This is a ``preproof'' accepted article for *Canadian Mathematical Bulletin*
This version may be subject to change during the production process.
DOI: 10.4153/S0008439525101409

NOT EVERY COMPLETE LATTICE CAN SUPPORT A UNITAL QUANTALE

SHENGWEI HAN AND XIN WANG

Abstract. Quantales can be regarded as a combination of complete lattices and semigroups. Unital quantales constitute a significant subclass within quantale theory, which play a crucial role in the theoretical framework of quantale research. It is well known that every complete lattice can support a quantale. However, the question of whether every complete lattice can support a unital quantale has not been considered before. In this paper, we first give some counter-examples to indicate that the answer to the above question is negative, and then investigate the complete lattices of supporting unital quantales.

1. Introduction

Originally, quantales were invented to formalize the logic of observables in quantum physics, which became useful to describe the "spaces" of C^*-algebras (see [3, 11, 12]) and non-commutative geometry (see [4, 14, 15, 16, 17, 19]). The literature on quantales often emphasizes that these structures are required to have units. Unital quantales, as a valued domain, have wide applications in theoretical computer science, logic, quantitative domain, enriched category and many-valued topological space (see [1, 6, 10, 18, 21, 22]). By definition, a *quantale* is a complete lattice Q with a semigroup multiplication & that distributes over arbitrary joins, that is,

$$a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i) \quad \text{and} \quad (\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a) \tag{1.1}$$

for all $a \in Q, \{b_i\}_{i\in I} \subseteq Q$. A quantale Q is said to be *unital* provided that there exists an element $e \in Q$ such that $q\&e = q = e\&q$ for all $q \in Q$. Q is said to be *commutative* provided that $a\&b = b\&a$ for all $a, b \in Q$.

Definition 1.1. Let Q be a complete lattice. Then we say that Q *supports a* (*unital*) *quantale* if there exists a semigroup multiplication & on Q such that $(Q, \&)$ becomes a (unital) quantale. An element e of Q *can become a unit* means that Q supports a unital quantale with unit e.

In [20], one can see that every complete lattice Q can support a quantale $(Q, \&_a)$ where $a \in Q$ and $\&_a$ is a semigroup multiplication determined by

$$\forall x, y \in Q,\ x\&_a y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ a, & \text{otherwise.} \end{cases} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 06F07, 06A06, 06B05.

Key words and phrases. Complete lattice, Semigroup, Quantale, Unital quantale.

This work was supported by the National Natural Science Foundation of China (Grant nos: 12471436, 12331016).

In [13], Paseka and Kruml investigated the construction of unital quantales from a given quantale, and they proved that every quantale Q can be embedded into a unital quantale $Q[e]$. For some special quantales, by addition of an isolated unit Gutiérrez García and Höhle considered the construction of unitally non-distributive quantales (see [8]). Based on the above work, Han and Lv further discussed the construction of unital quantales from a given quantale in [9]. According to the construction of unital quantale $Q[e]$, by (1.2) one can easily see that if a complete lattice L is isomorphic to $Q \times \mathbf{2}$ for some complete lattice Q, then L can support a unital quantale, where $\mathbf{2}$ is the two-element lattice. So, we have the following two questions.

Question 1: Does every complete lattice support a unital quantale?

If the answer to Question 1 is negative, then we have Question 2 to ask.

Question 2: Which class of complete lattices can support unital quantales?

In order to answer the above questions, we need to review some concepts and notations. Let L be a complete lattice, $a, b \in L$. Then L is said to be *trivial* if $| L |= 1$ and a is said to be *comparable* if $a \leq b$ or $b \leq a$ for all $b \in L$. Further, we let $[a, b] = \{x \in L \mid a \leq x \leq b\}$ and $(a, b] = \{x \in L \mid a < x \leq b\}$. a is called a *lower-cover* of b (or b is called an *up-cover* of a) if $a \neq b$ and $[a, b] = \{a, b\}$. A non-empty subset S of L is said to be a *sublattice* if S is closed under $\wedge$ and $\vee$. We denote by $L \times M$ the Cartesian product of complete lattices L and M. Then $L \times M$ is a complete lattice with pointwise order. In this paper, we denote the bottom element and the top element in a complete lattice by 0 and 1 respectively. For the notions and concepts, which are not explained, please refer to [2, 5, 7, 20].

2. Not every complete lattice can support a unital quantale

As we all know, every complete lattice can support a quantale. However, in this section we will prove that not every complete lattice can support a unital quantale, that is, the answer to Question 1 is negative. To see this, we need to consider the following complete lattices.

Lemma 2.1. *Let $(Q, \&)$ be a unital quantale with unit e. Then $a\&b \leq a \wedge b$ for all $a, b \in [0, e]$.*

Example 2.2. Let $N_5 = \{0, a, b, c, 1\}$ be a complete lattice which is determined by Figure 1. Then the top element 1 cannot become a unit.

