4

Higher-Dimensional Rectifiable Sets

Federer generalized most of Besicovitch's theory to higher dimensions in [199]. Most of the proofs, or sketches with further references, for the results of this section can be found in [203], [297] and [321].

4.1 Definitions and Area and Coarea Formulas

We now define

Definition 4.1 A set $E \subset \mathbb{R}^n$ is *m-rectifiable* if there are Lipschitz maps $f_i: \mathbb{R}^m \to \mathbb{R}^n, i = 1, 2, \dots$ such that

$$\mathcal{H}^m\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)\right) = 0.$$

A set $E \subset \mathbb{R}^n$ is *purely m-unrectifiable* if $\mathcal{H}^m(E \cap F) = 0$ for every m-rectifiable set $F \subset \mathbb{R}^n$.

Usually m and n will be integers with $0 < m < n$, but sometimes m can be 0; then 0-rectifiable means countable. We shall often also consider rectifiability of measures:

Definition 4.2 A measure μ on $\mathbb{R}^n$ is *m-rectifiable* if there are Lipschitz maps $f_i: \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)\right) = 0.$$

μ is *purely m-unrectifiable* if $\mu(f(\mathbb{R}^m)) = 0$ for every Lipschitz map $f: \mathbb{R}^m \to \mathbb{R}^n$.

Often the condition $\mu \ll \mathcal{H}^m$ is added, but in many places later it is better to have the definition without it. In some texts it is only required that $\mu(\mathbb{R}^n \setminus E) = 0$ for some m-rectifiable set E.

So the m-rectifiability of E means that $\mathcal{H}^m \llcorner E$ is m-rectifiable. We can uniquely decompose any $\mu \in \mathcal{M}(\mathbb{R}^n)$ as $\mu = \mu_r + \mu_u$, where μ_r is m-rectifiable and μ_u is purely m-unrectifiable.

It often is useful to use other sets in place of Lipschitz images. The following alternatives give equivalent definitions:

(1) Lipschitz images of arbitrary or compact subsets of $\mathbb{R}^m$.
(2) C^1 images of $\mathbb{R}^m$, or of arbitrary or compact subsets of $\mathbb{R}^m$.
(3) Lipschitz (or C^1) graphs over subsets of m-planes.
(4) m-Dimensional C^1 submanifolds of $\mathbb{R}^n$.
(5) Level sets of regular C^1 mappings $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$, that is, sets $\{x \in A : f(x) = y\}$, where $f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is C^1 and $Df(x)$ has rank $n - m$ for $x \in A$.

The proofs are routine verifications applying classical theorems of analysis: Lipschitz and Whitney's extension theorems and the implicit function theorem, and in particular Rademacher's theorem according to which a Lipschitz map f is differentiable at almost every point x. That is, there is a linear map $Df(x)$ such that

$$f(y) - f(x) = Df(x)(y - x) + |x - y|\varepsilon(|x - y|), \ \lim_{h\to 0} \varepsilon(h) = 0. \tag{4.1}$$

From this we can often go from properties of linear maps to Lipschitz maps $f : \mathbb{R}^m \to \mathbb{R}^n, m \le n$. For example, there is a Jacobian $Jf(x)$ defined in terms of the partial derivatives of f such that $\mathcal{H}^m(Df(x)(A)) = Jf(x)\mathcal{L}^m(A)$ for Lebesgue measurable sets $A \subset \mathbb{R}^m$. This leads (but not trivially, see [203, Section 3.2] or [189, Section 3.3]) to

$$\text{Area formula:} \quad \int \operatorname{card} A \cap f^{-1}\{y\}\, d\mathcal{H}^m y = \int_A Jf(x)\, d\mathcal{L}^m x. \tag{4.2}$$

The proof consists of splitting $A = \cup_{i=1}^{\infty} A_i \cup B$. In each A_i, the $Df(x)$ is injective and close to a linear map L_i. In B, $Jf(x) = 0$ and both sides of (4.2) are 0 for $A = B$.

There is also the Fubini-type coarea formula for Lipschitz maps $f : \mathbb{R}^n \to \mathbb{R}^m, m \le n$:

$$\text{Corea formula:} \quad \int \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, d\mathcal{L}^m y = \int_A Jf(x)\, d\mathcal{L}^n x. \tag{4.3}$$

Here the Jacobian $J(x)$ of f at x again is a kind of determinant of $Df(x)$ determined by the property that the coarea formula is valid for linear maps. See

[203, Section 3.2] or [189, Section 3.4] for the linear algebraic definition and the proofs.

With some analysis this gives the rectifiability of level sets, see [203, Theorem 3.2.15] or [397, Remark 12.8]:

Theorem 4.3 *If $f : \mathbb{R}^n \to \mathbb{R}^m, m \leq n$ is Lipschitz, then $f^{-1}\{y\}$ is $(n-m)$-rectifiable for $\mathcal{L}^m$ almost all $y \in \mathbb{R}^m$.*

Here 0-rectifiable means finite or countable.

The area and coarea formulas extend to Lipschitz maps between rectifiable sets, see Corollary 3.2.20 and Theorem 3.2.22 in [203].

4.2 Tangent Planes

For the tangential properties, we again define the cones

$$X(a, V, s) = \{x \in \mathbb{R}^n : d(x, V + a) < s|x - a|\}, \ a \in \mathbb{R}^n, V \in G(n, m), s > 0,$$

and the approximate tangent planes

Definition 4.4 A plane $V \in G(n, m)$ is an *approximate tangent plane* of a set $E \subset \mathbb{R}^n$ at a point $a \in \mathbb{R}^n$ if $\Theta^{*m}(E, a) > 0$ and for every $s > 0$,

$$\lim_{r \to 0} r^{-m} \mathcal{H}^m (E \cap B(a, r) \setminus X(a, V, s)) = 0.$$

We then denote $V = \mathrm{apTan}(E, a)$.

Observe that as with Hausdorff measure, approximate tangent plane is also a metric concept. We shall use exactly the same definition for other metrics.

