On a Class of Surfaces.

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§ 1. The surfaces here considered were first discussed by Monge as surfaces whose normals are tangents to given developables. Under the name general *surfaces moulures*, Darboux treats them as the surfaces traced out by a fixed curve on a plane which rolls on a developable. When the developable is a cylinder he gives the general coordinates in his *Leçons sur la Theorie Generale des Surfaces* (Volume I., page 105). This led me to take up the more general case, but I later found that Darboux had also considered this in an ingenious and elegant manner in his *Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes* (Tome 1, pages 26–34). Perhaps this quite different discussion will present some interesting points in analysis.

§ 2. The surface traced out by the curve being necessarily cut perpendicularly by the plane, and the normals to the surface intersecting, the curve is at the same time a geodesic and a line of principal curvature. Let us first find the fundamental forms of all the surfaces of which one system of the lines of curvature is geodesic.

If $x(u, v)$, $y(u, v)$, $z(u, v)$ are the coordinates of a point of a surface, and if the curves $u = \text{const.}$, $v = \text{const.}$ are the principal lines of curvature, the square of the element of length is given by

$$ds^2 = \mathrm{E}du^2 + \mathrm{G}dv^2$$

where
$$\mathrm{E} = \Sigma x_u^2, \quad \mathrm{G} = \Sigma x_v^2.$$

The radius R of curvature of a normal section is given by

$$\frac{ds^2}{\mathrm{R}} = \mathrm{L}du^2 + \mathrm{N}dv^2$$

where
$$\mathrm{L} = \frac{1}{\sqrt{\mathrm{EG}}}\begin{vmatrix} x_{uu} & x_u & x_v \\ y_{uu} & y_u & y_v \\ z_{uu} & z_u & z_v \end{vmatrix}, \quad \mathrm{N} = \frac{1}{\sqrt{\mathrm{EG}}}\begin{vmatrix} x_{vv} & x_u & x_v \\ y_{vv} & y_u & y_v \\ z_{vv} & z_u & z_v \end{vmatrix}.$$

Between the quantities E, G, L, N exist the Mainardi-Codazzi equations :—*

$$\frac{LN}{\sqrt{EG}}+\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial u}\right)=0\,;$$

$$\frac{\partial}{\partial v}\left(\frac{L}{\sqrt{E}}\right)=\frac{N}{G}\frac{\partial\sqrt{E}}{\partial v}; \qquad \frac{\partial}{\partial u}\left(\frac{N}{\sqrt{G}}\right)=\frac{L}{E}\frac{\partial\sqrt{G}}{\partial u}.$$

If E is a function of u alone the curves $v=$ const. are geodesics. We may then, by a change of variable without loss of generality, take E to be unity.

From the second of these equations L is a function of u alone. Let $N/\sqrt{G}=\psi$. Then from the first and the third equation

$$L\psi+\frac{\partial}{\partial u}\left(\frac{\psi_u}{L}\right)=0$$

or

$$\psi_{uu}-\frac{L_u}{L}\psi_u+L^2\psi=0.$$

As L from the second equation does not involve v, we may put $L=\dfrac{U'}{\sqrt{(1-U^2)}}$ where U is an arbitrary function of u and U′ its first derivative. We are guided to this substitution by the fact that if ψ in the above equation is considered known we find $L=\dfrac{\psi_u}{\sqrt{(1-\psi^2)}}$ as a solution.

The equation then becomes

$$\psi_{uu}-\left[\frac{U''}{U'}+\frac{UU'}{1-U^2}\right]\psi_u+\frac{U'^2}{1-U^2}\psi=0,$$

of which one solution is evidently $\psi=U$.

We have therefore

$$\psi=\frac{U}{\rho}+\frac{U}{\tau}\int\frac{dU}{U^2\sqrt{(1-U^2)}}$$

$$=\frac{U}{\rho}-\frac{\sqrt{(1-U^2)}}{\tau}$$

where ρ and τ are arbitrary functions of v.

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* (Differentialgeometrie, Bianchi. German translation, page 94).