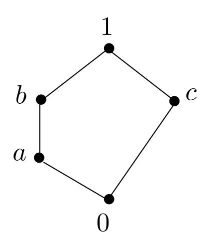

Figure 1. The partial order on N_5

To see this, we assume that 1 is the unit in quantale $(N_5, \&)$. Then by Lemma 2.1 we have that $b\&a \leq b \wedge a = a$ and $b\&c \leq b \wedge c = 0$, which implies that $b = b\&1 = b\&(a \vee c) = (b\&a) \vee (b\&c) = b\&a \leq a$, contradiction.

Example 2.3. Let $M_3 = \{0, a, b, c, 1\}$ be a complete lattice which is determined by Figure 2. Then the top element 1 cannot become a unit.

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

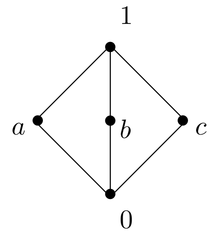

Figure 2. The partial order on M_3

To see this, we assume that 1 is the unit in quantale $(M_3, \&)$. Then by Lemma 2.1 we have that $a\&b \leq a \wedge b = 0$ and $a\&c \leq a \wedge c = 0$, which implies that $a = a\&1 = a\&(b \vee c) = (a\&b) \vee (a\&c) \leq 0$, contradiction.

Definition 2.4. Let L be a complete lattice and $x \in L\backslash\{0\}$. Then x is called an N_5-*element* (an M_3-*element*) of L if there exists some sublattice M of L such that x is the top element in M and M is isomorphic to N_5 (M_3).

Lemma 2.5. *Every* N_5*-element or* M_3*-element of a complete lattice* L *cannot become a unit of* L.

Proof. The proof follows from Examples 2.2 and 2.3. □

Proposition 2.6. *If every non-bottom element of a complete lattice* L *is either an* N_5*-element or an* M_3*-element, then* L *cannot support a unital quantale.*

Remark 2.7. It is easy to obtain an equivalent statement of Proposition 2.6, that is, if a complete lattice supports a unital quantale, then it must have a non-bottom element which is neither an N_5-element nor an M_3-element. Based on this equivalent statement, we see that the converse of Proposition 2.6 is in general not true (see Example 2.9 and Example 2.10).

Example 2.8. Let $L_1 = \{a_{ij} \mid 1 \leq j \leq 2^{i-1}, i = 1, 2, \cdots, n, \cdots\} \cup \{0\}$ be a complete lattice determined by Figure 3 in which every non-bottom element of L_1 has precisely two lower-covers, and every element of $L\backslash\{0, a_{11}\}$ has precisely one up-cover.

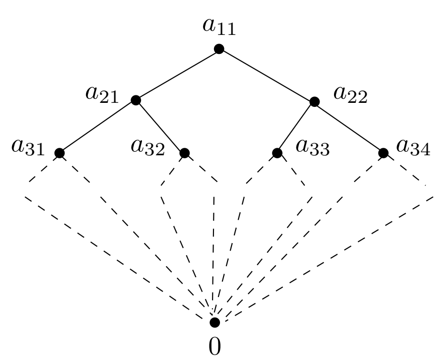

Figure 3. The partial order on L_1

It is easy to check that every non-bottom element of L_1 is an N_5-element. By Proposition 2.6, we have that L_1 cannot support a unital quantale.

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

Example 2.9. Let $L_2 = L_1 \cup \{a_{00}, a_{10}\}$ and the partial order on L_2 be determined by Figure 4. Then L_2 is a complete lattice. Further, one can see that every element of $L_2 \backslash \{a_{10}, 0\}$ is an N_5-element, and a_{10} is neither an N_5-element or an M_3-element. Surprisingly, L_2 can indeed support a unital quantale.

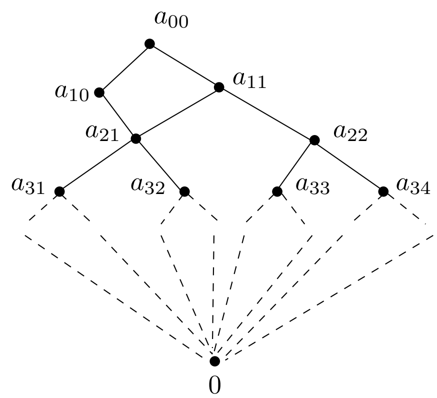

Figure 4. The partial order on L_2

To see this, we define now a binary operation & on L_2 as follows $\forall x, y \in L_2$,

$$x \& y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ 0, & \text{if } x \in (0, a_{21}], y \notin \{a_{10}, a_{00}\} \text{ or } x \notin \{a_{10}, a_{00}\}, y \in (0, a_{21}], \\ y, & \text{if } x = a_{10}, \\ x, & \text{if } y = a_{10}, \\ y, & \text{if } x = a_{00}, y \in (0, a_{21}], \\ x, & \text{if } x \in (0, a_{21}], y = a_{00}, \\ a_{00}, & \text{otherwise.} \end{cases}$$

Then $(L_2, \&)$ is a commutative unital quantale with unit a_{10}.