The characterization of rectifiability follows by similar arguments as in the one-dimensional case:

Theorem 4.5 *If E is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(E) < \infty$, then E is m-rectifiable if and only if it has an approximate tangent plane at $\mathcal{H}^m$ almost all of its points.*

So E is m-rectifiable if for almost all $a \in E$ there is an m-plane that approximates E well in all small balls $B(a, r)$. But what if we only have such an approximation in the weaker sense that the approximating plane is allowed to depend on the scale r? This is not enough, as easy examples show – not even if E would have positive lower density. One such example can be constructed as a subset of the modified von Koch snowflake curve where the angles go to zero but not too fast, see [146, Section 20]. But if we add the assumption that E has positive lower density at almost all of its points and the approximation is bilateral – not only are the points of E in $B(a, r)$ close to a plane W but also the points of $W \cap B(a, r)$ are close to E – then this weaker approximation implies rectifiability. This was proved by Marstrand in [308] for two-dimensional sets in $\mathbb{R}^3$ and generalized in [318] using Marstrand's fundamental ideas. We state this result in the following section in terms of tangent measures.

4.3 Tangent Measures

Tangent measures were introduced by Preiss in [382] to solve the density characterization of rectifiability, which we shall discuss below. They have turned out to be useful on many other occasions too.

Define

$$T_{a,r}(x) = (x - a)/r, \ \ x, a \in \mathbb{R}^n, r > 0.$$

So $T_{a,r}$ blows up the ball $B(a, r)$ to the unit ball. Now we also blow up measures.

Definition 4.6 Let μ be a Radon measure on $\mathbb{R}^n$. A non-zero Radon measure ν is called a *tangent measure* of μ at $a \in \mathbb{R}^n$ if there are sequences (c_i) and (r_i) of positive numbers such that $r_i \to 0$ and $c_i T_{a,r_i\#}\mu \to \nu$ weakly. We denote the set of tangent measures of μ at a by $\mathrm{Tan}(\mu, a)$.

Tangent measures tell us how the measure looks locally.

Notice that this definition requires rather little structure. It is enough to have a locally compact metric group (G, d) in place of $\mathbb{R}^n$ with dilations $\delta_r, r > 0$, which are group homomorphisms such that δ_1 is identity, $\delta_{rs} = \delta_r \circ \delta_s$ and $d(\delta_r(x), \delta_r(y)) = rd(x, y)$. Then we can use exactly the same definition for tangent measures. The following result was proved in [324] in this setting:

Theorem 4.7 *Let μ be a Radon measure on G. Then the following are equivalent:*

(1) *For μ almost all $a \in G$, μ has a unique (up to multiplication by a constant) tangent measure at a.*
(2) *For μ almost all $a \in G$ there is a closed subgroup H_a of G which is invariant under the δ_r such that* $\mathrm{Tan}(\mu, a) = \{c\lambda_a : 0 < c < \infty\}$, *where λ_a is a left Haar measure of H_a.*

The point here with respect to rectifiability is that if something is rectifiable, it should look the same at all small scales around typical points. This theorem then tells us how it should look. We shall come back to this in Section 5.6 and Chapter 8.

The following is essentially a restatement of Theorem 4.5:

Theorem 4.8 *If E is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(E) < \infty$, then E is m-rectifiable if and only if for $\mathcal{H}^m$ almost all $a \in E$ there is $V_a \in G(n,m)$ such that every measure in* $\operatorname{Tan}(\mathcal{H}^m \llcorner E, a)$ *is of the form $c\mathcal{H}^m \llcorner V_a$ for some $0 < c < \infty$.*

Let us call measures of the form $c\mathcal{H}^m \llcorner V$ for some $V \in G(n,m)$, $0 < c < \infty$, *m-flat*. The following is the bilateral approximation criterion mentioned above. We state it for general measures:

Theorem 4.9 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be such that $0 < \Theta^m_*(\mu, x) \le \Theta^{*m}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$. If for μ almost all $a \in \mathbb{R}^n$ every measure in* $\operatorname{Tan}(\mu, a)$ *is* m-flat*, then μ is m-rectifiable.*

The proof is a bit tricky, but here is an idea in the plane for $\mu = \mathcal{H}^1 \llcorner E$. If E were purely unrectifiable, it would project into a set of measure zero in almost all directions. In fact, we would not have to use the projection theorem since our assumptions combined with pure unrectifiability imply rather easily that all projections have measure zero, see [321, Lemma 16.1]. So given $L \in G(2,1)$, which approximates E well at some scale, it suffices to show that $\mathcal{H}^1(P_L(E)) > 0$. Suppose not and suppose $F \subset E$ is compact. Then we can find $a \in F$ such that all of F lies in a half-plane with a on its boundary, which is orthogonal to L. If we had a good bilateral approximation at a with some line, this line ought to be almost orthogonal to L. Of course, a now is very special, but choosing F suitably and using $\mathcal{H}^1(P_L(F)) = 0$, it is possible to find many such points. This will give much measure to many narrow rectangles orthogonal to L, and it leads to a contradiction with the fact that the upper density is bounded. Preiss gave a different proof in [382].

In Theorem 4.9 the assumption on positive lower density cannot be dropped. Preiss [382, 5.9(2)] constructed an example of a purely 1-unrectifiable Borel set $A \subset \mathbb{R}^2$ with finite $\mathcal{H}^1$ measure for which all tangent measures are 1-flat at $\mathcal{H}^1$ almost all points.

Some more recent interesting results on tangent measures were proven by Kenig, Preiss and Toro in [275]. Fragala and Mantegazza compared different notions of tangent measures in [209].

4.4 Densities

Now we give density criteria for rectifiability. The proofs in the higher-dimensional case are quite different from Besicovitch's arguments. We cannot use connectivity when $m > 1$. For instance, an m-dimensional analogue F_m of Example 3.6 is contained in a continuum C with $\mathcal{H}^m(C) < \infty$. This is easily seen by approximating F_m with unions of small cubes and connecting them with line segments. A more interesting case is explained in [203, 4.2.25].

Theorem 4.10 *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$. Then the following are equivalent:*

(1) *E is m-rectifiable.*
(2) $\Theta^m(E, x) = 1$ *for $\mathcal{H}^m$ almost all $x \in E$.*
(3) $\Theta^m(E, x)$ *exists for $\mathcal{H}^m$ almost all $x \in E$.*

That (1) implies (2), and hence (3), can be proven with the help of Rademacher's theorem, the area formula and the Lebesgue density theorem in $\mathbb{R}^m$. Of course, (2) implies (1) is a special case of (3) implies (1), but since the proof of the first is easier, although not easy, and it is based on different ideas, I say something about it too. That (2) implies (1) was proved by Marstrand in [308] for $m = 2, n = 3$, and generalized in [318] relying heavily on Marstrand's ideas. The proof of (3) implies (1) is due to Preiss [382]. For this purpose he introduced the tangent measures. In [159] De Lellis gives a very nice exposition of this proof.

