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RESEARCH ARTICLE

Complete positivity order and relative entropy decay

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Abstract
We prove that for a GNS-symmetric quantum Markov semigroup, the complete modified logarithmic Sobolev constant is bounded by the inverse of its complete positivity mixing time. For classical Markov semigroups, this gives a short proof that every sub-Laplacian of a Hörmander system on a compact manifold satisfies a modified log-Sobolev inequality uniformly for scalar and matrix-valued functions. For quantum Markov semigroups, we show that the complete modified logarithmic Sobolev constant is comparable to the spectral gap up to the logarithm of the dimension. Such estimates are asymptotically tight for a quantum birth-death process. Our results, along with the consequence of concentration inequalities, are applicable to GNS-symmetric semigroups on general von Neumann algebras.

Contents

1. Introduction

The time evolution of dynamical systems is a central topic in ergodic theory, probability theory, geometry and analysis. Similarly, decay properties of dissipative quantum systems also naturally arise in quantum many-body systems, quantum information theory and high energy physics. The aim of this article is to provide a new framework of decay estimates that applies for both classical and quantum systems in the *non-ergodic* setting. Here, ergodicity means the system admits a unique equilibrium state, also termed *primitive* in mathematical physics literature, whereas non-ergodic systems admit multiple equilibrium states.

Logarithmic Sobolev inequality (LSI) is a powerful functional inequality in deriving the mixing time of Markovian evolution. LSI was first introduced in the seminal works of Gross [35, 34] as an equivalent reformulation of Nelson's Hypercontractivity (HC) [58, 59]. It has been widely studied on manifolds and graphs for the deep connections to geometry and concentration phenomenon. However, attempts to translate the notion of hypercontractivity to the matrix-valued setting or the non-ergodic setting failed miserably [7], due to the lack of uniform convexity of certain noncommutative spaces [42]. This results in a roadblock for the standard argument connecting hypercontractivity, entropy decay and mixing time, as well as the lack of tensorization property used in many-body systems.

We propose a new, direct approach to entropy decay that also applies to fully noncommutative, non-ergodic setting. Let $T_t = e^{-Lt}$ be a quantum Markov semigroup on a finite von Neumann algebra $\mathcal{M}$ with generator L (i.e., a semigroup of completely positive trace-preserving maps). We aim to establish the exponential entropy decay,

$$D(T_t(\rho)\|E(\rho)) \ \le\ e^{-2\alpha_1 t} D(\rho\|E(\rho)) \tag{1.1}$$

$$\text{or equivalently} \qquad 2\alpha_1 D(\rho)\|E(\rho)) \le \tau\Big(L(\rho)(\ln\rho - \ln E(\rho))\Big),$$

where $D(\rho\|\sigma) = \tau(\rho\ln\rho - \rho\ln\sigma)$ is the quantum relative entropy for two density operators ρ,σ and τ can be any normal faithful trace on $\mathcal{M}$. The equilibrium state $E(\rho)$ associated to any initial density ρ is given by the ergodic mean $E(\rho) = \lim_{t\to\infty}\frac{1}{t}\int_0^t T_s(\rho)ds$. It turns out that the simple properties of relative entropy enable us to prove a direct link between *positivity order and entropy decay*. Indeed, let us for simplicity assume that the semigroup is trace symmetric

$$\tau(T_t(x)y) = \tau(xT_t(y)) \ \text{ for } \ x, y \in \mathcal{M}, t \ge 0.$$

Under this assumption, we discover the following *entropy difference lemma*:

$$D(\rho\|T_{2t}(\sigma)) \ \le\ D_{T_t}(\rho) + D(\rho\|\sigma), \quad \text{where} \quad D_\Phi(\rho) := \tau(\rho\ln\rho) - \tau(\Phi(\rho)\ln\Phi(\rho)). \tag{1.2}$$

The new quantity $D_\Phi(\rho)$ is the loss of von Neumann entropy under a channel map Φ. Our second ingredient is a *stability estimate* inspired by the positivity order condition by Gao and Rouzé [31] (see also [46]) that

$$(1-\varepsilon)E(x) \ \le\ T_t(x) \ \le\ (1+\varepsilon)E(x),\ \forall\, x\ge 0 \Longrightarrow D(\rho\|E(\rho)) \ \le\ C_\varepsilon D(\rho\|T_t(\rho)) \tag{1.3}$$

for some constant C_ε only depending on ε and the index of the ergodic mean projection E. Now, suppose the condition (1.3) holds for time $t(\varepsilon)$ and find

$$D(\rho\|E(\rho)) \le C_\varepsilon D(\rho\|T_{t(\varepsilon)}(\rho)) \le C_\varepsilon\Big(D_{T_{\frac{t(\varepsilon)}{2n}}}(\rho) + D(\rho\|T_{\frac{(n-1)t(\varepsilon)}{n}}(\rho))\Big) \ \le\ nC_\varepsilon D_{T_{t(\varepsilon)/2n}}(\rho),$$

where we apply (1.2) iteratively to the term $D(\rho\|T_{\frac{(n-1)t(\varepsilon)}{n}}(\rho))$. Taking the limit $n\to\infty$, we derive the inequality

$$D\big(\rho\|E(\rho)\big) \ \le\ \frac{t(\varepsilon)}{2}C_\varepsilon\tau(L(\rho)\ln\rho),$$

which is the differential version of (1.1) with $\alpha_1 = \frac{1}{C_\varepsilon t(\varepsilon)}$, called the *modified logarithmic Sobolev inequality* (in short, **MLSI**). The largest possible constant α_1 in (1.1) is called the MLSI constant.

1.1. MLSI for GNS-symmetric semigroups

Many dynamics in quantum information processing are not trace symmetric. One major application of open systems is state preparation by simulating time evolution governed by a Lindbladian

$$L(x) \ =\ i[h,x] + \textstyle\sum_j 2a_j^*xa_j - a_j^*a_jx - xa_j^*a_j. \tag{1.4}$$

A natural one is the Davies semigroup, which converges to the thermal Gibbs state $\phi = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$. For any finite inverse temperature $\beta>0$, the Davies semigroup is never trace symmetric but satisfies the following detailed balance condition

$$\phi(T_t(x)y) \ =\ \phi(xT_t(y)),\ \forall\, x,y$$

with respect to the Gibbs state ϕ, which we call **GNS-symmetry**. In this context, a breakthrough result of MLSI constant α_1 was made by Gao and Rouzé [31] that

$$\alpha_1 \ge \frac{\lambda(L)}{C(E)} \tag{1.5}$$

for every GNS symmetric semigroups in finite dimensions. Here, $\lambda(L)$ is the spectral gap of the semigroup generator L, and $C(E) = \inf\{\mu \mid x\le \mu E(x)\ ,\ \text{ for all } x\ge 0\}$ is the Pimsner-Popa index for the condition expectation $E = \lim_{t\to\infty} T_t$. An important consequence of Gao and Rouzé's estimate (1.5) is the positivity of the **complete MLSI** constant $\alpha_c(L) = \inf_n \alpha_1(L\otimes \mathrm{id}_{\mathbb{M}_n})$ (in short, **CMLSI** constant),

$$\alpha_c \ge \frac{\lambda}{C_{cb}(E)} > 0, \tag{1.6}$$

because the complete Pimsner-Popa index $C_{cb}(E) = \sup_n C(E\otimes \mathrm{id}_{\mathbb{M}_n})$ is finite in finite dimensions. The CMLSI constant is of particular interest because it satisfies the tensorization property $\alpha_c(T_t\otimes S_t) = \min\{\alpha_c(T_t), \alpha_c(S_t)\}$, while the MLSI constant α_1 does not.

Our 'positivity order implies entropy decay' argument above gives an exponential improvement to (1.6) in terms of the dimension constant $C_{cb}(E)$.

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Theorem 1.1 (cf. Theorem 3.2 and 4.10). *Let $T_t : \mathcal{M} \to \mathcal{M}$ be a quantum Markov semigroup GNS-symmetric to a faithful normal state ϕ. Then the* optimal CMLSI constant *satisfies*

$$\alpha_c \;\geq\; \frac{1}{2t_{cb}(0.1)},\ where\ \ t_{cb}(\varepsilon) := \inf\{t > 0 \mid (1-\varepsilon)E \leq_{cp} T_t \leq_{cp} (1+\varepsilon)E\}.$$

Here, $\Psi \leq_{cp} \Phi$ means $\Psi - \Phi$ is a completely positive map. Moreover, in finite dimensions,

$$\alpha_1 \geq \alpha_c \geq \frac{\lambda}{2\ln(10C_{cb}(E))}. \tag{1.7}$$

The quantity t_{cb}, called **CB return time**, is the mixing time in terms of complete positivity order. Similar terms of complete positivity have been also considered in the quantum setting for the study of approximate unitary t-design ([12]). In the fully non-ergodic noncommutative setting, t_{cb} was originally introduced in [29] via completely bounded (CB) $L_1 \to L_\infty$ norm, whose connection to complete positivity order and spectral gap relies heavily on operator space theory (see Section 3.2).

The proof to GNS-symmetric cases uses the ideas of Haagerup reduction [36], a method to derive results for type III von Neumann algebras by reducing them to cases of tracial von Neumann algebras. Thanks to this machinery, our estimate in trace-symmetric settings can be salvaged to a GNS-symmetric semigroup on general σ-finite von Neumann algebras, including both classical systems and quantum systems. A particular interesting example is a matrix version of the classical n-level death-birth process which admits an invariant state $\rho_n \propto (e^{-\beta k})_{k=0,..,n}$ and a Lindbladian given by nearest neighbor interactions. In this example, we show that both the spectral gap is are uniformly controlled, and

$$\lambda \sim \Theta(1)\ ,\ \alpha_1 \sim \alpha_c \sim \frac{1}{n},\ t_{cb} \sim \ln(C_{cb}(E)) \sim n.$$

Hence, both estimates in our Theorem 1.1 are asymptotically tight for this GNS-symmetric example.

1.2. MLSI for matrix-valued functions

Besides the quantum setting, our results also provide interesting MLSI and concentration inequalities for random matrices of arbitrary size. For a classical Markov semigroup $P_t : L_\infty(\Omega,\mu) \to L_\infty(\Omega,\mu)$ on some probability space (Ω,μ), the notion of CMLSI is basically a uniform MLSI for positive matrix-valued random variables $g : \Omega \to \mathbb{M}_n$ of all dimensions $n \geq 1$,

$$\mu \circ \mathrm{tr}(g\ln g - E_\mu(g)\ln E_\mu(g)) \leq \frac{1}{2\alpha}\mu \circ \mathrm{tr}((Lg)\ln g). \tag{1.8}$$

Here, $\mu(f) = \int f d\mu$ is the scalar valued mean, $E_\mu(g) = \int g d\mu \in \mathbb{M}_n$ is the matrix valued mean, and tr is the standard matrix trace. In this setting, the CB return time t_{cb} is simply the **L_∞-mixing time**

$$t_b(\varepsilon) = \{t > 0| \parallel T_t - E_\mu : L_1(\Omega) \to L_\infty(\Omega) \parallel \leq \varepsilon\},$$

which is accessible by kernel estimates derived from harmonic analysis. As a consequence of Theorem 1.1, we obtain CMLSI for all sub-Laplacians of Hörmander system.

Theorem 1.2. *Let (M,g) be a compact connected Riemannian manifold without boundary, and $\omega d\,\mathrm{vol}$ be a probability measure with a smooth density ω with respect to the volume form $d\,\mathrm{vol}$. Suppose $H = \{X_i\}_{i=1}^k \subset TM$ is a family of vectors fields satisfying the Hörmander's condition that at every point $x \in M$,*

$$T_xM = \mathrm{span}\{[X_{i_1},[X_{i_2},\cdots,[X_{i_{n-1}},X_{i_n}]]] \mid 1 \leqslant i_1, i_2\cdots i_n \leqslant k, n \geq 1\}.$$

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Then the horizontal heat semigroup $P_t = e^{-\Delta_H t}$ *generated by the sub-Laplacian*

$$\Delta_H = \sum_i X_i^* X_i = -\sum_i X_i^2 + (div_\mu(X_i) + X_i(\ln \omega))X_i$$

has CMLSI constant $\alpha_c(\Delta_H) > 0$*. Here,* X_i^* *is the adjoint operator with respect to* $L_2(M, \omega d\,\mathrm{vol})$.

For scalar-valued functions, the positivity of $\alpha_1(\Delta_H)$ was proved by Ługiewicz and Zegarliński [53], using a hypercontractive argument similar from [23]. Nevertheless, both [23] and [53] rely on the Rothaus Lemma [68, 3], a crucial step which does not apply for matrix-valued functions (see Section 3.6).

In this setting, our Theorem 1.1 gives a short proof of

$$\boxed{\text{Heat kernel estimate}} + \boxed{\text{Spectral gap}} \Longrightarrow \boxed{\text{LSI/MLSI}} \tag{1.9}$$

for scalar-valued function, and also extends to matrix-valued setting by replacing LSI with CMLSI. A particular interesting example, also covered in [28], is the Lie group $M = SU(2)$ with the canonical sub-Laplacian $\Delta_H = -X^2 - Y^2$, where the Lie algebra $\mathfrak{su}(2)$ is spanned by the Pauli matrices X, Y and $Z = \frac{1}{2}[X, Y]$. The CMLSI of heat semigroups (standard Laplacians) was obtained in [49, 14] using the Ricci curvature lower bound as a crucial tool. Nevertheless, in the sub-elliptic case the Ricci curvature in the degenerate direction of the vector field $H = \{X_i\}_{i=1}^k$ can be interpreted as $-\infty$. In [28], the curvature condition were substituted by a gradient estimate that was first introduced by Driver and Melcher [24] for Heisenberg group, later obtained for nilpotent Lie groups [54] and $SU(2)$ [8]. Our Theorem 1.2 obtains CMLSI for all sub-Laplacian of Hörmander systems, without using any curvature condition. It implies the following uniform CMLSI constant for trace symmetric Lindbladians as 'representation' of Hörmander system on Lie groups.

Corollary 1.3. *Let* G *be a compact Lie group and* $H = \{X_1, \cdots, X_k\}$ *be a generating set of its Lie algebra* $\mathfrak{g}$*. There exists a constant* $\alpha_c(\Delta_H) > 0$ *such that for all unitary representation* u*, the induced quantum Markov semigroup generated by*

$$L_H(\rho) = -\sum_{i=1}^k [d_u(X_i), [d_u(X_i), \rho]]$$

satisfies $\alpha_c(L_H) \geq \alpha_c(\Delta_H) > 0$*. Here,* d_u *is the Lie algebra homomorphism induced by* u*.*

1.3. Concentration inequalities

An important application of MLSI is to derive concentration inequalities. This was first discovered by Otto and Villani [61], later extended to the discrete case by Erbar and Maas [26], and more recently to the noncommutative setting in [69, 29, 16]. As an application of our MLSI estimate for GNS-symmetric semigroups, we derive concentration inequalities for a general faithful invariant state ϕ. Recall that the Lipschitz semi-norm

$$\|x\|_{\mathrm{Lip}} =: \max\{\|\Gamma_L(x,x)\|^{\frac{1}{2}},\ \|\Gamma_L(x^*,x^*)\|^{\frac{1}{2}}\}.$$

The Lipschitz semi-norm is defined through the *gradient form* (or Carré du Champ operator)

$$\Gamma_L(x,y) = \frac{1}{2}\Big(L(x^*)y + x^*L(y) - L(x^*y)\Big),\ \forall x, y \in \mathrm{dom}(L).$$

Theorem 1.4. *Let* $\mathcal{M}$ *be a* σ*-finite von Neumann algebra and let* $T_t = e^{-tL}$ *be a GNS-*ϕ*-symmetric quantum Markov semigroup with positive MLSI constant* $\alpha_1(L) > 0$*. Then there exists a universal constant* c *such that for* $2 \leq p < \infty$*,*

$$\alpha\|x - E(x)\|_{L_p(\mathcal{M},\phi)} \leq c\sqrt{p}\,\|x\|_{Lip}\,.$$

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Moreover, for any $t > 0$, there exists a projection e such that

$$\|e(x - E(x))e\|_\infty \le t \quad and \quad \phi(1 - e) \le 2\exp\left(-\frac{\alpha^2 t^2}{16ec^2\ \|x\|_{Lip}^2}\right).$$

As a special case, we obtain the following matrix concentration inequalities which can be compared to the work of Tropp [75].

Corollary 1.5. *Let $S_1, \cdots, S_n$ be an independent sequence of random $d \times d$-matrices such that $\| S_i - \mathbb{E}S_i \|_\infty \le M$, a.e. Then, we have the matrix Bernstein inequality that for the sum $Z = \sum_{k=1}^n S_k$,*

$$\mathbb{E}\ \| Z - \mathbb{E}Z \|_\infty \le 2ce^{-1/2}\sqrt{(v(Z) + M^2)\log d} \tag{1.10}$$

and the matrix Chernoff bound

$$P(|Z - \mathbb{E}Z| > t) \le 2d\exp\left(-\frac{t^2}{64ec^2(v(Z) + M^2)}\right),$$

where

$$v(Z) = \max\{\| \mathbb{E}((Z - \mathbb{E}Z)^*(Z - \mathbb{E}Z)) \|\ , \| \mathbb{E}((Z - \mathbb{E}Z)^*(Z - \mathbb{E}Z)) \|\}.$$

In particular, the inequality (1.10) improves the term $M\log d$ in Tropp's result [75] to $M\sqrt{\log d}$. For more details, see Example 5.18.

After the acceptance of this paper, we get to know the sub-Gaussian type estimate (1.10) of matrix concentration was obtained by Huang and Tropp [38, 39] via matrix-valued Poincare inequality and matrix-valued Bakry-Émery curvature condition. Actually, in the introduction of [39] they raise the question whether the sub-gaussian estimate can be obtained by matrix-valued Log-Sobolev inequality. Our result answers this question.

1.4. Outline of the paper

We organize our paper as follows to make it accessible for readers from different backgrounds. In Section 2, we provide a brief review of quantum information basics in the setting of tracial von Neumann algebras. We prove our key entropy difference lemma (Lemma 2.1) and an improved data processing inequality (Theorem 2.5). Building upon these results, we discuss the functional inequalities of symmetric quantum Markov semigroups in Section 3. We prove our main Theorem 1.1 in the trace symmetric case and its consequence Theorem 1.2 for classical Markov semigroups. We also illustrate the failure of the matrix-valued logarithmic Sobolev ineuqality in Proposition 3.15. The discussion up to this point does not involve much technicality beyond the basic concepts of finite von Neumann algebras. Readers from quantum information and classical analysis are welcome to consider examples such as the matrix algebra $\mathbb{M}_n$ and matrix-valued functions $L_\infty(\Omega, \mathbb{M}_n)$.

In Section 4, we dive into the GNS-symmetric cases. Here, we discuss the Haagerup reduction for channels and entropic quantities, deriving Theorem 1.1 (Theorem 4.10 and Corollary 4.13) in its full generality. Section 5 collects applications of our general results Theorem 1.4 and Corollary 1.5. We conclude the paper in Section 6 with some discussions on remaining open questions.

Notations. We use calligraphic letters $\mathcal{M}, \mathcal{N}$ for von Neumann algebras and denote $\mathbb{M}_n$ as the algebra of $n \times n$ as complex matrices. We use τ as the trace on von Neumann algebra, and tr as the standard matrix trace. The identity operator is denoted by 1, and the identity map between spaces is denoted as id, sometimes specified with subscript like $1_{\mathcal{M}}$ and $\mathrm{id}_{\mathbb{M}_n}$. We write a^* as the adjoint element of a and Φ_* for a pre-adjoint map of Φ.

2. Entropy contraction of symmetric Markov maps

2.1. States, channels and entropies

We briefly review some basic information-theoretic concepts in the noncommutative setting. Recall that a von Neumann algebra $\mathcal{M}$ is a unital $*$-subalgebra of $B(H)$ closed under weak*-topology. A linear functional $\phi : \mathcal{M} \to \mathbb{C}$ is called a state if it is positive, meaning $\phi(x^*x) \geq 0$ for any $x \in \mathcal{M}$, and additionally, $\phi(1) = 1$. We say ϕ is normal if ϕ is weak*-continuous. Throughout the paper, we will only consider normal states and denote $S(\mathcal{M})$ as the normal state space of $\mathcal{M}$. We write $s(\phi)$ as the support projection of a state ϕ, which is the minimal projection e such that $\phi(x) = \phi(exe)$, $\forall\, x \in \mathcal{M}$. A normal state ϕ is faithful if $s(\phi) = 1$. For two normal states ρ and σ, the relative entropy is defined as

$$D(\rho||\sigma) = \begin{cases} \langle \xi_\rho | \log \Delta(\rho/\sigma) | \xi_\rho \rangle, & \text{if } s(\rho) \leq s(\sigma) \\ +\infty, & \text{otherwise.} \end{cases}, \tag{2.1}$$

where ξ_ρ is a vector implementing the state ρ, and $\Delta(\rho/\sigma)$ is the relative modular operator. This form of definition (2.1) was introduced by Araki [2] for general von Neumann algebras.

In this section, we will focus on the case that $\mathcal{M}$ is a finite von Neumann algebra. Namely, $\mathcal{M}$ is equipped with a normal faithful tracial state τ. The tracial noncommutative L_p-space $L_p(\mathcal{M}, \tau)$ is defined as the completion of $\mathcal{M}$ with respect to the p-norm $\| a \|_p = \tau(|a|^p)^{1/p}$. We identify $L_\infty(\mathcal{M}) \cong \mathcal{M}$, and also $L_1(\mathcal{M}) \cong \mathcal{M}_*$ via the trace duality

$$d_\phi \in L_1(\mathcal{M}) \longleftrightarrow \phi \in \mathcal{M}_*, \ \phi(x) = \tau(d_\phi x).$$

We say $\rho \in L_1(\mathcal{M})$ is a density operator if $\rho \geq 0$ and $\tau(\rho) = 1$, which corresponds to a normal state in the above identification. We will often identify normal states with their density operators if no ambiguity. Via this identification, relative entropy reduces to the original definition of Umegaki [76],

$$D(\rho||\sigma) = \tau(\rho \log \rho - \rho \log \sigma),$$

provided this trace is well defined. For example, for ρ and σ in the bounded state space

$$S_B(\mathcal{M}) = \{\rho \in S(\mathcal{M}) \mid \mu_1 1 \leq \rho \leq \mu_2 1 \text{ for some } \mu_1, \mu_2 > 0\},$$

the Umegaki's formula is always well defined and finite. For this reason, we will mostly work with bounded states from $S_B(\mathcal{M})$ and derive results for general case $S(\mathcal{M})$ by approximation. When the second state $\sigma = 1$, this gives the entropy functional

$$H(\rho) := D(\rho||1) = \tau(\rho \log \rho).$$

Note that the standard convention of von Neumann entropy in quantum information literature is often with an additional negative sign .

We say a linear map $T : \mathcal{M} \to \mathcal{M}$ is a *quantum Markov map* if T is normal, unital and completely positive. Recall that T is unital if $T(1) = 1$. The pre-adjoint map $T_* : \mathcal{M}_* \to \mathcal{M}_*$ is called a quantum channel, which sends normal states to normal states. In the tracial setting, $T_* : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ given by

$$\tau(T_*(\rho)y) = \tau(\rho T(y)), \ \forall\, y \in \mathcal{M}, \rho \in L_1(\mathcal{M}),$$

is completely positive and trace-preserving (in short, CPTP). A fundamental inequality about quantum channel is the data processing inequality (also called monotonicity of relative entropy)

$$D(\rho||\sigma) \geq D(T_*(\rho)||T_*(\sigma)), \ \forall \rho, \sigma \in S(\mathcal{M}). \tag{2.2}$$

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The data processing inequality states that two quantum states cannot become more distinguishable under a quantum channel. The data processing inequality remains valid for T being positive but not necessarily completely positive; see [56, 27]. The main technical result of this work is an improved data processing inequality for quantum channels under symmetric conditions (Theorem 2.5).

2.2. Entropy contraction for unital quantum channels

We start our discussion on entropy contraction of unital quantum channels. The restriction of Φ on $\mathcal{M}$ is bounded and normal; thus, Φ can be viewed as the L_1-norm extension of its restriction $\Phi : \mathcal{M} \to \mathcal{M}$. By duality, its adjoint $\Phi^* : \mathcal{M} \to \mathcal{M}$ is a trace-preserving quantum Markov map and hence also extends to a unital quantum channel.

