GROUPS OF COMPLEXES OF A REPRESENTABLE LATTICE-ORDERED GROUP

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1. Introduction. In 1954 N. Kimura proved that each idempotent in a semigroup is contained in a unique maximal subgroup of the semigroup and that distinct maximal subgroups are disjoint [**13**] (or see [**6,** pp. 21–23]). This generalized earlier results of Schwarz [**14**] and Wallace [**15**]. These maximal subgroups are important in the study of semigroups. If G is a group, then the collection $S(G)$ of nonempty complexes of G is a semigroup and it is natural to inquire what properties of G are inherited by the maximal subgroups of $S(G)$. There seems to be very little literature devoted to this subject. In [**5,** Theorem 2], with certain hypotheses placed on an idempotent, it was shown that if G is a lattice-ordered group ("l-group") then a maximal subgroup of $S(G)$ containing an idempotent satisfying these conditions admits a natural lattice-order. The main result of this note (Theorem 1) is that if G is a representable l-group and E is a normal idempotent of $S(G)$ and a dual ideal of the lattice G, then the maximal subgroup of $S(G)$ containing E admits a representable lattice-order.

2. Notation and terminology. The collection $S(G)$ of all nonempty complexes (subsets) of a group G is a semigroup with respect to the binary operation $AB = \{ab \mid a \in A \text{ and } b \in B\}$ for $A, B \in S(G)$. If E is an idempotent in $S(G)$, then $H(E)$ denotes the maximal subgroup of $S(G)$ that contains E. A *normal* idempotent E of $S(G)$ is an idempotent of $S(G)$ and normal subset of G. If E is a normal idempotent of $S(G)$, then $T(E) = \{aE \mid a \in G\}$ is a subgroup of $H(E)$ and is isomorphic to G modulo the kernel of the mapping γ which sends x to xE for all $x \in G$. Moreover, if $A \in H(E)$, then $xAx^{-1} \in H(E)$ for all $x \in G$. If $1 \in E$, where E is a normal idempotent of $S(G)$, then $T(E) = H(E)$ [**4,** Proposition 4], and hence any property of G that is preserved by homomorphic images will be inherited by $H(E)$. Consequently, the "more interesting" cases of maximal subgroups of $S(G)$ occur when the identity element of G does not belong to the idempotent.

For the remainder of this note, we assume that G is an l-group, E is a normal idempotent of $S(G)$, and γ is the mapping of G into $H(E)$ given by $\gamma(x) = xE$ for all $x \in G$. For the standard definitions and results concerning l-groups the reader is referred to [**1**], [**10**], and [**12**]. An l-group G is said to be *representable* if there exists an l-isomorphism of G into a cardinal sum of totally ordered groups ("o-groups"). A *dual ideal* of G is a nonempty subset I of G such that $a, b \in I$, $x \in G$, and $x \geq a$ imply $a \wedge b$, $x \in I$. A *prime* subgroup of G is a convex l-subgroup such that if $a, b \in G^+ \backslash M$, then $a \wedge b \in G^+ \backslash M$.

Let E be a dual ideal of G. For an element A in $H(E)$, define $A \geq E$ if and only if $A \subseteq E$. Then $H(E)$ is an l-group with positive cone $\{A \mid A \geq E\}$, $T(E)$ is an l-subgroup of $H(E)$ with $aE \vee bE = (a \vee b)E$ and dually for all $a, b \in G$, and the kernel of γ is an l-ideal of G [**5,** Theorem 2]. It was shown in the proof of this theorem that if $A \in H(E)$, then $A \vee E = \cup (a \vee 1)E(a \in A)$.

If X and Y are sets, then $X\backslash Y$ denotes the set of elements in X but not in Y, and if G is an l-group, then $G^+=\{g\in G \mid g\geq 1\}$.

3. Representability of H(E). If M is an l-subgroup of G such that for every $g\in G$ with $g>1$ there exists an $a\in M$ with $g\geq a>1$, then M is said to be *dense* in G. If M is dense in G, then joins and meets in M agree with those in G [**2,** Lemma 10]. The proofs of the next two lemmas are straightforward and will be omitted.