Chlebik [95] generalized the part (2) implies (1), showing that there is $c(m) < 1$ such that rectifiability already follows from $\Theta^m_*(E, x) > c(m)$ for $\mathcal{H}^m$ almost $x \in E$. Notice that $c(m)$ is independent of n. In fact, his proof also works in infinite-dimensional Hilbert spaces, but the implication (3) $\Rightarrow$ (1) in Theorem 4.10 is false in infinite-dimensional Hilbert spaces, see Section 7.2. Preiss proved that the existence of density in (3) can be relaxed to $\Theta^m_*(E, x) > c(n, m)\Theta^{*m}(E, x)$ for $\mathcal{H}^m$ almost $x \in E$, where $c(n, m) < 1$. Very little is known of these constants except when $m = 1$; recall Besicovitch's 1/2-problem from the previous chapter.

Preiss's result (3) $\Rightarrow$ (1) in Theorem 4.10 can be stated more generally, but it actually is easily seen to be equivalent:

Theorem 4.11 *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ and the positive and finite limit*

$$\lim_{r \to 0} r^{-m} \mu(B(x, r))$$

exists for μ almost all $x \in \mathbb{R}^n$, then μ is m-rectifiable.

The equivalence to Theorem 4.10 follows from the fact that μ and the restriction of $\mathcal{H}^m$ to $\{x\colon 0 < \lim_{r\to 0} r^{-m}\mu(B(x,r)) < \infty\}$ are mutually absolutely continuous.

How do we get to rectifiability from density 1? The first step is Marstrand's reflection lemma.

Lemma 4.12 *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$ measurable with $\mathcal{H}^m(E) < \infty$ and with $\Theta^m(E,x) = 1$ for $\mathcal{H}^m$ almost all $x \in E$. If $\delta > 0$, then there are $r_0 > 0$ and a compact subset F of E such that $\mathcal{H}^m(E \setminus F) < \delta$ and $d(2a-b, E) < \delta|a-b|$ whenever $a, b \in F$ and $|a-b| < r_0$.*

So for any such pair a, b the symmetric point of b with respect to a is close to E. For one-dimensional sets this begins to look like rectifiability, but one can proceed from it also when $m > 1$, although with many complications.

I explain the idea behind Lemma 4.12 when $m = 1$. For the complete proof, see [318] or [297, Section 3.5]. Suppose for simplicity that we have found F such that $\mathcal{H}^1(E \cap B(x,r)) = 2r$ when $x \in F$ and $r < r_0$. Let $a, b \in F$ with $r\colon = |a-b| < r_0$ and let $\varepsilon > 0$ be much smaller than δ. Let $B_1 = B(a,(1-\varepsilon)r)$, $B_2 = B(b, \varepsilon r)$ and $B_3 = B(2a-b, \delta r)$. Then $d(B_1 \cup B_2 \setminus B_3) \le 2(1-\varepsilon)r$, so we may (almost) assume by Theorem 1.2 that $\mathcal{H}^1(E \cap (B_1 \cup B_2 \setminus B_3)) \le 2(1-\varepsilon)r$. If $E \cap B_3$ were empty, we would have $E \cap (B_1 \cup B_2) \subset E \cap (B_1 \cup B_2 \setminus B_3)$, whence

$$2r = 2(1-\varepsilon)r + 2\varepsilon r = \mathcal{H}^1(E \cap B_1) + \mathcal{H}^1(E \cap B_2) \\ \le \mathcal{H}^1(E \cap (B_1 \cup B_2 \setminus B_3)) \le 2(1-\varepsilon)r.$$

The lemma would follow from this contradiction.

Here are some basics behind Preiss's theorem that (3) implies (1), that is, Theorem 4.11. The power of tangent measures is that they turn limiting conditions to uniform equations or inequalities. In this case, for μ almost all $a \in \mathbb{R}^n$ every $\nu \in \operatorname{Tan}(\mu, a)$ is *m-uniform*, that is,

$$\nu(B(x,r)) = cr^m \text{ for all } x \in \operatorname{spt}\nu, r > 0.$$

If we could show that all m-uniform measures are m-flat, we would be done by Theorem 4.9. This is true for $m = 1, 2$ but is not easy to show, in particular for $m = 2$. However, it is false for $m > 2$. Preiss observed that $\mathcal{H}^3$ restricted to the cone $\{x \in \mathbb{R}^4\colon x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ is 3-uniform. To overcome this problem when $m > 2$, Preiss showed that if an m-uniform measure is not flat, then in a certain precise sense it is far from flat. Moreover, $\operatorname{Tan}(\mu, a)$ is connected. Hence it is enough to show that at almost all points, μ has some flat tangent measures.

This is an easier part of the argument and was already essentially proved by Marstrand in [309].

Suppose then that ν is an m-uniform measure such that $0 \in \operatorname{spt}\nu$ and

$$\nu(B(x,r)) = cr^m \text{ for } x \in \operatorname{spt}\nu, r > 0. \tag{4.4}$$

Then we have the identities

$$\int_{B(x,r)} (r^2 - |x-y|^2)^2 \, d\nu y = \int_{B(0,r)} (r^2 - |y|^2)^2 \, d\nu y, \ x \in \operatorname{spt}\nu, r > 0, \tag{4.5}$$

which are used to study b_r and Q_r defined by

$$b_r \cdot v = \int_{B(0,r)} (r^2 - |y|^2)(v \cdot y) \, d\nu y \Big/ \int_{B(0,r)} (r^2 - |y|^2) \, d\nu y, \ v \in \mathbb{R}^n, \tag{4.6}$$

$$Q_r(v) = \int_{B(0,r)} (v \cdot y)^2 \, d\nu y \Big/ \int_{B(0,r)} (r^2 - |y|^2) \, d\nu y, \ v \in \mathbb{R}^n. \tag{4.7}$$

It is shown that they have convergent subsequences $b_{r_i} \to b$ and $Q_{r_i} \to Q$ for which

$$\operatorname{spt}\nu \subset K := \{x \in \mathbb{R}^n : Q(x) - |x|^2 + 2b \cdot x = 0\}.$$

The proof for $m = 1, 2$ can be completed from this, but for $m > 2$ much more is needed. In particular, Preiss performed a detailed study of higher-order moments $\int (v \cdot y)^k e^{-s|y|^2} \, d\nu y$, $s > 0$, and their Taylor expansions.

