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HOMOTOPY ASSOCIATIVITY OF SPHERE EXTENSIONS

To the memory of Professor J. F. Adams

by N. IWASE

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Throughout this paper, we work in the category of (p-localized) spaces having the homotopy type of connected CW-complexes of finite type with base point. We consider a principal bundle

$$G_{n-1} \to X \to S^{2dn-1}, \tag{0.1}$$

where $G_n = SU(n)$, $U(n)$ or $Sp(n)$ and $d=1,1$ or 2 respectively. In this case, the bundle is obtained as an induced bundle by a mapping f of base space S^{2dn-1} from the classical group extension as follows:

$$\begin{array}{ccc} G_{n-1} & =\!=\!=\!=\!= & G_{n-1} \\ \cap & & \cap \\ X & \xrightarrow{\bar{f}} & G_n \\ \downarrow & & \downarrow \\ S^{2dn-1} & \xrightarrow{f} & S^{2dn-1}. \end{array}$$

We denote X by $M(n,\lambda)$ following Zabrodsky [19] when $\deg(f)=\lambda$. The problem is to describe, in terms of d, n and λ, the condition when $M(n,\lambda)$ becomes a homotopy associative H-space or more generally an A_m-space for $m \geqq 3$ (see Stasheff [17]). The case $m=2$ was studied by many authors (see Hilton–Roitberg [8], Stasheff [18], Curtis–Mislin [4], Sigrist–Suter [16] and Zabrodsky [19, 20, 21, 22]) and solved completely by 1972 as the following form.

Fact 1. *$M(n,\lambda)$ is an H-space if and only if one of the following three conditions is valid.*

(*a*) *λ is odd*

(*b*) *$dn \leqq 2$*

(*c*) *$\lambda = 0 \bmod 2d$ and $dn=4$.*

To avoid a confusion with an integer mod p, we adopt the notation "a property P at

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p", rather than "a mod p property P", for a localized property P at p. Now let us turn our attention to homotopy associativity (or an A_3-structure) of sphere extensions. At first, Sigrist–Suter show in [**15**] that $M(n,\lambda)$ is not A_3 in case $\lambda=0 \bmod 4$, $d=2$ and $n=2$ of Fact 1. By Hemmi [7], the result of Gonçalves [5] implies that $M(n,\lambda)$ is not A_3 at the prime 2 in case (c) of Fact 1. We summarize the above results.

Fact 2. *In case* (*c*) *of Fact 1*, $M(n,\lambda)$ *is not* A_3 *at the prime 2. In cases* (*a*) *and* (*b*) *of Fact 1*, $M(n,\lambda)$ *is an* A_∞*-space at the prime 2.*

Moreover Hemmi gives the necessary condition in [7] for $p=3$, that is, λ is prime to 6, when $dn=r\cdot 3^*$, $(r,3)=1$ and $r>3$, where we denote by 3^* a power of 3. On the other hand, the sufficiency condition is considered by M. Mimura and the author in Section 6 of [**14**] more generally as the construction of new (higher) homotopy associative H-spaces. The purpose of this paper is to describe the condition in terms of d, n and λ, working with a concept slightly stronger than homotopy associativity (or A_m-structure). Let Y be an A_m-space. Hopf's theorem implies that Y is rationally equivalent to a product of Eilenberg–MacLane spaces $\Pi_i K(Q,2n_i-1)$ which is a loop space. The space Y is defined to be A_m-primitive, following [**14**], if the rational equivalence preserves the A_m-structures, that is, it is an A_m-mapping.

Theorem A. *The following three conditions are equivalent for* $3\leqq m\leqq\infty$.

(1) $M(n,\lambda)$ *has an* A_m*-structure extending that of* G_{n-1}.

(2) $M(n,\lambda)$ *is an* A_m*-primitive* A_m*-space.*

(3) *For every prime* $p\leqq m$, *one of the following two is valid.*

  (*a*) λ *is prime to p*

  (*b*) $p\geqq dn$.

Remark 1. If m is not a prime, the primitivity condition in (2) is omittable. And if $dn\leqq 2p$, A_p-structure supports A_p-primitivity for dimensional reasons.

Theorem B. *Let p be an odd prime. Then the following three conditions are equivalent.*

(1) $M(n,\lambda)$ *is an* A_p*-primitive* A_p*-space at p.*

(2) $M(n,\lambda)$ *is an* A_∞*-space* (*loop space or monoid*) *at p.*

(3) λ *is prime to p or* $p\geqq dn$.

Remark 2. It is sufficient to prove for $G_n=U(n)$ and $Sp(n)$, because $U(n)$ has the homotopy type of $S^1\times SU(n)$. So, we may consider only for the cases $G_n=U(n)$ and $Sp(n)$.

Remark 3. (2) implies clearly (1). (3) implies that $M(n,\lambda)$ is homotopy equivalent to G_n at p. Hence (3) implies (2).

