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Asymptotic formula for shifted convolution sums involving the coefficients of an L-function

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Abstract Let f be a normalized Hecke eigenform of even weight $k \geq 2$ for $SL_2(\mathbb{Z})$. In this work, we establish an asymptotic formula for the shifted convolution sum of a general divisor function, where the sum involves the Fourier coefficients of a multi-folded L-function weighted with a kernel function.

Keywords: Eigenforms, automorphic L-functions, shifted sum

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1 Introduction and main results

Number-theoretic functions, also known as arithmetic functions, play a fundamental role in understanding the divisibility properties of integers, the distribution of prime numbers, and various other aspects of number theory. Several important arithmetic functions, such as the Möbius function $\mu(n)$, Euler's totient function $\varphi(n)$, the divisor function $d(n)$, and the sums of divisor function, have deep applications in analytic number theory, algebra, quantum computing, and even string theory.

A particularly interesting class of arithmetic functions was introduced by Ivić, Tenenbaum, and Erdős in 1986-87; namely, arithmetic functions with a squarefull kernel (cf. [20, 10]). Any integer $n \geq 1$ can be uniquely decomposed as $n = q(n)t(n)$, where $\gcd(q(n), t(n)) = 1$, the factor $q(n)$ is squarefree, and $t(n)$ is squarefull, meaning that $p^2 \mid t(n)$ whenever $p \mid t(n)$. A nonnegative integer-valued arithmetic function $\nu(n)$ is said to have a squarefull kernel, or be a 2-full kernel function, if it satisfies $\nu(n) = \nu(t(n))$ for all $n \geq 1$ and grows at most polynomially in any arbitrarily small exponent, i.e., $\nu(n) \ll n^{\epsilon}$ for any $\epsilon > 0$. The collection of such functions is denoted by K_2, as introduced by Ivić and Tenenbaum [20].

This concept extends naturally to higher orders. Given an integer $m \geq 2$, any integer $n \geq 1$ admits a unique factorization $n = q(n)t(n)$, where $q(n)$ is m-free and $t(n)$ is m-full, meaning that $p^m \mid t(n)$ whenever $p \mid t(n)$. The class of nonnegative integer-valued functions satisfying $\nu(n) = \nu(t(n))$ for all $n \geq 1$ and the growth condition $\nu(n) \ll n^{\epsilon}$ for any $\epsilon > 0$ is denoted by K_m. These functions, known as m-full kernel functions, generalize the framework developed for squarefull kernel functions. Note that K_m is not necessarily a class of multiplicative functions.

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One of the most well-known arithmetic functions exhibiting completely random behaviour is given by the Fourier coefficients of a Hecke eigenform for the full modular group. In this work, we establish an asymptotic result concerning the power-saving error term of an arithmetic function involving the coefficients of an L-function associated with a Hecke eigenform, combined with a weighted m-full kernel function $\nu(n) \in K_m$. Recently, Venkatasubbareddy and Sankaranarayanan [41, 40] established asymptotic formulae involving Hecke eigenvalues weighted by an m-full kernel function. To present our main results, we begin with an introduction to modular forms and the associated L-functions.

Let $M_k(1)$ denote the space of modular forms of even weight $k \geq 2$ for the full modular group $SL_2(\mathbb{Z})$, and let $S_k(1)$ be the subspace of cusp forms in $M_k(1)$. A cusp form $f \in S_k(1)$ is called a Hecke eigenform if it is a simultaneous eigenfunction for all Hecke operators.

Let $\lambda_f(n)$ be the normalized nth Fourier coefficient in the Fourier expansion of $f \in S_k(1)$ at infinity, given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$$

for all $z \in \mathbb{H}$, where $\mathbb{H}$ is the Poincaré upper half-plane. The function f is said to be normalized if $\lambda_f(1) = 1$. The set containing normalized $f \in S_k(1)$ is denoted by $S_k^*(1)$. For $f \in S_k^*(1)$, the sequence $\{\lambda_f(n)\}_{n\geq 1}$ defines a real multiplicative function satisfying Deligne's bound

$$|\lambda_f(n)| \leq d(n) \ll n^{\epsilon}$$

for any $\epsilon > 0$, where $d(n)$ is the divisor function.

For a Hecke eigenform $f \in S_k^*(1)$, we define the associated L-function $L(f,s)$ by

$$L(f,s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$, where for any prime p, there exist complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \text{and} \quad |\alpha_f(p)| = |\beta_f(p)| = 1, \quad \alpha_f(p)\beta_f(p) = 1. \tag{1.1}$$

For more details, see Deligne [9].

The average behaviour of arithmetic functions often exhibits inherent randomness, making them a subject of extensive study. Hecke [15], in 1927, established the bound

$$\sum_{n\leq X} \lambda_f(n) \ll X^{\frac{1}{2}}.$$

Subsequent works have refined this result, including

$$\sum_{n\leq X} \lambda_f(n) \ll_{f,\epsilon} \begin{cases} X^{\frac{11}{24}+\epsilon}, & \text{Kloosterman} \\ X^{\frac{4}{9}+\epsilon}, & \text{Davenport, Salié} \\ X^{\frac{5}{12}+\epsilon}, & \text{Weil} \\ X^{\frac{1}{3}+\epsilon}, & \text{Deligne} \\ X^{\frac{1}{3}}, & \text{Hafner and Ivić} \\ X^{\frac{1}{3}}(\log X)^{-0.0652+\epsilon}, & \text{Rankin} \\ X^{\frac{1}{3}}(\log X)^{-0.1185}, & \text{Wu} \end{cases}$$

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The second moment of Hecke eigenvalues was studied independently by Rankin [34] and Selberg [35] in 1930, who proved that

$$\sum_{n\leq X}\lambda_f^2(n)=c_fX+O(X^{\frac{3}{5}}),$$

where $c_f>0$ is a constant depending on f. More recently, Huang [18] in 2021 improved this estimate by showing that

$$\sum_{n\leq X}\lambda_f^2(n)=c_fX+O(X^{\frac{3}{5}-\delta}),\ \text{where } \delta\leq\frac{1}{560}.$$

Beyond classical summation results, one can also investigate the behaviour of Hecke eigenvalues along polynomial sequences. For a monic quadratic polynomial $q(x)$, Blomer [3] studied the first moment of $\lambda_f(n)$ along values of $q(x)$ and established that

$$\sum_{n\leq X}\lambda_f(q(n))\ll X^{\frac{6}{7}+\epsilon}$$

for any $\epsilon>0$. Similar results have been obtained for polynomials in two variables of degree 2, such as $q(x,y)=x^2+y^2$, in works by Banerjee and Pandey [2], and Acharya [1].