Example 2.10. Let I denote the unit open interval $(0, 1)$ with usual order $\leq$, $\Sigma = \{\sigma_r \mid r \in (0, 1)\}$ and $\Gamma = \{\tau_r \mid r \in (0, 1)\}$. Further, we let $r \in (0, 1)$, $X_r = \{x_{r,i} \mid i \in (0, 1)\}$, $Y_r = \{y_{r,j} \mid j \in (0, 1)\}$ and $L_3 = I \cup \Sigma \cup \Gamma \cup (\bigcup_{r \in I} (X_r \cup Y_r)) \cup \{\top, \bot\}$. Next, we define a binary relation $\leq_{L_3}$ on L_3 as follows: $\forall a, r, i, j \in I$, $x \in L_3$,

$$i \leq_{L_3} j \iff i \leq j, \ \sigma_i \leq_{L_3} \sigma_j \iff i \leq j \iff \tau_i \leq_{L_3} \tau_j,$$

$$x_{r,i} \leq_{L_3} x_{r,j} \iff i \leq j \iff y_{r,i} \leq_{L_3} y_{r,j},$$

$$a \leq_{L_3} \sigma_r \iff a \leq r \iff a \leq_{L_3} \tau_r,$$

$$a \leq_{L_3} x_{r,i} \iff a \leq r \iff a \leq_{L_3} y_{r,i},$$

$$x_{a,i} \leq_{L_3} \sigma_r \iff a \leq r \iff y_{a,j} \leq_{L_3} \tau_r,$$

$$\bot \leq_{L_3} x \leq_{L_3} \top.$$

It is easy to check that $(L_3, \leq_{L_3})$ is a complete lattice. Further, we see that every element of $\Sigma \cup \Gamma \cup \{\top\}$ is an N_5-element, and every element of $I \cup (\bigcup_{r \in I} (X_r \cup Y_r))$ is neither an N_5-element nor an M_3-element. However, L_3 indeed cannot support a unital quantale (see Proposition 2.12).

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

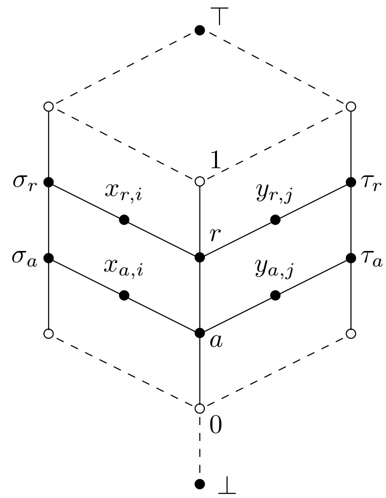

Figure 5. The partial order on L_3

For convenience, we use $\leq$ instead of $\leq_{L_3}$. The following lemma lists some properties of complete lattice L_3.

Lemma 2.11. *Let L_3 be the complete lattice in Example 2.10 and $a, r, i, j \in I$. Then*

(1) $\bigvee\limits_{b<r} b = r$, $\bigvee\limits_{r\in I} r = \top$ *and* $\bigvee\limits_{r\in I} \sigma_r = \top = \bigvee\limits_{r\in I} \tau_r$.
(2) $\bigwedge\limits_{j\in I} y_{r,j} = r = \bigwedge\limits_{i\in I} x_{r,i}$, $\bigvee\limits_{j\in I} y_{r,j} = \tau_r$ *and* $\bigvee\limits_{i\in I} x_{r,i} = \sigma_r$.
(3) $\bigvee\limits_{b<r} x_{b,j} = \sigma_r$ *and* $\bigvee\limits_{b<r} y_{b,j} = \tau_r$.
(4) *if* $a \neq r$, *then* $\sigma_r \vee a = \sigma_{r\vee a} = x_{r,i} \vee x_{a,j}$ *and* $\tau_r \vee a = \tau_{r\vee a} = y_{r,j} \vee y_{a,i}$.
(5) $\sigma_r \wedge a = r \wedge a = x_{r,i} \wedge y_{a,j}$ *and* $\tau_r \wedge a = r \wedge a = \sigma_r \wedge \tau_a = r \wedge \tau_a$.
(6) *if* $a \neq r$, *then* $x_{r,i} \wedge x_{a,j} = a \wedge r = y_{r,i} \wedge y_{a,j}$.
(7) *if* $a < r$, *then* $\sigma_a \wedge x_{r,i} = a$.
(8) *if* $x \in \Sigma \cup (\bigcup\limits_{r\in I} X_r), y \in \Gamma \cup (\bigcup\limits_{r\in I} Y_r)$, *then* $x \vee y = \top$.

Proposition 2.12. *L_3 cannot support a unital quantale.*

Proof. Assume that L_3 supports a unital quantale $(L_3, \&)$ with unit Υ. Then we have the following different cases for unit Υ.

Case 1. $\Upsilon \neq \perp$.

Assume that $\Upsilon = \perp$. Then by (1.1) one can see that $\perp = \perp \& \top = \Upsilon \& \top = \top$, contradiction.