For a state ρ with $H(\rho) < \infty$, we define the entropy difference of Φ,

$$D_\Phi(\rho) := H(\rho) - H(\Phi(\rho)).$$

Non-negativity of the entropy difference $D_\Phi(\rho) \geq 0$ follows from data processing inequality (2.2) and $\Phi(1) = 1$,

$$H(\rho) = D(\rho||1) \geq D(\Phi(\rho)||\Phi(1)) = H(\Phi(\rho)).$$

We start with the key lemma in our argument.

Lemma 2.1 (Entropy difference lemma). *Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ be a unital quantum channel and Φ^* be its adjoint. Then for two bounded states $\rho, \omega \in S_B(\mathcal{M})$,*

$$D(\rho\|\Phi^*\Phi(\omega)) \leq D_\Phi(\rho) + D(\rho\|\omega) \leq \tau((\mathrm{id} - \Phi^*\Phi)(\rho)\ln\rho) + D(\rho\|\omega).$$

Proof. By duality, Φ^* is also completely positive unital. Then,

$$\begin{aligned}
D(\rho\|\Phi^*\Phi(\omega)) &= \tau(\rho\ln\rho - \rho\ln\Phi^*\Phi(\omega)) \\
&= \tau(\rho\ln\rho - \Phi(\rho)\log\Phi(\rho)) + \tau(\Phi(\rho)\log\Phi(\rho) - \rho\ln\Phi^*\Phi(\omega)) \\
&= D_\Phi(\rho) + \tau\big(\Phi(\rho)\log\Phi(\rho) - \rho\ln\Phi^*\Phi(\omega)\big) \\
&\overset{(1)}{\leq} D_\Phi(\rho) + \tau\Big(\Phi(\rho)\log\Phi(\rho) - \rho\Phi^*\big(\ln\Phi(\omega)\big)\Big) \\
&= D_\Phi(\rho) + \tau\Big(\Phi(\rho)\log\Phi(\rho) - \Phi(\rho)\ln\Phi(\omega)\Big) \\
&= D_\Phi(\rho) + D(\Phi(\rho)\|\Phi(\omega)) \\
&\overset{(2)}{\leq} D_\Phi(\rho) + D(\rho\|\omega),
\end{aligned}$$

where (2) follows from the monotonicity of relative entropy. The inequality (1) uses the operator concavity [18] of logarithm function $t \mapsto \ln t$ that for any positive operator $x \geq 0$,

$$\Phi^*(\ln x) \leq \ln\Phi^*(x).$$

This proves the first inequality. For the second part, it suffices to notice that

$$D_\Phi(\rho) = \tau(\rho\log\rho - \Phi(\rho)\log\Phi(\rho)) \leq \tau(\rho\log\rho - \Phi(\rho)\Phi(\log\rho)) = \tau(\rho\log\rho - \Phi^*\Phi(\rho)\log\rho),$$

where we use the operator concavity $\Phi(\ln x) \leq \ln\Phi(x)$ again. □

We iterate the above lemma as follows:

$$D(\rho||(\Phi^*\Phi)^n(\rho)) \leq D_\Phi(\rho) + D(\rho||(\Phi^*\Phi)^{n-1}(\rho)) \leq nD_\Phi(\rho) + D(\rho||\rho) = nD_\Phi(\rho).$$

Then a relevant question is what would be the limit of $(\Phi^*\Phi)^n(\rho)$ as $n \to \infty$. This leads to the multiplicative domain of Φ. Recall that the multiplicative domain of a unital completely positive map Φ is

$$\mathcal{N}_\Phi = \{x \in \mathcal{M} \mid \Phi(y)\Phi(x) = \Phi(yx),\ \Phi(x)\Phi(y) = \Phi(xy), \forall y \in \mathcal{M}\}.$$

When Φ is normal, $\mathcal{N}_\Phi \subset \mathcal{M}$ is a von Neumann subalgebra ([52, Theorem 1]). A linear map $E : \mathcal{M} \to \mathcal{M}$ is called a conditional expectation if E is a unital completely positive map and idempotent $E \circ E = E$. When $\mathcal{M}$ is a finite von Neumann algebra, for any subalgebra $\mathcal{N} \subset \mathcal{M}$, there always exists a (unique) trace-preserving conditional expectation E onto $\mathcal{N}$ such that

$$\tau(xy) = \tau(xE(y)),\ x \in \mathcal{N}, y \in \mathcal{M}. \tag{2.3}$$

Such E is a unital quantum channel.

Proposition 2.2. *Let $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ be a unital quantum channel, and let $E : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto the multiplicative domain $\mathcal{N} := \mathcal{N}_\Phi$. Then*

i) *$\Phi : \mathcal{N} \to \Phi(\mathcal{N})$ is a $*$-isomorphism with inverse $\Phi^* : \Phi(\mathcal{N}) \to \mathcal{N}$. Moreover, $\Phi(\mathcal{N})$ is the multiplicative domain for Φ^*, and*

$$(\Phi^*\Phi) \circ E = E \circ (\Phi^*\Phi) = E,\ E_0 \circ \Phi = \Phi \circ E, \tag{2.4}$$

where $E_0 : \mathcal{M} \to \Phi(\mathcal{N})$ is the trace-preserving conditional expectation onto $\Phi(\mathcal{N})$.

ii) *Φ is an isometry on $L_2(\mathcal{N})$. If, in addition, $\| \Phi(\mathrm{id} - E) : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \|_2 < 1$, then $E = \lim_n (\Phi^*\Phi)^n$ as a map from $L_2(\mathcal{M})$ to $L_2(\mathcal{M})$.*

Proof. It is clear that Φ is a $*$-homomorphism on $\mathcal{N}$. For any $x, y \in L_2(\mathcal{N}) \subset L_2(\mathcal{M})$,

$$\tau(y(\Phi^* \circ \Phi)(x)) = \tau(\Phi(y)\Phi(x)) = \tau(\Phi(xy)) = \tau(xy).$$

Thus, $\Phi^* \circ \Phi|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$ is the identity map. This verifies $(\Phi^*\Phi) \circ E = E$. Since $E^* = E$, $E \circ (\Phi^*\Phi) = E$ follows from taking the adjoint. Thus, $\Phi : \mathcal{N} \to \Phi(\mathcal{N})$ is a $*$-isomorphism with inverse Φ^*. Denoting $\mathcal{N}_0$ as the multiplicative domain for Φ^*, we have $\Phi(\mathcal{N}) \subset \mathcal{N}_0$. Conversely, we also have $\Phi^*(N_0) \subset \mathcal{N}$ by switching the role of $\Phi = (\Phi^*)_*$. Then $\Phi(\mathcal{N}) = \mathcal{N}_0$ since Φ is bijective on $\mathcal{N}$. For ii), we note that by (2.4),

$$(\mathrm{id} - E)\Phi^*\Phi(\mathrm{id} - E) = (\mathrm{id} - E)(\Phi^*\Phi - E) = \Phi^*\Phi - E,\ (\Phi^*\Phi - E)^n = (\Phi^*\Phi)^n - E.$$

Therefore,

$$\begin{aligned}
&\| \Phi^*\Phi - E : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \| = \| \Phi(\mathrm{id} - E) \|_2^2 < 1, \\
&\| (\Phi^*\Phi)^n - E : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \| = \| (\Phi^*\Phi - E)^n : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \| = \| \Phi(\mathrm{id} - E) \|_2^{2n},
\end{aligned}$$

which goes to 0 as $n \to \infty$. □

In order to estimate entropic quantities, we will use the approximation in terms of complete positivity. Recall that for a density operator $\sigma \in S(\mathcal{M})$ with full support, the Bogoliubov-Kubo-Mori (BKM) metric for $X \in \mathcal{M}$ is defined by

$$\gamma_\sigma(X) := \int_0^\infty \tau(X^*(\sigma + s)^{-1} X(\sigma + s)^{-1}) ds.$$

The BKM metric is a Riemannian metric on the space of states with full support that is monotone under any quantum channel Ψ,

$$\gamma_{\Psi(\sigma)}(\Psi(X)) \le \gamma_\sigma(X), \forall X \in \mathcal{M}.$$

It connects to the relative entropy as follows ([31, Lemma 2.2]):

$$D(\rho||\sigma) = \int_0^1 \int_0^s \gamma_{\rho_t}(\rho - \sigma)dtds = \int_0^1 (1-t)\gamma_{\rho_t}(\rho - \sigma)dt, \tag{2.5}$$

where $\rho_t = t\rho + (1-t)\sigma$ for $t \in [0,1]$. It is proved in [31, Lemma 2.1 & 2.2] that if $\rho \le c\sigma$,

$$\begin{aligned} &c\gamma_\rho(X) \le \gamma_\sigma(X), \ \forall X \in \mathcal{M} \\ &k(c)\gamma_\sigma(\rho - \sigma) \le D(\rho||\sigma) \le \gamma_\sigma(\rho-\sigma), \end{aligned} \tag{2.6}$$

where $k(c) = \frac{c\ln c - c + 1}{(c-1)^2}$. The above discussion remains valid if $s(\rho) \le s(\sigma)$ and $X \in s(\sigma)\mathcal{M}s(\sigma)$. For two positive maps Ψ and Φ, we write $\Phi \le \Psi$ if $\Psi - \Phi$ is positive.

Lemma 2.3. *Let E be a conditional expectation (not necessarily trace-preserving) and Ψ be a quantum Markov map such that*

$$(1-\varepsilon)E \le \Psi \le (1+\varepsilon)E.$$

Assume that $E \circ \Psi = E$. Then for any $\rho \in S(\mathcal{M})$,

$$D(\rho||\Psi_*(\rho)) \ge \Big(\frac{1-\varepsilon}{1+\varepsilon} - \frac{\varepsilon}{(1-\varepsilon)k(2)}\Big) D(\rho||E_*(\rho)).$$

In particular, for $\varepsilon = \frac{1}{10}$,

$$D(\rho||\Psi_*(\rho)) \ge \frac{1}{2} D(\rho||E_*(\rho)).$$

Proof. By assumption, $\Psi_* = (1-\varepsilon)E_* + \varepsilon\Psi_0$ for some unital positive map $\Psi_0 \le 2E_*$. We denote $\sigma = E_*(\rho), \tilde{\sigma} = \Phi_*(\rho)$ and $\omega = \Psi_0(\rho)$. Then $\tilde{\sigma} = (1-\varepsilon)\sigma + \varepsilon\omega$. Note that for any bounded state $\sigma \in S_B(\mathcal{M})$, $X \mapsto \sqrt{\gamma_\sigma(X)}$ is a Hilbert space norm. Then by the triangle inequality,

$$\begin{aligned} \sqrt{\gamma(\rho - \tilde{\sigma})} &= \sqrt{\gamma(\rho - (1-\varepsilon)\sigma - \varepsilon\omega)} \\ &= \sqrt{\gamma((\rho - \sigma) + \varepsilon(\sigma - \omega))} \\ &\ge \sqrt{\gamma(\rho-\sigma)} - \varepsilon\sqrt{\gamma(\sigma - \omega)}, \end{aligned}$$

where γ can be γ_ϕ for any bounded state $\phi \in S_B(\mathcal{M})$. Then

$$\begin{aligned} \gamma(\rho - \tilde{\sigma}) &\ge \gamma(\rho - \sigma) - 2\varepsilon\sqrt{\gamma(\rho-\sigma)}\sqrt{\gamma(\sigma-\omega)} + \varepsilon^2\gamma(\sigma - \omega) \\ &\ge \gamma(\rho - \sigma) - 2\varepsilon\sqrt{\gamma(\rho-\sigma)}\sqrt{\gamma(\sigma-\omega)} \\ &\ge \gamma(\rho - \sigma) - \varepsilon\gamma(\rho - \sigma) - \varepsilon\gamma(\sigma - \omega) \\ &= (1-\varepsilon)\gamma(\rho - \sigma) - \varepsilon\gamma(\sigma - \omega). \end{aligned}$$

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Now take $\rho_t = t\rho + (1-t)\sigma$ and $\tilde{\rho}_t = t\rho + (1-t)\tilde{\sigma}$,

$$\begin{aligned} D(\rho||\tilde{\sigma}) &= \int_0^1 (1-t)\gamma_{\tilde{\rho}_t}(\rho - \tilde{\sigma})dt \\ &\geq (1-\varepsilon)\int_0^1 (1-t)\gamma_{\tilde{\rho}_t}(\rho-\sigma)dt - \varepsilon\int_0^1 (1-t)\gamma_{\tilde{\rho}_t}(\sigma-\omega)dt. \end{aligned}$$

For the first term, because $\tilde{\rho}_t \leq (1+\varepsilon)\rho_t$,

$$\int_0^1 (1-t)\gamma_{\tilde{\rho}_t}(\rho-\sigma)dt \geq (1+\varepsilon)^{-1}\int_0^1 (1-t)\gamma_{\rho_t}(\rho-\sigma)dtds = (1+\varepsilon)^{-1}D(\rho||\sigma).$$

For the second term, consider that $\tilde{\rho}_t \geq (1-\varepsilon)(1-t)\sigma$,

$$\begin{aligned} \int_0^1 (1-t)\gamma_{\tilde{\rho}_t}(\sigma-\omega)dt &\leq \frac{1}{(1-\varepsilon)}\int_0^1 \gamma_\sigma(\omega-\sigma)dt \\ &= \frac{1}{(1-\varepsilon)}\gamma_\sigma(\omega-\sigma) \\ &\overset{(1)}{\leq} \frac{1}{(1-\varepsilon)k(2)}D(\omega||\sigma) \\ &= \frac{1}{(1-\varepsilon)k(2)}D(\Psi_*(\rho)||\sigma) \overset{(2)}{\leq} \frac{1}{(1-\varepsilon)k(2)}D(\rho||\sigma). \end{aligned}$$

Here, the inequality (1) above uses $\omega \leq 2\sigma$ and (2.6). The inequality (2) above follows from the monotonicity of relative entropy and the fact $\Psi_*(\sigma) = \sigma$. Combining the estimated above, we obtained

$$D(\rho||\tilde{\sigma}) \geq \frac{1-\varepsilon}{1+\varepsilon}D(\rho||\sigma) - \varepsilon k(2)^{-1}D(\rho||\sigma) = \Big(\frac{1-\varepsilon}{1+\varepsilon} - \frac{\varepsilon}{(1-\varepsilon)k(2)}\Big)D(\rho||\sigma),$$

where $k(2) = 2\ln 2 - 1$. The above inequality is nontrivial for ε such that

$$\frac{1-\varepsilon}{1+\varepsilon} - \frac{\varepsilon}{(1-\varepsilon)k(2)} > 0.$$

Taking $\varepsilon = 0.1$, the above expression is approximately $0.53 > \frac{1}{2}$. □

Remark 2.4. This lemma is related to [47, Corollary 2.15] and is a variant of [31, Theorem 5.3], which proves for GNS symmetric Φ,

$$D(\rho||(\Phi_*)^2(\rho)) \geq (1-\varepsilon^2 k(2)^{-1})D(\rho||E_*(\rho)). \tag{2.7}$$

Compared with [47, Corollary 2.15], the above Lemma assumes a simpler condition and may achieve a stronger constant in certain regimes of interest. The above Lemma improves (2.7) from two points: 1) does not need any symmetric assumption; 2) remove the square in Φ_*^2. When $\Psi_* = \Phi_*^2$ is a square, (2.7) could yield better bound that for $\varepsilon = 0.4$,

$$(1-(0.4)^2k(2)^{-1}) > \frac{1}{2} > \frac{1}{4} > \frac{1-0.4^2}{1+0.4^2} - \frac{0.4^2}{(1-0.4^2)k(2)}.$$

Putting the above lemma together, we obtain the following entropy contraction of unital quantum channels.

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Theorem 2.5. *Let Φ be a unital quantum channel and let $E : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto the multiplicative domain $\mathcal{N}$ of Φ. Define the CB return time*

$$k_{cb}(\Phi) := \inf\{k \in \mathbb{N}^+ \mid 0.9E \leq_{cp} (\Phi^*\Phi)^k \leq_{cp} 1.1E\} . \tag{2.8}$$

Then for any state $\rho \in S(\mathcal{M})$,

$$D(\Phi(\rho)||\Phi \circ E(\rho)) \leq \Big(1 - \frac{1}{2k_{cb}(\Phi)}\Big) D(\rho||E(\rho)) . \tag{2.9}$$

Furthermore, for any finite von Neumann algebra $\mathcal{Q}$ and state $\rho \in S(\mathcal{M}\overline{\otimes}\mathcal{Q})$,

$$D(\Phi \otimes \mathrm{id}(\rho)||(\Phi \circ E) \otimes \mathrm{id}(\rho)) \leq \Big(1 - \frac{1}{2k_{cb}(\Phi)}\Big) D(\rho||E \otimes \mathrm{id}(\rho)). \tag{2.10}$$

Proof. It suffices to consider a bounded state $\rho \in S_B(\mathcal{M})$. Note that by the conditional expectation property (2.3),

$$D(\rho||E(\rho)) = \tau(\rho\log\rho - \rho\log E(\rho)) = \tau(\rho\log\rho) - \tau(E(\rho)\log E(\rho)) = H(\rho) - H(E(\rho)),$$
$$D(\Phi(\rho)||\Phi \circ E(\rho)) = D(\Phi(\rho)||E_0 \circ \Phi(\rho)) = H(\Phi(\rho)) - H(E_0 \circ \Phi(\rho)) = H(\Phi(\rho)) - H(\Phi \circ E(\rho)),$$

where we used the property $\Phi \circ E = E_0 \circ \Phi$ from Proposition 2.2. Moreover, $H(E(\rho)) = H(\Phi \circ E(\rho))$ as Φ is a trace-preserving $*$-isomorphism on $\mathcal{N}$. Thus, we have

$$D_\Phi(\rho) = H(\rho) - H(\Phi(\rho)) = D(\rho||E(\rho)) - D(\Phi(\rho)||\Phi \circ E(\rho)).$$

Iterating the entropy difference Lemma 2.1, we have

$$\begin{aligned} D(\rho\|(\Phi^*\Phi)^k(\rho)) &\leq D_\Phi(\rho) + D(\rho\|(\Phi^*\Phi)^{k-1}(\rho)) \\ &\leq kD_\Phi(\rho) + D(\rho\|\rho) \\ &= k(D(\rho||E(\rho)) - D(\Phi(\rho)||\Phi \circ E(\rho)). \end{aligned}$$

Now using Lemma 2.3, for $k = k_{cb}(\Phi)$,

$$D(\rho||E(\rho)) \leq 2D(\rho\|(\Phi^*\Phi)^k\rho)) \leq 2k\big(D(\rho||E(\rho)) - D(\Phi(\rho)||\Phi \circ E(\rho))\big).$$

Rearranging the terms gives the assertion. The general case $\rho \in S(\mathcal{M})$ can be obtained via approximation $\rho_\varepsilon = (1-\varepsilon)\rho + \varepsilon 1$ as [14, Lemma A.2]. The same argument applies to $\mathrm{id}_\mathcal{Q} \otimes \Phi$, because the CB return time $k_{cb}(\mathrm{id}_\mathcal{Q} \otimes \Phi) = k_{cb}(\Phi)$ is same as of Φ by the definition. □

The above theorem is an improved data processing inequality for the relative entropy between a state ρ and its conditional expectation $E(\rho)$. Here, $\mathcal{N}$ is the 'decoherence free' subalgebra. Indeed, for any two states $\sigma_1, \sigma_2 \in \mathcal{N}$,

$$D(\sigma_1||\sigma_2) \geq D(\Phi(\sigma_1)||\Phi(\sigma_2)) \geq D(\Phi^*\Phi(\sigma_1)||\Phi^*\Phi(\sigma_2)) = D(\sigma_1||\sigma_2)$$

does not decay. Outside the 'decoherence free' subalgebra $\mathcal{N}$, the relative entropy from a state ρ to its projection $E(\rho)$ on $\mathcal{N}$ is strictly contractive under every use of the channel Φ.

For Φ being a symmetric quantum Markov map, we have $\Phi = \Phi_*$. Moreover, Proposition 2.2 reduces to

$$\Phi \circ E = E \circ \Phi\ ,\ \ \Phi^2 \circ E = E \circ \Phi^2 = E .$$

Then

$$\begin{aligned} D(\Phi^2(\rho)\|E(\rho)) &= D(\Phi^2(\rho)\|\Phi^2\circ E(\rho)) = D(\Phi^2(\rho)\|\Phi\circ E\circ\Phi(\rho)) \\ &\le (1-\frac{1}{2k_{cb}(\Phi)})D(\Phi(\rho)\|E\circ\Phi(\rho)) = (1-\frac{1}{2k_{cb}(\Phi)})D(\Phi(\rho)\|\Phi\circ E(\rho)) \\ &\le (1-\frac{1}{2k_{cb}(\Phi)})^2 D(\rho\|E(\rho)). \end{aligned}$$

We can iterate the entropy contraction above and obtain the discrete time entropy decay,

$$D(\Phi^{2n}(\rho)||E(\rho)) \le (1-\frac{1}{2k_{cb}(\Phi)})^{2n} D(\rho||E(\rho)).$$

3. Complete modified log-Sobolev inequality for symmetric Markov semigroups

3.1. Functional inequalities

In this section, we discuss a continuous time relative entropy decay for symmetric quantum Markov semigroups. We first review some basics of quantum Markov semigroups. A quantum Markov semigroup $(T_t)_{t\ge 0} : \mathcal{M} \to \mathcal{M}$ is a family of maps satisfying

i) for each $t \ge 0$, T_t is a quantum Markov map (i.e., normal, completely positive and unital)
ii) $T_0 = \mathrm{id}_{\mathcal{M}}$ and $T_s \circ T_t = T_{s+t}$ for any $s, t \ge 0$.
iii) for $x \in \mathcal{M}$, $t \mapsto T_t(x)$ is weak*-continuous.

The generator of the semigroup is defined as

$$Lx = w^*\text{-}\lim_{t\to 0} \frac{1}{t}(x - T_t(x))$$

on the domain of L that the limit exists. In this section, we still consider $\mathcal{M}$ as a finite von Neumann algebra equipped with a normal faithful tracial state τ. Given $(T_t)_{t\ge 0}$ is *symmetric* (or more specifically, *trace-symmetric*), that is,

$$\tau(x^*T_t(y)) = \tau(T_t(x)^*y) \ , \ \forall x, y \in \mathcal{M}, t \ge 0,$$

the generator L is a positive, symmetric operator, densely defined on $L_2(\mathcal{M})$. Its kernel is the fixed-point subspace $\mathcal{N} := \ker(L) = \{x \in \mathcal{M} \mid T_t(x) = x, \forall t \ge 0\}$, which coincides with the common multiplicative domain of all T_t – hence a von Neumann subalgebra. Moreover, each T_t is an $\mathcal{N}$-bimodule map

$$T_t(axb) = aT_t(x)b, \ \forall\ a, b \in \mathcal{N}, x \in \mathcal{M}.$$

In particular, we have

$$T_t \circ E = E \circ T_t = E,$$

where $E : \mathcal{M} \to \mathcal{N}$ is the trace-preserving conditional expectation onto the fixpoint algebra $\mathcal{N}$. We say (T_t) is *ergodic* if $\mathcal{N} = \mathbb{C}1$ is trivial. Note that in the mathematical physics literature, it is common to use *primitive* instead of ergodic. In this case, the semigroup admits a unique invariant state – namely, the trace τ up to normalization. We will consider symmetric quantum Markov semigroups that are not necessarily ergodic.

Recall that a semigroup is equivalently determined by its *Dirichlet form*

$$\mathcal{E} : L_2(\mathcal{M}) \to [0,\infty] \ , \ \mathcal{E}(x,x) = \tau(x^*Lx).$$

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We write $\mathrm{dom}(L)$ for the domain of L and $\mathrm{dom}(\mathcal{E})$ for the domain of $\mathcal{E}$. The Dirichlet subalgebra is defined as $\mathcal{A}_{\mathcal{E}} := \mathrm{dom}(\mathcal{E}) \cap \mathcal{M}$. It was proved [21] that $\mathcal{A}_{\mathcal{E}}$ is a dense $*$-subalgebra of $\mathcal{M}$ and a core of $L^{1/2}$. We denote by

$$S(\mathcal{A}_{\mathcal{E}}) = S(\mathcal{M}) \cap \mathcal{A}_{\mathcal{E}},\ S_B(\mathcal{A}_{\mathcal{E}}) = S_B(\mathcal{M}) \cap \mathcal{A}_{\mathcal{E}}$$

the set of bounded density operators from $\mathcal{A}_{\mathcal{E}}$. We now introduce the formal definitions of functional inequalities for quantum Markov semigroups.

