal{M}) = \lim_{m\to\infty}\frac{\operatorname{trace}[P_{\mathcal{M}}(P_{\le m}\otimes I_{\mathcal{E}})]}{\operatorname{trace}[P_{\le m}]},$$

where $P_{\le m}$ is the orthogonal projection on the polynomials of degree $\le m$.

We remark that if $n_1=\cdots=n_k=1$, the result of corollary 6.3 is a twisted version of Fang's [9] commutative result for $H^2(\mathbb{D}^k)\otimes\mathcal{E}$ when $\mathbf{z}=\{1\}$.

The next result shows that the multiplicity invariant is lower semi-continuous.

Theorem 6.4 *Let $\boldsymbol{S}_{\mathcal{U}}$ be the $\mathcal{U}$-twisted multi-shift with $\mathcal{U}\subset B(\mathcal{H})$ and $\dim\mathcal{H}<\infty$, acting on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$. If $\mathcal{M}$ and $\mathcal{M}_p$ are invariant subspaces of $\boldsymbol{S}_{\mathcal{U}}$ and $I\otimes\mathcal{U}$ such that $P_{\mathcal{M}_p}\to P_{\mathcal{M}}$ in the weak operator topology, then*

$$\liminf_{p\to\infty} m(\mathcal{M}_p)\ge m(\mathcal{M})$$

and

$$\limsup_{p\to\infty}\operatorname{curv}(P_{\mathcal{M}_p^\perp}\boldsymbol{S}_{\mathcal{U}}|_{\mathcal{M}_p^\perp})\le \operatorname{curv}(P_{\mathcal{M}^\perp}\boldsymbol{S}_{\mathcal{U}}|_{\mathcal{M}^\perp}).$$

Proof. Let $B:=(B_1,\dots,B_k)$ with $B_i:=(B_{i,1},\dots,B_{i,n_i})$ and $B_{i,s}:=P_{\mathcal{M}^\perp}(\mathbf{S}_{\mathcal{U}})_{i,s}|_{\mathcal{M}^\perp}$. In a similar manner, we define $B^{(p)}:=(B_1^{(p)},\dots,B_k^{(p)})$. As in the proof of theorem 6.1, we have

$$\begin{aligned}\operatorname{trace}[\Phi_{B_1}^{q_1}\circ\cdots\circ\Phi_{B_k}^{q_k}(\Delta_B(I_{\mathcal{M}^\perp}))] &= \operatorname{trace}[P_{\mathcal{M}^\perp}(P_{(q_1,\dots,q_k)}\otimes I_{\mathcal{H}})]\\ &= n_1^{q_1}\cdots n_k^{q_k}\dim\mathcal{H}-\operatorname{trace}[P_{\mathcal{M}}(P_{(q_1,\dots,q_k)}\otimes I_{\mathcal{H}})]\end{aligned}$$

and a similar relation associated with $B^{(p)}$ holds. Since $P_{\mathcal{M}_p}\to P_{\mathcal{M}}$ in the weak operator topology, we have

$$\lim_{p\to\infty}\operatorname{trace}[P_{\mathcal{M}_m}(P_{(q_1,\dots,q_k)}\otimes I_{\mathcal{H}})] = \operatorname{trace}[P_{\mathcal{M}}(P_{(q_1,\dots,q_k)}\otimes I_{\mathcal{H}})].$$

Consequently, we obtain

$$\lim_{p\to\infty}\operatorname{trace}[\Phi_{B_1^{(p)}}^{q_1}\circ\cdots\circ\Phi_{B_k^{(p)}}^{q_k}(\Delta_{B^{(p)}}(I_{\mathcal{M}_p^\perp}))] = \operatorname{trace}[\Phi_{B_1}^{q_1}\circ\cdots\circ\Phi_{B_k}^{q_k}(\Delta_B(I_{\mathcal{M}^\perp}))].$$

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As in the proof of theorem 3.6, one can show that $\limsup_{p\to\infty} \operatorname{curv}(B^{(m)}) \leq \operatorname{curv}(B)$. On the other hand, using corollary 6.2, we have

$$m(\mathcal{M}) = \dim \mathcal{H} - \operatorname{curv}(B) \quad \text{and} \quad m(\mathcal{M}_p) = \dim \mathcal{H} - \operatorname{curv}(B^{(p)}).$$

Consequently, we obtain $\liminf_{p\to\infty} m(\mathcal{M}_p) \geq m(\mathcal{M})$. The proof is complete. □

Since the proof of the next result is straightforward, we should omit it.

Proposition 6.5. *Let $A = (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ be a $\mathcal{U}$-commuting tuple, and let $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace under A and $\mathcal{U}$. If $B := P_{\mathcal{M}^\perp} A|_{\mathcal{M}^\perp}$, then the following statements hold:*

(i) *B is $\mathcal{U}|_{\mathcal{M}^\perp}$-commuting.*
(ii) *$\Delta_{rB}(I_{\mathcal{M}^\perp}) = P_{\mathcal{M}^\perp} \Delta_{rA}(I_\mathcal{H})|_{\mathcal{M}^\perp}$ for any $r \in [0, 1]$.*
(iii) *If $A \in \boldsymbol{B}^\mathcal{U}(\mathcal{H})$, then $B \in \boldsymbol{B}^{\mathcal{U}|_{\mathcal{M}^\perp}}(\mathcal{M}^\perp)$.*
(iv) *If $A \in \boldsymbol{B}^\mathcal{U}_{reg}(\mathcal{H})$, then $B \in \boldsymbol{B}^{\mathcal{U}|_{\mathcal{M}^\perp}}_{reg}(\mathcal{M}^\perp)$.*

Theorem 6.6 *Let $A := (A_1, \ldots, A_k)$ be an element in the $\mathcal{U}$-twisted polyball $\boldsymbol{B}^\mathcal{U}(\mathcal{H})$ such that $\Delta_A(I)$ is a positive trace class operator. If $\mathcal{M}$ is an invariant subspace under A and $\mathcal{U}$ with $\dim \mathcal{M} < \infty$, then $P_{\mathcal{M}^\perp} A|_{\mathcal{M}^\perp} \in \boldsymbol{B}^{\mathcal{U}|_{\mathcal{M}^\perp}}(\mathcal{M}^\perp)$ has a positive trace class defect operator and*

$$\operatorname{curv}(A) = \operatorname{curv}(P_{\mathcal{M}^\perp} A|_{\mathcal{M}^\perp}).$$

Proof. Set $B := (B_1, \ldots, B_k)$ and $B_i := (B_{i,1}, \ldots, B_{i,n_i})$, where $B_{i,s} := P_{\mathcal{M}^\perp} A_{i,s}|_{\mathcal{M}^\perp}$ for $i \in \{1, \ldots, k\}$ and $s \in \{1, \ldots, n_i\}$. Note that $\Phi^{q_i}_{B_i}(I_{\mathcal{M}^\perp}) = P_{\mathcal{M}^\perp} \Phi^{q_i}_{A_i}(I_\mathcal{H})|_{\mathcal{M}^\perp}$ and $\Delta_B(I_{\mathcal{M}^\perp}) = P_{\mathcal{M}^\perp} \Delta_A(I_\mathcal{H})|_{\mathcal{M}^\perp} \geq 0$. Consequently, $\operatorname{trace} \Delta_B(I_{\mathcal{M}^\perp}) \leq \operatorname{trace} \Delta_A(I_\mathcal{H}) < \infty$. It is easy to see that, taking into account that

$$\| \left(id - \Phi^{q_1+1}_{A_1}\right) \circ \cdots \circ \left(id - \Phi^{q_k+1}_{A_k}\right)(I_\mathcal{H}) \| \leq 2^k,$$

we obtain

$$\begin{aligned}
&\operatorname{trace}\left[\left(id - \Phi^{q_1+1}_{A_1}\right) \circ \cdots \circ \left(id - \Phi^{q_k+1}_{A_k}\right)(I_\mathcal{H})\right] \\
&= \operatorname{trace}\left[P_{\mathcal{M}^\perp}\left(id - \Phi^{q_1+1}_{A_1}\right) \circ \cdots \circ \left(id - \Phi^{q_k+1}_{A_k}\right)(I_\mathcal{H})|_{\mathcal{M}^\perp}\right] \\
&\quad + \operatorname{trace}\left[P_\mathcal{M}\left(id - \Phi^{q_1+1}_{A_1}\right) \circ \cdots \circ \left(id - \Phi^{q_k+1}_{A_k}\right)(I_\mathcal{H})\right] \\
&\leq \operatorname{trace}\left[\left(id - \Phi^{q_1+1}_{B_1}\right) \circ \cdots \circ \left(id - \Phi^{q_k+1}_{B_k}\right)(I_{\mathcal{M}^\perp})\right] + 2^k \dim \mathcal{M}.
\end{aligned}$$