For $\ell\geq 1$, following the work of Garret and Harris [11], we consider the multi-folded L-function associated with the Hecke eigenform $f\in S_k^*(1)$, defined as

$$L(f\otimes\cdots\otimes_\ell f,s):=\sum_{n=1}^{\infty}\frac{\lambda_{f\otimes\cdots\otimes_\ell f}(n)}{n^s}=\prod_p\prod_\sigma\left(1-\alpha_f^{\sigma(1)}(p)\alpha_f^{\sigma(2)}(p)\cdots\alpha_f^{\sigma(\ell)}(p)p^{-s}\right)^{-1},$$

where σ runs over the set of maps from $\{1,2,\ldots,\ell\}$ to $\{1,2\}$ satisfying

$$\alpha_f^{\sigma(i)}(p)=\begin{cases}\alpha_f(p), & \text{if } \sigma(i)=1,\\ \beta_f(p), & \text{if } \sigma(i)=2.\end{cases}$$

It follows directly that the Fourier coefficient of the ℓ-fold product L-function satisfies

$$\lambda_{f\otimes f\otimes\cdots\otimes_\ell f}(p)=\lambda_f^\ell(p). \tag{1.2}$$

For $4\leq\ell\leq 8$, Venkatasubbareddy and Sankaranarayanan [39] investigated the error term in the asymptotic formula for $\sum_{n\leq X}\lambda_{f\otimes f\otimes\cdots\otimes_\ell f}(n)$. Recent progress in this direction has been made in [18, 28, 38, 16, 37], which serves as the primary motivation for this work.

Let $i\geq 1$ be a fixed integer. For $1\leq j\leq i$, let $\alpha_j\geq 1$ and $r_j\geq 1$ be positive integers. We define

$$\Lambda_{\ell,f}^i(n):=\lambda_{r_1,f\otimes\cdots\otimes_\ell f}^{\alpha_1}(n)\cdots\lambda_{r_i,f\otimes\cdots\otimes_\ell f}^{\alpha_i}(n), \tag{1.3}$$

where

$$\lambda_{r,f\otimes\cdots\otimes_\ell f}(n)=\sum_{n=n_1\cdots n_r}\lambda_{f\otimes\cdots\otimes_\ell f}(n_1)\cdots\lambda_{f\otimes\cdots\otimes_\ell f}(n_r).$$

Denote

$$R_i := \prod_{j=1}^{i} r_j^{\alpha_j} \quad \text{and} \quad A_i := \sum_{j=1}^{i} \alpha_j.$$

For $n \in \mathbb{N}$ and $r \leq n$, $\binom{n}{r}$ denotes the binomial coefficient with the convention $\binom{n}{r} = 0$ if $r < 0$. Let $[x]$ denote the greatest integer less than or equal to x. We now define the following terms:

$$A_{i,\ell,1} := R_i \left(\binom{\ell A_i}{\ell A_i/2} - \binom{\ell A_i}{\frac{\ell A_i}{2} - 1} \right), \quad A_{i,\ell,2} := R_i \left(\binom{\ell A_i}{\frac{\ell A_i}{2} - 1} - \binom{\ell A_i}{\frac{\ell A_i}{2} - 2} \right), \tag{1.4}$$

$$A_{i,\ell,3}(n) := R_i \left(\binom{\ell A_i}{n} - \binom{\ell A_i}{n-1} \right). \tag{1.5}$$

Similarly, we define

$$B_{i,\ell,1} := R_i \left(\binom{\ell A_i}{[\ell A_i/2]} - \binom{\ell A_i}{[\frac{\ell A_i}{2}] - 1} \right). \tag{1.6}$$

Let q be a positive integer. In this paper, we establish an asymptotic formula for the following power sum:

$$\sum_{\substack{n \leq X+1 \\ n \equiv 1(q)}} \Lambda_{\ell,f}^{i}(n), \tag{1.7}$$

where $\Lambda_{\ell,f}^{i}(n)$ is defined in (1.3). In particular, we prove the following result.

Theorem 1.1. *Let $f \in S_k(1)$ be a normalized Hecke eigenform of even weight $k \geq 2$. Let $\ell, i \geq 1$ and $q \geq 100$ be positive integers. For any $\epsilon > 0$ and sufficiently large X, the following hold:*

(i) *If ℓA_i is even and $q \ll X^{\mathfrak{A}_\ell - \epsilon}$, then we have*

$$\sum_{\substack{n \leq X+1 \\ n \equiv 1(q)}} \Lambda_{\ell,f}^{i}(n) = XP(\log X) \frac{\phi(q)^{A_{i,\ell,1}-1}}{q^{A_{i,\ell,1}}} + O\left(\frac{X^{1-\mathfrak{A}_\ell+\epsilon} q^{1+\epsilon}}{\phi(q)} \right),$$

uniformly, where $P(t)$ is a polynomial of degree $A_{i,\ell,1} - 1$ and

$$\mathfrak{A}_\ell = \frac{138}{46A_{i,\ell,1} + 201A_{i,\ell,2} + 69\sum_{n=0}^{[\ell A_i/2]-2} A_{i,\ell,3}(n)(\ell A_i - 2n + 1) + 46}.$$

Here, $A_{i,\ell,1}, A_{i,\ell,2}$, and $A_{i,\ell,3}(n)$ are given in (1.4) and (1.5).

(ii) *If ℓA_i is odd and $q \ll X^{\frac{1}{2\mathfrak{B}_\ell} - \epsilon}$, then we have*

$$\sum_{\substack{n \leq X+1 \\ n \equiv 1(q)}} \Lambda_{\ell,f}^{i}(n) \ll X^{1 - \frac{1}{2\mathfrak{B}_\ell} + \epsilon} q^{1+\epsilon},$$

uniformly, where

$$\mathfrak{B}_\ell = \frac{1}{2} B_{i,\ell,1} + \frac{1}{4} \sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n + 1).$$

Here, $B_{i,\ell,1}$ and $A_{i,\ell,3}(n)$ are given in (1.6) and (1.5).

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Next, we prove the following result on the shifted convolution of $\Lambda^i_{\ell,f}(n)$ involving an m-full kernel function.

Theorem 1.2. *Let $f \in S_k(1)$ be a normalized Hecke eigenform of even weight $k \geq 2$. Let $\ell, i \geq 1$ and $m \geq 2$ be fixed positive integers. Let $\nu(n) \in K_m$ be the m-full kernel function. For any $\epsilon > 0$ and sufficiently large X, the following hold:*

(i) *If ℓA_i is even, then we have*

$$\sum_{n \leq X} \nu(n)\Lambda^i_{\ell,f}(n+1) = X\widetilde{P}(\log X) + O\left(X^{1-\left(\frac{\mathfrak{A}_\ell}{3} - \frac{\mathfrak{A}_\ell}{3m}\right)+\epsilon}\right),$$

where $\widetilde{P}(t)$ is a polynomial of degree $A_{i,\ell,1} - 1$ and $\mathfrak{A}_\ell$ is given in Theorem 1.1.

(ii) *If ℓA_i is odd, then we have*

$$\sum_{n \leq X} \nu(n)\Lambda^i_{\ell,f}(n+1) \ll X^{1-\left(\frac{1}{6\mathfrak{B}_\ell} - \frac{1}{6m\mathfrak{B}_\ell}\right)+\epsilon},$$

where $\mathfrak{B}_\ell$ is given in Theorem 1.1.

To establish Theorem 1.1, we first use the orthogonality relation to decompose the sum defined in (1.7) over principal, primitive, and non-primitive characters modulo q. As a second step, we prove a decomposition result concerning the Dirichlet series associated with the coefficients of the power sum in (1.7), expressing it in terms of automorphic L-functions. Subsequently, we determine the error term using subconvex bounds, integral moments, and Hölder's inequality, along with Newton and Thorne's celebrated result on the automorphy of the symmetric power lift of $f \in S^*_k(1)$. To prove Theorem 1.2, we use Theorem 1.1 along with several summation techniques.