Definition 3.1. Let $T_t = e^{-Lt} : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and $E : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto its fixed point space. We say T_t satisfies

i) the Poincaré inequality (PI) for $\lambda > 0$ if

$$\lambda \,\|x - E(x)\|_2^2 \le \mathcal{E}(x, x),\ \forall x \in \mathcal{A}_{\mathcal{E}}, \tag{3.1}$$

ii) the log-Sobolev inequality (LSI) for $\alpha > 0$ if

$$\alpha\tau\big(|x|^2 \ln |x^2| - E(|x^2|) \ln E(|x^2|)\big) \le 2\mathcal{E}(x, x),\ \forall x \in \mathcal{A}_{\mathcal{E}}, \tag{3.2}$$

iii) the modified log-Sobolev inequality (MLSI) for $\alpha > 0$ if

$$2\alpha D(\rho||E(\rho)) \le \mathcal{E}(\rho, \ln \rho),\ \forall \rho \in S_B(\mathcal{A}_{\mathcal{E}}), \tag{3.3}$$

iv) the complete modified log-Sobolev inequality (CMLSI) for $\alpha > 0$ if $\mathrm{id}_{\mathcal{Q}} \otimes T_t$ satisfies α-MLSI inequality for any finite von Neumann algebra $\mathcal{Q}$.

The optimal (largest possible) constants for PI, LSI, MLSI and CMLSI will be denoted respectively as $\lambda(L), \alpha_2(L), \alpha_1(L)$ and $\alpha_c(L)$ (or $\lambda, \alpha_2, \alpha_1$ and α_c in short if the generator is clear).

The Poincaré inequality (3.1) is equivalent to the spectral gap of L as a positive operator. LSI (3.2) is equivalent to hypercontractivity [60]

$$\|\, T_t : L_2(\mathcal{M}) \to L_p(\mathcal{M})\,\| \le 1 \ \text{ if } \ p \le 1 + e^{2\alpha t}. \tag{3.4}$$

MLSI (3.3) is known to be equivalent to the exponential decay of relative entropy ([5, Theorem 3.2] and [14, Proposition A.3]) that

$$D(T_t(\rho)||E(\rho)) \le e^{-2\alpha t} D(\rho||E(\rho)),\ \forall \rho \in S(\mathcal{M}). \tag{3.5}$$

The equivalence of (3.3) and (3.5) is obtained by differentiating the relative entropy for T_t at 0, which leads to the entropy production on the R.H.S of MLSI

$$I_L(\rho) := \mathcal{E}(\rho, \ln \rho) = -\frac{d}{dt}|_{t=0} D(T_t(\rho)||E(\rho)) = \tau(L(\rho) \ln \rho).$$

It is well known that

$$\alpha_2 \le \alpha_1 \le \lambda.$$

The main motivation to consider CMLSI over MLSI and LSI is the tensorization property

$$\alpha_c(L_1 \otimes \mathrm{id} + \mathrm{id} \otimes L_2) = \min\{\alpha_c(L_1),\ \alpha_c(L_2)\}, \tag{3.6}$$

which in the quantum cases fails for α_1 [14, Section 4.4], and is only known to hold for α_2 for limited examples in small dimensions. The main result of this section is Theorem 1.1, which asserts a lower bound

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$$\alpha_c(L) \geq \frac{1}{2t_{cb}(L)}$$

by the inverse of CB return time

$$t_{cb}(L) = \inf\{t > 0 \mid (1-0.1)E \leq_{cp} T_t \leq_{cp} (1+0.1)E\}. \tag{3.7}$$

Here, we set $\varepsilon = 0.1$ for the notation $t_{cb}(\varepsilon)$ in Theorem 1.1 because of Lemma 2.3.

Theorem 3.2. *Let $T_t = e^{-Lt} : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and $E : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto the fixed point subalgebra $\mathcal{N}$. Define the CB return time as*

$$t_{cb}(L) = \inf\Big\{t > 0 \mid 0.9E \leq_{cp} T_t \leq_{cp} 1.1E\Big\}.$$

Then

$$\frac{1}{2t_{cb}(L)} \;\leq\; \alpha_c(L) \;\leq\; \alpha_1(L).$$

Proof. Let $t_m = t_{cb}(L)/2m$ for some $m \in \mathbb{N}_+$. Since T_t is symmetric, $T^*_{t_m}T_{t_m} = T_{t_m}T_{t_m} = T_{2t_m}$. Hence, T_{t_m} has discrete return time $k_{cb}(T_{t_m}) = m$. By the Lemma 2.1, for any $\rho \in S_B(\mathcal{M})$,

$$D(T_{t_m}(\rho)||E(\rho)) \leq (1-\frac{1}{2m})D(\rho||E(\rho)).$$

Write $t_{cb} = t_{cb}(L)$. Now assume further $\rho \in \cup_{t>0}T_t(\mathcal{M}) \subset \mathrm{dom}(L)$. We have by Theorem 2.5,

$$\begin{aligned} I(\rho) &= \lim_{t\to 0}\frac{D(\rho||E(\rho)) - D(T_t(\rho)||E(\rho))}{t} \\ &= \lim_{m\to\infty}\frac{D(\rho||E(\rho)) - D(T_{\frac{t_{cb}}{2m}}(\rho)||E(\rho))}{\frac{t_{cb}}{2m}} \\ &\geq \lim_{m\to\infty}\frac{\frac{1}{2m}D(\rho||E(\rho))}{\frac{t_{cb}}{2m}} = \frac{1}{t_{cb}}D(\rho||E(\rho)). \end{aligned}$$

The entropy decay for general $\rho \in S(\mathcal{M})$ can be obtained by approximation as in the Appendix [14, Appendix]). This proves $\alpha_1(L) \geq \frac{1}{2t_{cb}(L)}$. The same argument applies to $\mathrm{id}_{\mathcal{Q}} \otimes L$ yields the assertion $\alpha_c(L) \geq \frac{1}{2t_{cb}(L)}$. ◻

Remark 3.3. a) For LSI constant α_2, the $\Omega(\frac{1}{t_{cb}})$ lower bounds were obtained for **ergodic** semigroups in both classical [23] and quantum setting [74]. These bounds as well as our bound for α_c are asymptotic tight (See Example 5.6 and Section 5.3).

b) In [14], a similar estimate $\alpha_c \geq \Omega(\frac{1}{t_{cb}})$ was obtained for semigroups that admits non-negative entropic Ricci curvature lower bound. The entropy Ricci curvature lower bound for quantum Markov semigroup was introduced by Carlen and Mass [15] using λ-displacement convexity of entropy functionals H w.r.t to certain noncommutative Wasserstein distance, inspired from Lott and Villani [51], and Sturm's [71] work on metric measure spaces. For heat semigroups on Riemmannian manifold, the entropy Ricci curvature lower bound follows from a lower bound of the Ricci curvature tensor. Nevertheless, in the noncommutative case, these entropy Ricci curvature lower bounds for quantum Markov semigroup are in general hard to verify. So far, most examples rely on certain interwining relation $\nabla T_t = e^{-\lambda t}\tilde{T}_t\nabla$ between the semigroup T_t and a gradient operator ∇ (see [15, 13, 81]).

c) Our Theorem 1.1 here does not rely on any curvature conditions, which uses only information theoretic tools such as entropic quantities and inequalities. To the best of our knowledge, this direct proof

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is even novel in the classical setting. It is worth pointing out that the definition of relative entropy as well as its exponential decay of relative entropy is independent of the choice of the trace, which also shows the naturalness of our approach and the extension to non-tracial von Neumann algebras in Section 4.

3.2. CB return time

We now consider a common scenario where the CB return time t_{cb} is finite. The original motivation for the notion, despite defining using CP (completely positive) order (3.7), is the following characterization using CB (completely bounded) norm. Recall that a linear map $\Psi : \mathcal{M} \to \mathcal{M}$ is called a $\mathcal{N}$-bimodule map if

$$\Psi(axb) = a\Psi(x)b, \ \forall\, a, b \in \mathcal{N}, x \in \mathcal{M}.$$

Proposition 3.4. *Let $\mathcal{N} \subset \mathcal{M}$ be a subalgebra and $E : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation. Let $\Psi : \mathcal{M} \to \mathcal{M}$ be an $\mathcal{N}$-bimodule $*$-preserving map. For any $\varepsilon > 0$, the following two conditions are equivalent:*

i) $(1-\varepsilon)E \le_{cp} \Psi \le_{cp} (1+\varepsilon)E$;
ii) $\| \Psi - E : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \to L_\infty(\mathcal{M}) \|_{cb} \le \varepsilon$.

The condition ii) above is the completely bounded norm from the space $L_\infty^1(\mathcal{N} \subset \mathcal{M})$ to $\mathcal{M}$. $L_\infty^1(\mathcal{N} \subset \mathcal{M})$ is called a conditional L_∞ space, defined as the completion of $\mathcal{M}$ with respect to the norm

$$\| x \|_{L_\infty^1(\mathcal{N}\subset\mathcal{M})} = \sup_{a,b\in\mathcal{N}\ ,\ \|a\|_2=\|b\|_2=1} \| axb \|_1,$$

where the supremum takes over all $a, b \in L_2(\mathcal{N})$ with $\|a\|_2 = \|b\|_2 = 1$. The operator space structure of $L_\infty^1(\mathcal{N} \subset \mathcal{M})$ is given by the identification

$$\mathbb{M}_n(L_\infty^1(\mathcal{N} \subset \mathcal{M})) = L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))$$

(see [42] and [30, Appendix]). Proposition 3.4 is relatively self-evident in the ergodic case $\mathcal{N} = \mathbb{C}1$, $L_\infty^1(\mathcal{N} \subset \mathcal{M}) \cong L_1(\mathcal{M})$, which we illustrate below.

Example 3.5 (Classical case). Let (Ω, μ) be a probability space. Let $P : L_\infty(\Omega) \to L_\infty(\Omega)$ be a linear map with kernel $P(f)(x) = \int_\Omega k(x, y) f(y) d\mu(y)$. It is clear that P is $*$-preserving (i.e., $P(\bar{f}) = \overline{P(f)}$ if k is real); P is a positive map if and only if the kernel function $k \ge 0$. Recall the expectation map

$$E_\mu : L_\infty(\Omega) \to \mathbb{C}\mathbf{1}\ ,\ E(f) = (\int_\Omega f\mu)\mathbf{1},$$

where $\mathbf{1}$ is the unit constant function. The kernel of E_μ is the constant function $\mathbf{1}$ on the product space $\Omega \times \Omega$. The following equivalence is self-evident:

$$\begin{aligned}(1-\varepsilon)E \le P \le (1+\varepsilon)E &\iff \varepsilon E \le P - E \le \varepsilon E\\ &\iff \varepsilon\mathbf{1} \le k - \mathbf{1} \le \varepsilon\mathbf{1}\\ &\iff \| k - \mathbf{1} \|_{L_\infty(\Omega\otimes\Omega)} \le \varepsilon\\ &\iff \| P - E : L_1(\Omega) \to L_\infty(\Omega) \| \le \varepsilon. \end{aligned} \tag{3.8}$$

To see the equivalence in terms of complete positivity and completely bounded norm in Proposition 3.4, it suffices to notice that every positive (resp. bounded) map to $L_\infty(\Omega)$ is automatically completely positive (resp. completely bounded with same norm [70]).

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Example 3.6 (Quantum case). The above argument also applies to the noncommutative ergodic case $\mathcal{N} = \mathbb{C}1 \subset \mathcal{M}$. The correspondence between the map P and its kernel k generalizes to the isomorphism between the map T and its Choi operator $C_T \in \mathcal{M}^{op}\overline{\otimes}\mathcal{M}$

$$T(x) = \tau \otimes \mathrm{id}(C_T(x \otimes 1)),\ x \in L_1(\mathcal{M}) \cong (\mathcal{M}^{op})_*,$$

where $\mathcal{M}^{op}$ is the opposite algebra of $\mathcal{M}$. The isomorphism $T \mapsto C_T$ is not only positivity-preserving by Choi's Theorem (T is CP iff $C_T \geq 0$), but also isometric by Effros-Ruan Theorem (see [25, 10]),

$$\| T : L_1(\mathcal{M}) \to L_\infty(\mathcal{M}) \|_{cb} = \| C_T \|_{\mathcal{M}^{op}\overline{\otimes}\mathcal{M}} .$$

Then the equivalence in Proposition 3.4 follows as (3.8).

For the general case of a $\mathcal{N}$-bimodule map T with a nontrivial $\mathcal{N}$, the above isomorphism holds with more involved module Choi operator, which we refer to the discussion in Section 4.3 and also [6, Lemma 5.1] and [29, Lemma 3.15] for the complete proof of Proposition 3.4.

With Proposition 3.4, the CB-return time can be equivalently defined as

$$t_{cb}(L) := \inf\{\, t > 0 \mid \ \| T_t - E : L^1_\infty(\mathcal{N} \subset \mathcal{M}) \to L_\infty(\mathcal{M}) \|_{cb} \leq 0.1\}. \tag{3.9}$$

It is known that t_{cb} is finite whenever T_t satisfies the Poincaré inequality and one-point ultra-contractive estimate.

Proposition 3.7. *Let $T_t : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and $E : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto the fixed point space. Suppose*

i) *T_t satisfies the Poincaré inequality: $\lambda > 0$ such that $\| T_t - E : L_2(\mathcal{M}) \to L_2(\mathcal{M}) \| \leq e^{-\lambda t}$, $\forall t \geq 0$;*
ii) *There exists $t_0 \geq 0$ such that $\| T_{t_0} : L^1_\infty(\mathcal{N} \subset \mathcal{M}) \to L_\infty(\mathcal{M}) \|_{cb} \leq C_0$.*

Then $t_{cb} \ \leq \ \frac{1}{\lambda} \ln(10C_0) + t_0$.

Proof. This is now a standard argument similar to [14, Proposition 3.8] and [31, Lemma B.1]. □

Remark 3.8. For the special case of $t_0 = 0$ and $T_0 = \mathrm{id} : \mathcal{M} \to \mathcal{M}$, we consider

$$\| \mathrm{id} : L^1_\infty(\mathcal{N} \subset \mathcal{M}) \to L_\infty(\mathcal{M}) \|_{cb} = \inf\{\, \mu > 0 \mid \ \mathrm{id} \leq_{cp} \mu E\} := C_{cb}(E).$$

$C_{cb}(E)$ was introduced in [30] as the complete bounded version of Popa and Pimsner's subalgebra index [67],

$$C(E) := \inf\{\mu > 0 \mid \rho \leq \mu E\rho,\ \forall \rho \in \mathcal{M}_+\},\ C_{cb}(E) := \sup_n C(E \otimes \mathrm{id}_{\mathbb{M}_n}).$$

When $\mathcal{M}$ is finite dimensional, both the index $C(E)$ and $C_{cb}(E)$ are finite and admit the explicit formula [67, Theorem 6.1]. In this case, one can take $t_0 = 0$ in above Proposition 3.7 and yields

$$t_{cb} \ \leq \ \frac{\ln(10C_{cb}(E))}{\lambda}.$$

3.3. Classical Markov semigroups

In the remainder of this section, we focus on applications toward classical Markov map. We postpone the discussion of truly noncommutative semigroups to Section 4. Let $T_t = e^{-Lt} : L_\infty(\Omega, \mu) \to L_\infty(\Omega, \mu)$ be an ergodic Markov semigroup symmetric to the probability measure μ. Note that in the ergodic case $L^1_\infty(\mathbb{C}1 \subset L_\infty(\Omega)) = L_1(\Omega, \mu)$, and by Smith's lemma [70], any bounded map $T : L_1(\Omega, \mu) \to L_\infty(\Omega, \mu)$ is automatic completely bounded

$$\| T : L_1(\Omega, \mu) \to L_\infty(\Omega, \mu) \| = \| T : L_1(\Omega, \mu) \to L_\infty(\Omega, \mu) \|_{cb} .$$

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Then the CB return time t_{cb} reduces to the standard L_∞-mixing time

$$t_b(\varepsilon) = \inf\{ t > 0 \mid \| T_t - E : L_1(\Omega, \mu) \to L_\infty(\Omega, \mu) \| \le \varepsilon\}.$$

Then by a combination of Theorem 1.1 and Proposition 3.7, we obtain the following theorem.

Theorem 3.9. *Let $T_t = e^{-tL} : L_\infty(\Omega, \mu) \to L_\infty(\Omega, \mu)$ be an ergodic Markov semigroup symmetric to the probability measure μ. Suppose*

i) *T_t satisfies λ-Poincaré inequality for some $\lambda > 0$: for $f \in \mathrm{dom}(L^{1/2})$,*

$$\lambda\mu(|f - E_\mu(f)|^2) \le \int f(Lf)d\mu. \tag{3.10}$$

ii) *There exists $t_0 > 0$ such that*

$$\| T_{t_0} : L_1(\Omega, \mu) \to L_\infty(\Omega, \mu) \| \le C_0. \tag{3.11}$$

Then

$$\alpha_1 \ge \alpha_c \ge \frac{\lambda}{2(\lambda t_0 + \ln C_0 + 2)}. \tag{3.12}$$

This result can be compared to the bound of Diaconis and Saloff-Coste [23, Theoem 3.10], which states[1]

$$\alpha_1 \ge \alpha_2 \ge \frac{\lambda}{\lambda t_0 + \ln(C_0) + 1}. \tag{3.13}$$

In particular, $\alpha_1 \ge \alpha_2 \ge \frac{2}{t_b(e^{-2})}$ for the alternative L_∞-mixing time

$$t_b(e^{-2}) = \inf\{t > 0 \mid \| T_t - E_\mu : L_1(\Omega, \mu) \to L_\infty(\Omega, \mu) \| \le \frac{1}{e^2}\}.$$

By the comparison $e^{-3} < 0.1 < e^{-2}$, we have $t_b(e^{-2}) \le t_b(0.1) \le \frac{3}{2} t_b(e^{-2})$. Hence, in terms of lower bound for α_1, (3.13) and (3.12) are equivalent up to absolute constants. The difference is that (3.13) lower bounds the LSI constant α_2 and our estimate (3.12) bounds the CMLSI constant α_c.

For finite Markov chains with $|\Omega| < \infty$, we have finite index

$$C_{cb}(E_\mu) = C(E_\mu) = \inf\{C > 0 \mid f \le C\mu(f) \ \forall f \ge 0\} = \| \mu^{-1} \|_\infty,$$

where μ is a strictly positive probability density function. It was proved in [23] that

$$\frac{1}{t_b(e^{-2})} \le \lambda \le \frac{2 + \log \| \mu^{-1} \|_\infty}{2t_b(e^{-2})}, \tag{3.14}$$

$$\frac{1}{t_b(e^{-2})} \le \alpha_2 \le \frac{4 + \log\log \| \mu^{-1} \|_\infty}{2t_b(e^{-2})}. \tag{3.15}$$

Combined with our Theorem 1.1, we obtain the following:

Corollary 3.10. *For a finite Markov chain $T_t : l_\infty(\Omega, \mu) \to l_\infty(\Omega, \mu)$ symmetric to μ,*

$$\min\left\{\frac{4\alpha_2}{3(4 + \log\log \| \mu^{-1} \|_\infty)}, \frac{\lambda}{2\log(10 \| \mu^{-1} \|_\infty)}\right\} \le \alpha_c \le \alpha_1 \le \lambda.$$

[1] Note that the LSI constant in [23] is defined as half of our α_2 here.

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Proof. Note that $t_b(e^{-2}) \le t_{cb}(0.1) \le \frac{3}{2} t_b(e^{-2})$. Then by Theorem 1.1 and (3.15),

$$\alpha_c \ge \frac{1}{2t_{cb}(0.1)} \ge \frac{1}{3t_b(e^{-2})} \ge \frac{2\alpha_2}{3(4+\log\log\|\mu^{-1}\|_\infty)}.$$

The other lower bound $\alpha_c \ge \frac{\lambda}{2\log(10\|\mu^{-1}\|_\infty)}$ follows from Theorem 3.9 by choosing $t_0 = 0$. □

Example 3.11. When Ω is not finite, here is a simple example with $\alpha_c(L) > 0$, but the ultra-contractivity (3.11) is never satisfied for finite t_0. Take $L = I - E_\mu$. It generates the so-called depolarizing semigroup

$$T_t = e^{-t}\,\mathrm{id} + (1-e^{-t})E_\mu,\ T_t(f) = e^{-t} f + (1-e^{-t})\mu(f)\mathbf{1},$$

where $\mathbf{1}$ is the unit constant function. Then for any $t < \infty$,

$$\| T_t - E_\mu : L_1(\Omega,\mu) \to L_\infty(\Omega,\mu) \| = \| e^{-t}\,\mathrm{id} : L_1(\Omega,\mu) \to L_\infty(\Omega,\mu) \| = e^{-t} C(E_\mu),$$

which is infinite whenever $L_\infty(\Omega,\mu)$ is infinite dimensional. However, it follows from direct calculation that $\alpha_c(I - E_\mu) \ge \frac{1}{2}$.

3.4. *Hörmander system*

We now discuss the application to Markov semigroups on smooth manifolds generated by sub-Laplacians. Let (M, g) be a d-dimensional compact connected Riemannian manifold without boundary and let $d\mu = \omega d\,\mathrm{vol}$ be a probability measure with smooth density ω w.r.t the volume form $d\,\mathrm{vol}$. A family of vector fields $H = \{X_i\}_{i=1}^k \subset TM$ with $k \leqslant d$ is called a *Hörmander system* if at every point $x \in M$, the tangent space at x can be spanned by the iterated Lie brackets of X_i's

$$T_xM = \mathrm{span}\{[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{n-1}}, X_{i_n}]]] \mid 1 \leqslant i_1, i_2 \cdots i_n \leqslant k\}. \qquad \text{(Hörmander condition)}$$

By compactness, we can assume there is a global constant l_H such that for every point $x \in M$, we need at most l_Hth iterated Lie bracket in above expression (also called strong Hörmander condition). Denote $\nabla = (X_1, \cdots, X_k)$ and by X_i^* the adjoint of X_i on $L^2(M, d\mu)$. Under the Hörmander condition, the sub-Laplacian

$$\Delta_H = \nabla^*\nabla = \sum_i X_i^* X_i = -\sum_i X_i^2 + (\mathrm{div}_\mu(X_i) + X_i(\ln\omega))X_i$$

is a symmetric operator on $L^2(M, d\mu)$ which generates an ergodic Markov semigroup $P_t = e^{-\Delta_H t}$, often called the horizontal heat semigroup. Here, $\mathrm{div}_\mu(X)$ is the divergence of X w.r.t to μ. When $H = \{X_i\}_{i=1}^d$ forms an orthonormal frame to the Riemannian metric, $\Delta_H = \Delta$ recovers the (weighted) Laplace-Beltrami operator and $P_t = e^{-\Delta t}$ is the (weighted) heat semigroup on M.

The *gradient form* (Carré du Champ operator) of Δ_H is given by

$$\Gamma(f,g) := \frac{1}{2}(f\Delta_H(g) + \Delta_H(f)g - \Delta_H(fg)) = \sum_i \langle X_i f, X_i g\rangle.$$

It follows from the product rule of derivatives that Γ is diffusive (i.e., $\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h)$). For diffusion semigroups, it is known [4, Theorem 5.2.1] that the MLSI constant α_1 and the LSI constant α_2 coincide (i.e., $\alpha := \alpha_1 = \alpha_2$). The positivity

$$\alpha(\Delta_H) > 0$$

for any Hörmander's system $H = \{X_i\}_{i=1}^k$ on a compact connected Riemannian manifold without boundary was proved in [53, Theorem 3.1]. Our Theorem 1.2 improves this to $\alpha_c(\Delta_H) > 0$.