Hence, we deduce that

$$\left| \frac{\operatorname{trace}\left[\left(id-\Phi_{A_1}^{q_1+1}\right)\circ\cdots\circ\left(id-\Phi_{A_k}^{q_k+1}\right)(I_{\mathcal{H}})\right]}{\prod_{i=1}^k(1+n_i+\cdots+n_i^{q_i})} - \frac{\operatorname{trace}\left[\left(id-\Phi_{B_1}^{q_1+1}\right)\circ\cdots\circ\left(id-\Phi_{B_k}^{q_k+1}\right)(I_{\mathcal{M}^\perp})\right]}{\prod_{i=1}^k(1+n_i+\cdots+n_i^{q_i})}\right| \leq \frac{2^k\dim\mathcal{M}}{\prod_{i=1}^k(1+n_i+\cdots+n_i^{q_i})}.$$

Using corollary 3.4, we deduce that $\operatorname{curv}(A)=\operatorname{curv}(B)$. The proof is complete. □

According to [27], the reducing subspaces under the standard $\mathcal{U}$-twisted multi-shift $\mathbf{S}_{\mathcal{U}}$ on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$ are of the form $\mathcal{M}=\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{L}$, where $\mathcal{L}\subset\mathcal{H}$ is a reducing subspace under all the unitaries $U_{i,j}(s,t)$ in $\mathcal{U}$.

Using some results from [28], we prove the following.

Proposition 6.7. *Let $\boldsymbol{S}_{\mathcal{U}}$ be the $\mathcal{U}$-twisted multi-shift on $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$, and let $\mathcal{M}$ be an invariant subspace under $\boldsymbol{S}_{\mathcal{U}}$ and $\mathcal{U}$ which does not contain nontrivial reducing subspaces for $\boldsymbol{S}_{\mathcal{U}}$. Then, the compression $T:=P_{\mathcal{M}^\perp}\boldsymbol{S}_{\mathcal{U}}|_{\mathcal{M}^\perp}$ has a characteristic function if and only if $\mathcal{M}$ is a Beurling type invariant subspace.*

Proof. Under the given hypothesis, $\mathcal{M}^\perp$ is a cyclic subspace for $\mathbf{S}_{\mathcal{U}}$. Therefore, $\mathbf{S}_{\mathcal{U}}$ is a minimal isometric dilation of $T:=P_{\mathcal{M}^\perp}\mathbf{S}_{\mathcal{U}}|_{\mathcal{M}^\perp}$. On the other hand, according to [28] (see theorem 3.3 and its proof) if

$$K_T:\mathcal{M}^\perp\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}_T,\qquad \mathcal{D}_T:=\overline{\Delta_T(I)(\mathcal{M}^\perp)},$$

is the noncommutative Berezin kernel associated with T, then the subspace $K_T\mathcal{M}^\perp$ is co-invariant under each operator $(\mathbf{S}_{\mathcal{U}})_{i,s}$ for any $i\in\{1,\dots,k\}$, $j\in\{1,\dots,n_i\}$ and the dilation provided by the relation

$$T_{i,s}=K_T^*(\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_T}})_{i,s})K_T$$

is minimal. Due to the uniqueness of the minimal isometric dilation of pure elements in $\mathcal{U}$-twisted polyballs (see theorem 3.3 from [28]), there is a unitary operator

$$\Psi:\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{D}_T\to\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$$

such that $\Psi(\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_T}})_{i,s}=(\mathbf{S}_{\mathcal{U}})_{i,s})\Psi$ and $\Psi K_T=V$, where V is the injection of the subspace $\mathcal{M}^\perp$ into $\ell^2(\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+)\otimes\mathcal{H}$. Since Ψ is a unitary operator, we also

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deduce that $\Psi(\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_T}})^*_{i,s}) = (\mathbf{S}_{\mathcal{U}})^*_{i,s}\Psi$. Consequently,

$$\Psi(P_{\mathbb{C}} \otimes I_{\mathcal{D}_T}) = \Psi\Delta_{\mathbf{S}_{\mathcal{U}|_{\mathcal{D}_T}}}(I_{\mathcal{D}_T}) = \Delta_{\mathbf{S}_{\mathcal{U}}}(I_{\mathcal{H}})\Psi = (P_{\mathbb{C}} \otimes I_{\mathcal{H}})\Psi.$$

Hence, we deduce that $\Psi\mathcal{D}_T = \mathcal{H}$. Setting $\psi := \Psi|_{\mathcal{D}_T} : \mathcal{D}_T \to \mathcal{H}$ and taking into account that Ψ is multi-analytic, we deduce that $\Psi = I \otimes \psi$. Therefore, $\psi : \mathcal{D}_T \to \mathcal{H}$ is a unitary operator such that $(I \otimes \psi)K_T = V$. Consequently, $K_T K_T^* = (I \otimes \psi^*)P_{\mathcal{M}^\perp}(I \otimes \psi)$ and

$$\Delta_{\mathbf{S}_{\mathcal{U}}}(I - K_T K_T^*) = (I \otimes \psi^*)\Delta_{\mathbf{S}_{\mathcal{U}}}(P_{\mathcal{M}})(I \otimes \psi).$$

Due to the remarks preceding definition 5.7, $\mathcal{M}$ is a Beurling type invariant subspace if and only if $\Delta_{\mathbf{S}_{\mathcal{U}}}(P_{\mathcal{M}}) \geq 0$. Using the above identity, we deduce that $\Delta_{\mathbf{S}_{\mathcal{U}}}(I - K_T K_T^*) \geq 0$. Due to theorem 6.1 from [28], the later inequality is equivalent to T having a characteristic function. The proof is complete. □

THEOREM 6.8 *Let $\boldsymbol{S}$ be the standard k-tuple of doubly $I \otimes \mathcal{U}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}) \otimes \mathcal{H}$, where $\dim \mathcal{H} < \infty$. If $\mathcal{M}$ is a Beurling type invariant subspace under $\boldsymbol{S}$ and $I \otimes \mathcal{U}$ which does not contain non-trivial reducing subspace under $\boldsymbol{S}$, then the multiplicity operator $\Delta_{\boldsymbol{S}}(P_{\mathcal{M}})(N \otimes I)$ is trace class and*

$$m(\mathcal{M}) = \operatorname{trace}\left[\Delta_{\boldsymbol{S}}(P_{\mathcal{M}})(N \otimes I)\right].$$

In particular, this relation holds for Beurling type invariant subspace under $\boldsymbol{S_z} \otimes I_{\mathcal{E}}$ with $\dim \mathcal{E} < \infty$.