Due to Theorem 1.2 in the language of tangent measures, the assumption of density 1 corresponds to assuming in addition to (4.4) that $\nu(B(x,r)) \leq cr^m$ for $x \in \mathbb{R}^n$ and $r > 0$. Then one can show that $b = 0$ and $\operatorname{spt}\nu = K$, which is an m-plane. This gives a different proof for (2) implies (1) in Theorem 4.10.

The structure of m-uniform measures is a very interesting and in large part an open problem in itself. Christensen [112] introduced the more general class of *uniformly distributed* measures: $\nu(B(x,r)) = \nu(B(y,r))$ for $x, y \in \operatorname{spt}\nu, r > 0$. He showed that if such a measure has compact support, then it is contained in a sphere. Kirchheim and Preiss [277] proved that the support of any uniformly distributed measure in $\mathbb{R}^n$ is an analytic variety. We already noted that $\nu = \mathcal{H}^3 \llcorner \{x \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ is 3-uniform. Kowalski and Preiss proved in [281] that in addition to flat measures, this is the only (up to translations and rotations) example in $\mathbb{R}^4$, and in $\mathbb{R}^n, n > 4$, all $(n-1)$-uniform measures are either flat or of the form $\nu = \mathcal{H}^{n-1} \llcorner \{x \in \mathbb{R}^n : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$. For a long time no non-flat m-uniform measures were known when $2 < m < n-1$. Recently Nimer [368] produced many interesting examples. See also [367, 369, and 416] for other results.

We shall present a couple of results related to Theorem 4.11. In addition to that result, Preiss's paper contains a lot of information about general measures, including other characterizations of rectifiability. For example, see [382, Theorem 4.11]:

Theorem 4.13 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. Then*

$$\lim_{r\to 0} \mu(B(x,2r))/\mu(B(x,r)) \text{ exists for } \mu \text{ almost all } x \in \mathbb{R}^n \tag{4.8}$$

if and only if all tangent measures of μ at x are flat for μ almost all $x \in \mathbb{R}^n$.

Here the flat measures can be of any dimension, but under the density assumptions of the next corollary, they are m-flat and we can use Theorem 4.9 to get an extension of Theorem 4.11:

Corollary 4.14 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If $0 < \Theta^m_*(\mu,x) \leq \Theta^{*m}(\mu,x) < \infty$ for μ almost all $x \in \mathbb{R}^n$, then μ is m-rectifiable if and only if* (4.8) *holds.*

Tolsa and Toro proved in [421] a related result:

Theorem 4.15 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be such that $0 < \Theta^m_*(\mu,x) \leq \Theta^{*m}(\mu,x) < \infty$ for μ almost all $x \in \mathbb{R}^n$. Then the following are equivalent:*

(1) *μ is m-rectifiable.*
(2) $\int_0^1 \left|\frac{\mu(B(x,r))}{r^m} - \frac{\mu(B(x,2r))}{(2r)^m}\right|^2 r^{-1}dr < \infty$ *for μ almost all $x \in \mathbb{R}^n$.*
(3) $\lim_{r\to 0}\left(\frac{\mu(B(x,r))}{r^m} - \frac{\mu(B(x,2r))}{(2r)^m}\right) = 0$ *for μ almost all $x \in \mathbb{R}^n$.*

The main contribution here is the implication (1) $\Rightarrow$ (2). Its proof uses Calderón–Zygmund techniques. That (3) implies (1) follows from Corollary 4.14 because (3) clearly implies (4.8) under the density assumptions.

The uniform rectifiability version of the equivalence of (1) and (2) was proved earlier by Chousionis, Garnett, Le and Tolsa in [99], see Theorem 5.12. In [418] Tolsa showed that when $m = 1$, the equivalence of (1) and (2) holds, assuming only $\Theta^{*1}(\mu,x) > 0$ for μ almost all $x \in \mathbb{R}^n$. Hence it characterizes 1-rectifiability of general $\mathcal{H}^1$ measurable sets with $\mathcal{H}^1(E) < \infty$.

We shall discuss the rectifiability-densities question in metric spaces in Chapter 7, but let us briefly consider the case where $\mathbb{R}^n$ is equipped with the l^∞ norm. So the balls are cubes $Q(x,r)$ with sides parallel to the coordinate axis. This is widely open. Lorent [298] proved a partial result with a very complicated argument: if for a Radon measure μ in $\mathbb{R}^3$ we have $\mu(Q(x,r)) = r^2$ for all $x \in \operatorname{spt}\mu$, $r > 0$, then μ is 2-rectifiable.

I will say a few words about generalized densities. First, Marstrand proved in [309] that if $s > 0$ and for some n there is $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that the positive and finite limit $\lim_{r\to 0} r^{-s}\mu(B(x,r))$ exists for μ almost all $x \in \mathbb{R}^n$, then s is an integer. Preiss made a thorough and deep investigation of the corresponding and related questions for general density functions. The following is a special case of [382, Theorem 6.5]:

Theorem 4.16 *Let* $h\colon (0,\infty) \to (0,\infty)$ *be such that the limit* $\lim_{r\to 0} h(tr)/h(r)$ *exists for all* $t > 0$. *Then for some* n *there is* $\mu \in \mathcal{M}(\mathbb{R}^n)$ *such that the positive and finite limit* $\lim_{r\to 0} \mu(B(x,r))/h(r)$ *exists for* μ *almost all* $x \in \mathbb{R}^n$ *if and only if*

(1) *there is an integer* $m, 0 \le m \le n$ *such that* $0 < \lim_{r\to 0} r^{-m}h(r) < \infty$, *or*
(2) *there is an integer* $m, 1 \le m \le n-1$ *such that:*
- $\lim_{r\to 0} r^{-m}h(r) = 0$,
- $\lim_{r\to 0} h(tr)/h(r) = t^m$ *for all* $t > 0$, *and*
- $\lim_{r\to 0} \sup_{t\in(0,1]} h(tr)/h(r) = 1$.