Proof of Theorem 1.2. Recall the following Sobolev-type inequality (see, for example, [53, Lemma 2.1]):

$$\|f\|_q \ \leq \ C\big(\langle \Delta_H f, f\rangle + \| f \|_2^2 \,\big)^{1/2}, \tag{3.16}$$

where $q = \frac{2dl_H}{dl_H - 2} > 2$ and l_H is globoal Lie bracket length needed in the strong Hörmander condition. By Varopoulos' Theorem (see [77, Chapter 2]) on the dimension of semigroups, this implies the following ultra-contractive estimate:

$$\| e^{-\Delta_H t} : L_1(M,\mu) \to L_\infty(M,\mu) \| \leqslant C' t^{-m/2} \text{ for } 0 \leq t \leq 1 \text{ and some } C' > 0, \tag{3.17}$$

where $m = dl_H$. Also, it was proved in [53, Theorem 2.3] that Δ_H satisfies the Poincaré inequality: $\lambda(\Delta_H) > 0$. Combining these with our Theorem 3.9 yields the assertion. □

The Sobolev-type inequality (3.16) is also used in [53] by Lugiewicz and Zegarlínski to prove that $\alpha_2(\Delta_H) > 0$. Their proof relies on the Rothaus lemma, and so does the discrete case by Diaconis and Saloff-Coste [23]. However, we will see in Section 3.6 that this approach is out of scope for showing the CMLSI constant $\alpha_c(\Delta_H) > 0$.

Example 3.12. The special unitary group SU(2) is

$$\mathrm{SU}\,(2) = \{cI + xX + yY + zZ : c^2 + x^2 + y^2 + z^2 = 1, x, y, x, c \in \mathbb{R}\},$$

where X, Y, Z are the skew-Hermitian Pauli unitary

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

The Lie algebra is $\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\{X, Y, Z\}$ with Lie bracket rules as

$$[X, Y] = 2Z \ , [Y, Z] = 2X \ , [Z, X] = 2Y. \tag{3.18}$$

The canonical sub-Riemannian structure is given by $H = \{X, Y\}$, which is a generating set of $\mathfrak{g}$ because $[X, Y] = 2Z$. The associated sub-Laplacian is

$$\Delta_H = -(X^2 + Y^2). \tag{3.19}$$

The semigroup $P_t = e^{-\Delta_H t}$ on SU(2) has been studied as a prototype of horizontal heat semigroups. In particular, Baudoin and Bonnefont in [8] proved that

$$\Gamma(P_t f, P_t f) \leqslant Ce^{-4t} P_t(\Gamma(f, f)), \tag{3.20}$$

for some constant $C > 0$. In [31], Gao and Gordina based on (3.20) proved the CMLSI constant that

$$\alpha_c(\Delta_H) \geq (2\int_0^\infty Ce^{-4t} dt)^{-1} = \frac{2}{C}.$$

The gradient estimate (3.20), as a weaker variant of Bakry-Emery curvature dimension condition, has been found useful to derive CMLSI in [31]. Nevertheless, this weaker gradient estimate is only known for only a limited number of examples in the sub-Riemannian setting [24, 54]. Our result avoids this condition and obtains CMLSI for general Hörmander systems.

Example 3.13. Let $n \geq 3$. The special unitary group SU(n) is

$$\mathrm{SU}(n) = \{u \in \mathbb{M}_n \mid u^* u = 1, \ \det(u) = 1\}.$$

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The Lie algebra $\mathfrak{su}(n)$ is the space of all the skew-Hermitian matrices, and a natural basis $\mathfrak{su}(n)$ is given by $\{X_{j,k}, Y_{j,k}, Z_k \mid 1 \le j < k \le n\}$ where

$$X_{j,k} = e_{jk} - e_{kj}\ ,\ Y_{j,k} = i(e_{jk} + e_{kj})\ ,\ Z_k = i(e_{11} - e_{kk}),$$

which is $n^2 - 1$ dimensional. Let $V = \{1, \cdots, n\}$ be a vertex set and $E \subset V \times V$ as an edge set. The set

$$H_E = \{X_{j,k}, Y_{j,k} \mid (j,k) \in E\}$$

is a generating set if and only if (V, E) is a connected graph. The associated sub-Laplacian

$$\Delta_E = -\sum_{(j,k)\in E} X_{j,k}^2 + Y_{j,k}^2$$

is a generalization of (3.19). Theorem (1.2) implies that $\alpha_c(\Delta_E) > 0$ for all connected (V, E), despite the gradient estimate (3.20) is not known for this type of generator.

3.5. Transference semigroups

Let us discuss an immediate application of $\alpha_c(\Delta_H) > 0$ to symmetric Quantum Markov semigroups. Let G be a compact Lie group and $H = \{X_1, \cdots, X_k\}$ be a generating set of its Lie algebra $\mathfrak{g}$. Then $\{X_1, \cdots, X_k\}$ satisfies the Hörmander condition, and its sub-Laplacian $\Delta_H = -\sum_k X_i^2$ generates a Markov semigroup $P_t = e^{-\Delta_H t}$ symmetric to the Haar measure. Let $u : G \to \mathbb{M}_n$ be a finite dimensional unitary representation and $d_u : \mathfrak{g} \to i(\mathbb{M}_n)_{s.a.}$ be the corresponding Lie algebra morphism. $P_t = e^{-\Delta_H t}$ induces a quantum Markov semigroup $T_t = e^{-L_H t} : \mathbb{M}_m \to \mathbb{M}_m$ with generator in the Lindbladian form [50],

$$L_H(\rho) = -\sum_{i=1}^{k} [d_u(X_i), [d_u(X_i), \rho]].$$

T_t is called a transference semigroup of P_t by the following commuting diagram:

$$\begin{tikzcd} L_\infty(G, \mathbb{M}_m) \arrow[r, "P_t \otimes \mathrm{id}_{\mathbb{M}_m}"] & L_\infty(G, \mathbb{M}_m) \\ \mathbb{M}_m \arrow[u, "\pi_u"] \arrow[r, "T_t"] & \mathbb{M}_m, \arrow[u, "\pi_u"'] \end{tikzcd} \tag{3.21}$$

where the transference map π_u is a $*$-endomorphism given by

$$\pi_u : \mathbb{M}_m \to L_\infty(G, \mathbb{M}_m),\ \pi_u(\rho)(g) = u(g)^* \rho u(g),$$

which embeds $\mathbb{M}_m$ into $L^\infty(G, \mathbb{M}_m)$. Then the quantum semigroup T_t is the restriction of the matrix-valued extension of classical semigroup $P_t \otimes \mathrm{id}_{\mathbb{M}_m}$ on the image of $\pi(\mathbb{M}_m)$. Such a transference relation holds fro any unitary representation. We obtain the following dimension-free estimates both spectral gap and CMLSI constant (see [29, Section 4]):

$$\alpha_c(\Delta_H) \le \alpha_c(L_H), \lambda(\Delta_H) \le \lambda(L_H),$$

which are independent of the choice of the unitary representation. Then Corollary 1.3 follows immediately from Theorem 1.2.

3.6. Failure of matrix valued log-Sobolev inequality

As mentioned above, a standard analysis approach to MLSI through hypercontractivity or LSI relies on the Rothaus Lemma (see, for example, [68, 3])

$$H(|f|^2) \le H(|f - E_\mu(f)|^2) + \| f - E_\mu(f) \|_2^2 .$$

Here, we show that the Rothaus Lemma, LSI and hypercontractivity all fail for matrix-valued functions for any classical Markov semigroups. This is a strong indication that the approach by Diaconis and Saloff-Coste's hypercontractive [23] estimate (also used in [53]) cannot be used in proving lower bounds for the CMLSI constants.

The following lemma calculates the derivatives of the entropy functional $H(\rho) = \tau(\rho \log \rho)$. Recall the BKM metric of a operator $X \in \mathcal{M}$ at a base state ρ is

$$\gamma_\rho(X) = \int_0^\infty \tau(X^*(\rho + s)^{-1} X(\rho + s)^{-1}).$$

Lemma 3.14. *Let $t \mapsto \rho_t \in S_B(\mathcal{M}), t \in (a, b)$ be a smooth family of bounded density operator. Define the function $F(t) = H(\rho_t) = \tau(\rho_t \log \rho_t)$. Then*

$$F'(t) = \tau(\rho_t'(\log \rho_t + 1)),\ F''(t) = \tau(\rho_t''(\log \rho_t + 1)) + \gamma_{\rho_t}(\rho_t'),$$

where ρ_t' and ρ_t'' are the first and second order derivative of ρ_t.

Proof. The formula for F' follows from [79, Lemma 5.8]. For the second derivative, recall the noncommutative chain rule

$$\frac{d}{dt}(\log \rho_t) = \int_0^\infty (\rho_t + s)^{-1} \rho_t' (\rho_t + s)^{-1} ds.$$

By calculating the second derivative, we obtain the second assertion

$$F''(t) = \tau(\rho_t''(\log \rho_t + 1)) + \int_0^\infty \tau(\rho_t'(\rho_t + s)\rho_t'(\rho_t + s))ds \ = \ \tau(\rho_t''(\log \rho_t + 1)) + \gamma_{\rho_t}(\rho_t'). \quad \square$$

Proposition 3.15. *Let $T_t = e^{-tL} : L_\infty(\Omega, \mu) \to L_\infty(\Omega, \mu)$ be an ergodic symmetric Markov semigroup. Let $\alpha_R, \alpha_2, \alpha_h$ be the optimal (largest) constant such that the following inequalities hold for any $f \in L_\infty(\Omega, \mathbb{M}_2) \cap \mathcal{A}_\mathcal{E}$,*

$$\alpha_R\, D\Big(|f|^2 || E_\mu(|f|^2)\Big) \le D\Big(|\hat{f}|^2 || E_\mu(|\hat{f}|^2)\Big) + \| \hat{f} \|_2^2, \tag{Rothaus}$$

$$\alpha_2\, D(f^2 || E_\mu(f^2)) \le 2\mathcal{E}(f, f) \tag{LSI}$$

$$\| T_t f \|_{L_2(\mathbb{M}_2, L_{p(t)}(\Omega))} \le \| f \|_{L_2(\mathbb{M}_2, L_2(\Omega))} \ \ \textit{for } p(t) = 1 + e^{2\alpha_h t} \tag{Hypercontractivity}$$

where $E_\mu(f) = (\int f d\mu) 1_\Omega$ is the expectation map and $\hat{f} = f - E_\mu(f)$ is the mean zero part of f. Then $\alpha_R = \alpha_2 = \alpha_h = 0$.

Proof. We write $\tau(f) = \frac{1}{2}\int \mathrm{tr}(f) d\mu$ for the normalized trace on $L_\infty(\Omega, \mathbb{M}_2)$. We start with constant α_R in the Rothaus lemma. Without loss of generality, we may assume there is a measurable set $X \subset \Omega$ such that $\mu(X) = r$ for some $0 < r < 1$. Let $\eta \in (0, 1)$. Then $h_0 = (1 - r)1_X - r 1_{X^c}$ is a real mean zero function. Consider the matrix valued function $f_\varepsilon = f + \varepsilon h$ where

$$f = \begin{bmatrix} 1 + \eta & 0 \\ 0 & 1 - \eta \end{bmatrix} \mathbf{1},\ h = \begin{bmatrix} 0 & h_0 \\ h_0 & 0 \end{bmatrix},$$

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where f is a constant matrix valued function. Then $E_\mu f_\varepsilon = f$, $\hat{f}_\varepsilon = \varepsilon h$ and

$$f_\varepsilon^2 = (f+\varepsilon h)^2 = f^2 + \varepsilon(fh+hf) + \varepsilon^2 h^2 = f^2 + 2\varepsilon h + \varepsilon^2 h^2.$$

Then $E_\mu(|f_\varepsilon|^2) = f^2 + \varepsilon^2 h^2$ and $E_\mu(|\hat{f}_\varepsilon|^2) = E_\mu(h^2)\varepsilon^2$. Using Lemma 3.14, the Taylor expansion of the left-hand side of (LSI) is

$$\begin{aligned}
D\Big(|f|^2||E_\mu(|f|^2)\Big) &= D(f^2+2\varepsilon h+\varepsilon^2h^2||f^2+\varepsilon^2h^2)\\
&= H(f^2+2\varepsilon h+\varepsilon^2h^2) - H(f^2+\varepsilon^2h^2)\\
&= 2\tau(h\log f)\varepsilon + \big(2\tau(h^2(\log f+1)) + \gamma_f(2h)\big)\varepsilon^2 + O(\varepsilon^3)\\
&\quad - \tau(2E(h^2)(\log f+1))\varepsilon^2 + O(\varepsilon^3)\\
&= \gamma_f(2h)\varepsilon^2 + O(\varepsilon^3),
\end{aligned}$$

where we used the fact $\tau(h\log f) = 0$ and $\tau(h^2\log f - E_\mu(h^2)\log f) = 0$. For the right-hand side of the Rothaus lemma, we find

$$D(|\hat{f}_\varepsilon|^2||E_\mu(|\hat{f}_\varepsilon|^2)) = D(h^2||E_\mu(h^2))\varepsilon^2,\ \| \hat{f}_\varepsilon \|_2^2 = \| h \|_2^2\ \varepsilon^2.$$

While both $D(h^2||E_\mu(h^2))$ and $\| h \|_2^2$ are finite, we have

$$\begin{aligned}
\gamma_f(2h) &= 4\int_0^\infty \tau(h(f+s)^{-1}h(f+s)^{-1})ds\\
&= 4\int_0^\infty\int_\Omega \mathrm{tr}(\begin{bmatrix} \frac{1}{(1-\eta+s)(1+\eta+s)}h^2 & 0\\ 0 & \frac{1}{(1-\eta+s)(1+\eta+s)}h^2\end{bmatrix}\mathbf{1}_\Omega)d\mu ds\\
&= 4\Big(\int_0^\infty \frac{1}{(1-\eta+s)(1+\eta+s)}ds\Big)\| h \|_2^2\\
&= \frac{2}{\eta}\ln\frac{1+\eta}{1-\eta}\ \| h \|_2^2\ .
\end{aligned}$$

Note that we can choose $\eta\to 1$ and $\frac{1}{2\eta}\ln(\frac{1+\eta}{1-\eta}) \to +\infty$, which implies $\alpha_R = 0$. The same example applies to LSI by choosing a mean zero function h_0 such that $\mathcal{E}(h_0,h_0) < \infty$. For the hypercontractivity, for $p \geq 2$ we recall the norms

$$\begin{aligned}
\| f \|_{L_2(\mathbb{M}_2, L_2(\Omega))} &= \| f \|_{L_2(\mathbb{M}_2)\otimes L_2(\Omega)} = (\int \mathrm{tr}(f^* f)d\mu)^{1/2}.\\
\| f \|_{L_2(\mathbb{M}_2, L_p(\Omega))} &= \inf_{x,y\in(\mathbb{M}_2)_+,\ \| x \|_{2r} = \| y \|_{2r} = 1} \| x^{-1} f y^{-1} \|_{L_2(\mathbb{M}_2, L_2(\Omega))},
\end{aligned}$$

where the infimum takes over all positive invertible $x, y \in \mathbb{M}_2$ with unit $2r$-norm for $\frac{1}{r} = \frac{1}{2} - \frac{1}{p}$. Since T_t is a bimodule map for $\mathbb{C}1 \otimes \mathbb{M}_2 \subset L_\infty(\Omega, \mathbb{M}_2)$, we can equivalently consider the norm

$$\| T_t : L_2(\mathbb{M}_2, L_2(\Omega)) \to L_2(\mathbb{M}_2, L_p(\Omega)) \| = \| T_t : L_2(\mathbb{M}_2, L_2(\Omega)) \to L_2^a(\mathbb{M}_2, L_p(\Omega)) \|,$$

where the asymmetric amalgamated $L_2^a(\mathbb{M}_2, L_p(\Omega))$ space is equipped with norm

$$\| f \|_{L_2^a(\mathbb{M}_2, L_p(\Omega))} = \inf_{a\in(\mathbb{M}_2)_+,\ \| a \|_r = 1} \| f a^{-1} \|_{L_2(\mathbb{M}_2, L_2(\Omega))}\ .$$

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In particular,

$$\| f \|^2_{L_2^a(\mathbb{M}_2, L_p(\Omega))} = \| f^* f \|_{L_1(\mathbb{M}_2, L_{\frac{p}{2}}(\Omega))},$$

and we have

$$D(|f|^2 \| E(|f|^2)) = \lim_{q \to 1^+} \frac{\| |f|^2 \|_{L_1(\mathbb{M}_2, L_q(\Omega))} - \| |f|^2 \|_1}{q-1}.$$

Now define $p(t) = 2q(t) = 1 + e^{2\alpha_h t}$

$$G(t) = \| T_t f \|^2_{L_2^a(\mathbb{M}_2, L_p(t)(\Omega))} = \| |T_t f|^2 \|_{L_1(\mathbb{M}_2, L_{q(t)}(\Omega))}.$$

By assumption $G(t) \le 1$, we have

$$G'(0) = -2\mathcal{E}(f, f) + \alpha_h D(|f|^2 \| E(|f|^2)) \le 0,$$

which implies $\alpha_h \le \alpha_2 = 0$. Note, however, that $\alpha_h \ge 0$ because T_t is always contractive on $L_2(\mathbb{M}_2, L_2(\Omega))$. Hence, $\alpha_h = 0$, and the proof is complete. □

Remark 3.16. Similar to [7, Corollary 5.2], the above proposition implies that for $p \ne 2$, neither $L_2^a(\mathbb{M}_2, L_p(\Omega))$ or $L_2(\mathbb{M}_2, L_p(\Omega))$ are uniformly convex.

4. Entropy contraction for GNS symmetric quantum channels

4.1. State symmetric quantum channels

Let $\mathcal{M}$ be a von Neumann algebra and ϕ a normal faithful state. We have the GNS cyclic representation $\{\pi_\phi, H_\phi, \eta_\phi\}$, which is a $*$-isomorphism $\pi_\phi : \mathcal{M} \to H_\phi$ with a cyclic and separating vector η_ϕ such that

$$\phi(x) = \langle \eta_\phi, \pi_\phi(x)\eta_\phi \rangle, \ x \in \mathcal{M}.$$

By identifying $\mathcal{M} \cong \pi_\phi(\mathcal{M})$, the modular automorphism group α_t^ϕ for $t \in \mathbb{R}$ is defined as

$$\alpha_t^\phi : \mathcal{M} \to \mathcal{M} \ , \ \alpha_t^\phi(x) = \Delta^{it} x \Delta^{-it}, \ x \in \mathcal{M},$$

where Δ is the modular operator of ϕ, defined as follows:

$$\Delta = S^* \bar{S}, \ S(\pi_\phi(x)\eta_\phi) = \pi_\phi(x^*)\eta_\phi.$$

We consider the following two symmetric conditions with respect to a state ϕ.

Definition 4.1. We say a quantum Markov map $\Phi : \mathcal{M} \to \mathcal{M}$ is GNS-symmetric with respect to ϕ (in short, GNS-ϕ-symmetric) if

$$\phi(\Phi(x)y) = \phi(x\Phi(y)), \ \forall \, x, y \in \mathcal{M} \ ;$$

ii) We say Φ is KMS-symmetric with respect to ϕ (in short, KMS-ϕ-symmetric) if

$$\langle \Delta^{\frac{1}{4}} x \eta_\phi, \Delta^{\frac{1}{4}} \Phi(y)\eta_\phi \rangle = \langle \Delta^{\frac{1}{4}} \Phi(x)\eta_\phi, \Delta^{\frac{1}{4}} y \eta_\phi \rangle, \ \forall \, x, y \in \mathcal{M}.$$

Correspondingly, we call the pre-adjoint $\Phi_* : \mathcal{M}_* \to \mathcal{M}_*$ a GNS- or KMS-ϕ-symmetric quantum channel.

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Both definitions are generalizations of the detailed balance condition for classical Markov chains and imply that $\phi = \phi\circ\Phi = \Phi_*(\phi)$ is an invariant state of Φ. It is proven in [32, 80] that the GNS-ϕ-symmetric quantum Markov map is equivalent to KMS-ϕ-symmetric plus that Φ commutes with the modular group

$$\alpha_t^\phi \circ \Phi \;=\; \Phi \circ \alpha_t^\phi,\; t \in \mathbb{R}.$$

The commutation to modular group is also called Accardi-Cecchini condition in [32] for a study of quantum Bayes rule [62, 65, 63, 64].

For simplicity, we will consider a semifinite von Neumann algebra $\mathcal{M}$ equipped with a normal faithful semi-finite trace τ, but our discussion applies to general von Neumann algebras with proper interpretation of notations. In the tracial setting, we can write $\phi(x) = \tau(d_\phi x)$ using the density operator d_ϕ of ϕ. Then the modular automorphism group is given by

$$\alpha_t^\phi(x) \;=\; d_\phi^{-it} x d_\phi^{it},\; x \in \mathcal{M}, t \in \mathbb{R}.$$

Let $\Phi_* : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ be the pre-adjoint quantum channel via trace duality. The KMS-ϕ-symmetry is equivalent to

$$\Phi_*(d_\phi^{1/2} x d_\phi^{1/2}) \;=\; d_\phi^{1/2}\Phi(x) d_\phi^{1/2},\; \forall x \in \mathcal{M}. \tag{4.1}$$

For $1 \le p \le \infty$, the weighted L_p-space $L_p(\mathcal{M}, \phi)$ is the completion of $\mathcal{M}$ under the norm

$$\| x \|_{p,\phi} = \| d_\phi^{1/2p} x d_\phi^{1/2p} \|_p,$$

where $\| y \|_p = \tau(|y|^p)^{1/p}$ is the tracial p-norm. For $p = 2$, $L_p(\mathcal{M}, \phi)$ is a Hilbert space with KMS-inner product $\| x \|_{2,\phi}^2 = \langle \Delta^{\frac14} x\eta_\phi, \Delta^{\frac14} x\eta_\phi\rangle$. By equation (4.1), Φ is also a contraction on $L_1(\mathcal{M}, \phi)$, and hence a contraction on $L_p(\mathcal{M}, \phi)$ for all $1 \le p \le \infty$ by complex interpolation.

The lemma below is an analog of Proposition 2.2.

Proposition 4.2. *Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov map for a normal faithful state ϕ. Denote $\mathcal{N}$ as the multiplicative domain of Φ. Then*

i) *$\mathcal{N}$ is invariant under α_t^ϕ. Hence, there exists a ϕ-preserving normal conditional expectation $E : \mathcal{M} \to \mathcal{N}$.*
ii) *$\Phi|_\mathcal{N}$ is an involutive $*$-automorphism satisfying*

$$\Phi^2 \circ E = E \circ \Phi^2 = E,\; E \circ \Phi = \Phi \circ E. \tag{4.2}$$

Moreover, Φ^2 is a $\mathcal{N}$-bimodule map satisfying $\Phi^2(axb) = a\Phi^2(x)b$ for any $a, b \in \mathcal{N}$ and $x \in \mathcal{M}$.
iii) *Φ is an isometry on $L_2(\mathcal{N}, \phi)$. If, in addition,*

$$\| \Phi(\mathrm{id} - E) : L_2(\mathcal{M}, \phi) \to L_2(\mathcal{M}, \phi) \|_2 < 1,$$

then $E = \lim_n \Phi^{2n}$ as a map from $L_2(\mathcal{M}, \phi)$ to $L_2(\mathcal{M}, \phi)$.