Proof. According to proposition 6.7 and its proof, we have

$$\Delta_{\mathbf{S}_{\mathcal{U}}}(I - K_T K_T^*) = (I \otimes \psi^*)\Delta_{\mathbf{S}_{\mathcal{U}}}(P_{\mathcal{M}})(I \otimes \psi) \geq 0,$$

where $\psi : \mathcal{D}_T \to \mathcal{H}$ is a unitary operator. On the other hand, due to corollary 5.4 and its proof, we have

$$\operatorname{curv}(T) = \operatorname{rank} \Delta_T(I) - \operatorname{trace}\left[\Delta_{\mathbf{S}_{\mathcal{U}}}(I - K_T K_T^*)(N \otimes I_{\mathcal{D}_T})\right].$$

Consequently, using corollary 6.2, we deduce that

$$\begin{aligned} m(\mathcal{M}) &= \dim \mathcal{H} - \operatorname{curv}(T) \\ &= \operatorname{trace}\left[\Delta_{\mathbf{S}_{\mathcal{U}}}(I - K_T K_T^*)(N \otimes I_{\mathcal{D}_T})\right] \\ &= \operatorname{trace}\left[(I \otimes \psi^*)\Delta_{\mathbf{S}_{\mathcal{U}}}(P_{\mathcal{M}})(I \otimes \psi)(N \otimes I_{\mathcal{D}_T})\right) \\ &= \operatorname{trace}\left[\Delta_{\mathbf{S}}(P_{\mathcal{M}})(N \otimes I)\right]. \end{aligned}$$

The proof is complete. □

We remark that if $A = (A_1, \ldots, A_k)$ with $A_i = (A_{i,1}, \ldots, A_{i,n_i})$ is an element in $\mathbf{B}^{\mathcal{U}}_{reg}(\mathcal{H})$ and $\mathcal{M}$ is an invariant subspace under A and $\mathcal{U}$, then $A|_{\mathcal{M}}$ is not necessarily in the regular $\mathcal{U}|_{\mathcal{M}}$-twisted polyball, where $A|_{\mathcal{M}}$ is defined by taking the restrictions $A_{i,s}|\mathcal{M}$. However, we have the following result.

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Proposition 6.9. *Let* $A = (A_1, \ldots, A_k) \in B(\mathcal{H})^{n_1+\cdots+n_k}$ *be a* $\mathcal{U}$*-commuting tuple, and let* $\mathcal{M} \subset \mathcal{H}$ *be an invariant subspace under* A *and* $\mathcal{U}$*. Then the following statements hold:*

(i) $A|_\mathcal{M}$ *is* $\mathcal{U}|_\mathcal{M}$*-commuting.*
(ii) $\Delta_{rA|_\mathcal{M}}(I_\mathcal{M}) = \Delta_{rA}(P_\mathcal{M})|_\mathcal{M}$ *for any* $r \in [0,1]$.
(iii) If $A \in \boldsymbol{B}^\mathcal{U}(\mathcal{H})$*, then* $A|_\mathcal{M} \in \boldsymbol{B}^{\mathcal{U}|_\mathcal{M}}(\mathcal{M})$.
(iv) If $A \in \boldsymbol{B}^\mathcal{U}_{reg}(\mathcal{H})$*, then* $A|_\mathcal{M} \in \boldsymbol{B}^{\mathcal{U}|_\mathcal{M}}_{reg}(\mathcal{M})$ *if and only if*

$$\Delta_{rA}(P_\mathcal{M}) \geq 0, \qquad r \in [0,1).$$

If A *is pure, then the later condition is equivalent to* $\Delta_A(P_\mathcal{M}) \geq 0$.

Proof. Since $\mathcal{M}$ is an invariant subspace under A and $\mathcal{U}$, item (i) is straightforward and

$$\Delta_{rA|_\mathcal{M}}(I_\mathcal{M}) = \Delta_{rA}(P_\mathcal{M})|_\mathcal{M}, \qquad r \in [0,1).$$

Since item (iii) is clear, we prove (iv). Assume that $A \in \mathbf{B}^\mathcal{U}_{reg}(\mathcal{H})$, i.e. $\Delta_{rA|_\mathcal{M}}(I_\mathcal{M}) \geq 0$ for any $r \in [0,1)$. Let $h \in \mathcal{H}$, and consider the orthogonal decomposition $h = x+y$, with $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$. Using the fact that $\mathcal{M}^\perp$ is invariant subspace under each operator $A^*_{i,j}$, we have

$$\begin{aligned}\langle \Delta_{rA}(P_\mathcal{M})(x+y), x+y\rangle &= \langle \Delta_{rA}(P_\mathcal{M})x, x+y\rangle + \langle \Delta_{rA}(P_\mathcal{M})y, x+y\rangle \\ &= \langle \Delta_{rA}(P_\mathcal{M})x, x\rangle + \|y\|^2 \geq 0.\end{aligned}$$

Consequently, $\Delta_{rA}(P_\mathcal{M}) \geq 0$ for any $r \in [0,1)$. Conversely, if $\Delta_{rA}(P_\mathcal{M}) \geq 0$, then it is clear that $\Delta_{rA|_\mathcal{M}}(I_\mathcal{M}) \geq 0$. If, in addition, A is pure and $\Delta_A(P_\mathcal{M}) \geq 0$, then, since Φ_{A_1} is a positive linear map, we deduce that

$$\Phi^m_{A_1}\left(id - \Phi_{A_2}\right) \circ \cdots \circ \left(id - \Phi_{A_k}\right)(P_\mathcal{M}) \leq \left(id - \Phi_{A_2}\right) \circ \cdots \circ \left(id - \Phi_{A_k}\right)(P_\mathcal{M})$$

for any $m \in \mathbb{N}$. Taking into account that $\Phi^m_{A_1}(I) \to 0$ as $m \to \infty$, we deduce that $\left(id - \Phi_{A_2}\right)\circ\cdots\circ\left(id - \Phi_{A_k}\right)(P_\mathcal{M}) \geq 0$. Similarly, taking into account that $P_\mathcal{M} \in \mathcal{U}'$, we can use proposition 2.3 and show that $\left(id - \Phi_{A_1}\right)^{p_1}\circ\cdots\circ\left(id - \Phi_{A_k}\right)^{p_k}(P_\mathcal{M}) \geq 0$ for any $p_i \in \{0,1\}$. The later condition implies $\Delta_{rA}(P_\mathcal{M}) \geq 0$ for any $r \in [0,1)$. The proof is very similar to the proof of the implication (ii) $\Longrightarrow$ (iii) of proposition 1.2 from [28]. We include a proof for completeness. Since $\Delta_A(P_\mathcal{M}) \geq 0$, we deduce that $\Phi_{A_1}(\Delta_{(A_2,\ldots,A_k)}(P_\mathcal{M})) \leq \Delta_{(A_2,\ldots,A_k)}(P_\mathcal{M})$, where

$$\Delta_{(A_2,\ldots,A_k)}(P_\mathcal{M}) := (id - \Phi_{A_2}) \circ \cdots \circ (id - \Phi_{A_k})(P_\mathcal{M}) \geq 0.$$

Hence, $0 \leq \Phi_{rA_1}(\Delta_{(A_2,\ldots,A_k)}(P_\mathcal{M}) \leq \Delta_{(A_2,\ldots,A_k)}(P_\mathcal{M})$ for any $r \in [0,1)$, and consequently, we have $(id - \Phi_{rA_1}) \circ \cdots \circ (id - \Phi_{A_k})(P_\mathcal{M}) \geq 0$. Since $P_\mathcal{M} \in \mathcal{U}'$, we use proposition 2.3 to deduce that

$$(id - \Phi_{A_2}) \circ \cdots \circ (id - \Phi_{A_k}) \circ (id - \Phi_{rA_1})(P_\mathcal{M}) \geq 0. \tag{6.1}$$

A similar argument as above, starting with the inequality $(id - \Phi_{A_1}) \circ (id - \Phi_{A_3}) \circ \cdots \circ (id - \Phi_{A_k})(P_\mathcal{M}) \geq 0$, leads to $(id - \Phi_{A_3}) \circ \cdots \circ (id - \Phi_{A_k}) \circ (id - \Phi_{rA_1})(P_\mathcal{M}) \geq 0$.