Case 2. $\Upsilon \notin \Sigma \cup \Gamma \cup \{\top\}$.

Assume that $\Upsilon \in \Sigma \cup \Gamma \cup \{\top\}$. Since every element of $\Sigma \cup \Gamma \cup \{\top\}$ is an N_5-element, by Lemma 2.5 we have that Υ cannot become a unit, contradiction.

Case 3. $\Upsilon \notin I$.

Assume that $\Upsilon = r \in I$. Then we will prove that there exists $d < r$ such that $\sigma_d \leq x_{di}$, contradiction.

To see this, we first give the following asserts.

(1) For all $b \in I$ with $b < r$, we have that $b \leq \sigma_r \& b \leq \sigma_b$.

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

There exists an element $c \in I$ with $c < b$. By Lemma 2.11, we have $b = r\&b \leq \sigma_r\&b = (\sigma_c \vee r)\&b = (\sigma_c\&b) \vee (r\&b) \leq \sigma_c \vee b = \sigma_b$.

(2) There exists $r > b_0 \in I$ such that $b_0 < \sigma_r\&b_0 \leq \sigma_{b_0}$.

Assume that for all $b \in I$ with $b < r$, $b = \sigma_r\&b$. Then $r = \bigvee\limits_{b<r} b = \bigvee\limits_{b<r} (\sigma_r\&b) = \sigma_r\&(\bigvee\limits_{b<r} b) = \sigma_r\&r = \sigma_r$, which implies that $r = \sigma_r$, contradiction.

(3) For $b_0 < d < r$, we have that $\sigma_r\&d = \sigma_d$.

Since $b_0 < \sigma_r\&b_0 \leq \sigma_r\&d$, by (1) and Lemma 2.11 we have that $\sigma_r\&d = \sigma_d$.

By (3) and Lemma 2.11 we have that $\sigma_d = \sigma_r\&d = (x_{d,i} \vee r)\&d = (x_{d,i}\&d) \vee (r\&d) = (x_{d,i}\&d) \vee d \leq (x_{d,i}\&r) \vee d \leq x_{d,i} \vee d = x_{d,i}$, which implies that $\sigma_d \leq x_{d,i}$, contradiction. Thus, $\Upsilon \notin I$.

Case 4. $\Upsilon \notin \bigcup\limits_{r\in I} (X_r \cup Y_r)$.

Assume that $\Upsilon \in \bigcup\limits_{r\in I} (X_r \cup Y_r)$. Then we will prove that there exist $b, r \in I$ such that $b < r$, $(Y_r \cup \{\tau_r\}) \cap \{m \in L_3 \mid m \leq \tau_b\} \neq \emptyset$ or $(X_r \cup \{\sigma_r\}) \cap \{m \in L_3 \mid m \leq \sigma_b\} \neq \emptyset$, contradiction.

Without loss of generality, we suppose that $\Upsilon = x_{r,i_0} \in X_r$ for some $r \in I$. Then we have the following asserts.

(1) For all $b \in I$ with $b < r$, we have $\sigma_r\&b = b$ and $b\&\sigma_r = b$.

Since $b = x_{r,i_0}\&b \leq \sigma_r\&b = (\sigma_b \vee x_{r,i_0})\&b = (\sigma_b\&b) \vee (x_{r,i_0}\&b) \leq (\sigma_b\&x_{r,i_0}) \vee b = \sigma_b$, we have that $b \leq \sigma_r\&b \leq \sigma_b$. For all $x_{b,j}$, we have that $\sigma_r\&b = (x_{b,j} \vee x_{r,i_0})\&b = (x_{b,j}\&b) \vee (x_{r,i_0}\&b) \leq (x_{b,j}\&x_{r,i_0}) \vee b = x_{b,j}$. By Lemma 2.11, we have $b \leq \sigma_r\&b \leq \bigwedge\limits_{j\in I} x_{b,j} = b$, which implies that $\sigma_r\&b = b$. Similarly, we can show that $b\&\sigma_r = b$.

(2) $\sigma_r\&r = r = r\&\sigma_r$ and $r\&r = r$.

By (1) and Lemma 2.11, we have $r = \bigvee\limits_{b<r} b = \bigvee\limits_{b<r} (\sigma_r\&b) = \sigma_r\&(\bigvee\limits_{b<r} b) = \sigma_r\&r$. Similarly, we have $r = r\&\sigma_r$. For $b \in I$ with $b < r$, we have $\sigma_b\&r \leq \sigma_b \wedge r = b$ and $r = \sigma_r\&r = (\sigma_b \vee r)\&r = (\sigma_b\&r) \vee (r\&r) \leq b \vee (r\&r) \leq b \vee (r\&\sigma_r) \leq b \vee r = r$, which implies $r\&r = r$.

(3) For all $b \in I$ with $b < r$, we have $r\&b = b = b\&r$.