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Since $\frac{\partial \sqrt{G}}{\partial u} = \frac{\psi_u}{L}$ we have

$$\sqrt{G} = \frac{1}{\rho}\int \sqrt{(1-U^2)}du + \frac{1}{\tau}\int U du + \chi$$

where χ is an arbitrary function of v.

x, y, z are solutions of the equation

$$\frac{\partial^2\theta}{\partial u \partial v} = \frac{G_u}{2G}\frac{\partial\theta}{\partial v}.$$

[See Salmon, *Geometry of Three Dimensions*, p. 441, or Bianchi, p. 111.]

$$\therefore \; x = F_1(u) + \int \sqrt{(1-U^2)}du \int \frac{f_1(v)}{\rho}dv + \int U du \int \frac{f_1(v)}{\tau}dv + \int \chi f_1(v) dv.$$

y and z are obtained by replacing the suffix 1 by 2 and 3.

The three functions f_1, f_2, f_3 must be so chosen that

$$E = \Sigma x_u^2 = 1, \quad D = \Sigma x_u x_v = 0 \quad \text{and} \quad G = \Sigma x_v^2.$$

From the last $f_1^2 + f_2^2 + f_3^2 = 1$. Hence the three functions can be taken as the direction cosines (functions of v) of a line. Denote them by l, m, n.

Let
$$\int \frac{f_1}{\rho}dv = \alpha, \quad \int \frac{f_2}{\rho}dv = \beta, \quad \int \frac{f_3}{\rho}dv = \gamma,$$
$$\int \frac{f_1}{\tau}dv = \lambda, \quad \int \frac{f_2}{\tau}dv = \mu, \quad \int \frac{f_3}{\tau}dv = \nu.$$

From $\Sigma x_u x_v = 0$ we have

$$\Sigma\left\{F_r'(u) + \sqrt{(1-U^2)}\int \frac{f_r}{\rho}dv + U\int \frac{f_r}{\tau}dv\right\} f_r \sqrt{G} = 0.$$

We see that by arbitrarily varying u and v, F_1, F_2, F_3 are constants, and by changing the origin we can make them zero. Also we have

$$\alpha l + \beta m + \gamma n = 0,$$
$$\lambda l + \mu m + \nu n = 0.$$

From $\Sigma x_u^2 = 1$ we obtain

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{and} \quad \lambda^2 + \mu^2 + \nu^2 = 1.$$

The nine quantities are therefore the direction cosines of three mutually rectangular lines, satisfying the equations

$$\frac{d\alpha}{dv} = \frac{l}{\rho}, \quad \frac{d\lambda}{dv} = \frac{l}{\tau}, \quad l^2 = 1 - \alpha^2 - \lambda^2$$

or
$$\frac{dl}{dv} = -\frac{\alpha}{\rho} - \frac{\lambda}{\tau} \text{ with two corresponding sets.}$$

Therefore by Frenet's formulae ρ, τ are the radii of curvature and torsion of a curve of length v, and these nine quantities are the direction cosines of the tangent, principal normal and binormal.

The equations of the surface are therefore

$$x = a\int \sqrt{(1-U^2)}du + \lambda\int U du + \int l\chi dv,$$

$$y = \beta\int \sqrt{(1-U^2)}du + \mu\int U du + \int m\chi dv,$$

$$z = \gamma\int \sqrt{(1-U^2)}du + \nu\int U du + \int n\chi dv.$$

Since $\quad lx + my + nz = l\int l\chi dv + m\int m\chi dv + n\int n\chi dv,$

the curves $v =$ a constant are plane. Referred to two lines in this plane with direction cosines a, β, γ; λ, μ, ν, the equations of this curve are $\xi = \int \sqrt{(1-U^2)}du$, $\eta = \int U du$.

The plane of the curve is then a tangent plane to a developable.

Differentiating the equation of the plane twice with regard to v and solving, we find the edge of regression is given by

$$x = \int l\chi dv + \left\{a\frac{d}{dv}\left(\frac{\chi}{\tau}\right) - \lambda\frac{d}{dv}\left(\frac{\chi}{\rho}\right)\right\} \div \frac{1}{\tau^2}\frac{d}{dv}\left(\frac{\tau}{\rho}\right),$$

and two similar equations for y and z.