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Repeating the argument but starting with the inequality (6.1) shows that

$$(id - \Phi_{A_3}) \circ \cdots \circ (id - \Phi_{A_k}) \circ (id - \Phi_{rA_1}) \circ (id - \Phi_{rA_2})(P_{\mathcal{M}}) \geq 0.$$

Iterating this process, we conclude that $\Delta_{rA}(P_{\mathcal{M}}) \geq 0$ for any $r \in [0, 1)$. The proof is complete. □

PROPOSITION. *Let* $A := (A_1, \ldots, A_k) \in \boldsymbol{B}^{\mathcal{U}}_{reg}(\mathcal{H})$ *be a pure k-tuple, and let* $\mathcal{M} \subset \mathcal{H}$ *be a jointly invariant subspace under* A *and* $\mathcal{U}$*. Then, the following statements are equivalent:*

(i) $A|_{\mathcal{M}} \in \boldsymbol{B}^{\mathcal{U}|_{\mathcal{M}}}_{reg}(\mathcal{M})$.

(ii) $\mathcal{M}$ *is a Beurling type invariant subspace, i.e. there are a Hilbert space* $\mathcal{E}$*, a standard multi-shift* $\boldsymbol{S}_{\mathcal{W}}$ *of doubly* $I \otimes \mathcal{W}$*-commuting pure isometries on the Hilbert space* $\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}) \otimes \mathcal{E}$*, and a partial isometry* $\Psi : \ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}) \otimes \mathcal{E} \to \mathcal{H}$ *such that*

$$\Psi(\boldsymbol{S}_{\mathcal{W}})_{i,s} = A_{i,s}\Psi \text{ and } \Psi(I \otimes W_{i,j}(s,t)) = U_{i,j}(s,t)\Psi$$

and $\mathcal{M} = \Psi\left(\ell^2(\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}) \otimes \mathcal{E}\right)$.

Proof. The proof follows from proposition 6.9 and corollary 6.4 from [28]. □

THEOREM *Let* $A \in \boldsymbol{B}^{\mathcal{U}}(\mathcal{H})$ *be such that* $\Delta_A(I)$ *is a positive trace class operator. If* $\mathcal{M}$ *is an invariant subspace under* A *and* $\mathcal{U}$ *such that* $\dim \mathcal{M}^{\perp} < \infty$ *and* $\Delta_A(P_{\mathcal{M}}) \geq 0$*, then* $A|_{\mathcal{M}} \in \boldsymbol{B}^{\mathcal{U}|_{\mathcal{M}}}(\mathcal{M})$ *has a positive trace class defect and*

$$|\text{curv}(A) - \text{curv}(A|_{\mathcal{M}})| \leq \dim \mathcal{M}^{\perp} \prod_{i=1}^{k} (n_i - 1).$$

Proof. Since $\mathcal{M}$ is an invariant subspace under all the operators $A_{i,s}$, proposition 6.9 shows that $\Delta_{A|_{\mathcal{M}}}(I_{\mathcal{M}}) = \Delta_A(P_{\mathcal{M}})|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$. Taking into account that $\mathcal{M}^{\perp}$ is invariant under all $A^*_{i,s}$, we deduce that $\Delta_A(P_{\mathcal{M}})|_{\mathcal{M}^{\perp}} = 0$ and consequently, trace$[\Delta_{A|_{\mathcal{M}}}(I_{\mathcal{M}})] =$ trace$[\Delta_A(P_{\mathcal{M}})]$. The same proposition shows that if $\Delta_A(P_{\mathcal{M}}) \geq 0$, then $\Delta_{A|_{\mathcal{M}}}(I_{\mathcal{M}}) \geq 0$. On the other hand, taking into account that $\Delta_A(P_{\mathcal{M}}) = \Delta_A(I_{\mathcal{H}}) - \Delta_A(P_{\mathcal{M}^{\perp}})$ and $\dim \mathcal{M}^{\perp} < \infty$, we conclude that trace$[\Delta_A(P_{\mathcal{M}})] < \infty$. This shows that $A|_{\mathcal{M}}$ is in $\mathbf{B}^{\mathcal{U}|_{\mathcal{M}}}(\mathcal{M})$ and has a positive trace class defect. Therefore, curv$(A|_{\mathcal{M}})$ exists.

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Using again that $\mathcal{M}$ is an invariant subspace under all operators $A_{i,s}$, we deduce that

$$\begin{aligned}
&\operatorname{trace}\left[\left(id - \Phi_{A_1|_{\mathcal{M}}}^{q_1+1}\right) \circ \cdots \circ \left(id - \Phi_{A_k|_{\mathcal{M}}}^{q_k+1}\right)(I_{\mathcal{M}})\right] \\
&\quad = \operatorname{trace}\left[\left(id - \Phi_{A_1}^{q_1+1}\right) \circ \cdots \circ \left(id - \Phi_{A_k}^{q_k+1}\right)(P_{\mathcal{M}})\right] \\
&\quad = \operatorname{trace}\left[\left(id - \Phi_{A_1}^{q_1+1}\right) \circ \cdots \circ \left(id - \Phi_{A_k}^{q_k+1}\right)(I_{\mathcal{H}})\right] \\
&\qquad - \operatorname{trace}\left[\left(id - \Phi_{A_1}^{q_1+1}\right) \circ \cdots \circ \left(id - \Phi_{A_k}^{q_k+1}\right)(P_{\mathcal{M}^\perp})\right] \\
&\quad \leq \operatorname{trace}\left[\left(id - \Phi_{A_1}^{q_1+1}\right) \circ \cdots \circ \left(id - \Phi_{A_k}^{q_k+1}\right)(I_{\mathcal{H}})\right] \\
&\qquad + (1+n_1^{q_1+1}) \cdots (1+n_k^{q_k+1}) \operatorname{trace}[P_{\mathcal{M}^\perp}].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\left| \frac{\operatorname{trace}\left[\left(id - \Phi_{T_1}^{q_1+1}\right) \circ \cdots \circ \left(id - \Phi_{T_k}^{q_k+1}\right)(I_{\mathcal{H}})\right]}{\prod_{i=1}^k (1+n_i+\cdots+n_i^{q_i})} \right. \\
&\qquad \left. - \frac{\operatorname{trace}\left[\left(id - \Phi_{T_1|_{\mathcal{M}}}^{q_1+1}\right) \circ \cdots \circ \left(id - \Phi_{T_k|_{\mathcal{M}}}^{q_k+1}\right)(I_{\mathcal{M}})\right]}{\prod_{i=1}^k (1+n_i+\cdots+n_i^{q_i})} \right| \\
&\qquad \leq \frac{(1+n_1^{q_1+1}) \cdots (1+n_k^{q_k+1})}{\prod_{i=1}^k (1+n_i+\cdots+n_i^{q_i})} \operatorname{trace}[P_{\mathcal{M}^\perp}].
\end{aligned}$$

If $n_i = 1$ for some $i \in \{1, \ldots, k\}$, an application of corollary 3.4 shows that $\operatorname{curv}(A) = \operatorname{curv}(A|_{\mathcal{M}})$. On the other hand, if all $n_i \geq 2$, then, using the same corollary, we obtain the inequality in the theorem. The proof is complete. □