We will show in Section 1 that Theorem B implies Theorem A. So, we shall show that

(1) implies (3) to prove Theorem B in cases $G_n = U(n)$ and $Sp(n)$. To show this, we calculate that p-divisibility of Hubbuck operations (see [9, 10, 11]) on the projective space of $M(n,\lambda)$. Although the divisibility is not determined naturally and depends on the choice of a splitting of K-theory, the calculations on $BU(n)$ can be applied on the suspension space of $M(n,\lambda)$.

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1. Proof of Theorem A from Theorem B

Let Π be the set of all primes, $\mathbb{P}_1$ the set of primes p with $p \geqq dn$, $\mathbb{P}_2$ the set of primes p with $dn > p > m$ and $\mathbb{P}_3 = \Pi - \mathbb{P}_1 - \mathbb{P}_2$. Then G_n has the homotopy type of a product of spheres at $\mathbb{P}_1$. In particular, the bundle (0.1) is trivial. Hence, the pull-back $M(n,\lambda)$ is also trivial and homotopy equivalent to G_n at $\mathbb{P}_1$. Hence $M(n,\lambda)$ has an A_∞-structure extending that of G_{n-1} at $\mathbb{P}_1$. Secondly, we may write $\lambda = \lambda_1 \cdot \lambda_2$ where $(\lambda_1, \mathbb{P}_2) = 1$ and $(\lambda_2, \Pi - \mathbb{P}_2) = 1$. Let $q = \operatorname{Min} \mathbb{P}_2$ where we regard $\operatorname{Min} \phi = \infty$. Then $M(n,\lambda)$ is homotopy equivalent to $M(n,\lambda_2)$ at $\mathbb{P}_2$ which has an A_{q-1}-structure extending that of G_{n-1}, by Theorem 6.5 of [14]. Therefore, $M(n,\lambda)$ has an A_{q-1}-structure extending that of G_n, if and only if it has at $\mathbb{P}_3$, by the property (P7) of [14].

Firstly we assume (3). Then f is a homotopy equivalence between $M(n,\lambda)$ and G_n at $\mathbb{P}_3$ and G_n has an A_∞-structure extending that of G_{n-1}. This implies (1). Secondly we assume (1). Then by [13], it follows that the generators of $H^*(M(n,\lambda)); \mathbb{Q})$ are all A_{m-1}-primitive and therefore represented by A_{m-1}-mappings, by the property (P9) of [14]. By the proof of the Corollary in [13], the obstruction to be A_m-primitive is in $H^{2i}(M(n,\lambda)) * \ldots * M(n,\lambda); \mathbb{Q}) = H^{2i}(G_{n-1} * \cdots * G_{n-1}; \mathbb{Q})$, $i \leqq n$. (1) implies that the inclusion mapping $G_{n-1} \to M(n,\lambda)$ induces a homomorphism of spectral sequences of Stasheff's type (see [17, 13]). For dimensional reasons, the obstructions are mapped to 0 by the injective homomorphism induced from the inclusion. Hence the generators are represented by A_m-mappings and $M(n,\lambda)$ is A_m-primitive. This implies (2). Thirdly, we assume (2). Then by Fact 1 and Fact 2, it follows that λ is odd or $dn \leqq 2$, since m is greater than or equal to 3. For an odd prime $p \leqq m$, (2) implies that $M(n,\lambda)$ is an A_p-primitive space at p. Then by Theorem B, we obtain (3). This completes the proof of Theorem A.

2. Decomposition of BU and $\tilde{K}(BT^n)$ at an odd prime

Let R be the ring of localized integers at an odd prime p, $BU_{(p)}$ the localization of BU and $\tilde{K}(X) = \tilde{K}(X; R) = [X, BU_{(p)}]$. By Adams [3], $BU_{(p)}$ is decomposable to $p-1$ factors such as $BU_{(p)} \simeq BU^{(1)} \times \cdots \times BU^{(p-1)}$ and the Chern character is also decomposable to $p-1$ factors

$$ch^{(i)}: BU^{(i)} \to \Pi_{j \geqq 0} K(\mathbb{Q}, i + j(p-1)).$$

We denote by $\tilde{K}(X)^{(i)}$ the factor $[X, BU_{(p)}^{(i)}]$ of $\tilde{K}(X)$. Then it follows that $\tilde{K}(X) \simeq \tilde{K}(X)^{(1)} \oplus \cdots \oplus \tilde{K}(X)^{(p-1)}$. For $X = BT$, we have

$$\tilde{K}(BT)^{(i)} = R[[x^{p-1}]] \cdot x^i,$$

$$x \in \tilde{K}(BT)^{(1)}, ch^{(1)}(x) = \Sigma_{j \geqq 0} \frac{\bar{\alpha}_j}{(1 + j(p-1))!} y^{1+j(p-1)}, \tag{2.1}$$

with $\bar{\alpha}_0 = 1$ and $\bar{\alpha}_j \in Z$.