Making $a = \beta = \nu = n = 0$ we see that $\frac{1}{\rho} = 0$ and that the plane rotates round a cylinder whose generators are parallel to the axis of z. If then $\lambda = \cos v$, $\mu = \sin v$, $l = \sin v$, $m = -\cos v$, we get Darboux's formulae:—

$$x = \cos v\int U du + \int \chi \sin v dv,$$

$$y = \sin v\int U du - \int \chi \cos v dv,$$

$$z = \int \sqrt{(1-U^2)}du.$$

If in the general equations $\chi = 0$, the plane rolls on a cone. If both χ and n are zero, the surfaces become surfaces of revolution, because from Frenet's formulae γ and ν become constant.

§ 3. Suppose the position of the moving plane is given at every instant, that is, its direction cosines and its distance from the origin as functions of the time v are known.

Let its equation be

$$lx + my + nz = \mathrm{F}(v).$$

The direction cosines α, β, γ ; λ, μ, ν have first to be found.

$$\alpha^2 + \lambda^2 + l^2 = 1.$$

Therefore on differentiating with regard to v,

$$\frac{\alpha}{\rho} + \frac{\lambda}{\tau} = -l',$$

$$\frac{\beta}{\rho} + \frac{\mu}{\tau} = -m',$$

$$\frac{\gamma}{\rho} + \frac{\nu}{\tau} = -n'.$$

Squaring and adding, we have

$$\frac{1}{\rho^2} + \frac{1}{\tau^2} = l'^2 + m'^2 + n'^2 = \frac{1}{\mathrm{P}^2}, \text{ say.}$$

Let $$\frac{\mathrm{P}}{\rho} = \cos\theta, \quad \frac{\mathrm{P}}{\tau} = \sin\theta.$$

Then $$\alpha\cos\theta + \lambda\sin\theta = -\mathrm{P}l',$$

$$\beta\cos\theta + \mu\sin\theta = -\mathrm{P}m',$$

$$\gamma\cos\theta + \nu\sin\theta = -\mathrm{P}n'.$$

Therefore $$(n\beta - m\gamma)\cos\theta - (m\nu - \mu n)\sin\theta = \mathrm{P}(mn' - m'n),$$

or $$\lambda\cos\theta - \alpha\sin\theta = \mathrm{P}(mn' - m'n).$$

Therefore $$\alpha = -\mathrm{P}\{l'\cos\theta + (mn' - m'n)\sin\theta\},$$

$$\lambda = -\mathrm{P}\{l'\sin\theta - (mn' - m'n)\cos\theta\},$$

with similar results for β, μ and γ, ν.

Differentiating $\alpha\cos\theta + \lambda\sin\theta = -\mathrm{P}l'$, we have

$$\mathrm{P}^3 l'(l'l'' + m'm'' + n'n'') = \mathrm{P}l'' + l\left(\frac{\cos\theta}{\rho} + \frac{\sin\theta}{\tau}\right) - (\alpha\sin\theta - \lambda\cos\theta)\theta'$$

$$= \mathrm{P}l'' + \frac{l}{\mathrm{P}} + \mathrm{P}(mn' - m'n)\theta'.$$

Also, cyclically,

$$\mathrm{P}^3 m'(l'l'' + m'm'' + n'n'') = \mathrm{P}m'' + \frac{m}{\mathrm{P}} + \mathrm{P}(nl' - n'l)\theta'.$$

Dividing, we have

$$\theta' \mathrm{P}(nl'^2 - n'll' - mm'n' + m'^2 n) = \mathrm{P}(l''m' - l'm'') + \frac{1}{\mathrm{P}}(lm' - l'm).$$

Since $ll' + mm' + nn' = 0$, this reduces to

$$n\theta' = \mathrm{P}^2(l''m' - l'm'') + lm' - l'm.$$

Also
$$l\theta' = \mathrm{P}^2(m''n' - m'n'') + mn' - m'n,$$
$$m\theta' = \mathrm{P}^2(n''l' - n'l'') + nl' - n'l.$$

Multiplying by n, l, m and adding, we obtain

$$\theta' = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \div (ll'' + mm'' + nn''),$$

since $\dfrac{1}{\mathrm{P}^2} = l'^2 + m'^2 + n'^2 = -(ll'' + mm'' + nn'')$.