By an example in [382, Proposition 6.9], the assumption of the existence of the limit $\lim_{r\to 0} h(tr)/h(r)$ is needed. Preiss called functions h as above exact density functions. A surprising consequence of Theorem 4.16 is that $r/|\log r|$ is an exact density function but $r|\log r|$ is not. For rectifiability criteria with general density functions, see [382, Corollary 5.4].

4.5 Projections

Federer extended Besicovitch's projection theorem to general dimensions in [199]:

Theorem 4.17 *Let* $E \subset \mathbb{R}^n$ *be* $\mathcal{H}^m$ *measurable with* $\mathcal{H}^m(E) < \infty$. *Then* E *is purely* m*-unrectifiable if and only if* $\mathcal{H}^m(P_V(E)) = 0$ *for almost all* $V \in G(n,m)$.

Federer proved this first for $m = n-1$ using Besicovitch's three alternatives method, which we explained in the previous chapter. Then he used downward induction on m and some integral geometry to get the general case.

White [438] gave a different proof. He took Besicovitch's result in the plane for granted and showed that if $\mathcal{H}^m(P_V(E)) > 0$ for positively many V, then E is not purely unrectifiable. First he used induction on n to get the result for $m = 1$ and all n. To do this he applied the induction hypothesis to $P_W(E)$ on some suitably chosen hyperplane W. To get to $m > 1$ he used an elegant argument applying the case $m = 1$ to the intersections $X \cap E$ of E with appropriate affine $(n-m+1)$-planes X. The planes W and X are found by some integral geometry. This is only a rough imprecise idea.

Jones, Katz and Vargas [264] also gave another proof in the case $m = n-1$. They used induction on n beginning with Besicovitch's result.

O'Neil proved in [371] a local version of the projection theorem: if lower and upper m-densities of a measure μ are positive and finite and all projections on m-planes of the supports of the tangent measures are convex, then μ is m-rectifiable.

We also have the Crofton formula: if $E \subset \mathbb{R}^n$ is an m-rectifiable Borel set with $\mathcal{H}^m(E) < \infty$, then

$$\mathcal{H}^m(E) = c(n,m) \int_{G(n,m)} \int_V \operatorname{card}(E \cap P_V^{-1}\{a\})\, d\mathcal{H}^m a\, d\gamma_{n,m} V.$$

This can be stated in terms of the *integral-geometric measure* $\mathcal{I}_1^m$: $\mathcal{H}^m(E) = \mathcal{I}_1^m(E)$ for rectifiable sets. There is a continuum of integral-geometric measures $\mathcal{I}_t^m, 1 \le t \le \infty$, defined in [203, 2.10.5] by

$$\mathcal{I}_t^m(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \zeta_{n,m,t}(B_i) \colon A \subset \cup_{i=1}^{\infty} B_i, B_i \text{ Borel sets}, d(B_i) < \delta \right\},$$

where

$$\zeta_{n,m,t}(B) = c(n,m,t) \left(\int \mathcal{H}^m(P_V(B))^t \, d\gamma_{n,m} V \right)^{1/t} \text{ if } 1 \le t < \infty,$$

$$\zeta_{n,m,\infty}(B) = \operatorname{esssup}_V \mathcal{H}^m(P_V(B)).$$

For any Borel set $B \subset \mathbb{R}^n$,

$$\mathcal{I}_1^m(B) = c(n,m) \int_{G(n,m)} \int_V \operatorname{card}(B \cap P_V^{-1}\{a\})\, d\mathcal{H}^m a\, d\gamma_{n,m} V.$$

All these measures agree with $\mathcal{H}^m$ for m-rectifiable sets, and they have the same null-sets: $\mathcal{I}_t^m(A) = 0$ if and only if A is contained in a Borel set B such that $\mathcal{H}^m(P_V(B)) = 0$ for almost all $V \in G(n,m)$. Moreover, $\mathcal{I}_s^m \lesssim \mathcal{I}_t^m$ if $s \le t$. It is also known that if $\mathcal{I}_\infty^m(A) < \infty$, then all $\mathcal{I}_t^m(A)$ agree, see [203, 3.3.16].

In [319] a compact set $F \subset \mathbb{R}^2$ was constructed for which $\mathcal{I}_1^1(F) < \mathcal{I}_\infty^1(F) = \infty$. But it is not known if, for example, all $\mathcal{I}_t^m, 1 < t < \infty$, agree with $\mathcal{I}_1^m$ or with $\mathcal{I}_\infty^m$. Here are two theorems relevant for rectifiability. The first is often called structure theorem. It is essentially a restatement of what already was said above. The proof of the second requires still more work, see [203], 3.3.13 and 3.3.14.

Theorem 4.18 *If $E \subset \mathbb{R}^n$ with $\mathcal{H}^m(E) < \infty$ and $1 \le t \le \infty$, then $E = R \cup P$ where R is m-rectifiable and $\mathcal{I}_t^m(P) = 0$. In particular, E is m-rectifiable if and only if $\mathcal{H}^m(E) = \mathcal{I}_t^m(E)$.*

Theorem 4.19 *If $E \subset \mathbb{R}^n$ with $\mathcal{I}_\infty^m(E) < \infty$, then E is $\mathcal{I}_\infty^m$-rectifiable, that is,*

$\mathcal{I}^m_\infty$ almost all of $\mathbb{R}^n$ can be covered with countably many Lipschitz images of $\mathbb{R}^m$.

In addition to Hausdorff and integral-geometric measures, there are many other natural m-dimensional measures which all agree on m-rectifiable sets, see [203, Theorem 3.2.26] and [321, Theorem 17.11].

Brothers [81] proved the analogue of Theorem 4.17 and studied integral-geometric measures on n-dimensional manifolds X with a transitive group G of diffeomorphisms. Let Y be an $(n-m)$-dimensional submanifold. Then the condition that almost all projections of E have $\mathcal{H}^m$ measure zero can be stated as $E \cap g(Y) = \emptyset$ for almost all $g \in G$.

Hovila, E. and M. Järvenpää and Ledrappier [242] proved the projection theorem for more general transversal families of linear maps $\mathbb{R}^n \to \mathbb{R}^m$ and applied it to invariant measures of geodesic flows on surfaces. Besicovitch's three alternatives are still there in both of these approaches.