LEMMA 1. *Let G and M be l-groups, η an l-homomorphism of G onto M, and E a dual ideal of G. Then $E^*=\eta(E)$ is a normal idempotent of $S(M)$, a dual ideal of M, and if $A\in H(E)$, then $\eta(A)\in H(E^*)$. Let $\eta^*: H(E)\to H(E^*)$ be given by $\eta^*(A)=\eta(A)$ for all $A\in H(E)$, and let $\gamma^*: M\to H(E^*)$ be the l-homomorphism given by $\gamma^*(x)=xE^*$ for all $x\in M$. Then η^* is an l-homomorphism making the following diagram commute*

$$\begin{array}{ccc} G & \xrightarrow{\eta} & M \\ \gamma\downarrow & & \downarrow\gamma^* \\ H(E) & \xrightarrow{\eta^*} & H(E^*). \end{array}$$

LEMMA 2. *If E is a dual ideal of an o-group, then $H(E)$ is an o-group.*

LEMMA 3. *If E is a dual ideal of G, then $T(E)$ is dense in $H(E)$. Moreover, if $A\in H(E)$, then there is a $g\in G$ with $A\leq gE$.*

Proof. Let $A\in H(E)$ such that $E<A$ and let B be the inverse of A in $H(E)$. We may assume $A\notin T(E)$. Then we have $B<E<A$ and $B\notin T(E)$. Thus $B<bE$ for all $b\in B$. Now

$$B=B\wedge E<bE\wedge E=(b\wedge 1)E\leq E$$

for all $b\in B$. Suppose that $(b\wedge 1)E=E$ for all $b\in B$. Then $E=(b\wedge 1)E\leq bE$ for all $b\in B$ and so $B=BE=\bigcup_{b\in B} bE\subseteq E$, a contradiction. Thus $B<(b\wedge 1)E<E$ for some $b\in B$ and so $E<(b\wedge 1)^{-1}E<A$. Therefore $T(E)$ is dense in $H(E)$. The second statement of the lemma is clear.

An element b of G is called *basic* if $b>1$ and $\{x \mid x\in G$ and $1<x\leq b\}$ is totally ordered. A subset B of G is a *basis* if B is a maximal set of (pairwise) disjoint elements in G and each b in B is basic. An l-group G has a basis if and only if for every $g\in G$ with $g>1$, g exceeds a basic element [**7,** Theorem 5.1]. An *atom* is an element of G that covers 1.

COROLLARY 1. *If E is a dual ideal of G, γ is one-to-one, and b is basic (respectively, an atom), then bE is basic (respectively, an atom) in $H(E)$. Hence, if G has a basis, then $H(E)$ has a basis.*

THEOREM 1. *If E is a dual ideal of G and G is representable, then $H(E)$ is representable.*

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Proof. By [**8,** Lemma 3] it suffices to show that for each strictly positive element A of $H(E)$, there is an l-homomorphism η^* from $H(E)$ into an o-group such that $\eta^*(A)\neq 1$. Since $T(E)$ is dense in $H(E)$, it suffices to take $A=aE$ where $a>1$. We first consider the case in which the mapping γ is one-to-one. We assert that in this case $E\subseteq G^+$. If $b\in E$, then $bE\subseteq E$ and so $bE\geq E$. Thus $bE=bE\vee E=(b\vee 1)E$. Since γ is one-to-one, $b=b\vee 1\in G^+$. Now suppose that $a\leq e$ for every $e\in E$. Then, if $e\in E$, $e=e_1e_2$, where e_1, $e_2\in E$ and $a\leq e_1$. Hence $ae_2\leq e_1e_2=e$ and so $e\in aE$. Since aE is positive, $aE\subseteq E$. But then $aE=E$ contrary to the choice of aE. It follows that $a\wedge e<a$ for some $e\in E$. Since G is representable, there is a minimal prime subgroup M of G which is normal [**3,** Theorem 3.1], G/M is an o-group, and $(a\wedge e)M<aM$. Consider the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta} & G/M \\ \gamma\downarrow & & \downarrow\gamma^* \\ H(E) & \xrightarrow{\eta^*} & H(E^*) \end{array}$$

where η is the canonical mapping, $E^*=\eta(E)$, $\gamma^*(xM)=xME^*$ for every $xM\in G/M$, and $\eta^*(A)=\eta(A)$ for every $A\in H(E)$. By Lemma 1, η^* is an l-homomorphism and by Lemma 2, $H(E^*)$ is an o-group. Now $\eta(a)\wedge\eta(e)=\eta(a\wedge e)<\eta(a)$ and since G/M is totally ordered, we have that $\eta(e)<\eta(a)$. If $f\in E$, then $\eta(af)=\eta(a)\eta(f)\geq\eta(a)>\eta(e)$. Therefore $\eta(aE)\neq\eta(E)$ and so $\eta^*(aE)\neq E^*$. Thus $H(E)$ is representable.