Thus θ and the six remaining direction cosines are determined.

Since $l\int l\chi dv + m\int m\chi dv + n\int n\chi dv = lx + my + nz$,

we have $l\int l\chi dv + m\int m\chi dv + n\int n\chi dv = \mathrm{F}(v)$.

Let l, m, n be the direction cosines of the tangent to a curve of length v, and let l_1, m_1, n_1; l_2, m_2, n_2 be the direction cosines of the principal normal and binormal, r and σ being the radii of curvature and torsion.

Differentiating and using Frenet's formulae, we obtain

$$\Sigma l_1 \int l\chi dv = (\mathrm{F}' - \chi)r.$$

A second differentiation gives

$$\Sigma\left(\frac{l}{r} + \frac{l_2}{\sigma}\right)\int l\chi dv = (\chi' - \mathrm{F}'')r + (\chi - \mathrm{F}')r'.$$

Therefore
$$\Sigma l_2 \int l\chi dv = (\chi' - \mathrm{F}'')r\sigma + (\chi - \mathrm{F}')r'\sigma - \frac{\mathrm{F}\sigma}{r}.$$

A third differentiation gives

$$(\mathrm{F}' - \chi)\frac{r}{\sigma} = \Sigma\frac{l_1}{\sigma}\int l\chi dv = (\chi'' - \mathrm{F}''')r\sigma + (\chi' - \mathrm{F}'')r'\sigma + (\chi' - \mathrm{F}'')(r'\sigma + r\sigma')$$
$$+ (\chi - \mathrm{F}')(r''\sigma + r'\sigma') - \frac{\mathrm{F}'\sigma}{r} - \frac{\mathrm{F}\sigma'}{r} + \frac{\mathrm{F}\sigma r'}{r^2}.$$

Therefore $$r^2\sigma\chi'' + (2rr'\sigma + r^2\sigma')\chi' + (r + r''\sigma^2 + r'\sigma\sigma')\frac{r}{\sigma}\chi$$

$$= r^2\sigma F''' + (2rr'\sigma + r^2\sigma')F'' + (r + r''\sigma^2 + r'\sigma\sigma')\frac{r}{\sigma}F'$$

$$+ F'\sigma + F\sigma' - \frac{F\sigma r'}{r}.$$

If the curve is spherical, so that R being the radius of spherical curvature, $R^2 = r^2 + \sigma^2\left(\frac{dr}{dv}\right)^2$ or $r + \sigma^2 r'' + \sigma\sigma' r' = 0$, then

$$r^2\sigma\chi'' + (2rr'\sigma + r^2\sigma')\chi' = F'''r^2\sigma + (2rr'\sigma + r^2\sigma')F''$$
$$+ F'\sigma + F\sigma' - \frac{F\sigma r'}{r}.$$

This gives

$$r^2\sigma\chi' = r^2\sigma F'' + F\sigma - \int\frac{F\sigma r'}{r}dv$$

$$= r^2\sigma F'' + \int r\frac{d}{dv}\left(\frac{F\sigma}{r}\right)dv.$$

Hence $$\chi = F' + \int\left\{\frac{1}{r^2\sigma}\int r\frac{d}{dv}\left(\frac{F\sigma}{r}\right)dv\right\}dv.$$

If the curve is not spherical let ds be a new element of length, and choose v such a function of s that the new curve may be spherical.

From $\frac{dl}{dv} = \frac{l_1}{r}$ we have $\frac{dl}{ds} = \frac{l_1}{r\frac{ds}{dv}}$.

Thus we see that the direction cosines are not altered, but that the new radius of curvature $r_1 = r\frac{ds}{dv}$, and similarly the new radius of torsion σ_1 is $\sigma\frac{ds}{dv}$.