Proof. It suffices to explain i). The rest follows similar as Proposition 2.2 (see also [31, Lemma 2.5] for the finite dimensional case). Indeed, since Φ commutes with α_t^ϕ, for $x \in \mathcal{N}$,

$$\begin{aligned}\Phi\big(\alpha_t^\phi(x)y\big) &= \Phi\big(\alpha_t^\phi(x\alpha_t^\phi \circ \alpha_{-t}^\phi(y))\big) = \alpha_t^\phi \circ \Phi\big(x\alpha_t^\phi \circ \alpha_{-t}^\phi(y)\big)\\ &= \alpha_t^\phi\Big(\Phi(x)\Phi(\alpha_t^\phi \circ \alpha_{-t}^\phi(y))\Big) = \alpha_t^\phi \circ \Phi(x)\alpha_t^\phi \circ \Phi \circ \alpha_{-t}^\phi(y) = \Phi(\alpha_t^\phi(x))\Phi(y).\end{aligned}$$

Multiplicativity on the other side is similar, implying $\alpha_t^\phi(x) \in \mathcal{N}$. By Takesaki's theorem [72], there exists ϕ-preserving conditional expectation satisfying the defining property

$$\phi(xy) = \phi(xE(y)) \; \forall \, x \in \mathcal{N}, y \in \mathcal{M},$$

from which the GNS-ϕ-symmetry follows. □

4.2. Haagerup's reduction

A von Neumann algebra $\mathcal{M}$ is called type III if it does not admit a nontrivial semifinite trace. We briefly review the basics of Haagerup's construction and refer to [36] for more details. The key idea is to consider the additive subgroup $G = \bigcup_{n\in\mathbb{N}} 2^{-n}\mathbb{Z} \subset \mathbb{R}$ of the automorphism group. Let $\mathcal{M} \subset B(H)$ be a von Neumann algebra and ϕ be a normal faithful state. One can define the crossed product by the action $\alpha^\phi : G \curvearrowright \mathcal{M}$

$$\hat{\mathcal{M}} \;=\; \mathcal{M} \rtimes_{\alpha^\phi} G.$$

$\hat{\mathcal{M}}$ can be considered as the von Neumann subalgebra $\hat{\mathcal{M}} = \{\pi(\mathcal{M}), \lambda(G)\}'' \subset \mathcal{M}\overline{\otimes}B(\ell_2(G))$ generated by the embeddings

$$\begin{aligned} \pi : \mathcal{M} \to \mathcal{M} \rtimes_{\alpha^\phi} G, \pi(a) \;&=\; \sum_g \alpha_{g^{-1}}(a) \otimes |g\rangle\langle g| \\ \lambda : G \to \mathcal{M} \rtimes_{\alpha^\phi} G, \lambda(g)(|x\rangle \otimes |h\rangle) &= |x\rangle \otimes |gh\rangle \;,\; \forall \, |x\rangle \in H, |h\rangle \in \ell_2(G). \end{aligned} \tag{4.3}$$

Basically, π is the transference homomorphism $\mathcal{M} \to \ell_\infty(G, \mathcal{M})$, and λ is the left regular representation on $\ell_2(G)$. The set of finite sums $\{\sum_g a_g\lambda(g) \mid a_g \in \mathcal{M}\} \subset \hat{\mathcal{M}}$ forms a dense w^*-subalgebra of $\hat{\mathcal{M}}$. In the following, we identify $\mathcal{M}$ with $\pi(\mathcal{M}) \subset \hat{M}$ (resp. a with $\pi(a)$) and view $\mathcal{M} \subset \hat{\mathcal{M}}$ as a subalgebra. The state ϕ admits a natural extension as a normal faithful state on $\hat{\mathcal{M}}$

$$\hat{\phi}(\sum_g a_g\lambda(g)) \;=\; \phi(a_0).$$

Moreover,

$$E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M} \;,\; E_{\mathcal{M}}(\sum_g a_g\lambda(g)) = a_0$$

is the canonical normal conditional expectation such that $\phi \circ E_{\mathcal{M}} \;=\; \hat{\phi}$.

The main object in Haagerup's construction is an increasing family of subalgebras

$$\mathcal{M}_n \;=\; \hat{\mathcal{M}}_{\psi_n} := \{x \in \hat{\mathcal{M}} \mid \alpha_t^{\psi_n}(x) = x \;,\; \forall \, t \in \mathbb{R}\},$$

given by the centralizer algebra $\hat{\mathcal{M}}_{\psi_n}$ for a suitable family of states ψ_n so that $\bigcup_n \mathcal{M}_n$ is w^*-dense in $\hat{\mathcal{M}}$. The state ψ_n is defined via a Radon-Nikodym density w.r.t to $\hat{\phi}$

$$\psi_n(x) \;=\; \hat{\phi}(e^{-a_n}x) \;,\; a_n \;=\; -i2^n\mathrm{Log}(\lambda(2^{-n})).$$

Here, Log is the principal branch of the logarithmic function with $0 \leq \mathrm{Log}(z) < 2\pi$. Each subalgebra $\mathcal{M}_n$ contains $\lambda(G)$, and there exists normal conditional expectation $E_{\mathcal{M}_n} : \hat{\mathcal{M}} \to \mathcal{M}_n$. Indeed, by the definition of ψ_n, the modular group $\alpha_t^{\psi_n}$ is 2^{-n} periodic. The explicit form (see [36, Lemma 2.3]) is given by

$$E_{\mathcal{M}_n} = 2^n \int_0^{2^{-n}} \alpha_t^{\psi_n} dt.$$

The normalized state $\tau_n = \frac{\psi_n}{\psi_n(1)}$ is a normalized trace on $\mathcal{M}_n$. The key properties of $\mathcal{M}_n$ are summarized in [36, Theorem 2.1 & Lemma 2.7], which we state below.

Theorem 4.3. *With above notations, $\mathcal{M}_n$ is an increasing family of von Neumann subalgebras satisfying the following properties*

(1) *Each $(\mathcal{M}_n, \tau_n)$ is a finite von Neumann algebra.*
(2) *$\bigcup_{n\geq 1} \mathcal{M}_n$ is weak*-dense in $\hat{\mathcal{M}}$.*
(3) *There exists a $\hat{\phi}$-preserving normal faithful conditional expectation $E_{\mathcal{M}_n} : \hat{\mathcal{M}} \to \mathcal{M}_n$ such that*

$$\hat{\phi} \circ E_{\mathcal{M}_n} = \hat{\phi}\ ,\ \alpha_t^{\hat{\phi}} \circ E_{\mathcal{M}_n} = E_{\mathcal{M}_n} \circ \alpha_t^{\hat{\phi}}.$$

Moreover, $E_{\mathcal{M}_n}(x) \to x$ in σ-strong topology for any $x \in \hat{\mathcal{M}}$.

We now look at the Haagerup reduction on the states. For a state $\rho \in S(\mathcal{M})$, $\hat{\rho} = \rho \circ E_{\mathcal{M}}$ is the canonical extension on $\hat{\mathcal{M}}$. We denote $\rho_n := \hat{\rho}|_{\mathcal{M}_n} \in \mathcal{M}_{n,*}$ as the restriction state of $\hat{\rho}$ on the subalgebra $\mathcal{M}_n \subset \hat{\mathcal{M}}$. Note that the predual $\mathcal{M}_{n,*}$ can be viewed as a subspace of $\hat{\mathcal{M}}_*$ via the embedding

$$\iota_{n,*} : \hat{\mathcal{M}}_{n,*} \to \hat{\mathcal{M}}_*\ , \iota_{n,*}(\omega) = \omega \circ E_{\mathcal{M}_n}.$$

Via this identification, $\rho_n = \hat{\rho}|_{\mathcal{M}_n} \circ E_{\mathcal{M}_n} = \hat{\rho} \circ E_{\mathcal{M}_n} = E_{\mathcal{M}_n,*}(\hat{\rho}) \in \hat{\mathcal{M}}_*$. Moreover, by the weak*-density of the family $\mathcal{M}_n$, $\rho_n \to \hat{\rho}$ converges in the weak topology. An immediate consequence is the following approximation of relative entropy.

Lemma 4.4. *Let ρ and σ be two normal states of $\mathcal{M}$. Then*

$$D(\rho||\sigma) \ = \ D(\hat{\rho}||\hat{\sigma}) = \lim_{n\to\infty} D(\rho_n||\sigma_n).$$

Proof. Let $\iota : \mathcal{M} \subset \hat{\mathcal{M}}$ be the inclusion map. Because $\hat{\rho} = \rho \circ E_{\mathcal{M}}$ is an extension of ρ, $\iota_*(\hat{\rho}) = \hat{\rho}|_{\mathcal{M}} = \rho$, and similarly for σ. Both $\iota : \mathcal{M} \to \hat{\mathcal{M}}$ and $E_{\mathcal{M}} : \hat{\mathcal{M}} \to \mathcal{M}$ are quantum Markov maps. Then by the data processing inequality,

$$D(\rho||\sigma) = D(\iota_*(\hat{\rho})||\iota_*(\hat{\sigma})) \leq D(\hat{\rho}||\hat{\sigma}) = D(E_{\mathcal{M},*}(\rho)||E_{\mathcal{M},*}(\sigma)) \leq D(\rho||\sigma).$$

Thus, $D(\rho||\sigma) = D(\hat{\rho}||\hat{\sigma})$. As for the limit, we have

$$\begin{aligned} D(\hat{\rho}\|\hat{\sigma}) &\leq \liminf_n D(\rho_n\|\sigma_n) \\ &= \liminf_n D(E_{\mathcal{M}_n,*}(\rho_n)||E_{\mathcal{M}_n,*}(\sigma_{n,*})) \\ &\leq D(\hat{\rho}||\hat{\sigma}), \end{aligned}$$

where the equality follows from the lower semi-continuity of relative entropy (see, for example, [37, Theorem 2.7]). The second inequality is another use the data processing inequality. ◻

We shall also apply the Haagerup's reduction on GNS-symmetric maps. Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov map. Its canonical extension map

$$\hat{\Phi} : \hat{\mathcal{M}} \to \hat{\mathcal{M}}\ ,\ \hat{\Phi}(\sum_g a_g \lambda(g)) \ = \ \sum_g \Phi(a_g)\lambda(g)$$

is also a GNS-$\hat{\phi}$-symmetric quantum Markov map. Indeed, $\hat{\Phi} = \Phi \otimes \mathrm{id}_{B(\ell_2(G))}\,|_{\hat{\mathcal{M}}}$ is the restriction of $\Phi \otimes \mathrm{id}_{B(\ell_2(G))}$ on $\hat{\mathcal{M}} \subset \mathcal{M}\overline{\otimes}B(\ell_2(G))$. It is clear that $\hat{\Phi}$ has the multiplicative domain

$$\hat{\mathcal{N}} := \mathcal{N} \rtimes_{\alpha^{\phi}} G, \tag{4.4}$$

where $\mathcal{N}$ is the multiplicative domain of Φ. In particular, this crossed product is well defined because $\alpha_t^{\phi}(\mathcal{N}) = \mathcal{N}$. Moreover, the $\hat{\phi}$-preserving conditional expectation $\hat{E} : \hat{\mathcal{M}} \to \hat{\mathcal{N}}$ is nothing but the canonical extension of $E : \mathcal{M} \to \mathcal{N}$.

Recall that we write $E_{\mathcal{M}}$ and E_n as the normal conditional expectations from $\hat{\mathcal{M}}$ onto $\mathcal{M}$ and $\mathcal{M}_n$, respectively. The next lemma shows that the extension $\hat{\Phi}$ is well compatible with the approximation family $\mathcal{M}_n$.

Lemma 4.5. *Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov map. With the notations above,*

i) *$\hat{\Phi}$ commutes with $E_{\mathcal{M}}, \hat{E}$ and $E_{\mathcal{M}_n}$. In particular, $\hat{\Phi}(\mathcal{M}_n) \subset \mathcal{M}_n$.*
ii) *The restriction $\Phi_n = \hat{\Phi}|_{\mathcal{M}_n}$ is a normal unital completely positive map symmetric with respect to the tracial state τ_n.*
iii) *Let $\mathcal{N}_n \subset \mathcal{M}_n$ be the multiplicative domain for Φ_n. Then the restriction map $E_n := \hat{E}|_{\mathcal{M}_n} : \mathcal{M}_n \to \mathcal{N}_n$ is the τ_n-preserving conditional expectation.*

Proof. The relation $\hat{\Phi} \circ E_{\mathcal{M}} = E_{\mathcal{M}} \circ \hat{\Phi}$ is clear from the definition of $\hat{\Phi}$, and $\hat{\Phi} \circ \hat{E} = \hat{E} \circ \hat{\Phi}$ follows from Lemma 4.2. Recall that $\psi_n(x) = \hat{\phi}(e^{-a_n}x)$ with density operator $e^{-a_n} \in \lambda(G)''$ and $\lambda(G)$ is in the centralizer of $\hat{\phi}$ [36, Lemma 2.3]. Then

$$\alpha_t^{\psi_n} = u(t)^* \alpha_t^{\hat{\phi}} u(t) = \mathrm{ad}_{u(t)} \alpha_t^{\hat{\phi}}$$

for the unitary $u(t) = e^{-ita_n}$. Note that $\hat{\Phi}$ commutes with $\alpha_t^{\hat{\phi}}$ by GNS-$\hat{\phi}$-symmetry, and also commutes with $\mathrm{ad}_{u(t)}$ because $u(t) \in \lambda(G)''$ is in $\hat{\Phi}$'s multiplicative domain. Thus, $\hat{\Phi}$ commutes with $\alpha_t^{\psi_n}$ and hence the conditional expectation $E_{\mathcal{M}_n} = 2^{-n} \int_0^{2^{-n}} \alpha_t^{\psi_n}$. This proves i).

For ii), we note that for $x, y \in \mathcal{M}_n$,

$$\psi_n(x\Phi_n(y)) = \hat{\phi}(e^{-a_n}x\hat{\Phi}(y)) = \hat{\phi}(\hat{\Phi}(e^{-a_n}x)y) = \hat{\phi}(e^{-a_n}\hat{\Phi}(x)y) = \psi_n(\Phi_n(x)y),$$

where we use the fact that $\hat{\Phi}$ is GNS-$\hat{\phi}$-symmetric and $e^{-a_n} \in \lambda(G)''$ is in the fixed point subspace of $\hat{\Phi}$. Finally, iii) follows from applying i) and ii) to $\hat{E}$. □

To summarize the lemma above, we have the following commuting diagrams:

$$\begin{tikzcd}
\mathcal{M} \arrow[d, "\Phi"'] & \hat{\mathcal{M}} \arrow[l, "E_{\mathcal{M}}"'] \arrow[r, "E_{\mathcal{M}_n}"] \arrow[d, "\hat{\Phi}"] & \mathcal{M}_n \arrow[d, "\Phi_n"] \\
\mathcal{M} & \hat{\mathcal{M}} \arrow[l, "E_{\mathcal{M}}"] \arrow[r, "E_{\mathcal{M}_n}"'] & \mathcal{M}_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M} \arrow[d, "E"'] & \hat{\mathcal{M}} \arrow[l, "E_{\mathcal{M}}"'] \arrow[r, "E_{\mathcal{M}_n}"] \arrow[d, "\hat{E}"] & \mathcal{M}_n \arrow[d, "E_n"] \\
\mathcal{N} & \hat{\mathcal{N}} \arrow[l, "E_{\mathcal{N}}"] \arrow[r, "E_{\mathcal{N}_n}"'] & \mathcal{N}_n
\end{tikzcd}$$

Figure 1. *Haagerup reduction of quantum Markov map and conditional expectation.*

Basically, Φ_n is a family of trace symmetric channels approximating $\hat{\Phi}$, which is in turn a natural extension of Φ. The same picture holds for the conditional expectations $E_n, \hat{E}$ and E.

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4.3. Entropy contraction

We shall now discuss the entropy contraction of GNS-ϕ-symmetric channels. The first step is to extend the entropy difference Lemma 2.1. Define the state space that is bounded with respect to ϕ,

$$S_B(\mathcal{M}, \phi) = \{\rho \in S(\mathcal{M}) \mid c^{-1}\phi \leq \rho \leq c\phi\ ,\ \text{for some } c > 0\}.$$

For all $\rho \in S_B(\mathcal{M}, \phi)$, $D(\rho||\phi) < \infty$ is finite. Such $S_B(\mathcal{M}, \phi)$ is a dense subset of $S(\mathcal{M})$ because for any ρ and $0 < \varepsilon < 1$, $\rho_\varepsilon = (1-\varepsilon)\rho + \varepsilon\phi \in S_B(\mathcal{M})$. For $\rho \in S_B(\mathcal{M})$, we define the entropy difference for a GNS-ϕ-symmetric quantum channel Φ_* as

$$D_{\Phi_*}(\rho) := D(\rho||\phi) - D(\Phi_*(\rho)||\phi).$$

By data processing inequality and $\Phi_*(\phi) = \phi$, $D_{\Phi_*}(\rho) \geq 0$. In the trace symmetric case, $D_{\Phi_*}(\rho) = D(\rho||1) - D(\Phi_*(\rho)||1) = H(\rho) - H(\Phi_*(\rho))$ as in Section 2. Let E be the conditional expectation onto the multiplicative domain of Φ. By the chain rule [66, Theorem 2] that for any E invariant state $\psi \circ E = \psi$,

$$D(\rho||\psi) = D(\rho||E_*(\rho)) + D(E_*(\rho)||\psi),$$

we have the alternative expressions $D_{\Phi_*}(\rho) = D(\rho||E_*(\rho)) - D(\Phi_*(\rho)||\Phi_*E_*(\rho))$, where we used the property $\Phi_*E_* = E_*\Phi_*$ in Proposition 4.2.

Lemma 4.6. *Let Φ_* be a GNS-ϕ-symmetric quantum channel. For any state $\rho, \omega \in S_b(\mathcal{M}, \phi)$,*

$$D(\rho||\Phi_*^2(\omega)) \ \leq\ D_{\Phi_*}(\rho) + D(\rho||\omega).$$

Proof. Recall that we use $\rho_n = \hat{\rho}|_{\mathcal{M}_n} = E_{\mathcal{M}_n,*}(\hat{\rho})$ and $\omega_n = \hat{\omega}|_{\mathcal{M}_n} = E_{\mathcal{M}_n,*}(\hat{\omega})$ as the restriction states on finite von Neumann algebra $\mathcal{M}_n \subset \hat{\mathcal{M}}$ obtained from the Haagerup reduction. By Lemma 4.5, we know that $\Phi_n = \hat{\Phi}|_{\mathcal{M}_n}$ is a quantum Markov map symmetric with respect to the tracial state τ_n. Thus, by Lemma 2.1 in the tracial case,

$$D(\rho_n||\Phi_n^2(\omega_n)) \ \leq\ D_{\Phi_n}(\rho_n) + D(\rho_n||\omega_n),$$

where we identify $\Phi_n = \Phi_{n,*}$ by trace symmetry. Here, since $\Phi_n = \Phi|_{\mathcal{M}_n}$ is GNS-symmetric to $\phi_n = \phi|_{\mathcal{M}_n}$,

$$D_{\Phi_n}(\rho_n) = D(\rho_n||\tau_n) - D(\Phi_n(\rho_n)||\tau_n) = D(\rho_n||\phi_n) - D(\Phi_n(\rho_n)||\phi_n).$$

By the definitions of Φ_n and ρ_n, and the hat "^" notation for states on $\hat{\mathcal{M}}$,

$$\Phi_n(\rho_n) = \hat{\rho}|_{\mathcal{M}_n} \circ \Phi_n = \hat{\rho} \circ \hat{\Phi}|_{\mathcal{M}_n} = \widehat{\Phi_*(\rho)}|_{\mathcal{M}_n} = \Phi_*(\rho)_n\ ,$$
$$\Phi_n^2(\rho_n) = \Phi_n(\Phi_*(\rho)_n) = \Phi_*^2(\rho)_n.$$

Then by Lemma 4.4, we can approximate every entropic term

$$\begin{aligned}\lim_n D(\rho_n||(\Phi_n)^2(\omega_n)) &= \lim_n D(\rho_n||\Phi_*^2(\omega)_n) = D(\rho||\Phi_*^2(\omega))\ ,\\ \lim_n D_{\Phi_n}(\rho_n) &= \lim_n D(\rho_n||\phi_n) - D(\Phi_n(\rho_n)||\phi_n)\\ &= \lim_n D(\rho_n||\phi_n) - D(\Phi_*(\rho)_n||\phi_n) = D_{\Phi_*}(\rho)\ ,\\ \lim_n D(\rho_n||\omega_n) &= D(\rho||\omega).\end{aligned}$$

□

The next lemma shows the CB-return time is also compatible with Haargerup reduction.

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Lemma 4.7. *Let $\Psi : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov map and E be the conditional expectation on its multiplicative domain. Suppose*

$$(1-\varepsilon)E \le_{cp} \Psi \le_{cp} (1+\varepsilon)E. \tag{4.5}$$

Then for all $n \in \mathbb{N}$,

$$(1-\varepsilon)E_n \le_{cp} \Psi_n \le_{cp} (1+\varepsilon)E_n.$$

Moreover, if $0.9E \le_{cp} \Psi \le_{cp} 1.1E$ and $\Psi \circ E = E$, then for any $\rho \in S_B(\mathcal{M}, \phi)$,

$$\frac{1}{2}D(\rho||E_*(\rho)) \le D(\rho||\Psi_*(\rho)).$$

Proof. The CP order inequality follows from the fact that both maps E_n and Ψ_n are the restriction of $E \otimes \mathrm{id}$ and $\Psi \otimes \mathrm{id}$ on the subalgebra $\mathcal{M}_n \subset \hat{\mathcal{M}} \subset \mathcal{M}\overline{\otimes}B(\ell_2(G))$. Then the entropy inequality can be obtained by the tracial case Lemma 2.3 and approximation as in Lemma 4.6. □

We then extend the entropy contraction to the GNS-symmetric case.

Theorem 4.8. *Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov map and E be the ϕ-preserving conditional expectation onto its multiplicative domain $\mathcal{N}$. Define*

$$k_{cb}(\Phi) \ = \ \inf\{k \in \mathbb{N}^+ \mid 0.9E \le_{cp} \Phi^{2k} \le_{cp} 1.1E\ \}.$$

Then, for any σ-finite von Neumann algebra $\mathcal{Q}$, state $\rho \in S(\mathcal{M}\overline{\otimes}\mathcal{Q})$,

$$D(\Phi_* \otimes \mathrm{id}_{\mathcal{Q}}(\rho)||(\Phi_* \circ E_*) \otimes \mathrm{id}_{\mathcal{Q}}(\rho)) \ \le \ \Big(1 - \frac{1}{2k_{cb}(\Phi)}\Big)D(\rho||E_* \otimes \mathrm{id}_{\mathcal{Q}}(\rho)).$$

Proof. For $\rho \in S_B(\mathcal{M}, \phi)$, the proof is same as the tracial case Theorem 2.5 by using Lemma 4.6 and Lemma 4.7 above. The general case $\rho \in S(\mathcal{M})$ can be approximated by $\rho_\varepsilon = (1-\varepsilon)\rho + \varepsilon\phi$. □

Recall that in finite dimensions, the MLSI is defined as the supremum of α such that

$$2\alpha D(\rho||E_*(\rho)) \le I_L(\rho) := \tau(L_*(\rho)(\ln\rho - \ln\phi)).$$

The right-hand side $I_L(\rho)$ is the entropy production, and the equivalence to entropy decay relies on the de Bruijn identity

$$I_L(\rho) = -\frac{d}{dt}D(T_*(\rho)||E_*(\rho))|_{t=0}. \tag{4.6}$$

In infinite dimensions, the de Bruijn identity (4.6) is less justified even in $B(H)$ with $\dim(H) = +\infty$ (see discussions in [40, 44]). To avoid this issue, we define the MLSI on Type III von Neumann algebra as follows.

Definition 4.9. For a GNS-ϕ-symmetric quantum Markov semigroup $T_t = e^{-tL} : \mathcal{M} \to \mathcal{M}$, we define the modified log-Sobolev (MLSI) constant $\alpha_1(L)$ as the largest constant α such that

$$D(T_{t,*}(\rho)||E_*(\rho)) \ \le \ e^{-2\alpha t}D(\rho||E_*(\rho))\ ,\ \forall\rho \in S(\mathcal{M}), \tag{4.7}$$

where E is the ϕ-preserving conditional expectation onto the fixed point subalgebra $\mathcal{N}$. The complete MLSI constant is then defined as $\alpha_c(L) := \sup_{\mathcal{Q}} \alpha(L \otimes \mathrm{id}_{\mathcal{Q}})$, where the supremum is over all σ-finite von Neumann algebra $\mathcal{Q}$.

This definition of MLSI also does not depend on any choice of reference state ϕ (see Lemma 4.16). With this definition, we obtain the first half of Theorem 1.1, which is restated below.

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Theorem 4.10. *Let $T_t = e^{-tL} : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov semigroup. Denote $t_{cb} = \inf\{t > 0 \mid 0.9E \leq_{cp} T_t \leq_{cp} 1.1E\}$. Then*

$$\alpha_1 \geq \alpha_c \geq \frac{1}{2t_{cb}}.$$

Namely, for any σ-finite von Neumann algebra $\mathcal{Q}$ and state $\rho \in S(\mathcal{M}\overline{\otimes}\mathcal{Q})$, we have the exponential decay of relative entropy

$$D(T_{t,*} \otimes \mathrm{id}_{\mathcal{Q}}(\rho)||E_* \otimes \mathrm{id}_{\mathcal{Q}}(\rho)) \leq e^{-\frac{t}{t_{cb}}} D(\rho||E_* \otimes \mathrm{id}_{\mathcal{Q}}(\rho)) , \ t \geq 0.$$

Proof. This can be approximated using the tracial case Theorem 2.5 as Lemma 4.6 above. □

Remark 4.11. In the above Haagerup's reduction, both $\hat{\Phi}$ and Φ_n are always non-ergodic even given Φ is ergodic. From this point of view, our consideration for non-ergodic cases is essential even for ergodic Φ. It also indicates that Haagerup's reduction does not work for LSI/hypercontractivity.