We let $c \in I$ and $c < b$. Then for all $x_{c,j}$, by (1) we have $x_{c,j}\&b \leq x_{c,j}\&x_{r,i_0} = x_{c,j}$ and $x_{c,j}\&b \leq \sigma_c\&b \leq \sigma_r\&b = b$, which implies that $x_{c,j}\&b \leq x_{c,j} \wedge b = c$. Thus, $b = \sigma_r\&b = (x_{c,j} \vee r)\&b = (x_{c,j}\&b) \vee (r\&b)$, which implies $b = r\&b$. Similarly, we can prove that $b = b\&r$.

(4) For all $b \in I$ with $b < r$, we have $r\&\sigma_b = b = \sigma_b\&r$.

By (2), we have that $\sigma_b\&r \leq \sigma_b\&x_{r,i_0} = \sigma_b$ and $\sigma_b\&r \leq \sigma_r\&r = r$, which implies $\sigma_b\&r \leq \sigma_b \wedge r = b$. It follows from (3) that $b = b\&r \leq \sigma_b\&r \leq b$, that is, $b = \sigma_b\&r$. Similarly, we can show that $r\&\sigma_b = b$.

(5) For all $b \in I$ with $b < r$, we have $r\&x_{b,j} = b = x_{b,j}\&r$.

By (3) and (4), we have that $b = r\&b \leq r\&x_{b,j} \leq r\&\sigma_b = b$, that is, $r\&x_{b,j} = b$. Similarly, we have $b = x_{b,j}\&r$.

(6) For all $b \in I$ with $b < r$, we have $\tau_r\&b = \top\&b$ and $b\&\tau_r = b\&\top$.

By (1) and (3), we have that $\tau_r\&b = (r \vee \tau_b)\&b = (r\&b) \vee (\tau_b\&b) = b \vee (\tau_b\&b) = (\sigma_r\&b) \vee (\tau_b\&b) = (\sigma_r \vee \tau_b)\&b = \top\&b$. Similarly, we have $b\&\tau_r = b\&\top$.

(7) $r < \tau_r\&r = \top\&r \leq \tau_r$.

By (6), we have that $\tau_r\&r = \tau_r\&(\bigvee\limits_{b<r} b) = \bigvee\limits_{b<r} (\tau_r\&b) = \bigvee\limits_{b<r} (\top\&b) = \top\&(\bigvee\limits_{b<r} b) = \top\&r$. Further, one can see that $r = r\&r \leq x_{r,i_0}\&r \leq \top\&r = \tau_r\&r \leq \tau_r\&x_{r,i_0} = \tau_r$.

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

Assume that $r = \tau_r\&r$. It follows from (5) and Lemma 2.11 that for all $b < r$ we have that $r = \top\&r = (x_{b,j} \vee \tau_b)\&r = (x_{b,j}\&r) \vee (\tau_b\&r) = b \vee (\tau_b\&r) \leq b \vee (\tau_b\&x_{r,i_0}) \leq b \vee \tau_b = \tau_b$, which implies $r \leq \tau_b$, contradiction. Thus, $r < \tau_r\&r = \top\&r \leq \tau_r$.

By (1)-(7), for all $b < r$, we see that $r < \top\&r = (\sigma_b \vee \tau_b)\&r = (\sigma_b\&r) \vee (\tau_b\&r) = b \vee (\tau_b\&r) = (b\&r) \vee (\tau_b\&r) = (b \vee \tau_b)\&r = \tau_b\&r \leq \tau_r\&r \leq \tau_r$, which implies that $r < \tau_b\&r \leq \tau_r$, that is, $\tau_b\&r \in Y_r \cup \{\tau_r\}$. Obviously, $\tau_b\&r \leq \tau_b\&x_{r,i_0} \leq \tau_b$. However, $(Y_r \cup \{\tau_r\}) \cap \{m \in L_3 \mid m \leq \tau_b\} = \emptyset$, contradiction. Thus, $\Upsilon \notin \bigcup\limits_{r\in I} X_r$. Similarly, we can show that $\Upsilon \notin \bigcup\limits_{r\in I} Y_r$.

From Cases 1-4, it follows that $\Upsilon \notin L_3$, contradiction. Therefore, L_3 cannot support a unital quantale. □

By Example 2.8 and Proposition 2.12, we have the below conclusion.

Proposition 2.13. *Not every complete lattice can support a unital quantale.*

3. Complete lattices supporting unital quantales

In Section 2, we see that not all complete lattices can support unital quantales. In this section, we will investigate Question 2, that is, which class of complete lattices can support unital quantales.

An element x of quantale Q is called a *zero-factor* of Q provided that $x \neq 0$ and there exists an element $y \in Q\backslash\{0\}$ such that $x\&y = 0$ or $y\&x = 0$.

Proposition 3.1. *Every complete chain supports a commutative unital quantale without zero-factors.*

Proof. Let I be a complete chain, $e \in I\backslash\{0\}$. Then we have two ways to define a binary operation on I such that I becomes a commutative unital quantale without zero-factors.

(1) The first way to define a binary operation & on I is given as follows

$$\forall x, y \in I,\ x\&y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ x \wedge y, & \text{if } x, y \in (0, e], \\ x \vee y, & \text{otherwise.} \end{cases}$$

It is easy to check that $(I, \&)$ is a commutative unital quantale with unit e and L has no zero-factors.