Let $A := (A_1, \ldots, A_k)$ be an element in the $\mathcal{U}$-twisted polyball $\mathbf{B}^{\mathcal{U}}(\mathcal{H})$ such that $\Delta_A(I)$ is a positive trace class operator. If $\mathcal{M}$ is an invariant subspace under A and $\mathcal{U}$, we introduce the *multiplicity of* $\mathcal{M}$ *with respect to* A to be

$$m_A(\mathcal{M}) := \lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\operatorname{trace}\left[P_{\mathcal{M}} K_A^*(P_{(q_1,\ldots,q_k)} \otimes I_{\mathcal{H}}) K_A\right]}{\operatorname{trace}[P_{(q_1,\ldots,q_k)}]}.$$

A close look at the proof of theorem 6.1 reveals that one can replace the standard multi-shift $\mathbf{S}$ with A and the corresponding proof holds true showing that $m_A(\mathcal{M})$ exists. Moreover, one can obtain analogues of the asymptotic formulas from theorem 6.1 in the new setting and prove that the following index type formula:

$$m_A(\mathcal{M}) = \operatorname{curv}(A) - \operatorname{curv}(P_{\mathcal{M}^\perp} A|_{\mathcal{M}^\perp}).$$

Note that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are orthogonal invariant subspaces under A and $\mathcal{U}$, then

$$m_A(\mathcal{M}_1 \oplus \mathcal{M}_2) = m_A(\mathcal{M}_1) + m_A(\mathcal{M}_2).$$

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We remark that if $A = \mathbf{S}$, then $m_{\mathbf{S}}(\mathcal{M}) = m(\mathcal{M})$. Indeed, this is due to the fact that

$$K_{\mathbf{S}}^*(P_{(q_1,\dots,q_k)} \otimes I_{\mathcal{H}})K_{\mathbf{S}} = \Phi_{\mathbf{S}_1}^{q_1} \circ \cdots \circ \Phi_{\mathbf{S}_k}^{q_k}(\Delta_{\mathbf{S}}(I)) = P_{(q_1,\dots,q_k)} \otimes I_{\mathcal{H}}.$$

7. The range of the curvature and multiplicity invariants

In this section, we determine the range of the curvature and the multiplicity invariants. If $(n_1,\dots,n_k) \in \mathbb{N}^k$ is such that $n_j \geq 2$ for some j, we prove that the range of the curvature over the pure elements in the $\mathcal{U}$-twisted polyballs and the range of the multiplicity invariant coincide with $[0,\infty)$. We also show that the range of the curvature restricted to the class of doubly $\mathcal{U}$-commuting row isometries with trace class defect operator is $\mathbb{Z}_+$.

Let $S := (S_1,\dots,S_k)$ with $S_i := [S_{i,1} \cdots S_{i,n_i}]$ be the standard multi-shift on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ associated with $\mathcal{U} = \{1_{\mathbb{C}}\}$, and let $\mathbf{S} := (\mathbf{S}_1,\dots,\mathbf{S}_k)$ with $\mathbf{S}_i := [\mathbf{S}_{i,1} \cdots \mathbf{S}_{i,n_i}]$ be the standard k-tuple of doubly $I \otimes \mathcal{U}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H}$. For each $i \in \{1,\dots,k\}$ and $s \in \{1,\dots,n_i\}$, we define the block diagonal operator $D_{i,s} \in B(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{H})$ by setting

$$D_{i,s}(\chi_{(\alpha_1,\dots,\alpha_k)} \otimes h) := \chi_{(\alpha_1,\dots,\alpha_k)} \otimes \boldsymbol{U}_{i,1}(s,\alpha_1) \cdots \boldsymbol{U}_{i,i-1}(s,\alpha_{i-1})h, \qquad i \in \{2,\dots,k\},$$

and $D_{i,s} = I$ if $i = 1$. Note that $\mathbf{S}_{i,s} = (S_{i,s} \otimes I_{\mathcal{H}})D_{i,s}$.

LEMMA 7.1. *Let $\Omega \subset \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, and let $\mathcal{M}_\Omega \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ be the smallest invariant subspace under the multi-shift S generated by $\{\chi_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Omega\}$. If $\mathcal{N} \subset \mathcal{H}$ is an invariant subspace under $\{U_{i,j}(s,t)\}_{(i,j,s,t)\in\Gamma_-}$, where $\Gamma_- := \{(i,j,s,t) \in \Gamma, i > j\}$, then $\mathcal{M}_\Omega \otimes \mathcal{N}$ is an invariant subspace under the multi-shift $\boldsymbol{S}$.*

Proof. Note that

$$\mathcal{M}_\Omega = \overline{\text{span}}\{\chi_{(\beta_1\alpha_1,\dots,\beta_k\alpha_k)} : \boldsymbol{\alpha} \in \Omega, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\}$$

and if $h \in \mathcal{N}$, then

$$D_{i,s}(\chi_{(\beta_1\alpha_1,\dots,\beta_k\alpha_k)} \otimes h) = \chi_{(\beta_1\alpha_1,\dots,\beta_k\alpha_k)} \otimes \boldsymbol{U}_{i,1}(s,\beta_1\alpha_1) \cdots \boldsymbol{U}_{i,i-1}(s,\beta_{i-1}\alpha_{i-1})h \in \mathcal{M}_\Omega \otimes \mathcal{L}.$$

Since $(S_{i,s} \otimes I)(\mathcal{M}_\Omega \otimes \mathcal{L}) \subset \mathcal{M}_\Omega \otimes \mathcal{L}$ and $\mathbf{S}_{i,s} = (S_{i,s} \otimes I_{\mathcal{H}})D_{i,s}$, the proof is complete. □

THEOREM 7.2 *Let $(n_1,\dots,n_k) \in \mathbb{N}^k$ be such that $n_j \geq 2$ for some j. Then, the following statements hold:*

(i) For any $t \in [0,1]$, there is an invariant subspace $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ of the $\boldsymbol{z}$-twisted multi-shift $\mathbf{S}_{\mathbf{z}}$ on $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ such that

$$\text{curv}(P_{\mathcal{M}^\perp} \boldsymbol{S}_{\boldsymbol{z}}|_{\mathcal{M}^\perp}) = t \quad \textit{and} \quad m(\mathcal{M}) = 1 - t.$$

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(ii) *For any* $t \in [0, m]$, *there exists a pure element* A *in the* $\mathcal{U}$*-twisted polyball such that* $\operatorname{rank} A = m$ *and* $\operatorname{curv}(A) = t$.

(iii) *The range of the curvature on the* $\mathcal{U}$*-twisted polyballs is* $[0, \infty)$.

(iv) $P_{\mathcal{M}^\perp} \boldsymbol{S}_z|_{\mathcal{M}^\perp}$ *is unitarily equivalent to* $P_{\mathcal{M}^\perp} \boldsymbol{S}_{z'}|_{\mathcal{M}^\perp}$ *if and only if* $\boldsymbol{z} = \boldsymbol{z}'$.