As we have seen in Proposition 3.7 for the tracial case, a combination of heat kernel estimates and spectral gap allows us to bound CB return time. The same analysis remains valid in the GNS-ϕ-symmetric case. For $1 \leq p \leq \infty$, we define the ϕ-weighted conditional $L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)$ space as the completion of $\mathcal{M}$ under the norm

$$\| x \|_{L_\infty^p(\mathcal{N}\subset\mathcal{M},\phi)} = \sup\{\| axb \|_{p,\phi} \ \mid a, b \in \mathcal{N}, \ \| aa^* \|_{p,\phi} = \| b^*b \|_{p,\phi} = 1 \}.$$

For a GNS-symmetric $\mathcal{N}$-bimodule map $\Psi : \mathcal{M} \to \mathcal{M}$, the equivalence in Proposition 3.4 also holds,

$$(1-\varepsilon)E \leq_{cp} \Psi \leq_{cp} (1+\varepsilon)E \iff \| \Psi - E : L_\infty^1(\mathcal{N} \subset \mathcal{M}, \phi) \to L_\infty(\mathcal{M}) \|_{cb} \leq \varepsilon. \tag{4.8}$$

Based on that, we have an analog of Proposition 3.7.

Proposition 4.12. *Let $T_t : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov semigroup and $E : \mathcal{M} \to \mathcal{N}$ be the ϕ-preserving conditional expectation onto the fixed point space. Suppose*

i) *the λ-Poincaré inequality that $\| T_t - E : L_2(\mathcal{M}, \phi) \to L_2(\mathcal{M}, \phi) \| \leq e^{-\lambda t}$, $\forall t \geq 0$;*
ii) *there exists t_0 such that $\| T_{t_0} : L_\infty^1(\mathcal{N} \subset \mathcal{M}, \phi) \to L_\infty(\mathcal{M}) \|_{cb} \leq C_0$.*

Then $t_{cb} \leq \frac{1}{\lambda} \ln(10C_0) + t_0$. In particular, if $C_{cb}(E) < \infty$, $t_{cb} \leq \frac{1}{\lambda} \ln(10C_{cb}(E))$.

Proof. The argument is similar to the tracial cases by using the property of $L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)$ for general von Neumann algebra established in [42]. See also [6, Section 5] for the argument in finite dimensional GNS-symmetric cases. □

4.4. Applications to finite quantum Markov chains

Let $T_t = e^{-Lt} : \mathbb{M}_d \to \mathbb{M}_d$ be a quantum Markov semigroup on matrix algebra $\mathbb{M}_d$. Its generator L admits the following Lindbladian form ([33, 50]):

$$L(x) = i[h, x] + \sum_j \gamma_j (V_j^*[x, V_j] + [V_j^*, x]V_j),$$

where $h, V_j \in \mathbb{M}_d$ and $h = h^*$ is Hermitian. When T_t is GNS-symmetric, one has the following simplified form [1, 45] that

$$L(x) = \sum_j e^{-w_j/2}\Big(V_j^*[x, V_j] + [V_j^*, x]V_j\Big),$$

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where $\{V_j\} = \{V_j\}^*$ is an orthogonal set with respect to trace inner product and the eigenvector of modular group $\alpha_t^{\phi}(V_j) = e^{-iw_j t} V_j$. In finite dimensions, the completely Pimsner-Popa index $C_{cb}(E)$ is always finite. Combining Theorem 4.10 and Proposition 4.12, we obtain the second half of Theorem 1.1 restated as below.

Corollary 4.13. *For finite dimensional GNS-symmetric quantum Markov semigroups,*

$$\alpha_1 \geq \alpha_c \geq \frac{\lambda}{2\ln(10C_{cb}(E))}. \tag{4.9}$$

Corollary 4.13 improves the bound $\alpha_c \geq \frac{\lambda}{2C_{cb}(E)}$ in the previous work of Gao and Rouzé [31].

Remark 4.14. In the ergodic case $\mathcal{N} = \mathbb{C}1$, the conditional expectation $E_\phi(x) = \phi(x)1$ has index

$$C(E_\phi) = \| \phi^{-1} \|_\infty , C_{cb}(E_\phi) \leq \| \phi^{-1} \|_\infty^2 .$$

The above bound (4.9) gives

$$\alpha_1 \geq \alpha_c \geq \frac{\lambda}{2\ln 10 + 4\ln \| \phi^{-1} \|_\infty}.$$

This can be compared to the bound

$$\alpha_1 \geq \alpha_2 \geq \frac{2(1 - \frac{2}{\|\phi^{-1}\|_\infty})\lambda}{\ln(\| \phi^{-1} \|_\infty - 1)} \tag{4.10}$$

proved by Diaconis and Saloff-Coste [23] for symmetric classical Markov semigroups. In the quantum case, it is only obtained for unital semigroups [43] and $d = 2$ [9]. For both classical and quantum depolarizing semigroups $L(x) = x - \phi(x)1$, this bound is known to be optimal for α_2, which lower bounds α_1. Our results gives a general $\mathcal{O}(\frac{\lambda}{\|\phi^{-1}\|_\infty})$ lower bound for α_1 for non-ergodic cases and also the complete constant α_c.

Remark 4.15. The Corollary 3.10 shows that the CMLSI constant α_c for a classical Markov semigroup is lower bounded by LSI constant α_2 up to a $O(\log\log \| \mu^{-1} \|_\infty)$ term. This argument does not work for Quantum Markov semigroup $T_t : \mathbb{M}_d \to \mathbb{M}_d$ on matrix algebras, although (3.15) remains valid for ergodic quantum Markov semigroups. The difference is that for matrix algebra, the bounded return time

$$t_b(e^{-2}) := \frac{1}{2}\inf\{t > 0 \mid \ \|T_t - E : L_1(\mathbb{M}_d, \phi) \to L_\infty(\mathbb{M}_d) \| < 1/e^2\}$$

and the CB return time of completely bounded norm

$$t_{cb} = \inf\{t > 0 \mid \ \| T_t - E_\mu : L_1(\mathbb{M}_d, \phi) \to L_\infty(\mathbb{M}_d) \|_{cb} < 1/10 \ \}$$

are quite different. In the classical setting, we used the fact

$$\| T : L_1(\Omega) \to L_\infty(\Omega) \| = \| T : L_1(\Omega) \to L_\infty(\Omega) \|_{cb} .$$

So the $t_b(e^{-2})$ and $t_{cb}(0.1)$ are comparable by absolute constants. In the noncommutative setting, we only have

$$\| T_t - E_\mu : L_1(\mathbb{M}_d, \phi) \to L_\infty(\mathbb{M}_d) \|_{cb} \leq d \, \| T_t - E_\mu : L_1(\mathbb{M}_d) \to L_\infty(\mathbb{M}_d) \| .$$

In the trace symmetric case, $\| \mu^{-1} \|_\infty = d$ and $t_{cb}(0.1) \le \frac{3}{2} t_b(e^{-2}) + \ln d$,

$$\alpha_c \ge \frac{1}{2t_{cb}(0.1)} \ge \frac{1}{3t_b(e^{-2}) + 2\ln d} \sim O(\frac{\alpha_2}{\ln d}),$$

which is worse than the lower bound in the previous remark as $\alpha_2 \le \lambda$.

4.5. Independence of invariant state

The next lemma shows that the GNS-symmetry is also independent of the choice of invariant state ϕ.

Lemma 4.16. *Let $T : \mathcal{M} \to \mathcal{M}$ be a GNS-ϕ-symmetric quantum Markov map for a normal faithful state ϕ. Denote $E : \mathcal{M} \to \mathcal{N}$ as the ϕ-preserving conditional expectation onto the multiplicative domain. Suppose ψ is an another normal faithful state invariant under E (i.e., $\psi \circ E = \psi$). Then $T : \mathcal{M} \to \mathcal{M}$ is also GNS-ψ-symmetric.*

Proof. Without loss of generality, we assume $\psi \le C\phi$ for some $C > 0$. We first view them as the states on the subalgebra $\mathcal{N}$ by restriction. By [73, Theorem 3.17], there exists $h \in \mathcal{N}$ such that

$$\psi(x) = \phi(h^* x h) \ , \ \forall x \in \mathcal{N}.$$

This identity actually also holds for $y \in \mathcal{M}$. Indeed, because of $\phi \circ E = \phi$ and $\psi \circ E = \psi$,

$$\psi(y) = \psi(E(y)) = \phi(h^* E(y) h) = \phi(E(h^* y h)) = \phi(h^* y h) \ , \ \forall y \in \mathcal{M}.$$

Moreover, one can replace h by $T(h)$, because

$$\psi(x) = \psi(T(x)) = \phi(h^* T(x) h) = \phi \circ T(T(h^*) x T(h)) = \phi(T(h^*) x T(h)),$$

where we use the fact that $T^2(h) = h$. Thus, the GNS-symmetry with respect to ψ follows that for $x, y \in \mathcal{M}$,

$$\begin{aligned}\psi(xT(y)) &= \phi(h^* x T(y) h) = \phi(h^* x T(yT(h))) = \phi(T(h^* x) y T(h)) = \phi(T(h^*) T(x) y T(h)) \\ &= \psi(T(x) y),\end{aligned}$$

where we used the multiplicative property of $T(axb) = T(a)T(x)T(b)$ for $a, b \in \mathcal{N}$. The general case can be obtained via $\psi_\varepsilon = (1-\varepsilon)\psi + \varepsilon\phi$. □

We remark that if one has convergence $\lim_n \Phi^{2n} = E$ in L_2-norm, the above E-invariant condition $\phi \circ E = \phi$ can be replaced by $\phi = \Phi^2 \circ \phi$.

Note that the left-hand side of (4.8) only relies on complete positivity. Indeed, the $L^1_\infty(\mathcal{N} \subset \mathcal{M}, \phi)$ norm at the right hand-side is also independent of the choice of the invariant state $\phi = \phi \circ E$.

Lemma 4.17. *Let ϕ be a normal faithful state and $E : \mathcal{M} \to \mathcal{N}$ be a ϕ-preserving conditional expectation. Suppose $\psi = \psi \circ E$ is another normal faithful state preserved by E. Then,*

$$\| x \|_{L^p_\infty(\mathcal{N} \subset \mathcal{M}, \phi)} = \| x \|_{L^p_\infty(\mathcal{N} \subset \mathcal{M}, \psi)} \ , \ \forall x \in \mathcal{M}.$$

The identity extends to all $x \in L^p_\infty(\mathcal{N} \subset \mathcal{M}, \phi)$.

Proof. Note that if both ϕ and ψ are E invariant, then $d_\psi^{-\frac{1}{2p}} d_\phi^{\frac{1}{2p}}$ is affiliated to $\mathcal{N}$. Indeed, as argued in Lemma 4.16, if $\psi \le C\phi$, then $d_\psi = h d_\phi h^*$ for some $h \in \mathcal{N}$, and the general case follows from approximation $\psi \le \frac{1}{\varepsilon}((1-\varepsilon)\phi + \varepsilon\psi)$. Then we have

$$\| aa^* \|_{\phi,p} = \| d_\phi^{\frac{1}{2p}} aa^* d_\phi^{\frac{1}{2p}} \|_p = \| d_\psi^{-\frac{1}{2p}} d_\phi^{\frac{1}{2p}} aa^* d_\phi^{\frac{1}{2p}} d_\psi^{-\frac{1}{2p}} \|_{\psi,2p} \ .$$

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Denote $a_1 = d_\psi^{-\frac{1}{2p}} d_\phi^{\frac{1}{2p}} a$ and $b_1 = bd_\phi^{\frac{1}{2p}} d_\psi^{-\frac{1}{2p}}$. For $x \in \mathcal{M}$,

$$\begin{aligned}\|x\|_{L^p_\infty(\mathcal{N}\subset\mathcal{M},\phi)} &= \sup_{\|aa^*\|_{\phi,2p}=\|b^*b\|_{\phi,2p}=1} \|axb\|_{\phi,p} = \sup_{\|aa^*\|_{\phi,p}=\|b^*b\|_{\phi,p}=1} \|d_\psi^{-\frac{1}{2p}} d_\phi^{\frac{1}{2p}} axbd_\phi^{\frac{1}{2p}} d_\psi^{-\frac{1}{2p}}\|_{\psi,p} \\ &= \sup_{\|a_1a_1^*\|_{2p,\psi}=\|b_1b_1^*\|_{2p,\psi}=1} \|a_1xb_1\|_{\psi,p} = \|x\|_{L^p_\infty(\mathcal{N}\subset\mathcal{M},\psi)},\end{aligned}$$

where the supremum are for $a, b \in \mathcal{N}$. □

Remark 4.18. For finite $\mathcal{M}$, one particular invariant state of E used in [7, 6] is $\phi_{\mathrm{tr}} = E_*(1)$. This state is convenient because $\phi_{\mathrm{tr}}|_{\mathcal{N}}$ is a trace. Then by Lemma 4.17, we have

$$\|x\|_{L^p_\infty(\mathcal{N}\subset\mathcal{M},\phi)} = \|x\|_{L^p_\infty(\mathcal{N}\subset\mathcal{M},\phi_{\mathrm{tr}})} = \sup\{\|axb\|_{p,\phi_{\mathrm{tr}}} \mid a, b \in \mathcal{N}, \ \|a\|_{p,\phi_{\mathrm{tr}}} = \|b\|_{p,\phi_{\mathrm{tr}}} = 1\},$$

where we used the fact $L_p(\mathcal{N}, \phi_{\mathrm{tr}})$ is a tracial L_p-space. We will use this point to simplify the discussion in Section 5.4.

5. Applications and examples

5.1. Entropy contraction coefficients

In this section, we discuss the implications of our results on contraction coefficients studied in [23, 22, 57, 31]. These are analogs of functional inequalities for a single quantum channel.

Definition 5.1. Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a quantum Markov map GNS-ϕ-symmetric to a normal faithful state ϕ and $E : \mathcal{M} \to \mathcal{N}$ be the ϕ-preserving conditional expectation onto the multiplicative domain of Φ. We define

i) the L_2-contraction coefficient:

$$\lambda(\Phi) := \|\Phi(\mathrm{id} - E) : L_2(\mathcal{M}, \phi) \to L_2(\mathcal{M}, \phi)\|. \tag{5.1}$$

ii) the entropy contraction coefficient:

$$\alpha(\Phi) := \sup_\rho \frac{D(\Phi_*(\rho)||\Phi_* \circ E_*(\rho))}{D(\rho||E_*(\rho))}.$$

iii) the complete entropy contraction coefficient $\alpha_c(\Phi) := \sup_{\mathcal{Q}} \alpha(\mathrm{id}_{\mathcal{Q}} \otimes \Phi)$ where the supremum is over all σ-finite von Neumann algebras $\mathcal{Q}$.

The condition $\lambda(\Phi) < 1$ can be viewed as a Poincaré inequality for a quantum channel Φ, which implies the exponential convergence in L_2,

$$\|\Phi^n(X) - E(X)\|_{L_2(\mathcal{M},\phi)} \le \lambda(\Phi)^n \ \|X - E(X)\|_{L_2(\mathcal{M},\phi)} \to 0.$$

Similarly, the entropy contraction coefficient gives the convergence in relative entropy

$$D(\Phi^n(\rho)||\Phi^n \circ E(\rho)) \le \alpha(\Phi)^n D(\rho||E(\rho)).$$

The complete constant $\alpha_c(\Phi)$ controls not only the entropy contraction of Φ but also $\mathrm{id}_{\mathcal{Q}} \otimes \Phi$ with any environment system $\mathcal{Q}$. This leads to the tensorization property of α_c that for two GNS-symmetric quantum channels [31],

$$\alpha_c(\Phi_1 \otimes \Phi_2) = \max\{\alpha_c(\Phi_1), \alpha_c(\Phi_2)\}. \tag{5.2}$$

For classical Markov maps, the tensorization property (5.2) is known to also hold for the non-complete constant α. Nevertheless, for the quantum Markov map (channel), this is not the case, and $\alpha(\Phi)$ in general can be strictly less than $\alpha_c(\Phi)$ (see [14, Section 4.4]).

In finite dimensions, the existence of strictly contractive constant $\alpha_c(\Phi) < 1$ was obtained in [31, Theorem 4.1]. Our results give an explicit estimate for $\alpha_c(\Phi)$.

Corollary 5.2. *Let Φ be a GNS-symmetric quantum Markov map,*

$$\lambda(\Phi) \le \alpha(\Phi) \le \alpha_c(\Phi) \le (1 - \frac{1}{2k_{cb}(\Phi)}) \le \left(1 - \frac{-\ln \lambda(\Phi)}{\ln(10 C_{cb}(E))}\right).$$

Proof. The estimate follows from Theorem 4.8 and a discrete time analog of Proposition 4.12. ◻

Remark 5.3. In the ergodic trace symmetric case $\mathcal{N} = \mathbb{C}1$ and $\mathcal{M} = \mathbb{M}_d$, we have the trace map $E(x) = \mathrm{tr}(x)\frac{1}{d}$ and the CB-index $C_{cb}(E) = d^2$. The above estimate implies

$$\lambda(\Phi) \le \alpha(\Phi) \le \alpha_c(\Phi) \le (1 - \frac{-\ln \lambda(\Phi)}{\ln(10d^2)}). \tag{5.3}$$

This can be compared to [57, Theorem 4.2] and [43, Corollary 27],

$$\alpha(\Phi) \le 1 - \frac{1}{2}\alpha_2(\mathrm{id} - \Phi^*\Phi) \le 1 - \frac{(1-\lambda(\Phi)^2)^2(1-\frac{2}{d})}{\ln(d-1)}, \tag{5.4}$$

where $\alpha_2(\mathrm{id} - \Phi^2)$ is the LSI constant of $\mathrm{id} - \Phi^2$ as a generator of quantum Markov semigroup. The two upper bounds in (5.3) and (5.4) are comparable, as both are asymptotically $\Theta(\frac{-\ln\lambda(\Phi)}{\ln d})$. The strength of our results is that (5.3) also bounds the complete constant $\alpha_c(\Phi)$ which has the tensorization property.

Remark 5.4. Our Lemma 2.1 implies

$$1 - \alpha_1(\mathrm{id} - \Phi^2) \le \alpha(\Phi),$$

where α_1 is MLSI constant of the semigroup generator $(\mathrm{id} - \Phi^*\Phi)$. For a classical Markov map, it was proved by Del Moral, Ledoux and Miclo [22] that there exists a universal constant $0 < c < 1$ such that

$$1 - \alpha_1(\mathrm{id} - \Phi^*\Phi) \le \alpha(\Phi) \le 1 - c\alpha_1(\mathrm{id} - \Phi^*\Phi). \tag{5.5}$$

To the best of our knowledge, the above upper bound in (5.5) is open in the quantum case.

5.2. *Graph random walks*

Let $G = (V, E)$ be a finite undirected graph with $|V| = d$ and the edge set $E \subset V \times V$. The discrete time random walk on G is a finite Markov chain given by the stochastic matrix

$$K_G(u,v) = \begin{cases} \frac{1}{d(u)}, & \text{if } (u,v) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Here, $d(u)$ is the degree of vertex $u \in V$. Then $K_G : l_\infty(V) \to l_\infty(V)$ is a Markov map. The K_G admits a unique station distribution $\pi(u) = \frac{d(u)}{2m}$, where $|E| = m$. It is clear that K_G is symmetric to the measure π, also called reversible. Hence, K_G is an ergodic unital channel on $L_\infty(V, \pi)$ as $\pi(K_G(f)) = \pi(f)$. The expectation map is $E_\pi(f) = \pi(f)1$ whose index is

$$C_{cb}(E_\pi) = \| \pi^{-1} \|_\infty .$$

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K_G is connected and E_π are symmetric operators on $L_2(V,\pi)$ and

$$\lambda(K_G) = \| K_G - E_\pi : L_2(V,\pi) \to L_2(V,\pi) \| < 1$$

if K_G not bipartite (in the bipartite case K_G has eigenvalue -1). Then our results imply

$$\alpha(K_G) \le \alpha_c(K_G) \le (1 - \frac{1}{2k_{cb}(K_G)}) \le (1 - \frac{-\ln\lambda(K_G)}{\ln(10\,\|\pi^{-1}\|_\infty)}). \tag{5.6}$$

Example 5.5 (Cyclic graphs). Let us consider the cyclic graph $C_d = (V,E)$ with $d \ge 4$ where $V = \{1,\cdots,d\}$ and $E = \{(j,j+1)|j = 1,\cdots,d\}$. Here, the addition is understood in the sense of 'mod d'. Then

$$K_{C_d}(i,j) = \begin{cases} \frac{1}{2}, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

As C_d is 2-regular, K_{C_d} is symmetric to the uniform distribution $\pi(i) = 1/d$. It is known that K_{C_d} has spectrum

$$\lambda_j = \cos(\frac{2\pi j}{d}) \ , \ j = 0,\cdots,d-1.$$

The associated eigenvector is $e_j = \frac{1}{\sqrt{d}}(1,\omega^j,\omega^{2j},\cdots,\omega^{(d-1)j})$ where $\omega = \exp(\frac{2\pi i}{d})$. When $d = 2m+1$ is odd, π is the unique stationary measure, and E_π is the projection onto the vector e_0. We have

$$K_G^k - E_\pi = (K_G - E_\pi)^k = \sum_{j=1}^{2m} \lambda_j^k |e_j\rangle\langle e_j|.$$

By triangle inequality, we have

$$\begin{aligned} \| K_G^k - E_\pi : L_1(V,\pi) \to L_\infty(V,\pi) \| &\le \sum_{j=1}^{2m} |\lambda_j|^k = 2\sum_{j=1}^{m} \cos(\frac{\pi j}{d})^k \\ &\le 2\frac{d}{\pi}\int_0^{\pi/2} \cos^k(x)dx = 2\frac{d}{\pi}W_k \le 2Cd\sqrt{\frac{1}{2k\pi}}, \end{aligned}$$

where $C > 0$ is some absolute constant by fact that the Wallis integrals $W_k = \int_0^{\pi/2} \cos^k(x)dx \sim \sqrt{\frac{\pi}{2k}}$. Thus,

$$k_{cb}(K_{C_d}) \le \frac{(10Cd)^2}{\pi} \sim \mathcal{O}(d^2),$$

and (5.6) implies

$$\alpha(K_{C_d}) \ge \alpha_c(K_{C_d}) \ge 1 - \mathcal{O}(d^{-2}).$$

By Miclo's result (5.5), this is asymptotically tight because the MLSI constant $\alpha_1(I - K_G^2) \sim \mathcal{O}(d^{-2})$ (see Example 5.6 below for detials). The similar asymptotic estimate also holds for even circle $d = 2m$.

For the continuous time random walk, we consider $w : E \to (0,\infty)$ to be a positive weighted function on the edge set E. The (weighted) graph Laplacian is given by the matrix

$$L_G(u,v) = \begin{cases} \sum_{e=(u,u')\in E} w_e, & \text{if } u = v \\ -w_e, & \text{if } (u,v) \in E \\ 0, & \text{otherwise.} \end{cases}$$

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L_G generates the continuous time random walk $T_t = e^{-L_G t}$ as a Markov semigroup and is symmetric with to the uniform distribution π on V. T_t is ergodic if and only if $G = (V, E)$ is connected. The expectation map $E_\pi(f) = \pi(f)\mathbf{1}$ has index $C_{cb}(E_\pi) = d$. Then Corollary 4.13,

$$\frac{\lambda(L_G)}{2(\ln d + \ln 10)} \le \alpha_c(L_G) \le \alpha(L_G) \le \lambda(L_G). \tag{5.7}$$

This lower bound of $\alpha_c(L_G)$ has better dependence on the dimension d than [49, Lemma 5.2].