Then by (2.1), it follows that, for $X = BT^n$,

$$\tilde{K}(BT^n)^{(i)} = R\{X_{a_1}^{i_1 + j_1(p-1)} \times \cdots \times x_{a_m}^{i_m + j_m(p-1)}\}$$

where $j_t \geqq 0$, $i_1 + \cdots + i_m = i$, $1 \leqq a_1 < \cdots < a_m \leqq n$, $m \leqq n$ and $x_a \in \tilde{K}(BT^n)^{(1)}$ corresponds to the generator of the ath factor of BT^n.

Using x_a as above, we write K-algebras $K(BU(n))$ and $K(BSp(n))$ as follows:

$$K(BU(n)) = R[[c_1^K, \dots, c_n^K]] \cong R[[x_1, \dots, x_n]]^{\Sigma_n},$$

$$K(BSp(n)) = R[[c_2^K, \dots, c_{2n}^K]],$$

where $c_i^K \in K(BU(n))^{(i)}$ is mapped to $\sigma_i(x_1, \dots, x_n)$ by the monomorphism $K(BU(n)) \to K(BT^n)$, σ_i is the ith elementary symmetric polynomial and σ_n is the symmetric group on n letters.

Remark 2.1. c_i^K is the class obtained by modifying the γ-class so that $ch(c_i^K)$ lies in $\Pi_{j \geqq 0} H^{2i+2j(p-1)}(BU; R)$. Hence c_{2i+1}^K is mapped to 0 in $K(BSp(n))$.

3. Hubbuck operations in $K(BU(n))$

Let E be the fake $R \times BU_{(p)}$ such as $E = \Pi_{j \geqq 0} K(R, 2j)$ and $E(X) = [X, E]$. Then $E(BU(n)) \simeq R[[c_1, \dots, c_n]]$ and $E(BT_n) \simeq R[[y_1, \dots, y_n]]$, where c_1 is the ith Chern class and is mapped to $\sigma_i(y_1, \dots, y_n)$ by the ring monomorphism $E(BU(n)) \to E(BT^n)$.

To define Hubbuck operation, we need a splitting. Let us define the ring isomorphisms $J: E(BU(n)) \to K(BU(n))$ and $J_0: E(BT^n) \to K(BT^n)$ as follows:

$$J(c_i) = c_i^K,$$

$$J_0(y_i) = x_i.$$

We regard the algebras $K(BU(n))$ and $E(BU(n))$ as the subalgebra of $K(BT^n)$ and $E(BT^n)$, respectively. Then it follows that J can be regarded as the restriction of J_0 to $E(BU(n))$ and so we often denote J_0 by J. We wish to know the manner of Hubbuck operations on c_i^K in K-theory.

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Let us recall the Chern character on $K(BT)$. By (2.1), it follows that

$$ch(x)=\Sigma_{j\geqq 0}\frac{\alpha_j}{p^j}y^{j(p-1)+1},$$

where $\alpha_j=\bar{\alpha}_j\cdot p^j/(1+j(p-1))!$ is in R, because $(q+1)!$ divides $m(q)=\Pi_{\text{all primes}}p^{[q/(p-1)]}$ (see Adams [1]). To simplify notation, we introduce some functions in $\mathbb{Q}[[t]]$ where t is transcendental:

$$e(t)=\Sigma_j\frac{\alpha_j}{p^j}t^{j(p-1)+1}.$$

Since $d/dt(e(t))|_{t=0}=\alpha_0=\bar{\alpha}_0=1$, $e(t)$ has the inversion $\ell(t)$ in $\mathbb{Q}[[t]]$. We choose local integers β_j in R such that

$$\ell(t)=\Sigma_j\frac{\beta_j}{p^j}t^{j(p-1)+1}\quad\text{with } \beta_0=1.$$

Then it follows that

$$ch(x)=e(y)\quad\text{and}\quad ch(\ell(x))=\ell(e(y))=y. \tag{3.1}$$

We will describe Adams operations by using e and ℓ.

Firstly we will define a fake Adams operation Ψ^k on the fake K-theory E (see Hubbuck [9, 10, 11]) and reserve the symbol ψ^k for the genuine Adams operation.