(2) The second way to define a binary operation $\otimes$ on I is given as follows

$$\forall x, y \in I,\ x \otimes y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ x \wedge y, & \text{if } x, y \in (0, e], \\ y, & \text{if } x \in (0, e],\ y \in (e, 1], \\ x, & \text{if } y \in (0, e],\ x \in (e, 1], \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to check that $(I, \otimes)$ is a commutative unital quantale with unit e and L has no zero-factors. □

Proposition 3.2. *Let Q be a complete lattice and $e \in Q\backslash\{0\}$. If $[0, e]$ supports a* (*commutative*) *unital quantale with unit k and Q satisfies the condition* $(*)$

$$\forall x \in (0, e], y \in Q\backslash[0, e],\ x \leq y \Longleftrightarrow e \leq y, \tag{$*$}$$

then Q supports a (*commutative*) *unital quantale with unit k.*

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

Proof. Let $([0,e],\&)$ be a (commutative) unital quantale with unit k. Then $\&$ can be extended to a binary operation $\otimes$ on Q as follows

$$\forall x,y\in Q,\ x\otimes y=\begin{cases}0, & \text{if }\ x=0 \text{ or } y=0,\\ y, & \text{if }\ x\in(0,e],\ y\in Q\backslash[0,e],\\ x, & \text{if }\ x\in Q\backslash[0,e],\ y\in(0,e],\\ x\& y, & \text{if }\ x,y\in(0,e],\\ 1, & \text{if }\ x,y\in Q\backslash[0,e].\end{cases}$$

It can be verified that $(Q,\otimes)$ is a (commutative) unital quantale with unit k. Further, if $([0,e],\&)$ has no zero-factors, then $(Q,\otimes)$ has no zero-factors. □

Corollary 3.3. Let Q be a complete lattice and $e\in Q\backslash\{0\}$. If $[0,e]$ is a chain and Q satisfies the condition $(*)$, then Q supports a commutative unital quantale without zero-factors.

Proof. The proof follows immediately from Propositions 3.1 and 3.2. □

Corollary 3.4. Every complete lattice with non-bottom minimal element supports a commutative unital quantale without zero-factors.

Proof. Let Q be a complete lattice with non-bottom minimal element e. Then $[0,e]=\{0,e\}$ and Q satisfies the condition $(*)$. By Corollary 3.3, we have that Q supports a commutative unital quantale without zero-factors. □

Corollary 3.5. Every finite complete lattice supports a commutative unital quantale without zero-factors.

Proposition 3.6. *Let Q be a complete lattice and $a\in Q\backslash\{0,1\}$ be a comparable element. Then we have*

(1) *if $[a,1]$ is a chain, then Q supports a commutative unital quantale.*
(2) *if $[0,a]$ is a chain, then Q supports a commutative unital quantale without zero-factors.*

Proof. (1) Assume that $[a,1]$ is a chain. Then we can define a binary operation $\otimes$ on Q as follows

$$\forall x,y\in Q,\ x\otimes y=\begin{cases}0, & \text{if }\ x,y\in[0,a],\\ x\wedge y, & \text{otherwise.}\end{cases}$$

It is easy to verify that $(Q,\otimes)$ is a commutative unital quantale with unit 1.

(2) The proof follows directly from Propositions 3.1 and 3.2. □

Let Q be a complete lattice and $\{\perp,e,\top\}\cap Q=\emptyset$. Next, we will consider three types of extensions $Q_\perp,Q_e,Q_\top$ of Q, where $Q_\perp=Q\cup\{\perp\},Q_e=Q\cup\{e\},Q_\top=Q\cup\{\top\}$. We respectively define the relations $\le_{Q_\perp},\le_{Q_e},\le_{Q_\top}$ as follows

$$\le_{Q_\perp}=\le_Q\cup\{(\perp,x)\mid x\in Q_\perp\},$$
$$\le_{Q_e}=\le_Q\cup\{(0,e),(e,e),(e,1)\},$$
$$\le_{Q_\top}=\le_Q\cup\{(x,\top)\mid x\in Q_\top\}.$$

It is easily checked that $(Q_\perp,\le_{Q_\perp}),(Q_e,\le_{Q_e})$ and $(Q_\top,\le_{Q_\top})$ are complete lattices, called the *extensions* of Q. By Corollary 3.3 and Proposition 3.6, we have the following result.

Corollary 3.7. For any complete lattice Q, the extensions $Q_\perp,Q_e,Q_\top$ can all support unital quantales.

The proof of Theorem 3.8 can be found in Theorem 2.4 in [9]. For the convenience of readers, we will give the main details.