Example 5.6 (Cyclic graphs). Let us again consider the cyclic graph C_d with d vertices. For the uniformly weighted case $w_e \equiv 1$, L_{C_d} is a circulant matrix

$$L_{C_d}(i,j) = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $L_{C_d} = 2(I - K_{C_d})$ where K_{C_d} is the random walk kernel in Example 5.5, and L_{C_d} has spectrum $\lambda_j = 2(1 - \cos\frac{2\pi j}{d})$. As discussed in [23, Example 3.6],

$$\| T_t - E : L_1(V,\pi) \to L_\infty(V,\pi) \| \le 2\exp(-\frac{4t}{d^2})(\sqrt{1 + d^2/4t}).$$

Choosing $t_0 = d^2$, we have

$$\| T_t - E : L_1(V,\pi) \to L_\infty(V,\pi) \| \le 2e^{-4}\sqrt{5/4} < \frac{1}{10}.$$

Thus, by Theorem 1.1,

$$\frac{1}{2d^2} \le \alpha_c(L_{C_d}) \le \alpha_1(L_{C_d}) \le 2(1 - \cos\frac{2\pi}{d}) = \frac{8\pi^2}{d^2} + \mathcal{O}(\frac{1}{d^4}).$$

This shows that for this example, our inverse of t_{cb} bound for α_c is tight up to absolute constant. Note that the LSI constant $\alpha_2(L_{C_d})$ is also of $\Theta(\frac{1}{d^2})$.

We refer to [23, 11] more examples on spectral gap λ, Log-Sobolev constants α_2,α_1, and L_∞ mixing time t_b of finite Markov chains.

5.3. A noncommutative Birth-Death process

Let us illustrate our estimate with a noncommutative birth-death process. This example is a generalization of graph Laplacians on matrix algebras (see [49, 41] for similar constructions). To fix the notation, let $G = (V, E)$ be an undirected graph with $n = |V|$ vertices and edge set E. For each edge $(r, s) \in E$, we introduce the edge Lindbladian on $\mathbb{M}_n$,

$$\begin{aligned} L_{rs}(x) &= e^{\beta_{rs}/2} L_{e_{rs}}(x) + e^{-\beta_{rs}} L_{e_{sr}}(x) \\ &= e^{\beta_{rs}/2}(e_{ss}x + xe_{ss} - 2e_{sr}xe_{rs}) + e^{-\beta_{rs}/2}(e_{rr}x + xe_{rr} - 2e_{rs}xe_{sr}), \end{aligned}$$

where $e_{rs} \in \mathbb{M}_n$ is the matrix unit with 1 at the (r, s) position. The total Lindbladian is a weighted sum over the edge set E,

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$$\begin{aligned} L &= \sum_{(r,s)\in E} w(r,s)L_{rs} \\ &= 2\sum_{s\in V}\left(\sum_{(r,s)\in E} w(r,s)e^{\beta_{rs}/2}\right)(e_{ss}x + xe_{ss}) - 4\sum_{(r,s)\in E} w(r,s)e^{\beta_{rs}/2}e_{sr}xe_{rs}, \end{aligned}$$

where we assume $\beta_{rs} = -\beta_{sr}$ and $w(r,s) = w(s,r) > 0$ for the GNS-symmetry condition. Note that for $j \neq k$,

$$\begin{aligned} L(e_{jk}) &= 2(\sum_{(r,k)\in E} w(r,k)e^{\beta_{rk}/2} + \sum_{(r,j)\in E} w(r,j)e^{\beta_{r,j}/2})e_{jk}\ , \\ L(e_{jj}) &= 4\sum_{(r,j)\in E} w(r,j)(e^{\beta_{r,j}/2}e_{jj} - e^{-\beta_{r,j}/2}e_{rr}). \end{aligned}$$

Let us collect some relevant facts of such a Lindbladian L as noncommutative extension of graph Laplacian.

i) Denote $\ell_\infty(V) \subset \mathbb{M}_n$ as the diagonal subalgebra. $L(\ell_\infty(V)) \subset \ell_\infty(V)$, and $L|_{\ell_\infty(V)}$ is a weighted graph Laplacian;
ii) For $r \neq s$, the matrix unit e_{rs} is an eigenvector of L

$$L(e_{rs}) = \gamma_{rs}e_{rs},$$

where $\gamma_{rs} = 2(\sum_{(r,j)\in E} w(r,j)e^{\beta_{rj}/2} + \sum_{(k,s)\in E} w(k,s)e^{-\beta_{ks}/2})$.
iii) $\ker(L) \subset \ell_\infty(V)$, and $\ker(L) = \mathbb{C}1$ if $\mathcal{G} = (V,E)$ is connected.
iv) Let $\mu = (\mu_k) \in \ell_\infty(V)$ be a density operator in the diagonal subalgebra. Then L is GNS-μ-symmetric if $e^{\beta_{rs}} = \mu_s/\mu_r$ for any $s \neq r$.

Assume $L = \sum_{(s,r)\in E} L_{sr}$ is an ergodic graph Lindbladian satisfying GNS-μ-symmetric condition for a diagonal density operator μ. Denote E_d as the projection onto diagonal subalgebra. We can decompose the semigroup $T_t = e^{-tL}$ on the diagonal part and off diagonal part.

$$T_t = T_tE_d + T_t(\mathrm{id} - E_d) := T_t^{diag} + T_t^{off}\,. \tag{5.8}$$

It is clear from i) and ii) that T_tE_d is a classical graph random walk and $T_t(\mathrm{id} - E_d)$ is a Schur multiplier on $\mathbb{M}_n$. Using this decomposition, we consider the CB-return time of the semigroup

$$t_{cb}(\varepsilon) := \inf\{t > 0 \mid \ \|T_t - E_\mu : L_1(\mathbb{M}_n,\mu) \to \mathbb{M}_n\|_{cb} \leq \epsilon\}$$

satisfying

$$t_{cb}(2\varepsilon) \leq t_{cb}^{diag}(\varepsilon) + t_{cb}^{off}(\varepsilon),$$

where t_{cb}^{diag} and t_{cb}^{off} are the CB-return time for the diagonal part T_tE_d and off diagonal part $T_t(\mathrm{id} - E_d)$, respectively, where

$$\begin{aligned} t_{cb}^{diag}(\epsilon) &=: \inf\{t > 0 \mid \|T_tE_d - E_\mu : L_1(v,\mu) \to L_\infty(V)\|_{cb} \leq \epsilon\} \\ t_{cb}^{off}(\epsilon) &=: \inf\{t > 0 \mid \|T_t(\mathrm{id} - E_d) : L_1(\mathbb{M}_n,\mu) \to \mathbb{M}_n\|_{cb} \leq \varepsilon\}. \end{aligned}$$

For the diagonal part, $t_{cb}^{diag}(\epsilon)$ is a classical L_∞ mixing time, i.e. the smallest t such that

$$\| T_tE_d - E_\mu : L_1(V,\mu) \to L_\infty(V) \| \leq \varepsilon.$$

For the off-diagonal term, we deduce from the Effros-Ruan isomorphism that a Schur multiplier map

$$\| T_t(\mathrm{id}-E_d) : L_1(\mathbb{M}_n,\mu) \to \mathbb{M}_n \|_{cb} = \| \sum_{r\neq s} \mu_r^{-1/2} e^{-\gamma_{rs}t} \mu_s^{-1/2} e_{rs} \otimes e_{rs} \|_\infty$$
$$= \| \sum_{r\neq s} \mu_r^{-1/2} e^{-\gamma_{rs}t} \mu_s^{-1/2} e_{rs} \|_\infty .$$

Note that for each t,

$$A_t = \sum_{r\neq s} \mu_r^{-1/2} e^{-\gamma_{rs}t} \mu_s^{-1/2} e_{rs}$$

is a symmetric matrix with positive entry. A standard application of Schur's lemma for matrices with positive entries implies

$$\| A_t \|_\infty \le \sup_r \Big(\sum_s \mu_r^{-1/2} e^{-\gamma_{rs}t} \mu_s^{-1/2} \Big),$$

which gives us an estimate for the off diagonal term $t_{cb}^{off}(\epsilon)$.

Now we consider the birth-death process on a finite state space $V = \{1, \cdots, n\}$, which we denote as L_n^{BD}. The corresponding edge E set consists of only successive vertices $E = \{(j, j+1) | 1 \le j \le n-1\}$. The simplest case chooses the uniform weight $w(r,s) = 1$ for $(r,s) \in E$ and allows only one Bohr frequency $e^{-\beta} = \frac{\mu_j}{\mu_{j+1}}$, and the resulting stationary measure is the well-studied thermal state

$$\mu = Z_\beta^{-1} (e^{-\beta j})_{j=1}^n ,$$

where $Z_\beta = \sum_{j=1}^n e^{-\beta j}$ is the normalization constant. In this case, $\gamma_{rs} = 8(\cosh\beta)t$, and the off diagonal CB norm can be estimated by

$$\| A_t \|_\infty \le \sup_r \Big(\sum_{s=1}^n e^{\beta r/2} e^{\beta s/2} \Big) Z_\beta e^{-8(\cosh\beta)t}$$
$$\le e^{\beta \frac{n-2}{2}} \frac{1-e^{n\beta/2}}{1-e^{\beta/2}} \frac{1-e^{-n\beta}}{1-e^{-\beta}} e^{-8(\cosh\beta)t} .$$

Thus, $t_{cb}^{off}(\varepsilon) \le C_1(\beta) n$ for some constant $C_1(\beta)$ depending on β. For the classic part, we refer to [55] and [17] for the fact that the spectral gap is of order $O(1)$; that is,

$$c(\beta) \le \lambda(L_n^{diag}) \le C_2(\beta)$$

for all $n \in \mathbb{N}$. For the commutative system on the diagonal part, this implies (see also [23])

$$t_{cb}^{diag}(\varepsilon) \ \le\ 2c(\beta)^{-1}(2 + |\log \mu_n|) \ \le\ C_2(\beta) n,$$

(for $\varepsilon = e^{-2}$, but here, the actual value of ε does not change the asymptotic estimate). However, we have based on [55] that

$$t_{cb}^{diag}(0.1) \ge \alpha_1 (L_n^{diag})^{-1} \ \ge\ c(\beta) n.$$

Combining the diagonal and off diagonal part, we know $t_{cb}(L_n^{BD}) \sim n$. It turns out CMLSI constant has asymptotic $\alpha_c(L_n^{BD}) \sim \frac{1}{n}$, which indicates our estimate $\alpha_c \ge \frac{1}{2t_{cb}}$ is asymptotically tight for this example.

Theorem 5.7. *For $\beta > 0$, there exist constants $c(\beta), C(\beta) > 0$ such that the CMLSI constant of noncommutative birth-death process L_n^{BD} satisfies*

$$\frac{c(\beta)}{n} \leq \alpha_c(L_n^{BD}) \leq \alpha_1(L_n^{BD}) \leq \frac{C(\beta)}{n}.$$

The same $\Theta(\frac{1}{n})$ asymptotic holds for $t_{cb}(L_n^{BD})^{-1}$.

Proof. It suffices to show that

$$\alpha_c(L_n^{BD}) \leq \alpha_1(L_n^{BD}) \leq \frac{C(\beta)}{n}.$$

For this, we consider the function in the commutative system on the diagonal

$$f(k) = \frac{Z(\beta)}{n} e^{\beta k} \text{ and } \sum_{k=1}^{n} f(k)\mu(k) = \sum_{k=1}^{n} \frac{Z(\beta)}{n} e^{\beta k} \frac{1}{Z(\beta)} e^{-\beta k} = 1$$

so that $\rho := f\mu$ represents a probability density. The relative entropy term satisfies

$$D(\rho||\mu) = D(f\mu||\mu) = \sum_k \frac{e^{-\beta k}}{Z(\beta)} f(k)(\beta k + \ln Z(\beta) - \ln n) = \ln Z(\beta) - \ln n + \beta \frac{n+1}{2}.$$

Our density is $\rho \equiv (\frac{1}{n})$, and the reference density is $\mu(k) = \frac{e^{-\beta k}}{Z(\beta)}$.

Denote $a_k = |k\rangle\langle k+1|$. On the diagonal, we have

$$\begin{aligned}
\frac{1}{2} L_*(f) &= \sum_k e^{\beta/2}(a_k a_k^* f - a_k^* f a_k) + e^{-\beta/2}(a_k^* a_k f - a_k f a_k^*) \\
&= \sum_k e^{\beta/2}(e_k f(k) - f(k) e_{k+1}) + e^{-\beta/2}(f(k+1) e_{k+1} - f(k+1) e_k) \\
&= \frac{1}{Z(\beta) n}(e^{\beta/2}(e_0 - e_n) + e^{-\beta/2}(e_n - e_0)).
\end{aligned}$$

We have

$$L_{n,*}^{BD}(f)(k) = \begin{cases} 4(e^{\beta/2} - e^{-\beta/2}), & \text{if} \quad k = 1; \\ 0, & \text{if} \quad k = 2, n-1; \\ 4(e^{-\beta/2} - e^{\beta/2}), & \text{if} \quad k = n. \end{cases}$$

Note that

$$\ln \rho - \ln \mu = \ln f = \big(\beta k - \ln(Z(\beta) n)\big)_{k=1}^{n}.$$

Then we have the entropy production

$$I_{L_n^{BD}}(\rho) = \tau(L_{n,*}^{BD}(f) \ln f) \sim c(\beta)$$

for some constant $c(\beta)$ only depending on β. This holds for $n \geq n_0$ large enough. □

Remark 5.8. When $\beta > 0$, $\sum_{k=1}^{n} e^{-\beta k} = O(1)$ is a geometric series. In the case that $\beta = 0$, the above birth-death process reduces to a 'broken' version of the cyclic graph (linear graph) as in Example 5.6 with $\alpha_c(L_n) \sim 1/n^2$.

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5.4. Noncommutative concentration inequality

In this section, we show that CMLSI of a GNS-ϕ-symmetric semigroup implies concentration inequalities for the state ϕ. The key quantity in the discussion is the Lipschitz semi-norm

$$\|x\|_{\mathrm{Lip}}^2 \;=\; \max\{\|\,\Gamma_L(x,x)\,\|\;,\;\|\,\Gamma_L(x^*,x^*)\,\|\},$$

where the *gradient form* (or Carré du Champ operator) is

$$\Gamma_L(x,y) \;=\; \frac{1}{2}\Big(L(x^*)y + x^*L(y) - L(x^*y)\Big)\;,\;\forall x,y\in \mathrm{dom}(L).$$

Note that $\|\cdot\|_{\mathrm{Lip}}$ is a semi-norm (satisfying triangle inequality) because Γ_L is completely positive bilinear form. Our first lemma is to show that $\|x\|_{\mathrm{Lip}}$ can be approximated by Haagerup reduction.

Lemma 5.9. *Let $x\in\mathcal{M}$. Then for all $n\in\mathbb{N}$,*

$$\|E_{\mathcal{M}_n}(x)\|_{Lip} \;\le\; \|x\|_{Lip}.$$

Proof. Recall the conditional expectation $E_{\mathcal{M}_n}:\hat{\mathcal{M}}\to\mathcal{M}_n$ is given by

$$E_{\mathcal{M}_n}(x) = 2^n\int_0^{2^{-n}}\alpha_t^{\psi_n}(x)dt.$$

Note that $\alpha_t^{\psi_n}$ is an inner automorphism on $\mathcal{M}\rtimes_\alpha 2^{-n}\mathbb{Z}\cong L_\infty(\mathbb{T},\mathcal{M})$. We note that for a modular automorphism α_t such that $L\alpha_t=\alpha_t L$,

$$\Gamma_L(\alpha_t(x),\alpha_t(y)) \;=\; \alpha_t(\Gamma_L(x,y)),$$

which implies $\|\,x\,\|_{\mathrm{Lip}}=\|\,\alpha_t(x)\,\|_{\mathrm{Lip}}$. Here, both $\alpha_t^{\hat{\phi}}$ and $\alpha_t^{\psi_n}$ commute with $\hat{L}=\mathrm{id}_{\mathbb{T}}\otimes L$ by the GNS-symmetricness of $\hat{L}$. Then by triangle inequality,

$$\|\,E_{\mathcal{M}_n}(x)\,\|_{\mathrm{Lip}} = \Big\|2^n\int_0^{2^{-n}}\alpha_t^{\psi_n}(x)dt\Big\|_{\mathrm{Lip}} \le 2^n\int_0^{2^{-n}}\|\,x\,\|_{\mathrm{Lip}}\,dt = \|\,x\,\|_{\mathrm{Lip}}\;. \qquad \square$$

Lemma 5.10. *Let $\mathcal{M}_0,\mathcal{N}\subset\mathcal{M}$ be two subalgebras and ϕ be a normal faithful state. Suppose $E_0:\mathcal{M}\to\mathcal{M}_0$ and $E:\mathcal{M}\to\mathcal{N}$ are ϕ-preserving conditional expectations onto $\mathcal{M}_0$ and $\mathcal{N}$, respectively. Suppose $E\circ E_0=E_0\circ E$ satisfy the commuting square condition*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{E_0} & \mathcal{M}_0 \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E} \\ \mathcal{N} & \xrightarrow[E_0]{} & \mathcal{N}_0 \end{array}\;,$$

where $\mathcal{N}_0\subset\mathcal{N}$ is a subalgebra. Then for any $p\in[1,\infty]$ and any $x\in\mathcal{M}$,

$$\|\,E_0(x)\,\|_{L_\infty^p(\mathcal{N}_0\subset\mathcal{M}_0,\phi)} = \|\,E_0(x)\,\|_{L_\infty^p(\mathcal{N}\subset\mathcal{M},\phi)} \le \|\,x\,\|_{L_\infty^p(\mathcal{N}\subset\mathcal{M},\phi)}\;.$$

In other words, $L_\infty^p(\mathcal{N}_0\subset\mathcal{M}_0,\phi)\subset L_\infty^p(\mathcal{N}\subset\mathcal{M},\phi)$ as a 1-complemented subspace with projection E_0.

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Proof. We can assume $\phi = \phi_{\mathrm{tr}}$ in the Remark 4.18. Using commuting square assumption, we know $E_0(a) \in \mathcal{N}_0$ for $a \in \mathcal{N}$. By definition,

$$\begin{aligned}\| E_0(x) \|_{L_\infty^p(\mathcal{N}_0 \subset \mathcal{M}_0, \phi)} &= \sup_{a,b \in \mathcal{N}_0} \| aE_0(x)b \|_{\phi,p} \le \sup_{a,b \in \mathcal{N}_0} \| E_0(axb) \|_{\phi,p} \\ &\le \sup_{a,b \in \mathcal{N}_0} \| axb \|_{\phi,p} \le \| x \|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)},\end{aligned}$$

where the supremum is for all a, b in the corresponding subalgebra with $\| a \|_{\phi,p} = \| b \|_{\phi,p} = 1$. Now it suffices to show the other direction

$$\| x \|_{L_\infty^p(\mathcal{N}_0 \subset \mathcal{M}_0, \phi)} \ge \| x \|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)},$$

for $x \in \mathcal{M}_0$. For that, we revoke that for $\frac{1}{p} + \frac{1}{q} = 1$, $L_1^{p'}(\mathcal{N} \subset \mathcal{M}) \subset L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)^*$ is as a weak*-dense subspace [42, Proposition 4.5]. Here, for $x \in \mathcal{M}$,

$$\| y \|_{L_1^q(\mathcal{N} \subset \mathcal{M})} = \inf_{y = azb} \| a \|_{2p,\phi} \| y \|_{q,\phi} \| b \|_{2p,\phi},$$

where the infimum is over all factorization $y = azb$ with $a, b \in \mathcal{N}$, $z \in \mathcal{M}$. The duality pairing is given by the KMS inner product,

$$\langle x, y \rangle = \tau(x^* d_\phi^{1/2} y d_\phi^{1/2}) = \langle x, y \rangle_\phi.$$

Indeed, it was proved in [42, Corollary 3.13] that

$$E_0 : L_1^q(\mathcal{N} \subset \mathcal{M}) \to L_1^q(\mathcal{N}_0 \subset \mathcal{M}_0)$$

is a contraction by the commuting square condition. Therefore, for $x \in \mathcal{N}_0$, by the KMS-ϕ-symmetry of E_0,

$$\begin{aligned}\| x \|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)} &= \sup_{\| y \|_{L_1^q(\mathcal{N} \subset \mathcal{M})} = 1} \langle x, y \rangle_\phi \\ &= \sup_{\| y \|_{L_1^q(\mathcal{N} \subset \mathcal{M})} = 1} \langle x, E_0(y) \rangle_\phi \\ &\le \sup_{\| z \|_{L_1^q(\mathcal{N}_0 \subset \mathcal{M}_0)} = 1} \langle x, z \rangle_\phi = \| x \|_{L_\infty^p(\mathcal{N}_0 \subset \mathcal{M}_0, \phi)} .\end{aligned}$$

□

Lemma 5.11. *For $x \in \mathcal{M}$, $\lim_n \|E_{\mathcal{M}_n}(x)\|_{L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n, \psi_n)} = \|x\|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)}$.*

Proof. Recall the commuting square condition $E_{\mathcal{M}_n} \circ \hat{E} = \hat{E} \circ E_{\mathcal{M}_n}$. By Lemma 4.17 & 5.10,

$$\| E_{\mathcal{M}_n}(x) \|_{L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n, \psi_n)} = \| E_{\mathcal{M}_n}(x) \|_{L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n, \hat{\phi})} = \| E_{\mathcal{M}_n}(x) \|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \hat{\phi})} \le \| x \|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \hat{\phi})} .$$

The other direction follows from the weak*-convergence $E_{\mathcal{M}_n}(x) \to x$. Fix $\frac{1}{q} + \frac{1}{p} = 1$. For any $\varepsilon > 0$, there exists $a_0, b_0 \in \hat{\mathcal{N}}$ and $y_0 \in \hat{\mathcal{M}}$ such that

$$\| aa^* \|_{p,\hat{\phi}} = \| b^* b \|_{p,\hat{\phi}} = \| y \|_{\hat{\phi},q} = 1 , \qquad \hat{\tau}(d_{\hat{\phi}}^{1/2} axbd_{\hat{\phi}}^{1/2} y) \ge \| x \|_{L_\infty^p(\hat{\mathcal{N}} \subset \hat{\mathcal{M}}, \hat{\phi})} - \varepsilon.$$

By the weak*-density, we can choose n_1, n_2, n_3 and $n_4 \ge \max\{n_1, n_2, n_3\}$ inductively such that

$$\tau(d_\phi^{1/2} E_{\mathcal{M}_{n_1}}(a) E_{\mathcal{M}_{n_4}}(x) E_{\mathcal{M}_{n_2}}(b) d_\phi^{1/2} E_{\mathcal{M}_{n_3}}(y)) > \tau(d_\phi^{1/2} axbd_\phi^{1/2} y) - \varepsilon > \| x \|_{L_\infty^p(\hat{\mathcal{N}} \subset \hat{\mathcal{M}}, \hat{\phi})} - 2\varepsilon.$$

Since $E_{\mathcal{M}_n}(\hat{\mathcal{N}}) = \mathcal{N}_n$ (see the commuting diagram after Lemma 4.5), we have

$$\begin{aligned}
&\| E_{\mathcal{M}_{n_1}}(a)E_{\mathcal{M}_{n_1}}(a)^* \|_{\hat{\phi},p} \leq \| E_{\mathcal{M}_{n_1}}(aa^*) \|_{\hat{\phi},p} \leq \| aa^* \|_{\hat{\phi},p} = 1 \\
&\| E_{\mathcal{M}_{n_3}}(b^*)E_{\mathcal{M}_{n_3}}(b) \|_{\hat{\phi},p} \leq \| b^*b \|_{\hat{\phi},p} = 1 \ , \\
&\| E_{\mathcal{M}_{n_4}}(y) \|_{\hat{\phi},q} \leq \| y \|_{\hat{\phi},q} = 1
\end{aligned}$$

by the KMS-$\hat{\phi}$-symmetry of $E_{\mathcal{M}_n}$. Then, for $n \geq n_4 = \max\{n_1, n_2, n_3, n_4\}$,

$$\begin{aligned}
\| E_{\mathcal{M}_n}(x) \|_{L_\infty^p(\hat{\mathcal{N}}_n \subset \hat{\mathcal{M}}_n, \hat{\phi})} &\geq \| E_{\mathcal{M}_{n_4}}(x) \|_{L_\infty^p(\hat{\mathcal{N}}_n \subset \hat{\mathcal{M}}_n, \hat{\phi})} \\
&\geq \tau(d_{\hat{\phi}}^{1/2} E_{\mathcal{M}_{n_1}}(a) E_{\mathcal{M}_{n_4}}(x) E_{\mathcal{M}_{n_2}}(b) d_{\hat{\phi}}^{1/2} E_{\mathcal{M}_{n_3}}(y)) \\
&\geq \| x \|_{L_\infty^p(\hat{\mathcal{N}} \subset \hat{\mathcal{M}}, \hat{\phi})} - 2\varepsilon.
\end{aligned}$$

This proves

$$\lim_n \|E_{\mathcal{M}_n}(x)\|_{L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n, \psi_n)} = \|x\|_{L_\infty^p(\hat{\mathcal{N}} \subset \hat{\mathcal{M}}, \hat{\phi})}.$$

Finally, the assertion follows from

$$\|x\|_{L_\infty^p(\hat{\mathcal{N}} \subset \hat{\mathcal{M}}, \hat{\phi})} = \|x\|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)},$$

as a consequence of $E_0 \circ \hat{E} = \hat{E} \circ E_0$ by Lemma 5.10. ◻

Now we restate and prove Theorem 1.4.