Proof. Fix $a \in (0,1)$ and consider its n_j-arry representation $a = \sum_{p=1}^N \frac{d_p}{n_j^{k_p}}$, where $\{k_p\}_{p=1}^N$ is a sequence of natural numbers with $1 \leq k_1 < k_2 < \cdots$, $N \in \mathbb{N}$ or $N = \infty$, and $d_p \in \{1, 2, \dots, n_j - 1\}$. Consider the subsets of $\mathbb{F}_{n_j}^+$ defined by setting

$$\Omega_1 := \left\{ (g_1^j)^{k_1}, \dots, (g_{d_1}^j)^{k_1} \right\},$$
$$\Omega_p := \left\{ (g_1^j)^{k_p - k_{p-1}} (g_{n_j}^j)^{k_{p-1}}, (g_2^j)^{k_p - k_{p-1}} (g_{n_j}^j)^{k_{p-1}}, \dots, (g_{d_p}^j)^{k_p - k_{p-1}} (g_{n_j}^j)^{k_{p-1}} \right\},$$
$$p = 2, 3, \dots, N,$$

and let

$$\mathcal{M} := \overline{\operatorname{span}} \left\{ \chi_{(\alpha_1, \dots, \alpha_k)} : \ \alpha_j \in \cup_{p=1}^N \Omega_p \text{ and } \ \alpha_i \in \mathbb{F}_{n_i}^+ \text{if } i \neq j \right\}.$$

Due to lemma 7.1, $\mathcal{M}$ is an invariant subspace under the $\mathbf{z}$-twisted multi-shift $\mathbf{S}_\mathbf{z}$. Assume now that $k_p \leq q_j < k_{p+1}$ and $q_i \in \mathbb{Z}_+$ if $i \neq j$. Note that

$$\begin{aligned}
&\frac{\operatorname{trace}\left[P_\mathcal{M} P_{(q_1, \dots, q_k)} \right]}{\operatorname{trace}\left[P_{(q_1, \dots, q_k)} \right]} \\
&\qquad = \frac{1}{n_1^{q_1} \cdots n_k^{q_k}} \sum_{(\alpha_1, \dots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \left\langle P_\mathcal{M} P_{(q_1, \dots, q_k)} \chi_{(\alpha_1, \dots, \alpha_k)}, \chi_{(\alpha_1, \dots, \alpha_k)} \right\rangle \\
&\qquad = \frac{1}{n_1^{q_1} \cdots n_k^{q_k}} \sum_{\substack{(\alpha_1, \dots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \\ |\alpha_1| = q_1, \dots, |\alpha_k| = q_k}} \| P_\mathcal{M} \chi_{(\alpha_1, \dots, \alpha_k)} \|^2 \\
&= \frac{d_1 n_j^{q_j - k_1} \left(\prod_{i \in \{1, \dots, k\} \setminus \{j\}} n_i^{q_i} \right) + \cdots + d_p n_j^{q_j - k_p} \left(\prod_{i \in \{1, \dots, k\} \setminus \{j\}} n_i^{q_j} \right)}{n_1^{q_1} \cdots n_k^{q_k}} \\
&= \frac{d_1}{n_j^{k_1}} + \cdots + \frac{d_p}{n_j^{k_p}}.
\end{aligned}$$

Hence and using corollary 6.2, we deduce that

$$\operatorname{curv}(P_{\mathcal{M}^\perp} \mathbf{S}_\mathbf{z}|_{\mathcal{M}^\perp}) = 1 - \lim_{q_1, \dots, q_k \to \infty} \frac{\operatorname{trace}\left[P_\mathcal{M} P_{(q_1, \dots, q_k)} \right]}{\operatorname{trace}\left[P_{(q_1, \dots, q_k)} \right]} = 1 - \sum_{p=1}^N \frac{d_p}{n_j^{k_p}} = 1 - a.$$

On the other hand, we note that $\operatorname{curv}(S_\mathbf{z}) = 1$ (see corollary 5.6) and $\operatorname{curv}(A) = 0$ if A is the $\mathcal{U}$-twisted polyball over a finite dimensional Hilbert space. This proves item (i) of the theorem. To prove items (ii) and (iii), we use part (i), consider direct sums of $\mathcal{U}$-twisted polyballs and apply corollary 2.9.

Now, we prove item (iv). Since $\mathcal{M}^\perp$ is a cyclic subspace $\mathbf{S_z}$, the minimal isometric dilation of $P_{\mathcal{M}^\perp}\mathbf{S_z}|_{\mathcal{M}^\perp}$ is $\mathbf{S_z}$. If $P_{\mathcal{M}^\perp}\mathbf{S_z}|_{\mathcal{M}^\perp}$ is unitarily equivalent to $P_{\mathcal{M}^\perp}\mathbf{S_{z'}}|_{\mathcal{M}^\perp}$, then their minimal isometric dilations $\mathbf{S_z}$ and $\mathbf{S_{z'}}$, respectively, are unitarily equivalent. Using corollary 4.2, we complete the proof. □

What is the range of the curvature in the particular case when $n_1 = \cdots = n_k = 1$? At the moment, we know that the curvature invariant takes all the values in $\mathbb{Z}_+$. Whether these are the only values remains to be seen. We mention that, when $\mathcal{U} = \{1\}$, then according to [9], the curvature takes only integer values.

Corollary. The range of the $*$-curvature on the $\mathcal{U}$-twisted polyballs is $[0, \infty)$.

We remark that, due to corollary 6.2, if $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ is a proper invariant subspace of the $\mathbf{z}$-twisted multi-shift $\mathbf{S_z}$ with $\dim \mathcal{M}^\perp < \infty$, then

$$\operatorname{curv}(P_{\mathcal{M}^\perp}\mathbf{S_z}|_{\mathcal{M}^\perp}) = 0 \quad \text{and} \quad m(\mathcal{M}) = 1.$$

However, we have the following result when $\dim \mathcal{M}^\perp = \infty$.

Proposition 7.4. *If $(n_1, \ldots, n_k) \in \mathbb{N}^k$ with $n_j \geq 2$ for some $j \in \{1, \ldots, k\}$, then there exist invariant subspaces $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ of the $\mathbf{z}$-twisted multi-shift $\mathbf{S_z}$ with $\dim \mathcal{M}^\perp = \infty$ such that $\operatorname{curv}(P_{\mathcal{M}^\perp}\boldsymbol{S_z}|_{\mathcal{M}^\perp}) = 0$.*

Proof. Let

$$\mathcal{S} : \overline{\operatorname{span}} \left\{ \chi_{g_0^1, \ldots, g_0^{j-1}, (g_1^j)^p, g_0^{j+1}, \ldots, g_0^k)} : \ p \in \mathbb{Z}^+ \right\},$$

and note that $\mathcal{M} := \mathcal{S}^\perp$ is infinite codimensional and invariant under $\mathbf{S_z}$. Due to corollary 6.2, we have

$$\begin{aligned}
&\operatorname{curv}(P_{\mathcal{M}^\perp}\mathbf{S_z}|_{\mathcal{M}^\perp}) \\
&\quad = 1 - \lim_{q_1, \ldots, q_k \to \infty} \frac{1}{n_1^{q_1} \cdots n_k^{q_k}} \sum_{(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \\
&\qquad \left\langle P_{\mathcal{M}} P_{(q_1, \ldots, q_k)} \chi_{(\alpha_1, \ldots, \alpha_k)}, \chi_{(\alpha_1, \ldots, \alpha_k)} \right\rangle \\
&\quad = 1 - \lim_{q_1, \ldots, q_k \to \infty} \frac{1}{n_1^{q_1} \cdots n_k^{q_k}} \sum_{\substack{(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \\ |\alpha_1| = q_1, \ldots, |\alpha_k| = q_k}} \|P_{\mathcal{M}} \chi_{(\alpha_1, \ldots, \alpha_k)}\|^2 \\
&\quad = 1 - \lim_{q_1, \ldots, q_k \to \infty} \frac{n_1^{q_1} \cdots n_{j-1}^{q_{j-1}} (n_j^{q_j} - 1) n_{j+1}^{q_{j+1}} \cdots n_k^{q_k}}{n_1^{q_1} \cdots n_k^{q_k}} = 0.
\end{aligned}$$

The proof is complete. □

Theorem *Let $V := (V_1, \ldots, V_k)$ with $V_i := [V_{i,1} \cdots V_{i,n_i}]$ and $V_{i,s} \in B(\mathcal{H})$ be a k-tuple of doubly $\mathcal{U}$-commuting row isometries with a trace class defect operator. Then the following statements hold:*

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(i) $\operatorname{curv}(V) = \operatorname{trace}[\Delta_V(I)] = \operatorname{rank}\Delta_V(I)$ *and*

$$\operatorname{curv}(V) \neq 0 \textit{ if and only if } \bigcap_{\substack{i\in\{1,\ldots,k\}\\ s\in\{1,\ldots,n_i\}}} \ker V_{i,s}^* \neq \{0\}.$$

(ii) *For each* $m \in \mathbb{Z}_+$,

$$\operatorname{curv}(V) = m \quad \textit{if and only if} \quad \dim \bigcap_{\substack{i\in\{1,\ldots,k\}\\ s\in\{1,\ldots,n_i\}}} \ker V_{i,s}^* = m.$$

(iii) *If* $m \in \mathbb{Z}_+$, *there is a* k*-tuple* V *of doubly* $\mathcal{U}$*-commuting row isometries such that* $\operatorname{curv}(V) = m$.