Then R_1 being the radius of the sphere, we have

$$R_1^2 = r_1^2 + \sigma_1^2\left(\frac{dr_1}{ds}\right)^2.$$

Therefore $$\sigma_1\frac{dr_1}{ds} = \sqrt{\left\{R_1^2 - r^2\left(\frac{ds}{dv}\right)^2\right\}}.$$

But $$\frac{dr_1}{ds}=\frac{dr}{dv}+r\frac{d^2s}{dv^2}\cdot\frac{dv}{ds}.$$

Therefore $$\sigma_1\frac{dr_1}{ds}=\sigma\left\{\frac{dr}{dv}\frac{ds}{dv}+r\frac{d^2s}{dv^2}\right\}=\sqrt{\left\{R_1^2-r^2\left(\frac{ds}{dv}\right)^2\right\}},$$

or $$\frac{\frac{d}{dv}\left(r\frac{ds}{dv}\right)}{\sqrt{\left\{R_1^2-r^2\left(\frac{ds}{dv}\right)^2\right\}}}=\frac{1}{\sigma},$$

or $$\frac{ds}{dv}=\frac{R_1}{r}\sin\int\frac{dv}{\sigma}.$$

If then in the original equation for χ we change the independent variable from v to s and solve for $\chi\frac{dv}{ds}$, we get

$$\chi\frac{dv}{ds}=\frac{dF}{ds}+\int\left\{\frac{1}{r_1^2\sigma_1}\int r_1\frac{d}{ds}\left(\frac{F\sigma_1}{r_1}\right)ds\right\}ds,$$

or $$\chi=\frac{dF}{dv}+\frac{1}{r}\sin\int\frac{dv}{\sigma}\times\int\left[\frac{1}{\sigma\sin^2\int\frac{dv}{\sigma}}\int\left\{\left(\sin\int\frac{dv}{\sigma}\right)\times\frac{d}{dv}\left(\frac{F\sigma}{r}\right)\right\}dv\right]dv.$$

This seems complicated, but will reduce to two quadratures at most.

$\int\frac{dv}{\sigma}=\int\frac{ds}{\sigma_1}$, and for a spherical curve $R_1^2=r_1^2+\sigma_1^2\left(\frac{dr_1}{ds}\right)^2$ where R_1 is constant.

Therefore $$\int\frac{ds}{\sigma_1}=\int\frac{dr_1}{\sqrt{(R_1^2-r_1^2)}}=\sin^{-1}\left(\frac{r_1}{R_1}\right)+\text{constant}.$$

Hence $\int\frac{ds}{\sigma_1}$ for a spherical curve, and $\int\frac{dv}{\sigma}$ for any curve of length v, can be expressed in finite terms.

The equation for χ

$$A\int\chi A\,dv+B\int\chi B\,dv+C\int\chi C\,dv=F$$

where A, B, C, F are given functions of v, could be treated in a similar way. Divide by $\sqrt{(A^2+B^2+C^2)}$ and solve for $\chi\sqrt{(A^2+B^2+C^2)}$.

§ 4. *Example.*—Consider the developable formed by the tangents to the helix $x = a\cos\phi$, $y = a\sin\phi$, $z = b\phi$, where ϕ replaces v in the previous paragraphs.

The equation of a tangent plane is

$$\frac{b\sin\phi}{\sqrt{(a^2+b^2)}}x - \frac{b\cos\phi}{\sqrt{(a^2+b^2)}}y + \frac{a}{\sqrt{(a^2+b^2)}}z = \frac{ab\phi}{\sqrt{(a^2+b^2)}}$$

$$l = \frac{b\sin\phi}{\sqrt{(a^2+b^2)}},\quad m = -\frac{b\cos\phi}{\sqrt{(a^2+b^2)}},\quad n = \frac{a}{\sqrt{(a^2+b^2)}}$$

$$ll'' + mm'' + nn'' = -b^2/(a^2+b^2).$$

Therefore $$\theta' = \frac{1}{\sqrt{(a^2+b^2)}}\begin{vmatrix} b\sin\phi & -b\cos\phi & a \\ b\cos\phi & b\sin\phi & 0 \\ -b\sin\phi & b\cos\phi & 0 \end{vmatrix} \div (-b^2)$$