Definition 3.1. The fake Adams operation Ψ^k on $E(-)$ is defined by the following formula:

$$\Psi^k(x_n)=k^n\cdot x_n\quad\text{for}\quad x_n\in H^{2n}(X;R).$$

Then the Chern character commutes with (fake) Adams operations ψ^k and Ψ^k. Therefore the Adams operation preserves the mod p decomposition of (fake) K-theories. So, we may write for the generator x of $K(BT)$,

$$\psi^k(x)=\Sigma_{j\geqq 0}\bar{r}_j(k)\cdot x^{j(p-1)+1}.$$

where $\bar{r}_j(k)$ is a local integer in R, $\bar{r}_0(k)=k$ and $\bar{r}_1(p)=1 \bmod p$. On the other hand, we can compute the Adams operation by using e and ℓ as follows:

$$\psi^k(x)=\psi^k(e(\ell(x)))=e(\psi^k(\ell(x)))$$

and

$$ch(\psi^k(\ell(x)))=\Psi^k(ch(\ell(x)))=\Psi^k(y)=k\cdot y=k\cdot ch(\ell(x))=ch(k\cdot\ell(x)).$$

Therefore, we obtain $\psi^k(\ell(x)) = k \cdot \ell(x)$ and then it follows that

$$\psi^k(x) = e(k \cdot \ell(x))$$

$$= \Sigma_{j \geqq 0} \frac{\alpha_j}{p^j} \cdot k^{j(p-1)+1} \cdot \ell(x)^{j(p-1)+1}$$

$$= \Sigma_{q \geqq 0} \frac{k}{p^j} (\Sigma_{q=j+1} \alpha_j \cdot k^{j(p-1)} \cdot \beta_{j,i}) \cdot x^{q(p-1)+1},$$

where $\beta_{j,i} \in R$ is given by the formula

$$\ell(x)^{j(p-1)+1} = \Sigma_{j \geqq 0} \frac{\beta_{j,i}}{p^i} \cdot x^{(i+j)(p-1)+1}.$$

Using α_j and $\beta_{j,i}$, we can define more "stabilized" decomposition of the Adams operation ψ^k by the following formula

$$\psi^k(x) = k \cdot r(k; x)$$

where

$$r(k; t) = \Sigma_{j \geqq 0} \frac{r_j(k)}{p^j} \cdot t^{j(p-1)+1},$$

$$r_q(k) = \Sigma_{q=j+i} \alpha_j \cdot k^{j(p-1)} \cdot \beta_{j,i}.$$

We remark that $\bar{r}_j(k)$ and $r_j(k)$ has the following relation

$$\bar{r}_j(k) = \frac{k \cdot r_j(k)}{p^j}. \tag{3.2}$$

We have prepared to describe the Hubbuck operations on $K(BU(n))$. Let us define Q_i, S_i and $R(k)_i$ in the ring $Q[[t_1, \dots, t_n]]$ as follows:

$$\begin{aligned} Q_i(t_1, \dots, t_n) &= \sigma_i(e(t_1), \dots, e(t_n)) \\ &= \Sigma_{j \geqq 0} \frac{1}{p^j} \cdot Q_i^j(t_1, \dots, t_n), \\ S_i(t_1, \dots, t_n) &= \sigma_i(\ell(t_1), \dots, \ell(t_n)) \\ &= \Sigma_{j \geqq 0} \frac{1}{p^j} \cdot S_i^j(t_1, \dots, t_n), \end{aligned} \tag{3.3}$$

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$$R(k)_i(t_1,\dots,t_n)=\sigma_i(r(k;t_1),\dots,r(k;t_n))$$

$$=\Sigma_{j\geqq 0}\bar{R}^j(k)_i(t_1,\dots,t_n)$$

where Q_i^j, S_i^j and $\bar{R}^j(k)_i$ are in $R[t_1,\dots,t_n]^{\Sigma^n}$ and are written as polynomials of the elementary symmetric polynomials $\sigma_1,\dots,\sigma_n$ of $t_1,\dots,t_n$, if $(k,p)=1$. The following equations can be easily checked:

$$ch(c_i^K)=Q_i(y_1,\dots,y_n),$$

$$ch(S_i(x_1,\dots,x_n))=c_i$$

$$\psi^k(c_i^K)=k^i\cdot R(k)_i(x_1,\dots,x_n),\qquad \text{when } (k,p)=1.$$

Next we define R-endomorphisms of $K(BU(n))$ extending the following relation by the Cartan formula (see Hubbuck [**9**, **10**, **11**]):

$$Q^j(c_i^K)=Q_i^j(x_1,\dots,x_n),$$

$$S^j(c_i^K)=S_i^j(x_1,\dots,x_n),$$

$$\bar{R}^j(k)(c_i^K)=\bar{R}^j(k)_i(x_1,\dots,x_n),\qquad \text{when } (k,p)=1.$$

Using them, we define $Q(t)$, $S(t)$ and $R(k;t)$ in $\mathrm{End}_R(K(BU(n)))\otimes_R\mathbb{Q}[[t]]$, by the following formula

$$Q(t)=\Sigma_j\frac{1}{p^j}Q^j\cdot t^{j(p-1)},$$

$$S(t)=\Sigma_j\frac{1}{p^j}S^j\cdot t^{j(p-1)},$$

$$R(k;t)=\Sigma_j\bar{R}^j(k)\cdot t^{j(p-1)}.$$

Then by the definition, we obtain

Proposition 3.1. *The following four equations are valid*:

