Third Meeting, 8th January 1904.

Mr CHARLES TWEEDIE, President, in the Chair.

On the Fractional Infinite Series for cosec$x$, sec$x$, cot$x$, and tan$x$.

By D. K. PICKEN, M.A.

The infinite products for sin$x$ and cos$x$ are most conveniently obtained in a rigorous way from the well-known factorial expressions for sin$n\theta$ and cos$n\theta$ which, when n is an even integer, take the forms

$$\text{(i)}\ \sin n\theta = 2^{n-1}.\sin\theta\cos\theta\left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)\ldots\left(\sin^2\frac{\overline{n-2}.\pi}{2n} - \sin^2\theta\right)$$

$$\text{(ii)}\ \cos n\theta = 2^{n-1}.\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\ldots\left(\sin^2\frac{\overline{n-1}.\pi}{2n} - \sin^2\theta\right);$$

θ being put equal to $\frac{x}{n}$ and n made infinitely great.*

It is then usual to obtain the fractional infinite series for cot$x$ and tan$x$ by logarithmic differentiation—a process in which the treatment of the remainder is somewhat involved—and to deduce those for cosec$x$ and sec$x$ by the use of certain elementary trigonometrical identities.

It seems, however, a more fundamental process to obtain from (i) and (ii) expressions for cosθcosec$n\theta$, sec$n\theta$, secθcot$n\theta$, and secθtan$n\theta$ in partial fractions; the degree in sinθ of the denominator in each of these functions being higher than that of the numerator; and then to proceed to the limit as in the case of the products.

* Cf. *Hobson's Trigonometry*, Chap. XVII.

https://doi.org/10.1017/S0013091500001899 Published online by Cambridge University Press

I. When n is even

$$\frac{\cos\theta}{\sin\theta\sin n\theta} \text{ can be written in the form } \sum_0^{\frac{n}{2}-1} \frac{A_r}{\sin^2\frac{r\pi}{n} - \sin^2\theta},$$

where $A_0 = \left[-\frac{\sin\theta\cos\theta}{\sin n\theta}\right]_{\theta=0} = -\frac{1}{n}$

and $A_r = \left[\frac{\left(\sin^2\frac{r\pi}{n} - \sin^2\theta\right)\cos\theta}{\sin\theta . \sin n\theta}\right]_{\theta=\frac{r\pi}{n}}$

$$= \left[\frac{\left(\sin\frac{r\pi}{n} + \sin\theta\right)\cos\theta}{\sin\theta} . \frac{\sin\frac{r\pi}{n} - \sin\theta}{\sin n\theta}\right]_{\theta=\frac{r\pi}{n}}$$

$$= \frac{-2\cos^2\frac{r\pi}{n}}{n\cos r\pi} = \frac{(-)^{r-1}}{n} . 2\cos^2\frac{r\pi}{n}, \left(r = 1, 2, \dots \frac{n}{2} - 1\right);$$

$$\therefore \cos\theta \operatorname{cosec} n\theta = \frac{1}{n\sin\theta} + \frac{2\sin\theta}{n} \sum_1^{\frac{n}{2}-1} (-)^{r-1} \frac{\cos^2\frac{r\pi}{n}}{\sin^2\frac{r\pi}{n} - \sin^2\theta}.$$

[When n is odd,

$$\operatorname{cosec} n\theta = \frac{1}{n\sin\theta} + \frac{2\sin\theta}{n} \sum_1^{\frac{n-1}{2}} (-)^{r-1} . \frac{\cos\frac{r\pi}{n}}{\sin^2\frac{r\pi}{n} - \sin^2\theta}].$$

Putting $\theta = \frac{x}{n}$, we get

$$\cos\frac{x}{n} \operatorname{cosec} x = \frac{1}{n\sin\frac{x}{n}} + \frac{2}{n}\sin\frac{x}{n} \sum_1^k (-)^{r-1} . \frac{\cos^2\frac{r\pi}{n}}{\sin^2\frac{r\pi}{n} - \sin^2\frac{x}{n}} + (-)^k . R$$

where k is any integer less than $\left(\frac{n}{2} - 1\right)$.