Notations. Throughout the paper, the symbol ϵ represents an arbitrarily small positive quantity, which may vary from one occurrence to another but always remains $\epsilon > 0$. We use the notation $s := \sigma + it$.

2 Preliminaries

This section provides a brief overview of twisted L-functions and symmetric power L-functions associated with Hecke eigenforms, emphasizing their fundamental properties. We also examine the relationships between the normalized Fourier coefficients of Hecke cusp forms and symmetric power L-functions. Furthermore, we discuss key results concerning subconvex bounds and integral moments. Finally, we present an essential result that plays a crucial role in establishing the main theorems of this article. From now onward, $f \in S^*_k(1)$.

The function $\lambda_f(n)$ satisfies the following Hecke relation [21, Eq. (6.83)]:

$$\lambda_f(m)\lambda_f(n) = \sum_{d \mid \gcd(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{2.1}$$

for all positive integers m and n.

For a Dirichlet character χ modulo q, the twisted Hecke L-function is defined as

$$L(f \otimes \chi, s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

It has the Euler product, satisfies a nice functional equation, and admits an analytic continuation to the entire complex plane [21, Section 7.2].

For $j \geq 2$, the jth symmetric power L-function of degree $j+1$ associated with f is given by

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{0 \leq r \leq j} \left(1 - \frac{\alpha_f^{j-r}(p)\beta_f^r(p)}{p^s}\right)^{-1} =: \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$, where $\alpha_f(p)$, $\beta_f(p)$ are complex numbers satisfying (1.1), and $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. For each prime p, we have

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{n=0}^{j} \alpha_f(p)^{j-n} \beta_f(p)^n. \tag{2.2}$$

By Deligne's bound, it is well-known that

$$|\lambda_{\mathrm{sym}^j f}(n)| \leq d_{j+1}(n) \ll_\epsilon n^\epsilon,$$

for any $\epsilon > 0$, where $d_{j+1}(n)$ denotes the $(j+1)$-fold divisor function.

Let $\mathbb{A}_\mathbb{Q}$ be the ring of adeles of $\mathbb{Q}$. It is well known that $f \in S_k^*(1)$ corresponds to an automorphic cuspidal representation π_f of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, which in turn gives rise to an automorphic L-function $L(\pi_f, s)$ that coincides with $L(f, s)$.

Langlands functoriality conjecture asserts that π_f gives rise to a symmetric power lift $\mathrm{sym}^j \pi_f$, that is an automorphic representation whose L-function is the symmetric power L-function attached to f, i.e., $L(\mathrm{sym}^j \pi_f, s) = L(\mathrm{sym}^j f, s)$. For the known cases, the lifts are cuspidal, namely, there exists an automorphic cuspidal self-dual representation, denoted by $\mathrm{sym}^j \pi_f$ of $\mathrm{GL}_{j+1}(\mathbb{A}_\mathbb{Q})$ whose L-function is the same as $L(\mathrm{sym}^j f, s)$. For $1 \leq j \leq 8$, the Langlands functoriality conjecture has been rigorously established through a series of works by Gelbart and Jacquet [12], Kim [24], Kim and Shahidi [25, 26], Shahidi [36], and Clozel and Thorne [5, 6, 7]. More recently, for all $j \geq 1$, Newton and Thorne in [31, 32] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $\mathrm{GL}_{j+1}(\mathbb{A}_\mathbb{Q})$. In particular, the authors established the existence of the symmetric power liftings $\mathrm{sym}^j \pi_f$ for all $j \geq 1$ regarding the automorphy of the symmetric power lifting for $f \in S_k^*(1)$. Furthermore, by combining the results of Newton and Thorne with those of Cogdell and Michel [8], it follows that the L-function $L(\mathrm{sym}^j f, s)$ admits an analytic continuation as an entire function in the whole complex plane satisfying a functional equation of Riemann zeta-type. For further details, readers may refer to [21, Chapter 5].

Similar to the twisted Hecke L-function, for each $j \geq 2$, we define the twisted jth symmetric power L-function as follows:

$$L(\mathrm{sym}^j f \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s}.$$

This function shares similar properties with the jth symmetric power L-function. Let $\zeta(s)$ and $L(s, \chi)$ denote the Riemann zeta function and the Dirichlet L-function, respectively, i.e.,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \quad \text{for } \mathrm{Re}(s) > 1,$$

where χ is a Dirichlet character modulo q.

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We adopt the following conventions:

$$\begin{aligned} L(\mathrm{sym}^0 f, s) &= \zeta(s), \qquad & L(\mathrm{sym}^0 f \otimes \chi, s) &= L(s, \chi), \\ L(\mathrm{sym}^1 f, s) &= L(f, s), \qquad & L(\mathrm{sym}^1 f \otimes \chi, s) &= L(f \otimes \chi, s). \end{aligned} \tag{2.3}$$

The following two lemmas are crucial in handling the error terms in our results. For Lemma 2.1, we refer to [19, Theorem 8.3] and [4], while Lemma 2.2 follows from [13] and [23]. Their proofs are omitted.

Lemma 2.1. *For the twelfth integral moment of the Riemann zeta function, we have*

$$\int_1^T |\zeta(\tfrac{1}{2} + \epsilon + it)|^{12} \mathrm{d}t \ll T^{2+\epsilon} \tag{2.4}$$

uniformly for $T \geq 1$. *Furthermore, for* $\frac{1}{2} \leq \sigma \leq 2 + \epsilon$, *the Riemann zeta function satisfies the subconvex bound*

$$\zeta(\sigma + it) \ll (1 + |t|)^{\max\{\frac{13}{42}(1-\sigma), 0\} + \epsilon}. \tag{2.5}$$

Lemma 2.2. *For any* $\epsilon > 0$, *the following bound holds for the* L*-function associated with* $f \in S_k^*(1)$*:*

$$L(f, \sigma + it) \ll (1 + |t|)^{\max\{\frac{2}{3}(1-\sigma), 0\} + \epsilon}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ *and* $|t| \geq 1$. *Additionally, for the sixth integral moment, we have*

$$\int_1^T |L(f, \tfrac{1}{2} + \epsilon + it)|^6 \mathrm{d}t \ll T^{2+\epsilon}$$

uniformly for $T \geq 1$.

We now state the following lemma, which will be used in the sequel. It follows from [14], [33], and [42].

Lemma 2.3. *Let* χ *be a primitive character modulo* q. *If* $q \ll T^2$, *then we have the bound:*

$$L(\sigma + iT, \chi) \ll (q(1 + |T|))^{\max\{\frac{1}{3}(1-\sigma), 0\} + \epsilon} \tag{2.6}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$. *Moreover, the fourth integral moment satisfies*

$$\int_1^T |L(\sigma + it, \chi)|^4 \mathrm{d}t \ll (qT)^{2(1-\sigma)+\epsilon} \tag{2.7}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ *and* $T \geq 1$.

Lemma 2.4. *Let* χ *be a primitive character modulo* q. *If* $q \ll T^2$, *then we have the subconvex bounds:*

$$L(\mathrm{sym}^2 f, \sigma + iT) \ll (1 + |T|)^{\max\{\frac{6}{5}(1-\sigma), 0\} + \epsilon} \tag{2.8}$$

and

$$L(\mathrm{sym}^2 f \otimes \chi, \sigma + iT) \ll (q(1 + |T|))^{\max\{\frac{67}{46}(1-\sigma), 0\} + \epsilon} \tag{2.9}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ *and* $|T| \geq 1$.