We next consider the general case. Let K denote the kernel of γ and consider the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta} & G/K \\ \gamma\downarrow & & \downarrow\gamma^* \\ H(E) & \xrightarrow{\eta^*} & H(E^*) \end{array}$$

where η, E^*, γ^*, and η^* are as given above. We assert that γ^* is one-to-one. If $x,y\in G$ with $\gamma^*\eta(x)=\gamma^*\eta(y)$, then $\eta(x)E^*=\eta(y)E^*$ and so $xKE=yKE$. Since $KE=E$, we have that $xE=yE$ and so $\eta(x)=\eta(y)$. Now G/K is representable and therefore, by the previous case, there is an l-homomorphism θ of $H(E^*)$ into an o-group such that $\theta\eta^*(aE)\neq 1$. It now follows that $H(E)$ is representable.

An l-group G is said to be *epi-Archimedean* (or *hyper-Archimedean*) if each l-homomorphic image of G is Archimedean. In [**11,** Theorem 1.1] five conditions are given each of which is equivalent to the epi-Archimedean property.

THEOREM 2. *If E is a dual ideal of G and G is epi-Archimedean, then $H(E)$ is Archimedean.*

Proof. Let $A,B\in H(E)$ such that $A>E$ and $B>E$. By Lemma 3, there exist $a,b\in G^+$ such that $A\geq aE>E$ and $bE\geq B$. Since $T(E)$ is an l-homomorphic image of G, $T(E)$ is Archimedean. Thus there exists a positive integer n such that $a^nE\not\leq bB$. Hence $A^n\not\leq B$ and so $H(E)$ is Archimedean.

4. Examples. In this section we give some examples to illustrate the scope and limitations of our results.

Example 1. Let $G=\sum_{i=1}^{n} \mathbf{Q}$ be the cardinal product of n copies of the rational numbers $\mathbf{Q}$ and let $E=\{(a_1, a_2, \ldots, a_n) \mid a_i>0,\ i=1, 2, \ldots, n\}$. Then E is a normal idempotent and a dual ideal of G. In this case $H(E)$ is l-isomorphic to $\sum_{i=1}^{n} \mathbf{R}$, the cardinal sum of n copies of the real numbers.

In the above example E is the collection of units. In any l-group the collection of units is a normal subset, and, when nonempty, is a dual ideal. However, in general, the collection of units will not be an idempotent.

Example 2. Let M be a normal proper prime subgroup of G such that G/M has no atoms. Then $E=\cup(xM)$ $(xM>M)$ is a normal idempotent and a dual ideal. Moreover, M is the kernel of γ. If G is Archimedean and M is not a maximal ideal of G, then $H(E)$ is not Archimedean.

If G is an l-subgroup of an l-group M, then M is said to be an *a-extension* of G provided that for every $1<a\in M$ there exists $g\in G$ and there exist positive integers m and n such that $g\le a^m$ and $a\le g^n$. If G is an l-subgroup of M, then M is an a-extension of G if and only if the mapping $C\to C\cap G$ is a one-to-one mapping of the lattice of convex l-subgroups of M onto the lattice of convex l-subgroups of G[**9,** Theorem 2.1]. Our next example shows that even when γ is one-to-one, $H(E)$ need not be an a-extension of G. The example also shows that when γ is one-to-one and G is epi-Archimedean, then $H(E)$ need not be epi-Archimedean, and thus Theorem 2 is in some sense the best possible.

Example 3. Let $G=\{f:\mathbf{Z}^+\to\mathbf{R} \mid$ there exists $N\in\mathbf{Z}^+$ with $f(n)=f(N)$ for all $n\ge N\}$, with $f\le g$ if $f(n)\le g(n)$ for all $n\in\mathbf{Z}^+$, and function addition is the group operation. Let $E=\{f\in G \mid f(n)>0$ for all $n\in\mathbf{Z}^+\}$. Then E is a normal idempotent and a dual ideal. In fact, E is the collection of units of G. Let $A=\{f\in G \mid f(n)>1/(n+1)$ for all $n\in\mathbf{Z}^+\}$. Then $A\in H(E)$ and $E<A$. If there exists $g\in G$ and a positive integer m with $gE\le A^m$ then there is a positive integer N with $g(k)\le 0$ if $k\ge N$. But if $g(k)\le 0$ for $k\ge N$, then $A\nleq g^nE$ for all $n\in\mathbf{Z}^+$. Thus $H(E)$ is not an a-extension of G.

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