https://doi.org/10.1017/S0013091500001899 Published online by Cambridge University Press

It is obvious that R is positive and less than

$$\frac{2\sin\frac{x}{n}\cos^2\frac{(k+1)\pi}{n}}{n\left\{\sin^2\frac{(k+1)\pi}{n}-\sin^2\frac{x}{n}\right\}}$$

provided n is so great that k can be chosen greater than $\frac{x}{\pi}$; for the angles $\frac{r\pi}{n}$ are increasing *acute* angles and therefore the terms of R are in descending order of numerical magnitude.

$$\therefore\ \cos\frac{x}{n}\operatorname{cosec}x=\frac{1}{n\sin\frac{x}{n}}+2n\sin\frac{x}{n}\,.\,\sum_1^k(-)^{r-1}\,.\,\frac{\cos^2\frac{r\pi}{n}}{n^2\left\{\sin^2\frac{r\pi}{n}-\sin^2\frac{x}{n}\right\}}$$

$$+(-)^k\,.\,\epsilon\,\frac{2n\sin\frac{x}{n}\cos^2\frac{(k+1)\pi}{n}}{n^2\left\{\sin^2\frac{(k+1)\pi}{n}-\sin^2\frac{x}{n}\right\}},$$

where ϵ is a positive proper fraction;

$\therefore$ proceeding to the limit when n becomes infinitely great,

$$\operatorname{cosec}x=\frac{1}{x}+\sum_1^k(-)^{r-1}\,.\,\frac{2x}{r^2\pi^2-x^2}+(-)^k\,.\,\epsilon_1\,.\,\frac{2x}{(k+1)^2\pi^2-x^2},$$

ϵ_1 being the limiting positive fractional value of ϵ.

Hence the greater we make the finite number k the more nearly is $\operatorname{cosec}x$ equal to $\frac{1}{x}+\sum_1^k(-)^{r-1}\,.\,\frac{2x}{r^2\pi^2-x^2}$, and the difference vanishes when k becomes infinitely great.

i.e., $\operatorname{cosec}x=\frac{1}{x}+\sum_1^\infty(-)^{r-1}\,.\,\frac{2x}{r^2\pi^2-x^2}$, an absolutely convergent series;

$=\frac{1}{x}+\frac{1}{\pi-x}-\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}+\ldots$, a semi-convergent series; for all real values of x, except $x=\pm r\pi$.

II. When n is even,

$$\sec n\theta = \sum_{1}^{\frac{n}{2}} \frac{A_r}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta},$$

where $A_r = \left[\dfrac{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}{\cos n\theta}\right]_{\theta=\frac{(2r-1)\pi}{2n}}$

$$= \frac{2\sin\frac{(2r-1)\pi}{2n}\cos\frac{(2r-1)\pi}{2n}}{n\sin\frac{(2r-1)\pi}{2}} = \frac{(-)^{r-1}}{n}\,.\,\sin\frac{(2r-1)\pi}{n};$$

$$\therefore\ \sec n\theta = \frac{1}{n}\sum_{1}^{\frac{n}{2}}(-)^{r-1}\,.\,\frac{\sin\frac{(2r-1)\pi}{n}}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta};$$

[and when n is odd,

$$\cos\theta\,.\,\sec n\theta = \frac{1}{n}\sum_{1}^{\frac{n-1}{2}}(-)^{r-1}.\frac{\sin\frac{(2r-1)\pi}{n}\,.\,\cos\frac{(2r-1)\pi}{2n}}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}\ ].$$