(1) $\mathrm{Q}=\mathrm{J}\circ\mathrm{ch}$ and $\mathrm{ch}\circ\mathrm{S}=\mathrm{J}^{-1}$,

(2) $\mathrm{S}\circ\mathrm{Q}=\mathrm{Q}\circ\mathrm{S}=$ Identity *and* $\psi^k\circ S\circ J(w_i)=k^i\cdot S\circ J(w_i)$,

(3) $\psi^k\circ J(w_i)=k^i\cdot R(k)\circ J(w_i)$,

(4) $\bar{R}^j(k)=1/p^j\Sigma_{i=0}^j k^{i(p-1)}\cdot S^{j-i}\circ Q^i$,

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where w_i *is in* $H^{2i}(BU(n); R)$.

Proof. (1) and (3) are obtained directly by the definitions of the Hubbuck operations on c_i^K together with the Cartan formulae. Firstly we show (2). By (1), $J \circ ch \circ S = \text{Identity}$. This implies that $Q \circ S = \text{Identity}$ and therefore, $S \circ Q = Q \circ S = \text{Identity}$. Similarly, we have $ch \circ S \circ J = \text{Identity}$. This implies that $ch \circ \psi^k \circ S \circ J = \Psi^k \circ ch \circ S \circ J = \Psi^k$ and $ch \circ \psi^k \circ S \circ J(w_i) = \psi^k(w_i) = k^i \cdot w_i = k^i \cdot ch \circ S \circ J(w_i)$. Hence $\psi^k \circ S \circ J(w_i) = k^i \cdot S \circ J(w_i)$. To show (4), it suffices to show

$$R(k) \circ J(w_i) = \Sigma_q (1/p^q) \cdot \Sigma_{q \geq j \geq 0} k^{j(p-1)} \cdot S^{q-j} \circ Q^j \circ J(w_i),$$

because both $R^q(k)$ and $S^{q-j} \circ Q^j$ increase the same weight $q(p-1)$. By (3), we obtain that

$$\begin{aligned} k^i \cdot R(k) \circ J(w_i) &= \psi^k \circ J(w_i) \\ &= \psi^k \circ S \circ Q \circ J(w_i) \\ &= \Sigma_j \frac{1}{p^j} \cdot \psi^k \circ S \circ Q^j \circ J(w_i). \end{aligned}$$

Here, $Q^j \circ J(w_i)$ has the weight $i + j(p-1)$ and then by (3), we proceed as follows:

$$\begin{aligned} k^i \cdot R(k) \circ J(w_i) &= \Sigma_j \frac{k^{i+j(p-1)}}{p^j} \cdot S \circ Q^j \circ J(w_i) \\ &= k^i \cdot \Sigma_q (1/p^q) \cdot (\Sigma_{q \geq j \geq 0} k^{j(p-1)} \cdot S^{q-j} \circ Q^j) \circ J(w_i). \end{aligned}$$

This completes the proof of Proposition 3.1.

4. p-divisibility

Before starting to prove Theorem B, we will show the key lemma of this paper. We denote by v_p the valuation of the ring of p-localized integers R, that is, $v_p(m)$ is the largest power of p dividing m.

From now we assume that $k = p - 1$. Then by Adams [2] or Hubbuck [**10**, Lemma 4.3], it follows that

$$v_p(k^{j(p-1)} - 1) = v_p(j) + 1.^* \tag{4.1}$$

Firstly we show the p-divisibility of Hubbuck operations in $K(BU(n))$.

*If we take $k = 2$, this equality fails for $p = 1093$ (see [**6**]).

Lemma 4.1 *Let $i\cdot n=m+j(p-1)$, $i\geqq 1$, $n\geqq m\geqq 1$. If $v_p(m)\leqq v_p(n)$, then the coefficient of $(c_n^K)^i$ in $\bar{R}^j(k)(c_m^K)$ is divisible by n/m in R.*

Proof. We write $\bar{R}^j(k)_m(t_1,\dots,t_n)=P_m^j(\sigma_1,\dots,\sigma_n)$ in $R[\sigma_1,\dots,\sigma_n]$, where σ_1 is the ith symmetric polynomial of $t_1,\dots,t_n$. Then the desired coefficient is given by $P_m^j(0,\dots,0,1)=\bar{R}^j(k)_m(\xi,\xi^2,\dots,\xi^n)$, where ξ is the primitive nth root of unity in the complex number field. Using the definition (3.3), we write

$$\bar{R}^j(k)_m(t_1,\dots,t_n)=\Sigma_{j_1,\dots,j_m}\beta_{j_1}\cdot\ldots\cdot\beta_{j_m}\cdot L_m^{j_1,\dots,j_m}(t_1,\dots,t_n),$$

where $j_1,\dots,j_m$ run over all integers such that $0\leqq j_1\leqq\cdots j_m\leqq j=j_1+\cdots+j_m$ and the polynomial $L_m^{\cdots}$ is given by