4.6 Multiscale Approximations

In Chapters 5 and 6 we shall discuss more extensively multiscale approximations in terms of Jones-type square functions, recall Section 3.4, but now let us just state the following result of Azzam and Tolsa from [417] and [42]. Define

$$\beta_E^{m,2}(x,r) = \inf_{V \text{ affine } m\text{-plane}} \left(r^{-m} \int_{E \cap B(x,r)} \left(\frac{d(y,V)}{r} \right)^2 \, d\mathcal{H}^m y \right)^{1/2}. \tag{4.9}$$

Theorem 4.20 *If $E \subset \mathbb{R}^n$ is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(E) < \infty$, then E is m-rectifiable if and only if*

$$\int_0^1 \beta_E^{m,2}(x,r)^2 r^{-1} dr < \infty \text{ for } \mathcal{H}^m \text{ almost all } x \in E. \tag{4.10}$$

The proof is very technical and complicated. The part that (4.10) implies rectifibility was proved in [42]. It uses stopping time arguments where the rough idea is the following. The assumption (4.10) tells us that a large part of E is well approximated by m-planes at most scales, so we start with some finite family of generalized dyadic cubes where this happens. Then we go to smaller subcubes and stop when the approximation is not good enough. The stopping cubes contain only a small part of E. In the others the approximation becomes better and better. That allows us to build Lipschitz graphs which in the limit tend to a Lipschitz graph that meets E in a set of positive measure. This is a vastly oversimplified sketch. In fact, this type of scheme is not typical only to the above paper. It is quite commonly used in particular in connection with uniform rectifiability, see Section 5.5. The other direction, proved in [417], is also based on stopping time arguments. There Tolsa first proves the corresponding result with α numbers, see (5.4). The proofs would be much easier if E had positive lower density; now the stopping also takes place when the density ratios get too small.

Edelen, Naber and Valtorta [184] proved a sufficient condition for rectifiability which extends that part of Theorem 4.20, see Theorem 6.3. Naber and Valtorta proved in [356] and [359] closely related quantitative results and applied them to harmonic maps between manifolds and to stationary varifolds, see Chapter 15.

Many results on rectifiability and uniform rectifiability hold with the exponent 2 in the β numbers replaced by a range of exponents p. Rather surprisingly, Tolsa showed in [419] that Theorem 4.20 holds only for $p = 2$. With additional density conditions other exponents p work too, see [377] and [50].

In [41] Azzam and Schul obtained higher-dimensional analogues of both Theorems 3.16 and 3.17 using β-integrals defined in terms of Hausdorff content. For related results, see [427], [44] and [247]. Hilbert space versions were proven by Hyde in [246].

4.7 Reifenberg-Type Results

For $0 < m < n, x \in \mathbb{R}^n, r > 0$ and $E \subset \mathbb{R}^n$ define the β number

$$\beta_E^m(x,r) = \inf_V \sup_{y \in E \cap B(x,r)} d(y,V)/r, \tag{4.11}$$

and the bilateral β number

$$b\beta_E^m(x,r) = \inf_V \left(\sup_{y \in E \cap B(x,r)} d(y,V)/r + \sup_{y \in V \cap B(x,r)} d(y,E)/r \right), \tag{4.12}$$

where the infima are taken over all m-planes in $\mathbb{R}^n$. Reifenberg proved in [389]

Theorem 4.21 *For any $0 < \alpha < 1$, there is $\delta = \delta(n,\alpha) > 0$ such that if $E \subset B^n(0,2)$ is closed and $b\beta_E^m(x,r) < \delta$ when $x \in E$ and $B(x,r) \subset B^n(0,2)$, then there is an α-bi-Hölder map from $B^m(0,1)$ onto $E \cap B^n(0,1)$.*

A nice exposition of the proof and related matters is given by Naber in [354].

Suppose that E satisfies this condition locally for every $\delta > 0$; $b\beta_E^m(x,r) < \delta$ when $x \in E, 0 < r < r(\delta)$. Then it follows that $\dim E = m$, but E need not have σ-finite $\mathcal{H}^m$ measure, in particular, it need not be m-rectifiable. This is easily seen by von Koch snowflake-type examples for which the angle between the segments in consecutive generations goes to zero slowly, recall Section 4.2. Then the approximating lines can turn around infinitely many times.

So to get rectifiability from Reifenberg-type assumptions, involving approximation by planes, we need something more. In a way many results discussed earlier are of this sort, but here I restrict 'Reifenberg type' to mean that in addition to fixing the dimension of the planes, we do not make any other dimensional assumptions, such as with densities. Jones's travelling salesman Theorem 3.16 is of this type. But Reifenberg type could also refer to results with bijective parametrizations.

Now we give a Reifenberg-type result of Simon, see [400, Section 4.2], which he used to prove rectifiability of singularities of minimal surfaces and harmonic maps. We shall return to this in Chapter 15.

The formulation of Simon's theorem is a bit complicated, so we state a simpler special case. The assumptions of the actual result allow at each scale a small exceptional set which seems to be essential in the applications.

Theorem 4.22 *For any $0 < \delta < 1$, there is $\varepsilon = \varepsilon(n, \delta) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n$ be closed such that $\beta_E^m(x, r) < \varepsilon$ when $x \in E$ and $0 < r < 1$ and suppose that E has the following property. Let $x_0 \in E, 0 < r_0 < 1$ and $V \in G(n, m)$ for which $d(x, V + x_0) \leq \varepsilon r_0$ for all $x \in E \cap B(x_0, r_0)$. If $x \in E \cap B(x_0, r_0)$ and $0 < r < r_0$ are such that $E \cap B(y, \delta s) \neq \emptyset$ for all $y \in (V + x) \cap B(x, s), r \leq s \leq r_0$, then $d(y, V + x) \leq \varepsilon s$ for all $y \in E \cap B(x, s), r \leq s \leq r_0$. Then E is m-rectifiable.*

That is, the main assumptions say something like this: if E is ε-well approximated in some ball with some plane, then it continues to be ε-well approximated in smaller balls with translates of the same plane as long as there are no δ-gaps, that is, as long as there is bilateral δ-approximation. So if at a generic point the approximating plane can turn (wildly, as for von Koch–type examples) only because of the gaps, then the set is rectifiable (and the plane cannot turn wildly).