Proof. The subconvex bounds in (2.8) and (2.9) follow from the Phragmén-Lindelöf convexity principle and results by Lin, Nunes, and Qi [27], and Huang [18], respectively. □

Let $\mathfrak{L}(s, \mathcal{A})$ be a general L-function of degree $2A$ in the sense of Perelli [33].

Lemma 2.5. *For the general L-function $\mathfrak{L}(s, \mathcal{A})$ of degree $2A$, we have the subconvex bound*

$$\mathfrak{L}(\sigma + it, \mathcal{A}) \ll (|t| + 1)^{\max\{A(1-\sigma),0\}+\epsilon} \tag{2.10}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and $|t| \geq 1$. Assume that the coefficients a_n of $\mathfrak{L}(s, \mathcal{A})$ satisfy $\sum_{n \leq X} |a_n|^2 \ll X^{1+\epsilon}$ for any $\epsilon > 0$. Then

$$\int_T^{2T} |\mathfrak{L}(\sigma + it, \mathcal{A})|^2 \mathrm{d}t \ll T^{2A(1-\sigma)+\epsilon} \tag{2.11}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and $T \geq 1$.

Proof. The results (2.10) and (2.11) follow from the maximum modulus principle and [33], respectively. □

Lemma 2.6. *Let χ be a primitive character modulo q. For the general L-function $\mathfrak{L}(s, \mathcal{A} \otimes \chi)$ of degree $2A$, we have*

$$\int_T^{2T} |\mathfrak{L}(\sigma + it, \mathcal{A} \otimes \chi)|^2 \mathrm{d}t \ll T(qT)^{\omega(\sigma)+\epsilon} + (qT)^{2A(1-\sigma)+\epsilon} \tag{2.12}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and $T \geq 1$, where

$$\omega(\sigma) = \begin{cases} 1, & \text{if } \sigma = 1/2, \\ 0, & \text{if } 1/2 < \sigma \leq 1. \end{cases}$$

Furthermore, we have

$$\mathfrak{L}(\sigma + it, \mathcal{A} \otimes \chi) \ll (q|t| + 1)^{\max\{A(1-\sigma),0\}+\epsilon}, \tag{2.13}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \epsilon$ and $|t| \geq 1$.

Proof. The bound in (2.12) follows from Matsumoto [29, Section 3] and Perelli [33, Theorem 4]. The subconvex bound in (2.13) can be found in Jiang [22, Lemma 2.4]. □

In the following lemma, we provide the decomposition of the Dirichlet series associated with the coefficients $\Lambda^i_{\ell,f}(n)$ twisted by χ in terms of general L-functions.

Lemma 2.7 (Decomposition)**.** *Define the Dirichlet series*

$$R^i_\ell(s, \chi) := \sum_{n=1}^{\infty} \frac{\Lambda^i_{\ell,f}(n)\chi(n)}{n^s}$$

for $Re(s) \gg 1$. Then,

$$R^i_\ell(s, \chi) = L^i_\ell(s, \chi) U^i_\ell(s),$$

where

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(i) *If ℓA_i is even, then*

$$L_\ell^i(s,\chi) = L(s,\chi)^{A_{i,\ell,1}} L(\mathrm{sym}^2 f\otimes\chi,s)^{A_{i,\ell,2}} \prod_{n=0}^{[\ell A_i/2]-2} L(\mathrm{sym}^{\ell A_i-2n} f\otimes\chi,s)^{A_{i,\ell,3}(n)},$$

where $A_{i,\ell,1}$, $A_{i,\ell,2}$, and $A_{i,\ell,3}(n)$ are given in (1.4) and (1.5). The function $U_\ell^i(s)$ converges absolutely and uniformly for $\sigma > \frac{1}{2}$ with $U_\ell^i(s) \neq 0$ when $\sigma = 1$.

(ii) *If ℓA_i is odd, then*

$$L_\ell^i(s,\chi) = L(f\otimes\chi,s)^{B_{i,\ell,1}} \prod_{n=0}^{[\ell A_i/2]-1} L(\mathrm{sym}^{\ell A_i-2n} f\otimes\chi,s)^{A_{i,\ell,3}(n)},$$

where $B_{i,\ell,1}$ and $A_{i,\ell,3}(n)$ are given in (1.6) and (1.5). The function $U_\ell^i(s)$ converges absolutely and uniformly for $\sigma > \frac{1}{2}$ with $U_\ell^i(s) \neq 0$ when $\sigma = 1$.

Proof. Recall that

$$R_i := \prod_{j=1}^{i} r_j^{\alpha_j} \quad \text{and} \quad A_i := \sum_{j=1}^{i} \alpha_j.$$

Using (1.3) and from [17, (24)], we have

$$\Lambda_{\ell,f}^i(p) = \lambda_{r_1,f\otimes\cdots\otimes_\ell f}^{\alpha_1}(p)\cdots\lambda_{r_i,f\otimes\cdots\otimes_\ell f}^{\alpha_i}(p) = \left(\prod_{j=1}^{i} r_j^{\alpha_j}\right)\lambda_f(p)^{\left(\ell\sum_{j=1}^i \alpha_j\right)}$$

and

$$\begin{aligned}\lambda_f(p)^{\left(\ell\sum_{j=1}^i \alpha_j\right)} &= \sum_{n=0}^{[\ell\sum_{j=1}^i \alpha_j/2]} \left(\binom{\ell\sum_{j=1}^i \alpha_j}{n} - \binom{\ell\sum_{j=1}^i \alpha_j}{n-1}\right)\lambda_{\mathrm{sym}^{(\ell\sum_{j=1}^i \alpha_j)-2n} f}(p)\\ &= \sum_{n=0}^{[\ell A_i/2]} \left(\binom{\ell A_i}{n} - \binom{\ell A_i}{n-1}\right)\lambda_{\mathrm{sym}^{\ell A_i-2n} f}(p).\end{aligned}$$

This implies

$$\Lambda_{\ell,f}^i(p) = R_i \sum_{n=0}^{[\ell A_i/2]} \left(\binom{\ell A_i}{n} - \binom{\ell A_i}{n-1}\right)\lambda_{\mathrm{sym}^{\ell A_i-2n} f}(p).$$

From [16], $\lambda_{f\otimes\cdots\otimes_\ell f}(n)$ is multiplicative, so its r-fold Dirichlet convolution $\lambda_{r,f\otimes\cdots\otimes_\ell f}(n)$ is also multiplicative. Hence, by definition, $\Lambda_{\ell,f}^i(n)$ is multiplicative, and since $\chi(n)$ is multiplicative, $R_\ell^i(s,\chi)$ has an Euler product expansion:

$$\begin{aligned}R_\ell^i(s,\chi) &= \prod_p \left(1 + \sum_{k\geq 1} \frac{\Lambda_{\ell,f}^i(p^k)\chi(p^k)}{p^{ks}}\right)\\ &= \prod_{n=0}^{[\ell A_i/2]} L(\mathrm{sym}^{\ell A_i-2n} f\otimes\chi,s)^{R_i\left(\binom{\ell A_i}{n}-\binom{\ell A_i}{n-1}\right)} U_\ell^i(s)\\ &=: L_\ell^i(s,\chi)U_\ell^i(s),\end{aligned}$$

for $\mathrm{Re}(s) > 1$, where $U_\ell^i(s)$ is absolutely and uniformly convergent for $\sigma > \frac{1}{2}$, and satisfies $U_\ell^i(s) \neq 0$ when $\sigma = 1$. Then, using (2.3), we get the result. □