$$L_m^{j_1,\dots,j_m}(t_1,\dots,t_n)=\Sigma t_{a_1}^{j'_1(p-1)+1}\dots t_{a_m}^{j'_m(p-1)+1},$$

where $j'_1,\dots,j'_m$ and $a_1,\dots,a_m$ run over the set Δ given by $\{(a_1,\dots,a_m;j'_1,\dots,j'_m)\,|\,1\leqq a_1<\cdots<a_m\leqq n,\ (j'_1,\dots,j'_m)=(j_1,\dots,j_m)$ if we ignore the ordering$\}$, by the following calculation:

$$\sigma_m(\ell(t_1),\dots,\ell(t_n))=\Sigma_{1\leqq a_1>\cdots>a_m\leqq n}\ell(t_{a_1})\cdot\ldots\cdot\ell(t_{a_m})$$

$$=\Sigma_j\frac{1}{p^j}\Sigma_{j=j_1+\cdots+j_m}\beta_{j_1}\cdot\ldots\cdot\beta_{j_m}\cdot\Sigma_{1\leqq a_1>\cdots>a_m}t_{a_1}^{j_1(p-1)+1}\dots t_{a_m}^{j_m(p-1)+1}$$

$$=\Sigma_j\frac{1}{p^j}\Sigma_{j_1,\dots,j_m}\beta_{j_1}\cdot\ldots\cdot\beta_{j_m}\cdot L_m^{j_1,\dots,j_m}(t_1,\dots,t_n).$$

Hence we obtain that

$$P_m^j(0,\dots,0,1)=\Sigma_{j_1,\dots,j_m}\beta_{j_1}\cdot\ldots\cdot\beta_{j_m}\cdot L_m^{j_1,\dots,j_m}(\xi,\dots,\xi^n).$$

$$L_m^{j_1,\dots,j_m}(\xi,\dots,\xi^n)=\Sigma_{\delta\in\Delta}\xi^{g(\delta)},$$

where $j_1,\dots,j_m$ run over $0\leqq j_1\leqq\cdots\leqq j_m\leqq j=j_1+\cdots+j_m$, $g(\delta)=a_1(j'_1(p-1)+1)+\cdots+a_m(j'_m(p-1)+1)$ in the cyclic group of order n and $\delta=(a_1,\dots,a_m;j'_1,\dots,j'_m)$. We remark here that $L_m^{\cdots}$ is a localized integer. So, we are left to show that the localized integer $L_m^{\cdots}$ is divisible by n/m if $v_p(n)\geqq v_p(m)$.

Let τ be the element of Σ_n such that $\tau(a)=a+1 \bmod n$ and σ the element of Σ_m such that $1\leqq\tau(a_{\sigma(1)})<\cdots<\tau(a_{\sigma(m)})\leqq n$. Then $\sigma=$Identity or $\sigma(i)=i-1 \bmod m$ for all i. We remark that σ depends on both τ and $(a_1,\dots,a_m)$. Let $\delta=(a_1,\dots,a_m;j'_1,\dots,j'_m)$ and $\tau\cdot\delta=(a_{\sigma(1)},\dots,a_{\sigma(m)};j'_{\sigma(1)},\dots,j'_{\sigma(m)})$. Then we obtain that

$$g(\tau\cdot\delta)=g(\delta)\quad\text{in } Z/nZ,$$

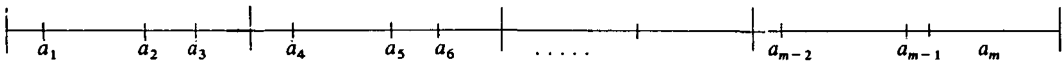

FIGURE 1

because $i\cdot n=j(p-1)+m=(j_1(p-1)+1)+\cdots+(j_m(p-1)+1)$. Therefore we obtain the following equation

$$\Sigma_{\delta\in\Delta}\xi^{g(\delta)}=\Sigma_{[\delta]\in\Delta'}n(\tau,\delta)\cdot\xi^{g(\delta)},$$

where Δ' is the quotient set of Δ by the action of Z/nZ and $n(\tau,\delta)$ is the cardinality of the set $\{\delta,\tau\delta,\ldots,\tau^{n-1}\delta\}$. Hence, $n(\tau,\delta)$ divides n.