(iv) *If* $\operatorname{curv}(V) \neq 0$ *and* $n_j \geq 2$ *for some* $j \in \{1,\ldots,k\}$, *then for any* $t \in [0, \operatorname{curv}(V)]$, *there is an invariant subspace* $\mathcal{M} \subset \mathcal{H}$ *under* V *and* $\mathcal{U}$ *such that* $\operatorname{curv}(P_{\mathcal{M}^\perp} V|_{\mathcal{M}^\perp}) = t$.

Proof. According to the Wold decomposition of theorem 1.5 from [27], there exist 2^k subspaces $\{\mathcal{H}_\Omega\}_{\Omega\subset\{1,\ldots,k\}}$ (some of them may be trivial) such that $\mathcal{K}$ admits a unique orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\Omega\subset\{1,\ldots,k\}} \mathcal{H}_\Omega$$

with the property that, for each subset $\Omega \subset \{1,\ldots,k\}$,

(i) the subspace $\mathcal{H}_\Omega$ is reducing for all the isometries $V_{i,m}$, where $i \in \{1,\ldots,k\}$ and $m \in \{1,\ldots,n_i\}$;
(ii) if $i \in \Omega$, then $V_i|_{\mathcal{H}_\Omega} := [V_{i,1}|_{\mathcal{H}_\Omega} \cdots V_{i,n_i}|_{\mathcal{K}_\Omega}]$ is a pure row isometry;
(iii) if $i \in \Omega^c$, then $V_i|_{\mathcal{H}_\Omega} := [V_{i,1}|_{\mathcal{H}_\Omega} \cdots V_{i,n_i}|_{\mathcal{H}_\Omega}]$ is a Cuntz row isometry;
(iv) the subspace $\mathcal{H}_\Omega$ is reducing for all the unitary operators in $\mathcal{U}$, and the row isometries

$$V_i|_{\mathcal{H}_\Omega} := [V_{i,1}|_{\mathcal{H}_\Omega} \cdots V_{i,n_i}|_{\mathcal{H}_\Omega}], \qquad i \in \{1,\ldots,k\},$$

are doubly $\mathcal{U}|_{\mathcal{H}_\Omega}$-commuting, where $\mathcal{U}|_{\mathcal{H}_\Omega} := \{U_{i,j}(s,t)|_{\mathcal{H}_\Omega}\}_{(i,j,s,t)\in\Gamma}$.

Consequently, we have

$$\operatorname{curv}(V) = \sum_{\Omega\subset\{1,\ldots,k\}} \operatorname{curv}(V|_{\mathcal{H}_\Omega}) \text{ and } \Delta_V(I) = \bigoplus_{\Omega\subset\{1,\ldots,k\}} \Delta_{V|_{\mathcal{H}_\Omega}}(I_{\mathcal{H}_\Omega}). \tag{7.1}$$

If $\Omega = \{1,\ldots,k\}$ and $\mathcal{H}_{\{1,\ldots,k\}} \neq \{0\}$, then $V|_{\mathcal{H}_{\{1,\ldots,k\}}}$ is a tuple of pure row isometries which, according to theorem 2.4 and theorem 1.6 from from [27], is unitarily equivalent to the standard $I \otimes U|_{\mathcal{L}_{\{1,\ldots,k\}}}$-twisted multi-shift $\mathbf{S}$ on the

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Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L}_{\{1,\dots,k\}}$, where

$$\mathcal{L}_{\{1,\dots,k\}} := \bigcap_{\substack{i\in\{1,\dots,k\}\\ s\in\{1,\dots,n_i\}}} \ker V_{i,s}^*.$$

Hence, we deduce that

$$\begin{aligned}\operatorname{curv}(V|_{\mathcal{H}_{\{1,\dots,k\}}}) &= \operatorname{rank} \Delta_{V|_{\mathcal{H}_{\{1,\dots,k\}}}}(I_{\mathcal{H}_{\{1,\dots,k\}}}) = \operatorname{curv}(\mathbf{S})\\ &= \operatorname{trace}(\mathbf{S}) = \operatorname{rank}\left(\Delta_{\mathbf{S}}(I)\right) = \dim \bigcap_{\substack{i\in\{1,\dots,k\}\\ s\in\{1,\dots,n_i\}}} \ker V_{i,s}^*.\end{aligned}$$

On the other hand, if $\Omega \subset \{1,\dots,k\}$ and $\Omega \neq \{1,\dots,k\}$, then, for any $i \in \Omega^c$, $V_i|_{\mathcal{H}_\Omega}$ is a Cuntz isometry, i.e. $\sum_{s=1}^{n_i} (V_{i,s}V_{i,s}^*)|_{\mathcal{H}_\Omega} = I_{\mathcal{H}_\Omega}$, which implies $\operatorname{curv}(V|_{\mathcal{H}_\Omega}) = 0$ and $\Delta_{V|_{\mathcal{H}_\Omega}}(I_{\mathcal{H}_\Omega}) = 0$. Using relation (7.1), we deduce that

$$\operatorname{curv}(V) = \operatorname{curv}(V|_{\mathcal{H}_{\{1,\dots,k\}}}) = \dim \bigcap_{\substack{i\in\{1,\dots,k\}\\ s\in\{1,\dots,n_i\}}} \ker V_{i,s}^*$$

and

$$\operatorname{rank} \Delta_V(I) = \operatorname{rank} \Delta_{V|_{\mathcal{H}_{\{1,\dots,k\}}}}(I_{\mathcal{H}_{\{1,\dots,k\}}}).$$

This proves items (i) and (ii). We already know that if $\mathbf{S}$ is the standard k-tuple of doubly $I\otimes\mathcal{U}$-commuting row isometries on the Hilbert space $\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)\otimes\mathcal{L}$, where $\dim\mathcal{L} < \infty$, then $\operatorname{curv}(\mathbf{S}) = \dim\mathcal{L}$. This proves part (iii).

Now, assume that $m := \operatorname{curv}(V) \neq 0$. Due to the Wold decomposition mentioned above, we may assume that $\mathcal{H} = \left[\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{L}_{\{1,\dots,k\}}\right] \oplus \mathcal{H}'$ and $V = \mathbf{S} \oplus V'$, where V' is a doubly $\mathcal{U}|_{\mathcal{H}'}$-commuting tuple of row isometries with $\Delta_{V'}(I_{\mathcal{H}'}) = 0$ and $\mathbf{S} = \bigoplus_{p=1}^m \mathbf{S}_{\mathbf{z}^{(p)}}$, where $m := \dim \mathcal{L}_{\{1,\dots,k\}}$. Therefore, we may assume that the multi-shift $\mathbf{S}$ is a direct sum of multi-shifts $\mathbf{S}_{\mathbf{z}} \in B(\ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+))$ with the scalar weights $\mathbf{z} = (z_{i,j}(s,t))_{(i,j,s,t)\in\Gamma}$, where $z_{i,j}(s,t) \in \mathbb{T}$ and $z_{j,i}(t,s) = \overline{z_{i,j}(s,t)}$ for $(i,j,s,t) \in \Gamma$.