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3 Proof of Theorem 1.1

From Deligne's bound (resp. Weil's bound), we know that $\Lambda^i_{\ell,f}(n)\chi(n) \ll n^\epsilon$ for any $\epsilon > 0$. Let χ be any Dirichlet character modulo q. By the orthogonality relation, we have

$$\begin{aligned}\sum_{\substack{n\le X+1\\ n\equiv 1(q)}} \Lambda^i_{\ell,f}(n) =& \frac{1}{\phi(q)}\sum_{\chi(q)}\sum_{n\le X+1}\Lambda^i_{\ell,f}(n)\chi(n)\\ =& \frac{1}{\phi(q)}\sum_{n\le X+1}\Lambda^i_{\ell,f}(n)\chi_0(n) + \frac{1}{\phi(q)}\sum_{n\le X+1}\sum_{\substack{\chi\ne\chi_0\\ \chi \text{ is primitive}}}\Lambda^i_{\ell,f}(n)\chi(n)\\ &+ \frac{1}{\phi(q)}\sum_{n\le X+1}\sum_{\substack{\chi\ne\chi_0\\ \chi \text{ is non-primitive}}}\Lambda^i_{\ell,f}(n)\chi(n)\\ =:& \frac{1}{\phi(q)}\left(\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3\right). \end{aligned} \tag{3.1}$$

For $\sum_1$, by Lemma 2.7 and applying Perron's formula (see [30, Pg 67]), we have

$$\sum_{n\le X+1}\Lambda^i_{\ell,f}(n)\chi_0(n) = \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT} R^i_\ell(s,\chi_0)\frac{(X+1)^s}{s}\mathrm{d}s + O\left(\frac{X^{1+\epsilon}}{T}\right).$$

Shifting the line of integration parallel to $\mathrm{Re}(s) = \frac{1}{2} + \epsilon =: \sigma_0$ and applying Cauchy's residue theorem, we obtain

$$\begin{aligned}\sum_{n\le X+1}\Lambda^i_{\ell,f}(n)\chi_0(n) =& \operatorname*{Res}_{s=1} R^i_\ell(s,\chi_0)\frac{(X+1)^s}{s}\\ &+ \frac{1}{2\pi i}\left\{\int_{\sigma_0-iT}^{\sigma_0+iT} + \int_{\sigma_0+iT}^{1+\epsilon+iT} + \int_{1+\epsilon-iT}^{\sigma_0-iT}\right\} R^i_\ell(s,\chi_0)\frac{(X+1)^s}{s}\mathrm{d}s + O\left(\frac{X^{1+\epsilon}}{T}\right)\\ =:& \operatorname*{Res}_{s=1} R^i_\ell(s,\chi_0)\frac{(X+1)^s}{s} + I_1 + I_2 + I_3 + O\left(\frac{X^{1+\epsilon}}{T}\right).\end{aligned}$$

It is easy to observe the following:

$$L(s,\chi_0) = \zeta(s)\prod_{p|q}\left(1 - \frac{1}{p^s}\right)$$

and

$$L(\mathrm{sym}^j f\otimes\chi_0, s) = L(\mathrm{sym}^j f, s)\prod_{p|q}\prod_{0\le m\le j}\left(1 - \frac{\alpha_f^{j-m}(p)\beta_f^m(p)}{p^s}\right).$$

for $j \ge 1$.

Let χ be a non-primitive character modulo q, and let χ^* be a primitive character modulo $q_1(\ne q)$ induced by χ. Then we have

$$L(s,\chi) = L(s,\chi^*)\prod_{\substack{p|q,\\ p\nmid q_1}}\left(1 - \frac{\chi^*(p)}{p^s}\right),$$

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$$L(\mathrm{sym}^j f\otimes\chi,s)=L(\mathrm{sym}^j f\otimes\chi^*,s)\prod_{\substack{p|q,\\ p\nmid q_1,\\ 0\le m\le j}}\left(1-\frac{\alpha_f^{j-m}(p)\beta_f^m(p)\chi^*(p)}{p^s}\right)\ \text{for } j\ge 1,$$

$$\prod_{\substack{p|q,\\ p\nmid q_1}}\left(1-\frac{\chi^*(p)}{p^s}\right)\ll q^\epsilon \quad \text{for } \frac{1}{2}+\epsilon<\mathrm{Re}(s)<1+\epsilon$$

and

$$\prod_{\substack{p|q,\\ p\nmid q_1,\\ 0\le m\le j}}\left(1-\frac{\alpha_f^{j-m}(p)\beta_f^m(p)\chi^*(p)}{p^s}\right)\ll q^\epsilon \quad \text{for } \frac{1}{2}+\epsilon<\mathrm{Re}(s)<1+\epsilon.$$

This implies that, for $\sum_2$ and $\sum_3$, it is enough to consider the sum $\sum_2$. By Perron's formula, we get

$$\sum_{\substack{n\le X+1\\ \chi\neq\chi_0\\ \chi \text{ is primitive}}}\Lambda_{\ell,f}^i(n)\chi(n)=\frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT}R_\ell^i(s,\chi)\frac{(X+1)^s}{s}\mathrm{d}s+O\left(\frac{X^{1+\epsilon}}{T}\right).$$

Moving the line of integration parallel to $\mathrm{Re}(s)=\frac{1}{2}+\epsilon=:\sigma_0$, using Cauchy's residue theorem and from Lemma 2.7, we have

$$\sum_{\substack{n\le X+1\\ \chi\neq\chi_0\\ \chi \text{ is primitive}}}\Lambda_{\ell,f}^i(n)\chi(n)=\frac{1}{2\pi i}\left\{\int_{\sigma_0-iT}^{\sigma_0+iT}+\int_{\sigma_0+iT}^{1+\epsilon+iT}+\int_{1+\epsilon-iT}^{\sigma_0-iT}\right\}R_\ell^i(s,\chi)\frac{(X+1)^s}{s}\mathrm{d}s+O\left(\frac{X^{1+\epsilon}}{T}\right)$$

$$=:I_4+I_5+I_6+O\left(\frac{X^{1+\epsilon}}{T}\right).$$

Now, we estimate the integrals I_j $(1\le j\le 6)$ based on the parity of ℓA_i.