Putting $\theta = \frac{x}{n}$, we get

$$\sec x = \frac{1}{n}\sum_{1}^{k}(-)^{r-1}\,.\,\frac{\sin\frac{(2r-1)\pi}{n}}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\frac{x}{n}} + (-)^k\,.\,R$$

and, as before, if n is so great that $(2k-1)$ can be taken greater than $\frac{2x}{\pi}$

$$R = \frac{\epsilon}{n}\,.\,\frac{\sin\frac{(2k+1)\pi}{n}}{\sin^2\frac{(2k+1)\pi}{2n} - \sin^2\frac{x}{n}},\ \text{for}$$

$$\sin\frac{(2r-1)\pi}{n}\left\{\sin^2\frac{(2r+1)\pi}{2n} - \sin^2\frac{x}{n}\right\}$$
$$-\sin\frac{(2r+1)\pi}{n}\left\{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\frac{x}{n}\right\}$$

https://doi.org/10.1017/S0013091500001899 Published online by Cambridge University Press

$$=2\sin\frac{\pi}{n}\left\{\sin\frac{(2r-1)\pi}{2n}\cdot\sin\frac{(2r+1)\pi}{2n}+\sin^2\frac{x}{n}\cdot\cos\frac{2r\pi}{n}\right\}$$

$$>2\sin\frac{\pi}{n}\left\{\sin^2\frac{(2r-1)\pi}{2n}-\sin^2\frac{x}{n}\right\},$$

and $\therefore$ the terms of R are in descending order of magnitude.

Hence, $\sec x$

$$=\frac{1}{n}\sum_1^k(-)^{r-1}\cdot\frac{\sin\frac{(2r-1)\pi}{n}}{\sin^2\frac{(2r-1)\pi}{2n}-\sin^2\frac{x}{n}}+(-)^k\cdot\frac{\epsilon}{n}\cdot\frac{\sin\frac{(2k+1)\pi}{n}}{\sin^2\frac{(2k+1)\pi}{2n}-\sin^2\frac{x}{n}}$$

and proceeding to the limit, we get, exactly as in I,

$$\sec x=4\pi\sum_1^\infty(-)^{r-1}\cdot\frac{2r-1}{(2r-1)^2\pi^2-4x^2},\text{ a semi-convergent series};$$

$$=2\left\{\frac{1}{\pi-2x}+\frac{1}{\pi+2x}-\frac{1}{3\pi-2x}-\frac{1}{3\pi+2x}+\ldots\ldots\right\};$$

for all real values of x except $x=\pm\frac{(2r-1)\pi}{2}$.

III. When n is even

$$\sec\theta\cdot\cot n\theta=\frac{\prod_1^{\frac{n}{2}}\left\{\sin^2\frac{(2r-1)\pi}{2n}-\sin^2\theta\right\}}{2^{n-1}\cdot\sin\theta(1-\sin^2\theta)\prod_1^{\frac{n}{2}-1}(\sin^2\frac{r\pi}{n}-\sin^2\theta)};$$

$$\therefore\quad\frac{\cos n\theta}{\sin\theta\cos\theta\sin n\theta}=\sum_0^{\frac{n}{2}}\frac{A_r}{\sin^2\frac{r\pi}{n}-\sin^2\theta},$$

$$\text{where }A_0=\left[-\frac{\sin\theta\cos n\theta}{\cos\theta\sin n\theta}\right]_{\theta=0}=-\frac{1}{n},$$

$$A_r=\left[\frac{\cos n\theta\left(\sin^2\frac{r\pi}{n}-\sin^2\theta\right)}{\sin\theta\cos\theta\cdot\sin n\theta}\right]_{\theta=\frac{r\pi}{n}}=-\frac{2}{n},$$

if $r=1,\ 2,\ \ldots\ldots,\ \frac{n}{2}-1,$

https://doi.org/10.1017/S0013091500001899 Published online by Cambridge University Press

and $A_{\frac{n}{2}} = \left[\dfrac{\cos n\theta \cos\theta}{\sin\theta \sin n\theta}\right]_{\theta=\frac{\pi}{2}} = -\dfrac{1}{n}$;

$$\therefore\ \sec\theta\cot n\theta = \frac{1}{n\sin\theta} - \frac{2\sin\theta}{n}\sum_{1}^{\frac{n}{2}-1}\frac{1}{\sin^2\frac{r\pi}{n} - \sin^2\theta} - \frac{\sin\theta}{n\cos^2\theta}.$$