Due to theorem 7.2, for each $t_p \in (0,1]$, $p \in \{1,\dots,m\}$, there is an invariant subspace $\mathcal{M}_p \subset \ell^2(\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+)$ under $\mathbf{S}_{\mathbf{z}^{(p)}}$ such that $\operatorname{curv}(P_{\mathcal{M}_p^\perp} \mathbf{S}_{\mathbf{z}^{(p)}}|_{\mathcal{M}_p^\perp}) = t_p$. Note that $(\bigoplus_{p=1}^m \mathcal{M}_p^\perp) \bigoplus \mathcal{H}'$ is invariant under $\left(\bigoplus_{p=1}^m \mathbf{S}_{\mathbf{z}^{(p)}}^*\right) \oplus V'^*$ and $\mathcal{U}$. Then $\mathcal{M} := \left((\bigoplus_{p=1}^m \mathcal{M}_p^\perp) \bigoplus \mathcal{H}'\right)^\perp$ is an invariant subspace under $V = \left(\bigoplus_{p=1}^m \mathbf{S}_{\mathbf{z}^{(p)}}\right) \oplus V'$ and $\mathcal{U}$ and

$$\operatorname{curv}(P_{\mathcal{M}^\perp} V|_{\mathcal{M}^\perp}) = \sum_{p=1}^m \operatorname{curv}(P_{\mathcal{M}_p^\perp} \mathbf{S}_{\mathbf{z}^{(p)}}|_{\mathcal{M}_p^\perp}) + \operatorname{curv}(V') = \sum_{p=1}^m t_p.$$

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On the other hand, using a similar argument and proposition 7.4, we find a non-trivial invariant subspace $\mathcal{M} \subset \mathcal{H}$ under V such that $\mathrm{curv}(P_{\mathcal{M}^\perp} V|_{\mathcal{M}^\perp}) = 0$. This completes the proof. □

We recall the following result which is needed in what follows. If $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+) \otimes \mathcal{K}$ is an invariant subspace under the multi-shift $\mathbf{S}_{\mathcal{U}}$ and $I \otimes \mathcal{U}$, then $\mathbf{S}_{\mathcal{U}}|_{\mathcal{M}}$ is in the regular $(I \otimes \mathcal{U})|_{\mathcal{M}}$-twisted polyball if and only if $\mathcal{M}$ is a Beurling type invariant subspace for $\mathbf{S}_{\mathcal{U}}$ and $I \otimes \mathcal{U}$.

Proposition. *Let $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ be a proper invariant subspace of the **z**-twisted multi-shift $\mathbf{S}_{\mathbf{z}}$. Then $\mathbf{S}_{\mathbf{z}}|_{\mathcal{M}}$ is in the $(I \otimes \mathbf{z})|_{\mathcal{M}}$-twisted polyball and $\mathrm{curv}(P_{\mathcal{M}^\perp} \boldsymbol{S}_z|_{\mathcal{M}^\perp}) = 0$ if and only if there is an inner sequence $\{\Psi_s\}_{s=1}^\infty$ for $\mathcal{M}$, i.e. Ψ_s are isometric multipliers of $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ with respect to $\boldsymbol{S}_{\boldsymbol{z}}$ such that*

$$P_{\mathcal{M}} = \sum_{s=1}^{\infty} \Psi_s \Psi_s^*,$$

where the convergence is in the strong operator topology and

$$\lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{1}{n_1^{q_1}\cdots n_k^{q_k}} \sum_{|\alpha_1|=q_1,\ldots,|\alpha_k|=q_k} \sum_{s=1}^{\infty} \|\Psi_s(\chi_{\alpha_1,\cdots,\alpha_k})\|^2 = 1.$$

Proof. According to the remarks preceding this proposition, $\mathbf{S}_{\mathbf{z}}|_{\mathcal{M}}$ is in the $(I \otimes \mathbf{z})|_{\mathcal{M}}$-twisted polyball if and only if $\mathcal{M}$ is a Beurling type invariant subspace under $\mathbf{S}_{\mathbf{z}}$. Therefore, there exist multi-analytic operators $\Psi_p \in B(\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ such that $P_{\mathcal{M}} = \sum_{s=1}^\infty \Psi_s \Psi_s^*$. Due to theorem 6.1 and corollary 6.2, condition $\mathrm{curv}(P_{\mathcal{M}^\perp} \mathbf{S}|_{\mathcal{M}^\perp}) = 0$ is equivalent to

$$\lim_{(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} \frac{\mathrm{trace}\left[\left(\sum_{s=1}^\infty \Psi_s\Psi_s^*\right) P_{(q_1,\ldots,q_k)}\right]}{\mathrm{trace}\left[P_{(q_1,\ldots,q_k)}\right]} = 1.$$

Now, one can easily complete the proof. □

Proposition 7.7. *If $\mathcal{M} \subset \ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$ is a proper Beurling type invariant subspace of multi-shift $\mathbf{S}_{\mathbf{z}}$, then*

$$0 \le \mathrm{curv}(P_{\mathcal{M}^\perp} \mathbf{S}_{\mathbf{z}}|_{\mathcal{M}^\perp}) < 1 \quad \textit{and} \quad 0 < m(\mathcal{M}) \le 1.$$

Proof. As in the proof of proposition 6.7, there is a unitary operator $\psi : \Delta_A(I)(\mathcal{M}^\perp) \to \mathbb{C}$ such that $(I \otimes \psi) K_A = V$, where K_A is the Berezin kernel associated with $A := P_{\mathcal{M}^\perp} \mathbf{S}_{\mathbf{z}}|_{\mathcal{M}^\perp}$ and V is the injection of $\mathcal{M}^\perp$ into $\ell^2(\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+)$. Hence, we deduce that $K_A K_A^* = (I \otimes \psi^*) P_{\mathcal{M}^\perp} (I \otimes \psi)$ and

$$\Delta_{\mathbf{S}_{\mathbf{z}}}(I - K_A K_A^*) = (I \otimes \psi^*) \Delta_{\mathbf{S}_{\mathbf{z}}}(P_{\mathcal{M}})(I \otimes \psi).$$

Since $\mathcal{M}$ is a Beurling type invariant subspace, we have $\Delta_{\mathbf{S}_{\mathbf{z}}}(P_{\mathcal{M}}) \ge 0$, which implies $\Delta_{\mathbf{S}_{\mathbf{z}}}(I - K_A K_A^*) \ge 0$. Now, assume that $\mathrm{curv}(A) = 1$. t Then, $\mathrm{curv}(A) =$

https://doi.org/10.1017/prm.2024.109 Published online by Cambridge University Press

rank A and, due to corollary 5.5, A is unitarily equivalent to $\mathbf{S_z}$. Hence, we deduce that $\mathcal{M}^{\perp}$ is an invariant subspace for $\mathbf{S_z}$. This shows that $\mathcal{M}^{\perp}$ is a reducing subspace for $\mathbf{S_z}$. Since the C^*-algebra $C^*(\mathbf{S_z})$ is irreducible (see [27]), we get a contradiction. This completes the proof. □

We remark that proposition 7.7 implies that $P_{\mathcal{M}^{\perp}}\mathbf{S_z}|_{\mathcal{M}^{\perp}}$ is not unitarily equivalent to $\mathbf{S_z}$. It remains an open question whether proposition 7.7 remains true for arbitrary proper invariant subspace of $\mathbf{S_z}$, which is the case when $k=1$ and $\mathcal{U} = \{1_{\mathbb{C}}\}$ (see [20]).

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