Theorem 5.12. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra and $T_t = e^{-tL}$ be a GNS-ϕ-symmetric quantum Markov semigroup. Suppose T_t satisfies MLSI with parameter $\alpha > 0$. There exists an universal constant c such that for $2 \leq p < \infty$,*

$$\alpha\|x - E(x)\|_{L_p(\mathcal{M}, \phi)} \leq \alpha\|x - E(x)\|_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)} \ \leq \ c\sqrt{p}\ \|x\|_{Lip}\ .$$

Proof. We first show that if T_t satisfies α-MLSI, so does the approximation semigroup.

$$T_{n,t} = \hat{T}_t|_{\mathcal{M}_n} : \mathcal{M}_n \to \mathcal{M}_n.$$

Indeed, as we see in the discussion above, $\mathcal{M}_n \subset \mathcal{M} \rtimes_{\alpha_t^\phi} 2^{-n}\mathbb{Z} \cong L_\infty(\mathbb{T}, \mathcal{M})$, and the extension $\hat{T}_t = T_t \otimes \mathrm{id}_{\mathbb{T}}$ has α-MLSI (because $L_\infty(\mathbb{T})$ is a commutative space). Note that since $\mathcal{M}_n \subset \mathcal{M} \rtimes_{\alpha_t^\phi} 2^{-n}\mathbb{Z} \subset \mathcal{M} \rtimes_{\alpha_t^\phi} G$, the restriction $E_{\mathcal{M}_n} : \mathcal{M} \rtimes_{\alpha_t^\phi} 2^{-n}\mathbb{Z} \to \mathcal{M}_n$ is also a conditional expectation. Then for any $\rho, \sigma \in S(\mathcal{M}_n)$, we have

$$D(E_{\mathcal{M}_n,*}\rho||E_{\mathcal{M}_n,*}\sigma) \leq D(\rho||\sigma) = D(\rho|_{\mathcal{M}_n}||\sigma|_{\mathcal{M}_n}) \leq D(E_{\mathcal{M}_n,*}\rho||E_{\mathcal{M}_n,*}\sigma).$$

Using the commutation relation $T_{n,t} \circ E_{\mathcal{M}_n} = E_{\mathcal{M}_n} \circ \hat{T}_t$ and $E_{\mathcal{M}_n} \circ \hat{E} = E_n \circ E_{\mathcal{M}_n}$, we have for $\rho \in S(\mathcal{M}_n)$

$$\begin{aligned}
D(T_{t,n,*}\rho||E_{n,*}\rho) &= D(E_{\mathcal{M}_n,*}T_{t,n,*}\rho||E_{\mathcal{M}_n,*}E_{n,*}\rho) = D(\hat{T}_{t,*}E_{\mathcal{M}_n,*}\rho||\hat{E}_*E_{\mathcal{M}_n,*}\rho) \\
&\leq e^{-2\alpha t}D(E_{\mathcal{M}_n,*}\rho||\hat{E}_*E_{\mathcal{M}_n,*}\rho) \\
&= e^{-2\alpha t}D(E_{\mathcal{M}_n,*}\rho||E_{\mathcal{M}_n,*}E_{n,*}\rho) = e^{-2\alpha t}D(\rho||E_{n,*}\rho).
\end{aligned}$$

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Thus, $T_{n,t}$ has α-MLSI on $\mathcal{M}_n$. Note that $T_{n,t}$ is both GNS-$\hat{\phi}$-symmetric for the extension state $\hat{\phi}$ and also symmetric for the trace ψ_n. Now, we may use the tracial version of the concentration inequality [29, Theorem 6.10] that for $x \in \mathcal{M}_n$,

$$\alpha \parallel E_{\mathcal{M}_n}(x) - E_n E_{\mathcal{M}_n}(x) \parallel_{L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n)} \leq C\sqrt{p} \parallel E_{\mathcal{M}_n}(x) \parallel_{\mathrm{Lip}} .$$

Now by the approximation of Lemma 5.11 and independence of $L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n)$ on the reference state, for $x \in \mathcal{M}$,

$$\begin{aligned} \parallel x - E(x) \parallel_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)} &= \lim_n \parallel E_{\mathcal{M}_n}(x - E(x)) \parallel_{L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n, \psi_n)} \\ &= \lim_n \parallel E_{\mathcal{M}_n}(x) - E_n E_{\mathcal{M}_n}(x) \parallel_{L_\infty^p(\mathcal{N}_n \subset \mathcal{M}_n, \psi_n)} \\ &\leq C\sqrt{p} \parallel E_{\mathcal{M}_n}(x) \parallel_{\mathrm{Lip}} \leq C\sqrt{p} \parallel x \parallel_{\mathrm{Lip}} . \end{aligned}$$

The other inequality

$$\parallel y \parallel_{L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)} \geq \parallel y \parallel_{L_p(\mathcal{M}, \phi)}$$

is clear from definition of $L_\infty^p(\mathcal{N} \subset \mathcal{M}, \phi)$. □

For Gaussian type concentration property, we introduce the following definition.

Definition 5.13. For an operator O, we say that

$$\mathrm{Prob}_\phi(|O| > t) \;\leq\; \varepsilon$$

if there exists a projection e such that

$$\|eOe\|_\infty \leq t \quad \text{and} \quad \phi(1-e) \leq \varepsilon.$$

The next lemma is a Chebyshev inequality for ϕ-weighted L_p norm.

Lemma 5.14. *Let $x \in L_p(\mathcal{M}, \phi)$ and $1 < p < \infty$. Then*

$$Prob_\phi(|x| > t) \;\leq\; 2\left(\frac{t}{4}\right)^{-p} \|x\|_{p,\phi}^p .$$

Proof. We start with a positive element $x = y^2$ and assume $\|x\|_{p,\phi} = M$. Then we have

$$M = \|x\|_{p,\phi} \;=\; \|d_\phi^{1/2p} x d_\phi^{1/2p}\|_p \;=\; \|y d_\phi^{1/2p}\|_{2p}^2 .$$

Recall the asymmetric Kosaki L_p-space

$$\parallel y \parallel_{L_{2p}^c(\mathcal{M}, \phi)} := \|y d_\phi^{1/2p}\|_{2p},$$

and the complex interpolation relation [42]

$$L_{2p}^c(\mathcal{M}, \phi) = [\mathcal{M}, L_2^c(\mathcal{M}, \phi)]_{1/p},$$

and the relation between real and complex interpolation

$$L_{2p}^c(\mathcal{M}, \phi) = [\mathcal{M}, L_2^c(\mathcal{M}, \phi)]_{1/p} \subset [\mathcal{M}, L_2^c(\mathcal{M}, \phi)]_{1/p,\infty}.$$

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By the definition of real interpolation space, for every $s > 0$, we have a decomposition $y = y_1 + y_2$ such that

$$\|y_1\|_\infty + s\,\|y_2\|_{L_2^c(\mathcal{M},\phi)} \le s^{1/p} M^{1/2}.$$

Then by Chebychev's inequality for the spectral projection $e = e_{[0,a]}(y_2^* y_2)$, we have

$$a\phi(1-e) \le \phi(y_2^* y_2) \le s^{2/p-2} M \quad \text{and} \quad \|y_2 e\|_\infty^2 = \|e y_2^* y_2 e\|_\infty \le a.$$

Choose $a = s^{2/p} M$ and deduce that

$$\|exe\|_\infty = \|ye\|_\infty^2 \le (\|y_1 e\|_\infty + \|y_2 e\|_\infty)^2 \le (s^{1/p} M^{1/2} + s^{1/p} M^{1/2})^2 = 4a.$$

Then for $t = 4a$ and

$$\phi(1-e) \le a^{-1} s^{2/p-2} M = s^{-2} = (\frac{t}{4M})^{-p} = (\frac{t}{4})^{-p} M^p.$$

For an arbitrary x, we may write $x = x_1 x_2$ such that

$$\|d_\phi^{1/2p} x_1\|_{2p} = \|x_2 d_\phi^{1/2p}\|_{2p} = \|x\|_{p,\phi} = M.$$

Then for each $s > 0$, we have decomposition

$$x_1 = x_{11} + x_{12}\,,\; x_2 = x_{21} + x_{22}$$

with

$$\|x_{11}\|_\infty + s\,\|x_{12}\|_{L_2^c(\mathcal{M},\phi)} \le s^{1/p} M^{1/2}\,,\; \|x_{21}\|_\infty + s\,\|x_{22}\|_{L_2^c(\mathcal{M},\phi)} \le s^{1/p} M^{1/2}.$$

We then use the Chebychev inequality for $e = e_{[0,a]}(x_{12}^* x_{12} + x_{22}^* x_{22})$,

$$a\phi(1-e) \le \phi(x_{12}^* x_{12} + x_{22}^* x_{22}) \le 2 s^{2/p-2} M.$$

Take $a = s^{2/p} M$,

$$\begin{aligned}\|exe\|_\infty = \|e(x_1 x_2)e\|_\infty &= \|e(x_{11} + x_{12})(x_{21} + x_{22})e\|_\infty \\ &\le \|x_{11} x_{22}\|_\infty + \|e x_{12} x_{21}\|_\infty + \|x_{11} x_{22} e\|_\infty + \|e x_{12} x_{22} e\|_\infty \\ &\le 4 s^{2/p} M.\end{aligned}$$

Thus, for $t = 4 s^{2/p} M$, by Chebychev's inequality for e,

$$\phi(1-e) \le \frac{1}{a}\phi(x_{12}^* x_{12} + x_{22}^* x_{22}) \le a^{-1} 2 s^{2/p-2} M = 2 s^{-2} = 2(\frac{t}{4M})^{-p}. \qquad \square$$

Corollary 5.15. *Let $T_t = e^{-tL}$ be a GNS-ϕ-symmetric quantum Markov semigroup. Suppose T_t satisfies α-MLSI. Then for any $x \in \mathcal{M}$ and $t > 0$,*

$$Prob_\phi(|x - E_{fix}(x)| > t) \le 2\exp\Big(-\frac{2}{e}\Big(\frac{\alpha t}{4c\,\|x\|_{Lip}}\Big)^2\Big),$$

where c is a universal constant as in Theorem 5.12.

Proof. By Lemma 5.14 and Theorem 5.12, we have

$$\mathrm{Prob}_\phi(|x - E(x)| > t) \le 2(t/4)^{-p}\,\|x - E(x)\|_{L_p(\mathcal{M},\phi)}^p \le 2\Big(\frac{4c\,\|x\|_{\mathrm{Lip}}\sqrt{p}}{\alpha t}\Big)^p.$$

Minimizing over p gives $p = \frac{1}{e}(\frac{\alpha t}{4c\|x\|_{\mathrm{Lip}}})^2$, which implies

$$\mathrm{Prob}_\phi(|x - E(x)| > t) \leq 2\exp(-\frac{\alpha^2 t^2}{16ec^2 \, \| x \|_{\mathrm{Lip}}^2}). \qquad \square$$

Remark 5.16. In the ergodic case, the above results can be compared to [69, Theorem 8], which states that for self-adjoint $x = x^*$,

$$\phi(e_{\{|x-E(x)|>t\}}) \leq \exp\Big(- \frac{\alpha t^2}{8 \, \| d_\phi^{-1/2} x d_\phi^{1/2} \|_{\tilde{\mathrm{Lip}}}^2} \Big)$$

with a different Lipschitz norm $\| \cdot \|_{\tilde{\mathrm{Lip}}}^2$. Our Corollary 5.15 here uses a more natural definition of the Lipschitz norm and applies to non-ergodic cases. Nevertheless, the projection we have for

$$\mathrm{Prob}_\phi(|x - E(x)| > t)$$

is not necessarily a spectral projection $e_{\{|x-E(x)|>t\}}$ and will depend on the state ϕ.

Remark 5.17. In the operator valued setting, let $\mathcal{Q}$ be any finite von Neumann algebra and $T_t \otimes \mathrm{id}_{\mathcal{Q}}$ be the amplification semigroup on $\mathcal{Q}\overline{\otimes}\mathcal{M}$. The conditional expectation for $T_t \otimes \mathrm{id}_{\mathcal{Q}}$ is $E \otimes \mathrm{id}_{\mathcal{Q}}$. Note that by Lemma 4.17, $T_t \otimes \mathrm{id}_{\mathcal{Q}}$ is GNS-symmetric to the product state $\phi \otimes \sigma$, for any state $\sigma \in S(\mathcal{Q})$ and any invariant state $\phi \in E_*(S(\mathcal{M}))$. This means we obtain

$$\mathrm{Prob}_{\phi\otimes\sigma}(|x - E_{fix}(x)| > t) \leq 2e^{-\frac{\alpha^2 t^2}{C\|x\|_{\mathrm{Lip}}^2}}$$

for any product state $\phi \otimes \sigma$ of this specific form. The projection of course depends on both ϕ and σ.

We illustrate our result with a special case as matrix concentration inequalities.

Example 5.18 (Matrix concentration inequality). Let $S_1, \cdots, S_n$ be an independent sequence of random $d \times d$-matrices $S_1, \cdots, S_n$ such that

$$\| S_i - \mathbb{E}S_i \|_\infty \leq M \ , \ a.e.$$

Tropp in [75, Corollary 6.1.2] proved the following matrix Bernstein inequality that for the sum $Z = \sum_{k=1}^n S_k$,

$$\mathbb{E} \, \| Z - \mathbb{E}Z \|_\infty \leq \sqrt{2v(Z)\log(2d)} + \frac{1}{3}M\log(2d)$$

and the matrix Chernoff bound

$$P(|Z - \mathbb{E}Z| > t) \leq 2d\exp\big(- \frac{t^2}{v(Z) + \frac{t}{3}M}\big),$$

where

$$v(Z) = \max\{\| \mathbb{E}((Z - \mathbb{E}Z)^*(Z - \mathbb{E}Z)) \| \, , \| \mathbb{E}((Z - \mathbb{E}Z)^*(Z - \mathbb{E}Z)) \|\}.$$

Now to apply our results, we recall that the depolarizing semigroup with generator $L(f) := (I - E_\mu)(f) = f - \mu(f)\mathbf{1}_\Omega$ on any probability space (Ω, μ) has $\alpha_c \geq \frac{1}{2}$ (a simple fact by convexity of relative entropy). For a random matrix $f : \Omega \to \mathbb{M}_d$, the Lipschitz norm is

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$$\begin{aligned}\|f\|_{\mathrm{Lip}}^2 &= \frac{1}{2}\max\{\|\hat{f}^*\hat{f}+E_\mu(\hat{f}^*\hat{f})\|_\infty\ ,\ \|\hat{f}\hat{f}^*+E_\mu(\hat{f}\hat{f}^*)\|_\infty\}\\ &\le \frac{1}{2}(\|f\|_\infty^2+v(f)),\end{aligned}\tag{5.9}$$

where $\hat{f}=f-E_\mu(f)$ is the mean zero part.

Now we consider for each $k=1,\cdots,n$, $S_k:\Omega_k\to M_d$ as a random matrix on (Ω_k,μ_k). Then on the product space $(\Omega,\mu)=(\Omega_1,\mu_1)\times\cdots\times(\Omega_n,\mu_n)$, we have by Theorem 5.12 for $Z=\sum_k S_k$

$$\mathbb{E}\,\|Z-\mathbb{E}Z\|_\infty\le\Big(\frac{1}{d}\mathbb{E}\,\|Z-\mathbb{E}Z\|_p^p\Big)^{1/p}\le d^{1/p}\,\|Z-\mathbb{E}Z\|_{L_\infty(M_d,L_p(\Omega))}\le 2cd^{1/p}\sqrt{p}\,\|Z\|_{\mathrm{Lip}},$$

where $\|\cdot\|_p$ is the p-norm for the normalized trace $(\mathrm{tr}(1)=1)$. Applying (5.9) and optimizing p gives

$$\mathbb{E}\,\|Z-\mathbb{E}Z\|_\infty\le 2ce^{-1/2}\sqrt{(v(Z)+M^2)\log d}.$$

For the matrix Chernoff bound, we use Corollary 5.15

$$P(|Z-\mathbb{E}Z|>t)\le d\mathrm{Prob}_{\mu\otimes\frac{\mathrm{tr}}{d}}(|Z-\mathbb{E}Z|>t)\le 2d\exp\Big(-\frac{t^2}{64ec^2(v(Z)+M^2)}\Big).$$

6. Final discussion

1. Positivity and complete positivity. The central quantity in this work is the CB return time t_{cb} and k_{cb} defined via complete positivity. Alternatively, one can consider positive maps and positivite mixing time. Indeed, the entropy difference Lemma 2.1

$$D(\rho\|\Phi^*\Phi(\omega))\le D_\Phi(\rho)+D(\rho\|\omega)$$

holds for a positive unital trace-preserving map Φ. This is because the operator concavity

$$\Phi(\ln x)\le\ln\Phi(x),\qquad\forall x\ge 0$$

of the logarithmic function holds for any unital positive map Φ [18], and the monotonicity of relative entropy

$$D(\rho\|\sigma)\ge D(\Phi(\rho)\|\Phi(\sigma))$$

was proved for any positive trace-preserving map Φ in [56] (see also [27]). Thus, both inequalities used in the proof of Lemma 2.1 hold for positive maps. Also, the conditions in Lemma 2.3 also only require positivity order

$$(1-\varepsilon)E\le\Psi\le(1+\varepsilon)E,\tag{6.1}$$

where $\Phi\ge\Psi$ means $\Phi-\Psi$ is a positive map but not necessarily completely positive. Combining these two relaxed lemmas for positive maps, we have an analog of Theorem 1.1.

Theorem 6.1. i) *For a positive unital trace-preserving map* $\Phi:\mathcal{M}\to\mathcal{M}$,

$$\alpha(\Phi)\le 1-\frac{1}{2k(\Phi)}\quad\textit{where}\quad k(\Phi):=\inf\{k\in\mathbb{N}^+\mid 0.9E\le\Phi^{2k}\le 1.1E\}.$$

ii) *For a trace symmetric positive unital semigroup* $T_t=e^{-tL}:\mathcal{M}\to\mathcal{M}$*,3*

$$\alpha(L)\ge\frac{1}{2t(L)}\quad\textit{where}\quad t(L):=\inf\{t\in\mathbb{N}^+\mid 0.9E\le T_t\le 1.1E\}.$$

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Applying the above theorem to $\Phi \otimes \mathrm{id}_{\mathcal{Q}}$ and $T_t \otimes \mathrm{id}_{\mathcal{Q}}$ for any finite von Neumann algebra $\mathcal{Q}$ actually yields our main Theorem 1.1 for trace symmetric cases. It remains open whether this observation holds for GNS-symmetric cases.

Problem 6.2. Does Theorem 6.1 with positivity conditions hold for GNS-symmetric cases?

The obstruction is that in the Haagerup reduction, we need the complete positivity and CB return time $k_{cb}(\Phi)$ of Φ to imply positivity and positivity mixing time $k(\Phi)$ of the extension $\hat{\Phi}$, similar for the semigroup T_t. One possible approach is to avoid using Haagerup reduction, and prove Lemma 4.6 directly.

The comparison between positivity and complete positivity has a deep root in the entanglement theory of quantum physics (see [20]). From the mathematical point of view, although the positivity looks a more flexible condition, it lacks connection to CB norms as Proposition 3.4. Indeed, there is no non-complete analog of Choi's theorem [19]

$$C_T \in (\mathcal{M} \otimes \mathcal{M}^{op})_+ \iff T(x) = \tau \otimes \mathrm{id}(C_T(x \otimes 1)) \text{ is } CP.$$

Therefore, despite that the estimate of $\alpha_1(L)$ only requires $t(L)$, our kernel estimate Proposition 3.7 only applies to $t_{cb}(L)$.

2. GNS and KMS symmetry. Both GNS-symmetry and KMS-symmetry are noncommutative generalizations for the detailed balance condition of classical Markov chains. As observed in [15], GNS-symmetry is the strongest generalization of detail balance condition, and KMS is the weakest, which means the assumption of GNS-symmetry is the most restrictive. It is natural to ask whether our main results (c.f. Theorem 4.10 & 4.8) can be obtained for KMS-symmetric channels or semigroups.

Problem 6.3. Do entropy decay results Theorem 4.10 and 4.8 or the entropy difference Lemma 4.6 hold for KMS-symmetric maps?

The key property of a GNS-symmetric map Φ is the commutation with modular group $\Phi \circ \alpha_t^\phi = \alpha_t^\phi \circ \Phi$. This has been used to ensure the compatibility of Haagerup reduction with channel and semigroups (see Lemma 4.5). One can ask whether the same commuting diagram Figure 1 can be obtained for KMS Markov maps. That will allow us to use Haagerup reduction to obtain the entropy difference Lemma (4.6) for KMS-symmetric channels. Another approach is, again, to avoid using Haagerup reduction and prove the KMS-case directly. At the moment of writing, this is not unclear to us even on finite dimensional matrix algebras.

From a mathematical physics perspective, it is also interesting to explore the relative entropy decay beyond GNS symmetry. For instance, one has a Lindbladian of the form $x \mapsto i[h, x] + L(x)$ such that L is GNS-symmetric and the adjoint action $\mathrm{ad}(e^{-iht})$ commutes with L. Then the associated semigroup is $e^{-iht}e^{-tL}(\cdot)e^{iht}$, which has the same entropy decay as e^{-tL}. Such Lindbladians are considered in [48]. Indeed, there is also numerical and theoretical evidence that adding an nonzero Hamiltonian part can destroy the exponential entropy decay. We refer to [48] for more discussion on entropy decay beyond symmetry conditions.

3. MLSI and CMLSI constant. By the results of this work and also previous works [49, 14, 28], we now know the positivity of CMLSI constant $\alpha_c > 0$ for many cases of classical Markov semigroups with the (non-complete) MLSI constant $\alpha > 0$. That is, $\alpha \geq \alpha_c > 0$ for

i) finite Markov chains [49, 28];
ii) heat semigroups on manifold with curvature lower bound [14];
iii) sub-Laplacians of Hörmander system on a compact Riemannian manifold.

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It remains open whether MLSI constant α and CMLSI constant α_c coincide for classical semigroups. This would be in the similar spirit that the bounded norm (resp. positivity) and the complete bounded norm (resp. complete positivity) coincide for a classical map on $L_\infty(\Omega, \mu)$.

Problem 6.4. Does $\alpha = \alpha_c$ for a classical symmetric Markov semigroup $T_t : L_\infty(\Omega, \mu) \to L_\infty(\Omega, \mu)$?

For a quantum Markov semigroup, a counterexample is the qubit depolarizing semigroup

$$T_t : \mathbb{M}_2 \to \mathbb{M}_2 \ , \ T_t(\rho) = e^{-t}\rho + (1 - e^{-t})\frac{1}{2},$$

which has $\frac{1}{2} \le \alpha_c(T_t) < \alpha(T_t) = 1$ because of entangled states [14, Section 4.3]. It is natural to ask whether $\alpha_c < \alpha$ also holds for classical depolarizing channel.

Another interesting example is the heat semigroup on the unit torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$,

$$P_t : L_\infty(\mathbb{T}) \to L_\infty(\mathbb{T}) \ , P_t(z^n) = e^{-n^2 t} z^n.$$

It was proved by [78] that $\alpha(P_t) = \lambda(P_t) = 1$. The best known bound for CMLSI is $\alpha_c(P_t) \ge \frac{1}{6}$. It is open whether the gap can be closed.

Problem 6.5. Does the heat semigroup P_t on the torus $\mathbb{T}$ have $\alpha_c(P_t) = \alpha(P_t) = 1$?

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