Case (i): Suppose ℓA_i is even. From Lemma 2.7 and using (2.4), (2.5), (2.8), (2.11), and applying Hölder's inequality, we get

$$\begin{aligned}|I_1|&\ll X^{\sigma_0}+\int_1^T|L_\ell^i(\sigma_0+it,\chi_0)|X^{\sigma_0}t^{-1}dt\\
&\ll X^{\sigma_0}+\frac{X^{\sigma_0}}{T}\left\{\sup_{1\le T_1\le T}\left(\max_{\frac{T_1}{2}\le t\le T_1}\left|\zeta\left(\frac{1}{2}+\epsilon+it\right)^{A_{i,\ell,1}-6}\right|\left|L\left(\mathrm{sym}^2 f,\frac{1}{2}+\epsilon+it\right)^{A_{i,\ell,2}}\right|\right)\right.\\
&\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|\zeta\left(\frac{1}{2}+\epsilon+it\right)\right|^{12}dt\right)^{\frac{1}{2}}\\
&\times\left.\left(\int_{\frac{T_1}{2}}^{T_1}\left|\prod_{n=0}^{[\ell A_i/2]-2}L\left(\mathrm{sym}^{\ell A_i-2n}f,\frac{1}{2}+\epsilon+it\right)^{A_{i,\ell,3}(n)}\right|^2dt\right)^{\frac{1}{2}}\right\}\\
&\ll X^{\frac{1}{2}+\epsilon}+\frac{X^{\frac{1}{2}+\epsilon}}{T}\left\{T^{\frac{13}{42}\times\frac{1}{2}(A_{i,\ell,1}-6)+\frac{6}{5}\times\frac{1}{2}\times A_{i,\ell,2}+\frac{2}{2}+\frac{1}{4}\left(\sum_{n=0}^{[\ell A_i/2]-2}A_{i,\ell,3}(n)(\ell A_i-2n+1)\right)+\epsilon}\right\}\end{aligned}$$

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$$\ll X^{\frac{1}{2}+\epsilon} T^{\frac{13}{84}A_{i,\ell,1}+\frac{3}{5}A_{i,\ell,2}+\frac{1}{4}\left(\sum_{n=0}^{[\ell A_i/2]-2} A_{i,\ell,3}(n)(\ell A_i-2n+1)\right)+\frac{1}{14}-1+\epsilon}.$$

One can notice that $|I_2| + |I_3| \ll |I_1|$. The function $R_\ell^i(s, \chi_0)$ has a simple pole at $s = 1$ of order $A_{i,\ell,1}$. Therefore, we have

$$\sum_{n\le X+1} \Lambda_{\ell,f}^i(n)\chi_0(n) = X P(\log X)\frac{\phi(q)^{A_{i,\ell,1}}}{q^{A_{i,\ell,1}}} + O\left(X^{\frac{1}{2}+\epsilon} T^{\gamma_\ell^i-1+\epsilon}\right) + O\left(\frac{X^{1+\epsilon}}{T}\right),$$

where $P(\log X)$ is a polynomial of degree $A_{i,\ell,1} - 1$ and

$$\gamma_\ell^i = \frac{13}{84}A_{i,\ell,1} + \frac{3}{5}A_{i,\ell,2} + \frac{1}{4}\left(\sum_{n=0}^{[\ell A_i/2]-2} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\right) + \frac{1}{14}.$$

Now, we compute the vertical segment using (2.6), (2.7), (2.9), (2.12), and applying Hölder's inequality

$$\begin{aligned}
|I_4| &\ll X^{\sigma_0} + \int_1^T |L_\ell^i(\sigma_0+it,\chi|X^{\sigma_0}t^{-1}dt \\
&\ll X^{\sigma_0} + \frac{X^{\sigma_0}}{T}\left\{\sup_{1\le T_1\le T}\left(\max_{\frac{T_1}{2}\le t\le T_1}\left|L\left(\frac{1}{2}+\epsilon+it,\chi\right)^{A_{i,\ell,1}-2}\right|\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{1}{2}+\epsilon+it\right)^{A_{i,\ell,2}}\right|\right)\right. \\
&\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\epsilon+it,\chi\right)\right|^4 dt\right)^{\frac{1}{2}} \\
&\times\left.\left(\int_{\frac{T_1}{2}}^{T_1}\left|\prod_{n=0}^{[\ell A_i/2]-2} L\left(\mathrm{sym}^{\ell A_i-2n}f\otimes\chi,\frac{1}{2}+\epsilon+it\right)^{A_{i,\ell,3}(n)}\right|^2 dt\right)^{\frac{1}{2}}\right\} \\
&\ll X^{\frac{1}{2}+\epsilon} + \frac{X^{\frac{1}{2}+\epsilon}}{T}\left\{(qT)^{\frac{1}{3}\times\frac{1}{2}(A_{i,\ell,1}-2)+\frac{67}{46}\times\frac{1}{2}(A_{i,\ell,2})+\frac{1}{2}+\frac{1}{4}\left(\sum_{n=0}^{[\ell A_i/2]-2} A_{i,\ell,3}(n)(\ell A_i-2n+1)\right)+\epsilon}\right\} \\
&\ll X^{\frac{1}{2}+\epsilon}(qT)^{\frac{1}{6}A_{i,\ell,1}+\frac{67}{92}A_{i,\ell,2}+\frac{1}{4}\left(\sum_{n=0}^{[\ell A_i/2]-2} A_{i,\ell,3}(n)(\ell A_i-2n+1)\right)+\frac{1}{6}+\epsilon} T^{-1}.
\end{aligned}$$

Also, one can check that $|I_5| + |I_6| \ll |I_4|$. Hence, from (3.1), we have

$$\begin{aligned}
\sum_{\substack{n\le X+1\\ n\equiv 1(q)}} \Lambda_{\ell,f}^i(n) =& X P(\log X)\frac{\phi(q)^{A_{i,\ell,1}-1}}{q^{A_{i,\ell,1}}} + O\left(\frac{X}{\phi(q)T}\right) \\
&+ O\left(X^{\frac{1}{2}+\epsilon}(qT)^{\frac{1}{6}A_{i,\ell,1}+\frac{67}{92}A_{i,\ell,2}+\frac{1}{4}\left(\sum_{n=0}^{[\ell A_i/2]-2} A_{i,\ell,3}(n)(\ell A_i-2n+1)\right)+\frac{1}{6}+\epsilon} T^{-1}\right).
\end{aligned}$$

Take $T = \frac{X^{\mathfrak{A}_\ell}}{q}$ to make the above error terms comparable. This yields

$$\sum_{\substack{n\le X+1\\ n\equiv 1(q)}} \Lambda_{\ell,f}^i(n) = X P(\log X)\frac{\phi(q)^{A_{i,\ell,1}-1}}{q^{A_{i,\ell,1}}} + O\left(\frac{X^{1-\mathfrak{A}_\ell+\epsilon}q^{1+\epsilon}}{\phi(q)}\right)$$

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for $q \ll X^{\mathfrak{A}_\ell - \epsilon}$, where

$$\mathfrak{A}_\ell = \frac{138}{46A_{i,\ell,1} + 201A_{i,\ell,2} + 69\big(\sum_{n=0}^{[\ell A_i/2]-2} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\big) + 46}$$

and $P(t)$ is a polynomial of degree $A_{i,\ell,1} - 1$.