To see why this might be true, observe first that if there are no gaps at all, there is approximation at all scales with planes parallel to a fixed plane. This easily implies that E is contained in a Lipschitz graph. Secondly, if $E \cap B(x, r)$ is contained in an εr neighbourhood of an m-plane through x and there is a δ-gap, then, since ε is much smaller than δ, $E \cap B(x, r)$ can be covered with balls B_i for which $\sum_i d(B_i)^m < \lambda(2r)^m$ with $\lambda < 1$ depending on δ. Thus gaps at many places and scales lead to small measure.

David and Toro found in [151] conditions which, when added to the assumptions of Theorem 4.21, guarantee that the map f can be chosen to be bi-Lipschitz. One such condition is that $\sum_k \beta_E^m(x, 10^{-k})^2$ is bounded. They also discussed relations to uniform rectifiability. The methods are partially based on the earlier work of Toro [423]. In [341] Merhej, generalizing a result from [423], established sufficient conditions for the bi-Lipschitz parametrization of a codimension one AD-regular rectifiable set in terms of the Poincaré inequality and quadratic oscillation of the unit normal. A related result of Azzam [30] tells us that AD-regularity, together with the Poincaré inequality, implies uniform rectifiability.

Naber and Valtorta used in [356] a β assumption to get a $W^{1,p}, p > n$, parametrization. Edelen, Naber and Valtorta [184] gave a Reifenberg-type result for measures, see Theorem 6.4. Results of this type have been applied to the structure of singularities, see Chapter 15.

4.8 Lebesgue Null-Sets and Singular Measures

Many of the results of this section are described in [3] and [4], but the full proofs have not yet been published.

This theory began when Preiss [383] discovered a set A in the plane of zero Lebesgue measure such that every Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable at some point of A. Since then, a lot of work has been done by Alberti, Csörnyei, Preiss and others on the differentiability properties of Lipschitz maps on null-sets. I don't go into that here, but see the surveys [3] and [4], and [5]. We shall concentrate on geometric properties of Lebesgue null-sets and general singular measures.

Preiss's set is simple to state: any G_δ null-set of the plane containing the countable set of lines with rational coordinates is fine. Then for a given Lipschitz function there are a lot of directional derivatives. But to get differentiability, one should be able to combine them to a derivative mapping. The following definition has turned out to be relevant for differentiability and other questions:

Definition 4.23 A Borel mapping $\tau\colon E \to G(n,m)$ is a *weak m-tangent field* of a set $E \subset \mathbb{R}^n$ if for every m-rectifiable set F with $\mathcal{H}^m(F) < \infty$, $\mathrm{apTan}(F,x) = \tau(x)$ for $\mathcal{H}^m$ almost all $x \in E \cap F$.

Let E be $\mathcal{H}^m$ measurable and $\mathcal{H}^m(E) < \infty$. If E is m-rectifiable, then E has an $\mathcal{H}^m$ unique weak m-tangent field, while if E is purely m-unrectifiable, every $\tau\colon E \to G(n,m)$ is a weak m-tangent field of E. In general, a weak m-tangent field is unique up to purely m-unrectifiable sets.

Alberti, Csörnyei and Preiss have proven

Theorem 4.24 *Any set $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(E) = 0$ admits a weak $(n-1)$-tangent field.*

In the words of the authors of [4]: 'This result can be understood as saying the rather mysterious fact that one can prescribe in which direction an $(n-1)$-surface meets a null set E, without knowing the surface itself'.

Integral representations with rectifiable measures play an important role in the investigations of singular measures. They were used by Alberti in [2] for the proof of his rank one Theorem 12.15 for BV-functions, where he established deep analytic and geometric properties of singular measures. Nowadays they are called Alberti representations by many authors.

Alberti, Csörnyei and Preiss presented the following general definitions and results in [4]:

Definition 4.25 A measure $\nu \in \mathcal{M}(\mathbb{R}^n)$ is called *m-rectifiably representable* if it can be written as $\nu = \int \mu_t \, dPt$, where $\mu_t \ll \mathcal{H}^m \llcorner E_t$ for some m-rectifiable set E_t with $\mathcal{H}^m(E_t) < \infty$ and P is some probability measure. For such a ν we say that $\tau \colon \mathbb{R}^n \to G(n,m)$ is an *m-tangent field* of ν if $\operatorname{apTan}(E_t, x) = \tau(x)$ for $\mathcal{H}^m$ almost all $x \in E_t$ and P almost all t.

Theorem 4.26 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. Then*

μ is m-rectifiably representable if and only if $\mu(E) = 0$ for every purely m-unrectifiable set $E \subset \mathbb{R}^n$.

If μ is $(n-1)$-rectifiably representable, then it admits an $(n-1)$-tangent field if and only if it is singular.

μ has a unique decomposition as $\mu = \mu_n + \mu_{n-1} + \cdots + \mu_0$, where each μ_m is m-rectifiably representable and it lives on a purely $(m+1)$-unrectifiable set.

We shall return to tangent fields and Alberti representations in Chapter 7.

Csörnyei, Preiss and Tiser [123] and Maleva and Preiss [303] introduced large subclasses of purely 1-unrectifiable sets such that for any set in these classes some Lipschitz function is non-differentiable at every point of it. They used them to describe many other detailed (non-)differentiability properties of Lipschitz functions too.

Alberti and Marchese [5] introduced the decomposition bundle $V(\mu, x)$, $x \in \mathbb{R}^n$, of any measure $\mu \in \mathcal{M}(\mathbb{R}^n)$: $V(\mu, x)$ is the smallest linear subspace of $\mathbb{R}^n$ with the following property: if $\nu = \int \mu_t \, dPt$ is 1-rectifiably representable, as in Definition 4.25, and $\nu \ll \mu$, then $\operatorname{apTan}(E_t, x) \subset V(\mu, x)$ for $\mathcal{H}^1$ almost all $x \in E_t$ and P almost all t. They used it to obtain interesting Lipschitz differentiability results and they also applied it to normal currents.