[When n is odd,

$$\sec\theta\cot n\theta = \frac{1}{n\sin\theta} - \frac{2\sin\theta}{n}\sum_{1}^{\frac{n-1}{2}}.\frac{1}{\sin^2\frac{r\pi}{n} - \sin^2\theta}\ ].$$

Putting $\theta = \dfrac{x}{n}$, we get

$\sec\dfrac{x}{n}\cot x$

$$= \frac{1}{n\sin\frac{x}{n}} - \frac{\sin\frac{x}{n}}{n\cos^2\frac{x}{n}} - 2n\sin\frac{x}{n}\sum_{1}^{k}\frac{1}{n^2\left\{\sin^2\frac{r\pi}{n} - \sin^2\frac{x}{n}\right\}} - 2n\sin\frac{x}{n}\,.\,R,$$

where $R = \displaystyle\sum_{k+1}^{\frac{n}{2}-1}\frac{1}{n^2\left\{\sin^2\frac{r\pi}{n} - \sin^2\frac{x}{n}\right\}}$

$$< \sum_{k+1}^{\frac{n}{2}-1}\frac{1}{n^2\left\{\frac{4r^2}{n^2} - \frac{x^2}{n^2}\right\}},\ \text{since } \phi > \sin\phi > \frac{2\phi}{\pi} \text{ if } 0 < \phi < \frac{\pi}{2};$$

provided that n is so great that $2k$ can be taken greater than x.

$\therefore\ R < \displaystyle\sum_{k+1}^{\frac{n}{2}-1}\frac{1}{4r^2 - x^2} < \sum_{k+1}^{\infty}\frac{1}{4r^2 - x^2}$, the remainder after k terms

of a convergent infinite series.

Proceeding to the limit

$$\cot x = \frac{1}{x} - \sum_{1}^{k}\frac{2x}{r^2\pi^2 - x^2} - 2x\,.\,R_1;$$

and R_1, the limiting value of R, can be made as small as we please by choosing k great enough.

$$\therefore \ \cot x = \frac{1}{x} - \sum_1^\infty \frac{2x}{r^2\pi^2 - x^2} \text{ an absolutely convergent series ;}$$

$$= \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \ldots\ldots, \text{ a semi-conver-}$$

gent series ; for all real values of x, except $x = \pm r\pi$.

IV. When n is even,

$$\sec\theta \operatorname{cosec}\theta \,.\, \tan n\theta = \sum_1^{\frac{n}{2}} \frac{A_r}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta},$$

$$\text{where } A_r = \left[\frac{\sin n\theta\left\{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta\right\}}{\sin\theta\cos\theta\cos n\theta}\right]_{\theta = \frac{(2r-1)\pi}{2n}} = \frac{2}{n};$$

$$\therefore \ \tan n\theta = \frac{2\sin\theta\cos\theta}{n} \sum_1^{\frac{n}{2}} \frac{1}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}.$$

[When n is odd,

$$\tan n\theta = \frac{1}{n}\tan\theta + \frac{2\sin\theta\cos\theta}{n} \sum_1^{\frac{n-1}{2}} \frac{1}{\sin^2\frac{(2r-1)\pi}{2n} - \sin^2\theta}].$$

Hence, exactly as in III.,

$$\tan x = \sum_1^\infty \frac{8x}{(2r-1)^2\pi^2 - 4x^2}, \text{ an absolutely convergent series ;}$$

$$= 2\left\{\frac{1}{\pi - 2x} - \frac{1}{\pi + 2x} + \frac{1}{3\pi - 2x} - \frac{1}{3\pi + 2x}\ldots\right\}, \text{ a semi-con-}$$

vergent series ; for all real values of x, except $x = \pm\frac{(2r-1)\pi}{2}$.

The continuity of the algebraic expressions ensures that these results still hold good when x has complex values.

https://doi.org/10.1017/S0013091500001899 Published online by Cambridge University Press