$$= -\frac{a}{\sqrt{(a^2+b^2)}}.$$

$$\theta = -\frac{a\phi}{\sqrt{(a^2+b^2)}}.$$

$$\frac{1}{\mathrm{P}^2} = l'^2 + m'^2 + n'^2 = \frac{b^2}{(a^2+b^2)}.$$

$$\alpha = -\left\{\cos\phi\cos\frac{a\phi}{\sqrt{(a^2+b^2)}} + \frac{a}{\sqrt{(a^2+b^2)}}\sin\phi\sin\frac{a\phi}{\sqrt{(a^2+b^2)}}\right\}.$$

$$\beta = -\left\{\sin\phi\cos\frac{a\phi}{\sqrt{(a^2+b^2)}} - \frac{a}{\sqrt{(a^2+b^2)}}\cos\phi\sin\frac{a\phi}{\sqrt{(a^2+b^2)}}\right\}.$$

$$\gamma = \frac{b}{\sqrt{(a^2+b^2)}}\sin\frac{a\phi}{\sqrt{(a^2+b^2)}}.$$

$$\lambda = \cos\phi\sin\frac{a\phi}{\sqrt{(a^2+b^2)}} - \frac{a}{\sqrt{(a^2+b^2)}}\sin\phi\cos\frac{a\phi}{\sqrt{(a^2+b^2)}}.$$

$$\mu = \sin\phi\sin\frac{a\phi}{\sqrt{(a^2+b^2)}} + \frac{a}{\sqrt{(a^2+b^2)}}\cos\phi\cos\frac{a\phi}{\sqrt{(a^2+b^2)}}.$$

$$\nu = \frac{b}{\sqrt{(a^2+b^2)}}\cos\frac{a\phi}{\sqrt{(a^2+b^2)}}.$$

Then for the curve, the direction cosines of whose tangents are l, m, n, we have

$$\frac{l_1}{r} = \frac{b\cos\phi}{\sqrt{(a^2+b^2)}},\quad \frac{m_1}{r} = \frac{b\sin\phi}{\sqrt{(a^2+b^2)}},\quad \frac{1}{r} = \frac{b}{\sqrt{(a^2+b^2)}}.$$

Therefore $\qquad l_1 = \cos\phi,\ \ m_1 = \sin\phi,\ \ n_1 = 0.$

$$-\sin\phi = -\frac{l}{r} - \frac{l_2}{\sigma},\ \cos\phi = -\frac{m}{r} - \frac{m_2}{\sigma},\ 0 = -\frac{n}{r} - \frac{n_2}{\sigma},$$

$$\frac{1}{r^2} + \frac{1}{\sigma^2} = 1 \quad \text{and} \quad \frac{1}{\sigma} = \frac{a}{\sqrt{(a^2+b^2)}}.$$

The curve is not spherical and $\displaystyle\int\frac{d\phi}{\sigma} = \frac{a\phi}{\sqrt{(a^2+b^2)}}.$

$$\mathrm{F} = \frac{ab\phi}{\sqrt{(a^2+b^2)}}.$$

$$\chi = \frac{ab}{\sqrt{(a^2+b^2)}} + \frac{b}{\sqrt{(a^2+b^2)}}\sin\frac{a\phi}{\sqrt{(a^2+b^2)}} \times$$

$$\int\left[\frac{a}{\sqrt{(a^2+b^2)}\sin^2\dfrac{a\phi}{\sqrt{(a^2+b^2)}}}\int\sin\frac{a\phi}{\sqrt{(a^2+b^2)}} \times \frac{b^2 d\phi}{\sqrt{(a^2+b^2)}}\right]d\phi$$

$$= \frac{ab}{\sqrt{(a^2+b^2)}} - \frac{b^3}{(a^2+b^2)}\sin\frac{a\phi}{\sqrt{(a^2+b^2)}}\int\cot\frac{a\phi}{\sqrt{(a^2+b^2)}}\operatorname{cosec}\frac{a\phi}{\sqrt{(a^2+b^2)}}d\phi$$

$$= \frac{ab}{\sqrt{(a^2+b^2)}} + \frac{b^3}{a\sqrt{(a^2+b^2)}}.$$

$$= \frac{b}{a}\sqrt{(a^2+b^2)}.$$