Case (ii): Suppose ℓA_i is odd. Similar to Case (i), we get the following by making use of Lemma 2.2, (2.11), and applying Hölder's inequality

$$\begin{aligned}
|I_1| &\ll X^{\sigma_0} + \int_1^T |L_\ell^i(\sigma_0 + it, \chi_0)| t^{-1} X^{\sigma_0} dt \\
&\ll X^{\frac{1}{2}+\epsilon} + X^{\frac{1}{2}+\epsilon} \frac{1}{T} \Bigg\{ \sup_{1 \le T_1 \le T} \Bigg(\max_{\frac{T_1}{2} \le t \le T_1} \Bigg| L\Big(f, \frac{1}{2} + \epsilon + it\Big)^{B_{i,\ell,1}-3} \Bigg| \Bigg) \\
&\times \Bigg(\int_{\frac{T_1}{2}}^{T_1} \Bigg| L\Big(f, \frac{1}{2} + \epsilon + it\Big) \Bigg|^6 dt \Bigg)^{\frac{1}{2}} \Bigg(\int_{\frac{T_1}{2}}^{T_1} \Bigg| \prod_{n=0}^{[\ell A_i/2]-1} L\Big(\mathrm{sym}^{\ell A_i - 2n} f, \frac{1}{2} + \epsilon + it\Big)^{A_{i,\ell,3}(n)} \Bigg|^2 dt \Bigg)^{\frac{1}{2}} \Bigg\} \\
&\ll X^{\frac{1}{2}+\epsilon} + X^{\frac{1}{2}+\epsilon} \frac{1}{T} \Bigg\{ T^{\frac{2}{3} \times \frac{1}{2}(B_{i,\ell,1}-3)+\epsilon} \times T^{1+\epsilon} \times T^{\frac{1}{4}\Big(\sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\Big)+\epsilon} \Bigg\} \\
&\ll X^{\frac{1}{2}+\epsilon} T^{\frac{1}{3}B_{i,\ell,1} + \frac{1}{4}\Big(\sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\Big) - 1 + \epsilon}.
\end{aligned}$$

One can check that $|I_2| + |I_3| \ll |I_1|$. Since the function $R_\ell^i(s, \chi_0)$ is analytic in the obtained region. Therefore, we have

$$\sum_{n \le X+1} \Lambda_{\ell,f}^i(n)\chi_0(n) \ll X^{\frac{1}{2}+\epsilon} T^{\frac{1}{3}B_{i,\ell,1} + \frac{1}{4}\Big(\sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\Big) - 1 + \epsilon} + \frac{X^{1+\epsilon}}{T},$$

Using Lemma 2.6 and applying Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
|I_4| &\ll X^{\sigma_0} + \int_1^T |L_\ell^i(\sigma_0 + it, \chi)| t^{-1} X^{\sigma_0} dt \\
&\ll X^{\frac{1}{2}+\epsilon} + X^{\frac{1}{2}+\epsilon} \frac{1}{T} \Bigg\{ \sup_{1 \le T_1 \le T} \Bigg(\int_{\frac{T_1}{2}}^{T_1} \Bigg| L\Big(f \otimes \chi, \frac{1}{2} + \epsilon + it\Big)^{B_{i,\ell,1}} \Bigg|^2 dt \Bigg)^{\frac{1}{2}} \\
&\times \Bigg(\int_{\frac{T_1}{2}}^{T_1} \Bigg| \prod_{n=0}^{[\ell A_i/2]-1} L\Big(\mathrm{sym}^{\ell A_i - 2n} f \otimes \chi, \frac{1}{2} + \epsilon + it\Big)^{A_{i,\ell,3}(n)} \Bigg|^2 dt \Bigg)^{\frac{1}{2}} \Bigg\} \\
&\ll X^{\frac{1}{2}+\epsilon} + X^{\frac{1}{2}+\epsilon} \frac{1}{T} \Bigg\{ (qT)^{\frac{1}{2}B_{i,\ell,1} + \frac{1}{4}\Big(\sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\Big)+\epsilon} \Bigg\} \\
&\ll X^{\frac{1}{2}+\epsilon} (qT)^{\frac{1}{2}B_{i,\ell,1} + \frac{1}{4}\Big(\sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\Big)+\epsilon} T^{-1}.
\end{aligned}$$

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One can check that $|I_5| + |I_6| \ll |I_4|$. Therefore,

$$\sum_{\substack{n \le X+1 \\ \chi \ne \chi_0 \\ \chi \text{ is primitive}}} \Lambda^i_{\ell,f}(n)\chi(n) \ll X^{\frac{1}{2}+\epsilon}(qT)^{\frac{1}{2}B_{i,\ell,1}+\frac{1}{4}\left(\sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\right)+\epsilon} T^{-1} + \frac{X^{1+\epsilon}}{T}.$$

Now, take $T = \frac{X^{\frac{1}{2\mathfrak{B}_\ell}}}{q}$ to optimize the error term. This gives

$$\sum_{\substack{n \le X+1 \\ n \equiv 1(q)}} \Lambda^i_{\ell,f}(n) \ll X^{1-\frac{1}{2\mathfrak{B}_\ell}+\epsilon} q^{1+\epsilon}$$

for $q \ll X^{\frac{1}{2\mathfrak{B}_\ell}-\epsilon}$, where

$$\mathfrak{B}_\ell = \frac{1}{2}B_{i,\ell,1} + \frac{1}{4}\Big(\sum_{n=0}^{[\ell A_i/2]-1} A_{i,\ell,3}(n)(\ell A_i - 2n+1)\Big).$$

□

4 Proof of Theorem 1.2

Recall that

$$\nu(n) = \nu(t(n)) \ll n^\epsilon \quad \text{(by definition)} \quad \text{and} \quad \Lambda^i_{\ell,f}(n) \ll n^\epsilon.$$

Let $1 \le H \le X^{\frac{\mathfrak{A}_\ell}{3}}$ be a parameter to be chosen appropriately later. Then, we have

$$\begin{aligned}
\sum_{n \le X} \nu(n)\Lambda^i_{\ell,f}(n+1) &= \sum_{\substack{n \le X \\ t(n) \le H}} \nu(n)\Lambda^i_{\ell,f}(n+1) + \sum_{\substack{n \le X \\ t(n) > H}} \nu(n)\Lambda^i_{\ell,f}(n+1) \\
&= \sum_{\substack{n \le X \\ t(n) \le H}} \nu(n)\Lambda^i_{\ell,f}(n+1) + O\left(\sum_{H < t(n) \le X} \nu(t(n)) \sum_{\substack{q(n) \le \frac{X}{t(n)} \\ (q(n),t(n))=1}} \Lambda^i_{\ell,f}(t(n)q(n)+1)\right) \\
&= \sum_{\substack{n \le X \\ t(n) \le H}} \nu(n)\Lambda^i_{\ell,f}(n+1) + O\left(\sum_{H < t(n) \le X} t(n)^\epsilon \sum_{\substack{q(n) \le \frac{X}{t(n)} \\ (q(n),t(n))=1}} (t(n)q(n))^\epsilon\right) \\
&= \sum_{\substack{n \le X \\ t(n) \le H}} \nu(n)\Lambda^i_{\ell,f}(n+1) + O\left(\sum_{H < t(n) \le X} t(n)^\epsilon \left(\frac{X}{t(n)}\right)^{1+\epsilon}\right) \\
&= \sum_{\substack{n \le X \\ t(n) \le H}} \nu(n)\Lambda^i_{\ell,f}(n+1) + O\left(X^{1+\epsilon} \sum_{H < t(n) \le X} \frac{1}{t(n)}\right) \\
&= \sum_{\substack{n \le X \\ t(n) \le H}} \nu(n)\Lambda^i_{\ell,f}(n+1) + O(X^{1+\epsilon} H^{\frac{1}{m}-1}). \qquad (4.1)
\end{aligned}$$