On the other hand, the equation $\xi^{n(\tau,\delta)}\delta=\delta$ implies $a_{\sigma'(i)}+n(\tau,\delta)=a_i \bmod n$ and $\sigma'(i)=i-m(\tau,\delta) \bmod m$ for some $\sigma'\in\Sigma_m$ and $1\leqq m(\tau,\delta)\leqq m$. Therefore, we obtain the equation

$$\begin{aligned} m(\tau,\delta) &= \#(\{a_1,\ldots,a_m\}\cap[1,n(\tau,\delta)]) \\ &= \#(\{a_1,\ldots,a_m\}\cap[n(\tau,\delta)+1,2n(\tau,\delta)]) \\ &\ \ \vdots \\ &= \#(\{a_1,\ldots,a_m\}\cap[n-n(\tau,\delta)+1,n]). \end{aligned}$$

This implies $m(\tau,\delta)$ divides m and $m/m(\tau,\delta)=n/n(\tau,\delta)$. Hence $n(\tau,\delta)=n\cdot m(\tau,\delta)/m$ and

$$L_m^{j_1,\ldots,j_m}(\xi,\ldots,\xi^n)=(n/m)\cdot\Sigma_{[\delta]\in\Delta'}m(\tau,\delta)\cdot\xi^{g(\delta)}$$

in the ring R if $v_p(n)\geqq v_p(m)$. This implies the lemma.

5. Proof of Theorem B

We assume (1). To construct a Hubbuck operation on the projective spaces $P(m)$ and $\bar{P}(m)$ for G_n and $M(n,\lambda)$, we need a splitting from the fake K-theory E^* to K-theory K^*. Let us recall that $K^*(P(m))=M\oplus S_m$ and $K^*(\bar{P}(m))=\bar{M}\oplus\bar{S}_m$ where M and $\bar{M}$ are polynomial algebras truncated at height $m+1$ and S_m and $\bar{S}_m$ are ideals (see [13]). By the definition of S_m and $\bar{S}_m$, it follows that $\psi^k(S_m)\subset S_m$ and $\psi^k(\bar{S}_m)\subset\bar{S}_m$. In the proof given in [13], it is required that the K-theory of H-spaces has no torsion and that H-spaces are A_m-primitive. No other assumption is required. So, we obtain the following isomorphisms similarly to [13]: $E^*(P(m))=N\oplus T_m$ and $E^*(\bar{P}(m))=\bar{N}\oplus\bar{T}_m$ where N and $\bar{N}$ are polynomial algebras truncated at height $m+1$ and T_m and $\bar{T}_m$ are ideals.

Let η_n be the canonical n-bundle over BG_n (complex or quaternionic). Then M is generated by $c_{di}^K(\eta_n)$ for $i\leqq n$ and N is generated by $c_{di}(\eta_n)$ for $i\leqq n$. Let QM and $Q\bar{M}$ be the indecomposable quotients of M and $\bar{M}$, respectively. Then by [13], it follows that $QM\simeq QK^*(G_n)$ and $Q\bar{M}\simeq QK^*(M(n,\lambda))$ whose generators are corresponding to each other by the homomorphism induced by $\bar{f}$ except for the generators in exact filtration degree $2dn-1$. In the filtration degree $2dn-1$, the generators are spherical and $\bar{f}^!$ times λ on the generators. Hence we obtain

Proposition 5.1.

(1) *M is generated by $u_i = c_{di}^K(\eta_n)$ for $i \leqq n$,*
(2) *$\bar{M}$ is generated by $\bar{u}_i$ for $i \leqq n$,*
(3) *$\bar{u}_i = \Sigma \bar{f}^!(u_i)$ in $Q\bar{M}$ for $i < n$ and*
(4) *$\lambda \cdot \bar{u}_n = \Sigma \bar{f}^!(u_n)$ in $Q\bar{M}$.*

Proposition 5.2.

(1) *N is generated by $v_i = c_{di}(\eta_n)$ for $i < n$,*
(2) *$\bar{N}$ is generated by $\bar{v}_i$ for $i \leqq n$,*
(3) *$\bar{v}_i = \Sigma \bar{f}^!(v_i)$ in $Q\bar{N}$ for $i < n$ and*
(4) *$\lambda \cdot \bar{v}_n = \Sigma \bar{f}^!(v_n)$ in $Q\bar{N}$.*

Then we define the splittings J and $\bar{J}$ by the following equations:

$$J(v_i) = u_i, \quad i \leqq n \quad \text{and}$$

$$\bar{J}(\bar{v}_i) = \bar{u}_i, \quad i \leqq n.$$

The mapping $\bar{f}$ induces the following homomorphism ϕ:

$$\begin{aligned} &\phi(u_i) = \bar{u}_i, \quad i < n, \quad \text{and} \\ &\phi(u_n) = \lambda \cdot \bar{u}_n \quad \text{in } Q\bar{M}. \end{aligned} \tag{5.1}$$

Remark 5.3. If one extends ϕ as a ring homomorphism, then ϕ does not commute with Adams operations, even if $\bar{f}$ is an A_m-mapping. Also $\bar{f}$ induces the following homomorphism ϕ_0:

$$\begin{aligned} &\phi_0(v_i) = \bar{v}_i, \quad i < n, \quad \text{and} \\ &\phi_0(v_n) = \lambda \cdot \bar{v}_n, \quad \text{in } Q\bar{N}. \end{aligned} \tag{5.2}$$

By Hubbuck [**9**, **10**], these splitting J and $\bar{J}$ determine K-theory operations S_J^h, $S_{\bar{J}}^h$, Q_J^h, $Q_{\bar{J}}^h$, $\bar{R}_J^h$ and $\bar{R}_{\bar{J}}^h$, which now satisfy

$$\bar{R}_{\bar{J}}^h \circ \phi_0 = \phi_0 \circ \bar{R}_J^h \quad \text{in } QN, \tag{5.3}$$

since $\phi \circ J = \bar{J} \circ \phi_0$ by (5.1) and (5.2).