Theorem 3.8. *Let Q be a complete lattice with $Q = M \times K$. If M or K supports a commutative unital quantale without zero-factors, then Q supports a unital quantale.*

Proof. Without loss of generality, we assume that $(K, \&)$ is a commutative unital quantale without zero-factors and e is the unit. By (1.2), we have that $(M, \&_a)$ is a quantale with $a \in M$. Next, we define a binary operation $\&_*$ on Q as follows: $\forall (m, k), (n, k') \in M \times K$,

$$(m, k)\&_*(n, k') = ((m\&_a n) \vee (m * k') \vee (n * k), k\&k'),$$

where $*$ is a module operator from $M \times K$ into M defined by

$$\forall m \in M, k \in K,\ m * k = \begin{cases} 0, & \text{if } \ k = 0, \\ m, & \text{otherwise.} \end{cases}$$

It can be verified that $(Q, \&_*)$ is a unital quantale with unit $(0, e)$. □

Corollary 3.9. Let Q be a complete lattice with $Q = M \times K$ satisfying one of the following conditions:

(1) M or K is a non-trivial complete chain;
(2) M or K is a non-trivial finite complete lattice;
(3) M or K has a comparable element e such that $[0, e]$ is a chain.

Then Q supports a unital quantale.

Proof. The proof follows from Proposition 3.1, Corollary 3.5, Proposition 3.6(2) and Theorem 3.8. □

In the following, we will give a complete lattice which does not belong to any of the aforementioned types of complete lattices supporting unital quantales.

Example 3.10. Let I denote the unit open interval $(0, 1)$ with usual order $\leq$, $\Sigma = \{\sigma_r \mid r \in (0, 1)\}$. Further, we let $X = \{x_r \mid r \in (0, 1)\}$ and $L_4 = I \cup \Sigma \cup X \cup \{\top, \bot\}$. We define a binary relation $\leq_{L_4}$ on L_4 as follows: $\forall r, i, j \in I$, $x \in_{L_4}$,

$$i \leq_{L_4} j \Longleftrightarrow i \leq j \Longleftrightarrow \sigma_i \leq_{L_4} \sigma_j,$$

$$i \leq_{L_4} \sigma_j \Longleftrightarrow i \leq j \Longleftrightarrow i \leq_{L_4} x_j,$$

$$x_r \leq_{L_4} \sigma_j \Longleftrightarrow r \leq j,$$

$$\bot \leq_{L_4} x \leq_{L_4} \top.$$

It is easy to check that $(L_4, \leq_{L_4})$ is a complete lattice.

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

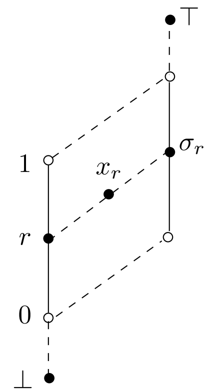

Figure 6. The partial order on L_4

We let $I_r = \{a \in I \mid a \leq r\}, X_r = \{x_a \in X \mid a \leq r\}$ and $\Sigma_r = \{\sigma_a \in \Sigma \mid a \leq r\}$. Next, we define a binary operation $\otimes$ on L_4 as follows: $\forall x, y \in L_4$,

$$x \otimes y = \begin{cases} \perp, & \text{if } x = \perp \text{ or } y = \perp, \\ x \wedge y, & \text{if } x, y \in I_r, \\ x_{a\wedge c}, & \text{if } x = x_a \in X_r, y = x_c \in X_r, \\ \sigma_{a\wedge c}, & \text{if } x = \sigma_a \in \Sigma_r, y = \sigma_c \in \Sigma_r, \\ a \wedge c, & \text{if } x = a \in I_r,\ y = x_c \in X_r \text{ or } x = x_a \in X_r, y = c \in I_r, \\ a \wedge c, & \text{if } x = a \in I_r,\ y = \sigma_c \in \Sigma_r \text{ or } x = \sigma_a \in \Sigma_r, y = c \in I_r, \\ \sigma_{a\wedge c}, & \text{if } x = x_a \in X_r, y = \sigma_c \in \Sigma_r \text{ or } x = \sigma_c \in \Sigma_r, y = x_a \in X_r, \\ y, & \text{if } x \in I_r \cup \Sigma_r \cup X_r,\ y \in L_4 \backslash (I_r \cup \Sigma_r \cup X_r), \\ x, & \text{if } y \in I_r \cup \Sigma_r \cup X_r,\ x \in L_4 \backslash (I_r \cup \Sigma_r \cup X_r), \\ \top, & \text{otherwise.} \end{cases}$$

It is easy to verify that $(L_4, \otimes)$ is a commutative unital quantale with unit x_r and L_4 has no zero-factors.

Further, we will obtain more interesting observations for complete lattice L_4.

(1) Since every element of Σ is an N_5-element, it follows from Lemma 2.5 that $\sigma_r \in \Sigma$ cannot become a unit.

(2) Every element of $I \cup X$ is neither an N_5-element nor an M_3-element. Based on the above description, we see that every element x_r of X can become a unit. However, every element of I cannot become a unit (see the proof of Case 3 in Proposition 2.12).

(3) The top element $\top$ is neither an N_5-element nor an M_3-element. Also, $\top$ cannot become a unit.