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Define $g(l) := \sum_{\alpha d^m = l} \mu(d)$, so that $g(q(n)) = 1$. Then, we obtain

$$\begin{aligned}
\sum_{\substack{n\le X\\ t(n)\le H}} \nu(n)\Lambda^i_{\ell,f}(n+1) &= \sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{q(n)\le \frac{X}{t(n)}\\ (q(n),t(n))=1}} \Lambda^i_{\ell,f}(t(n)q(n)+1)\\
&= \sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{q(n)\le \frac{X}{t(n)}\\ (q(n),t(n))=1}} g(q(n))\Lambda^i_{\ell,f}(t(n)q(n)+1)\\
&= \sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{q(n)\le \frac{X}{t(n)}\\ (q(n),t(n))=1}} \sum_{\alpha(n)d^m(n)=q(n)} \mu(d(n))\Lambda^i_{\ell,f}(t(n)q(n)+1)\\
&= \sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{d(n)\le \left(\frac{X}{t(n)}\right)^{\frac{1}{m}}\\ (d(n),t(n))=1}} \mu(d(n)) \sum_{\substack{\alpha(n)\le \frac{X}{t(n)d^m(n)}\\ (\alpha(n),t(n))=1}} \Lambda^i_{\ell,f}(t(n)\alpha(n)d^m(n)+1)\\
&=: \textstyle\sum_1^* + \sum_2^*, \qquad (4.2)
\end{aligned}$$

where

$$\textstyle\sum_1^* = \displaystyle\sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{d(n)\le H^{\frac{1}{m}}\\ (d(n),t(n))=1}} \mu(d(n)) \sum_{\substack{\alpha(n)\le \frac{X}{t(n)d^m(n)}\\ (\alpha(n),t(n))=1}} \Lambda^i_{\ell,f}(t(n)\alpha(n)d^m(n)+1),$$

and

$$\textstyle\sum_2^* = \displaystyle\sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{H^{\frac{1}{m}}<d(n)\le \left(\frac{X}{t(n)}\right)^{\frac{1}{m}}\\ (d(n),t(n))=1}} \mu(d(n)) \sum_{\substack{\alpha(n)\le \frac{X}{t(n)d^m(n)}\\ (\alpha(n),t(n))=1}} \Lambda^i_{\ell,f}(t(n)\alpha(n)d^m(n)+1).$$

For $\sum_2^*$, we see that

$$\begin{aligned}
\textstyle\sum_2^* &\ll \sum_{t(n)\le H} t(n)^{\epsilon} \sum_{d(n)\ge H^{\frac{1}{m}}} \sum_{\substack{\alpha(n)\le \frac{X}{t(n)d^m(n)}\\ (\alpha(n),t(n))=1}} (t(n)\alpha(n)d^m(n))^{\epsilon}\\
&\ll \sum_{t(n)\le H} t(n)^{2\epsilon} \sum_{d(n)\ge H^{\frac{1}{m}}} d(n)^{m\epsilon}\left(\frac{X}{t(n)d^m(n)}\right)^{1+\epsilon}\\
&\ll X^{1+\epsilon}H^{\frac{1}{m}-1}. \qquad (4.3)
\end{aligned}$$

Since $\sum_{\delta(n)|(\alpha(n),t(n))} \mu(\delta(n)) = 1$, we can express $\sum_1^*$ as

$$\textstyle\sum_1^* = \displaystyle\sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{d(n)\le H^{\frac{1}{m}}\\ (d(n),t(n))=1}} \mu(d(n)) \sum_{\delta(n)|t(n)} \mu(\delta(n)) \sum_{\alpha_1(n)\delta(n)t(n)d^m(n)\le X} \Lambda^i_{\ell,f}(t(n)\alpha_1(n)\delta(n)d^m(n)+1).$$

Write

$$\sum_{\alpha_1(n)\delta(n)t(n)d^m(n)\le X} \Lambda^i_{\ell,f}(t(n)\alpha_1(n)\delta(n)d^m(n)+1) := \sum_{\substack{n\le X+1\\ n\equiv 1 (t(n)\delta(n)d^m(n))}} \Lambda^i_{\ell,f}(n).$$

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From this point onward, we divide the proof into two cases based on the parity of ℓA_i.

Case (i): Suppose ℓA_i is even. By looking at Theorem 1.1(i), we split the sum as $\sum_1^* = \sum_1' + \sum_1''$, where

$$\sum_1' = \sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{d(n)\le H^{\frac{1}{m}} \\ (d(n),t(n))=1}} \mu(d(n)) \sum_{\delta(n)|t(n)} \mu(\delta(n)) \times XP(\log X) \frac{\phi(t(n)\delta(n)d^m(n))^{A_{i,\ell,1}-1}}{(t(n)\delta(n)d^m(n))^{A_{i,\ell,1}}},$$

and

$$\sum_1'' = O\left(\sum_{t(n)\le H} \nu(t(n)) \sum_{\substack{d(n)\le H^{\frac{1}{m}} \\ (d(n),t(n))=1}} \mu(d(n)) \sum_{\delta(n)|t(n)} \mu(\delta(n)) \times X^{1-\mathfrak{A}_\ell+\epsilon} \frac{(t(n)\delta(n)d^m(n))^{1+\epsilon}}{\phi(t(n)\delta(n)d^m(n))} \right).$$

Note that $\frac{q}{\phi(q)} \ll \log(\log q)$. So, we see that

$$\sum_1'' \ll X^{1-\mathfrak{A}_\ell+\epsilon} H^{\frac{2}{m}} \tag{4.4}$$

and

$$\sum_1' = X\widetilde{P}(\log X) + O\left(X^{1+\epsilon} H^{\frac{1}{m}-1} \right). \tag{4.5}$$

Combining (4.1)-(4.5), it follows that

$$\sum_{n\le X} \nu(n)\Lambda_{\ell,f}^i(n+1) = X\widetilde{P}(\log X) + O\left(X^{1+\epsilon} H^{\frac{1}{m}-1} \right) + O\left(X^{1-\mathfrak{A}_\ell+\epsilon} H^{\frac{2}{m}} \right).$$

By Theorem 1.1, we have $q = \delta(n)t(n)d^m(n)$ with $\delta(n), t(n), d^m(n) \le H$, which implies $q \le H^3$. Moreover, by Theorem 1.1, q satisfies

$$q \ll X^{\mathfrak{A}_\ell - \epsilon}.$$

We choose the optimal value of $H = X^{\frac{\mathfrak{A}_\ell}{3}-\epsilon}$ to obtain

$$\sum_{n\le X} \nu(n)\Lambda_{\ell,f}^i(n+1) = X\widetilde{P}(\log X) + O\left(X^{1-(\frac{\mathfrak{A}_\ell}{3}-\frac{\mathfrak{A}_\ell}{3m})+\epsilon} \right), \text{ as desired.}$$

Case (ii): Suppose ℓA_i is odd. We proceed with similar arguments as in the previous case, with slight modifications. As a result, we have

$$\sum_{n\le X} \nu(n)\Lambda_{\ell,f}^i(n+1) = O\left(X^{1-\left(\frac{1}{6\mathfrak{B}_\ell}-\frac{1}{6m\mathfrak{B}_\ell}\right)+\epsilon} \right),$$

as desired. □

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Declarations

Conflict of interest There is no conflict of interest.

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