We will write the Hubbuck operations by S^h, Q^h, R^h and $\bar{R}^h$ when the formula is valid independently of the choice of a splitting. The following formulae are due to Hubbuck (see [**9**, **10**]):

Proposition 5.4.

(1) $\bar{R}^h$ *is an integral operation,*

(2) $\bar{R}^h$ *satisfies the Cartan formula* $\bar{R}^h(x\cdot y)=\Sigma_{i+j=h}\bar{R}^i(x)\cdot\bar{R}^j(y)$,

(3) $S^{di}(v_i)=v_i^p \bmod p$,

(4) $S^{di}(\bar{v}_i)=\bar{v}_i^p \bmod p$,

(5) $(1-k^{q(p-1)})\cdot S^q=\Sigma_{h=1}^{q-1}k^{(q-h)(p-1)}\cdot p^h\cdot\bar{R}^h\circ S^{q-h}+p^q\cdot\bar{R}^q$, *where* $k=p-1$.

Remark 5.5. By the definition of J, S_J^h, Q_J^h and $\bar{R}_J^h$ coincide with the restriction to $P(m)$ of S^h, Q^h and $\bar{R}^h(k)$ respectively given in Section 4, if we identify K-theory with fake K-theory by the splitting J above.

Assuming that $\lambda=0 \bmod p$ and $dn>p$, we are led to a contradiction.

Let $a=v_p(dn)$. Then by a simple computation, $(a+1)(p-1)<dn$. By Proposition 5.4, we obtain the following proposition similarly to Hubbuck–Mimura [**12**].

Proposition 5.6. *The following two statements are valid in QM:*

(1) $p^{a+1}\cdot v_n\in p^h\cdot\bar{R}_J^h(QN^{dm}) \bmod I+(p^{a+2})$

(2) $p^{a+1}\cdot\bar{v}_n\in p^{h'}\cdot\bar{R}_{\bar{J}}^{h'}(Q\bar{N}^{dm}) \bmod \bar{I}+(p^{a+2})$

for some $1\leqq h,\ h'\leqq a+1$, *where* $m=n-h(p-1)/d$, $m'=n'-h'(p-1)/d$, $I=(v_1,\dots,v_{n-1})$ *and* $\bar{I}=(\bar{v}_1,\dots,\bar{v}_{n-1})$.

Proof. The formulae given in (5.4) imply that

$$(1-k^{dn(p-1)})\cdot v_n^i\in p^h\cdot\bar{R}_J^h(QN^{dm}) \bmod I+(p^{a+2})$$

$$(1-k^{dn(p-1)})\cdot\bar{v}_n^{i'}\in p^{h'}\cdot\bar{R}_{\bar{J}}^{h}(Q\bar{N}^{dm'}) \bmod \bar{I}+(p^{a+2})$$

for some $1\leqq h,\ h'\leqq a+1$ such that $in=m+h(p-1)/d$, $i'n=m'+h'(p-1)/d$ and $1\leqq i,\ i'\leqq p$. For dimensional reasons, in the formula above, we obtain that $i=i'=1$. Then by (4.1), Proposition 5.6 follows.

By Lemma 4.1 and Proposition 5.6, it follows that

$$p^{a+1}\cdot v_n\in p\cdot\bar{R}_J^1(QN^{dn-(p-1)}) \bmod p^{a+2}\quad \text{in } QN^{dn}.$$

Also by (5.3) together with Lemma 4.1 and Proposition 5.6, it follows that

$$p^{a+1}\cdot\bar{v}_n\in p\cdot\bar{R}_{\bar{J}}^1(Q\bar{N}^{dn-(p-1)}) \bmod p^{a+2}\quad \text{in } Q\bar{N}^{dn}.$$

However, if $(\lambda,p)\neq 1$, then by (5.3) together with Lemma 4.1 and Remark 4.2, it follows that $\bar{R}_{\bar{J}}^1(Q\bar{N}^{dn-(p-1)})=\lambda\cdot\phi_0\bar{R}_J^1(QN^{dn-(p-1)})=0 \bmod p^{a+1}$ and hence, $p^{a+1}\cdot\bar{v}_n=0 \bmod p^{a+2}$. It is a contradiction and this completes the proof of Theorem B.

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