We have by a fairly easy result of [5, Proposition 2.9]:

Proposition 4.27 *μ is purely 1-unrectifable if and only if $V(\mu, x) = \{0\}$ for μ almost all $x \in \mathbb{R}^n$.*

Del Nin and Merlo [171] found a nice application of the decomposition bundles. They proved a dichotomy in Fourier restriction:

Theorem 4.28 *If $0 < s \leq n$, $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $0 < \Theta^{*s}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$ and the restriction inequality*

$$\|\hat{f}\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{4.13}$$

holds when $q = sp'/n$, with $p' = p/(p-1)$, then either $q = p'$, that is, $s = n$, and so $\mu \ll \mathcal{L}^n$, or μ is purely 1-unrectifiable.

The value $q = sp'/n$ is the endpoint, the largest value for which this restriction inequality could hold under the density assumption.

To prove Theorem 4.28 one first checks by direct computation that (4.13) is preserved for tangent measures. At μ almost all points x the tangent measures are shown to be of the form $\nu \otimes \mathcal{H}^{k(x)} \llcorner V(\mu, x)$, $\nu \in \mathcal{M}(V(\mu, x)^\perp)$, where $k(x) = \dim V(\mu, x)$. If $k(x) > 0$, the restriction estimate for $\mathcal{H}^{k(x)} \llcorner V(\mu, x)$ follows, which, by trivial scaling, is only possible if $q = p'$. Hence the theorem follows from Proposition 4.27.

Another application of of the decomposition bundles was found by Marchese and Merlo in [305]. They showed that the Lusin-type approximation property (1.1) holds for a Radon measure μ on $\mathbb{R}^n$ in place of the Lebesgue measure if and only if μ is a sum of absolutely continuous (with respect to the corresponding Hausdorff measures) rectifiable measures of various dimensions.

4.9 Minkowski Content and Discrete Energies

For $0 < m < n$, the *lower and upper Minkowski contents* are defined for $A \subset \mathbb{R}^n$ by

$$\mathcal{M}^m_*(A) = \liminf_{\delta \to 0} \alpha(n-m)^{-1} \delta^{m-n} \mathcal{L}^n(\{x \in A : d(x, A) < \delta\}),$$

$$\mathcal{M}^{*m}(A) = \limsup_{\delta \to 0} \alpha(n-m)^{-1} \delta^{m-n} \mathcal{L}^n(\{x \in A : d(x, A) < \delta\}).$$

If they agree, their common value $\mathcal{M}^m(A)$ is the Minkowski content of A.

It is easy to see that $\mathcal{H}^m(A) \lesssim \mathcal{M}^m_*(A)$ and that there are compact sets A for which $\mathcal{H}^m(A) = 0$ and $\mathcal{M}^m_*(A) = \infty$. In particular, when m is an integer, such an A is m-rectifiable. Federer proved in [203, Theorem 3.2.39]

Theorem 4.29 *If* $0 < m < n$ *are integers and a compact set* $F \subset \mathbb{R}^n$ *is a Lipschitz image of a bounded subset of* $\mathbb{R}^m$*, then* $\mathcal{M}^m(F) = \mathcal{H}^m(F)$.

Borodachov, Hardin and Saff used in [72] and [73] this connection to prove a very general result on the asymptotics of discrete s-energies. For $s > 0$ and a compact subset F of $\mathbb{R}^n$, define

$$\mathcal{E}_s(F, N) = \min\left\{\sum_{i\neq j} |x_i - x_j|^{-s} : x_1, \ldots, x_N \subset F\right\}.$$

For a given F the question how does $\mathcal{E}_s(F, N)$ behave when $N \to \infty$ has been extensively studied. The most classical cases are when F is a sphere and $s = n - 2$, when $n \geq 3$, and $|x|^{-s}$ is replaced by $-\log|x|$, when $n = 2$. See the book [74] for a huge amount of generalizations and related questions, connections and applications to a wide variety of topics. Here I only discuss one result related to rectifiability.

Let $m = \dim F$ be the Hausdorff dimension of F. The cases $s < m$ and $s \geq m$ are quite different. In the first case by classical potential theory, see [74, Theorem 4.2.2],

$$\lim_{N\to\infty} N^{-2}\mathcal{E}_s(F, N) = \iint |x - y|^{-s}\, d\mu_{m,F}x\, d\mu_{m,F}y,$$

where $\mu_{s,F} \in \mathcal{M}(F)$ is an equilibrium probability measure that minimizes the energies $I_s(\mu) := \iint |x - y|^{-s}\, d\mu x\, d\mu y$ among all probability measures $\mu \in \mathcal{M}(F)$. The second case is much more delicate because $I_m(\mu) = \infty$ for all $\mu \in \mathcal{M}(F)$, at least if $\mathcal{H}^m(F) > 0$. We have by [72] and [73], see also [74, Theorem 8.5.2],

Theorem 4.30 *Let* $F \subset \mathbb{R}^n$ *be compact and* $s > m = \dim F$*. If* F *is* m*-rectifiable and* $\mathcal{M}^m(F) = \mathcal{H}^m(F)$*, then*

$$\lim_{N\to\infty} N^{-1-s/m}\mathcal{E}_s(F, N) = c(s, m)\mathcal{H}^m(F)^{-s/m}.$$

Moreover, if $\mathcal{H}^m(F) > 0$ *and* $\{x_{N,1}, \ldots, x_{N,N}\}$ *is a minimizing configuration for* $\mathcal{E}_s(F, N)$*, then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{x_{N,i}} \to \frac{1}{\mathcal{H}^s(F)}\mathcal{H}^s \llcorner F \text{ weakly as } N \to \infty. \tag{4.14}$$

Since the constant $c(s, m)$ does not depend on F, one can derive a lot of information about it looking, for example, at the case where F is an m-sphere or a cube in $\mathbb{R}^m$.

When $s = m$, similar results hold for compact subsets of C^1 submanifolds of $\mathbb{R}^n$, and a little more generally, see [74, Theorem 9.5.4]. Then $N^{1+s/m}$ is replaced by $N^2 \log N$. It seems to be unknown whether they are valid under rectifiability assumptions as in Theorem 4.30.

In the case of $s < m$, the cluster points of the extremal measures as in (4.14) are equilibrium measures $\mu_{s,F}$, which in general are not multiples of Hausdorff measures. For example, when F is a ball in $\mathbb{R}^m$ and $m - 2 < s < m$, the equilibrium measure is absolutely continuous, with density going to infinity at the boundary.

For recent related results, see [232].