Assume that $\top$ can become a unit in quantale $(L_4, \&)$. For $d, r \in I$, we have that $d\&\sigma_r \leq d \wedge \sigma_r \leq r$, which implies that $\sigma_r = \top\&\sigma_r = (\bigvee_{r<d} d)\&\sigma_r = \bigvee_{r<d}(d\&\sigma_r) \leq r$, contradiction.

(4) L_4 does not belong to any of the aforementioned types of complete lattices supporting unital quantales in Propositions 3.1, 3.2, 3.6 and Theorem 3.8.

To see this, L_4 is obviously not a chain and also L_4 has no non-trivial comparable elements. Further, we see that there are no non-bottom elements e in L_4 such that L_4 satisfies the condition $(*)$, and L_4 is not isomorphic to any product lattice $M \times K$ for all non-trivial complete lattices M and K.

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

4. Conclusion

From Remark 2.7, we see that if a complete lattice supports a unital quantale, then it must have a non-bottom element which is neither an N_5-element nor an M_3-element. However, the converse is in general not true. From Example 2.10 and Example 3.10, one can see that for the non-bottom elements of a complete lattice which are neither an N_5-element nor an M_3-element, some can become a unit while others cannot. So, it is not easy to determine whether a complete lattice can support a unital quantale. In other words, it is very difficult to give a necessary and sufficient condition for a complete lattice to support a unital quantale.

Acknowledgements. The authors are very grateful to the referees for the very careful reading of the manuscript and very useful comments.

References

[1] S. Abramsky and S. Vickers, *Quantale, observational logic and process semantics*. Math. Struct. Comput. Sci. **3**(1993), no. 2, 161–227.
[2] G. Birkhoff, *Lattice theory*. American Mathematical Society, New York, Providence, RI, 1940.
[3] F. Borceux, J. Rosický, G. Van den Bossche, *Quantales and C^*-algebras*. J. London Math. Soc. **40**(1989), no. 3, 398–404.
[4] A. Connes, *Noncommutative Geometry*. Academic Press, San Diego, 1994.
[5] P. Eklund, J. Gutiérrez García, U. Höhle, and J. Kortelainen, *Semigroups in Complete Lattices: Quantales, Modules and Related Topics*. Developments in Mathematics, vol. 54. Springer, Berlin, 2018.
[6] L. Fan, *A new approach to quantitative domain theory*. Electron. Notes Theor. Comput. Sci. **45**(2001), 77–87.
[7] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott, *A Compendium of Continuous Lattices*. Springer-Verlag, New York, 1980.
[8] J. Gutiérrez García, U. Höhle, *Unitally nondistributive quantales*. Port. Math. **82**(2025), no. 3-4, 259-280.
[9] S. Han, J. Lv, *Construction of unital quantales*. Canad. Math. Bull. **68**(2025), no. 3, 761-776.
[10] D. Hofmann and P. Waszkiewicz, *Approximation in quantale-enriched categories*. Topol. Appl. **158**(2011), no. 8, 963–977.
[11] D. Kruml, P. Resende, *On quantales that classify C^*-algebras*. Cah. Topol. Géom. Différ. Catég. **45**(2004), no. 4, 287–296.
[12] D. Kruml and J. Paseka, *Algebraic and categorical aspects of quantales*. Handbook of Algebra **5**(2008), 323–326.
[13] J. Paseka and D. Kruml, *Embeddings of quantales into simple quantales*. J. Pure Appl. Algebra **148**(2000), no. 2, 209–216.
[14] C.J. Mulvey, J.W. Pelletier, *On the quantisation of points*. J. Pure Appl. Algebra **159**(2001), no. 2-3, 231–295.
[15] C.J. Mulvey, J.W. Pelletier, *On the quantisation of spaces*. J. Pure Appl. Algebra **175**(2002), no. 1-3, 289–325.
[16] C.J. Mulvey, P. Resende, *A noncommutative theory of Penrose tilings*. Internat. J. Theoret. Phys. **44**(2005), no. 6, 655–689.
[17] M.C. Protin, P. Resende, *Quantales of open groupoids*. J. Noncommut. Geom. **6**(2012), no. 2, 199–247.
[18] P. Resende, *Quantales, finite observation and strong bismulation*. Theoret. Comput. Sci. **254**(2001), no. 1-2, 95–149.
[19] P. Resende, *Étale groupoids and their quantales*. Adv. Math. **208**(2007), no. 1, 147–209.
[20] K.I. Rosenthal, *Quantales and their Applications*. Longman Scientific & Technical, New York, 1990.
[21] D.N. Yetter, *Quantales and (noncommutative) linear logic*. J. Symb. Log. **55**(1990), no. 1, 41–64.
[22] D. Zhang, *An enriched category approach to many valued topology*. Fuzzy Sets Syst. **158**(2007), 349–366.

https://doi.org/10.4153/S0008439525101409 Published online by Cambridge University Press

Department of Mathematics, Shaanxi Normal University, Xi'an, 710119, China
Email address: hansw@snnu.edu.cn

Email address: wangxin2020@snnu.edu.cn