The Vibrations of a Particle about a Position of Equilibrium—Part IV.

The Convergence of the Trigonometric Series of Dynamics.

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1. It has been shown in the previous three parts of this work, that the whole question of the convergence of the series solution, for the particular dynamical system under consideration, has turned upon the cubic equation

$$4\alpha^2 x^3 - (4\alpha^2 + s^2)\,x^2 + (\alpha^2 + 2sg)\,x - g^2 = 0\,. \quad \ldots\ldots\ldots\ldots \quad (1)$$

Before proceeding to generalize the results it will be shown how this cubic equation may be derived in a slightly different fashion.

The two integrals of the differential equations of motion of the system were *

$$s_1 q_1 + s_2 q_2 + \alpha q_1 q_2^{\frac{1}{2}} \cos(2p_1 - p_2) = h \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

and

$$q_1 + 2q_2 = c\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

In accordance with the preceding work, we put

$$s_1 = 1,\ 2s_1 - s_2 = s,\ \therefore\ s_2 = 2 - s,\ h = 1 - g,\ c = 1, \text{ giving}$$

$$q_1 + (2 - s)q_2 + \alpha q_1 q_2^{\frac{1}{2}} \cos(2p_1 - p_2) = 1 - g \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$q_1 + 2q_2 = 1\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

The elimination of q_1 between equations (4) and (5) gives

$$\alpha\,(1 - 2q_2)\,q_2^{\frac{1}{2}} \cos(2p_1 - p_2) = sq_2 - g\,;$$

or, squaring and reducing,

$$4\alpha^2 \cos^2(2p_1 - p_2)\,q_2^3 - \{4\alpha^2 \cos^2(2p_1 - p_2) + s^2\}\,q_2^2$$
$$+ \{\alpha^2 \cos^2(2p_1 - p_2) + 2sg\}\,q_2 - g^2 = 0\,. \quad \ldots\ldots\ldots \quad (6)$$

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* Part I., § 2, p. 36, eqns. 5 and 6.

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Writing $\alpha' = \alpha \cos(2p_1 - p_2)$, this becomes

$$4\alpha'^2 q_2^3 - (4\alpha'^2 + s^2) q_2^2 + (\alpha'^2 + 2sg) q_2 - g^2 = 0 . \quad \ldots\ldots\ldots \quad (7)$$

This equation is seen to be identical with the cubic (1) except that in (7) we have α' in place of α.

For any particular value of $(2p_1 - p_2)$ the roots of equation (6) or (7) represent the three possible values of q_2 corresponding to this value of $(2p_1 - p_2)$. The roots of the cubic equation (1) there fore represent the three possible values of q_2 corresponding to values of p_1 and p_2 for which $\cos(2p_1 - p_2) = \pm 1$.

It is further apparent that if the roots of the cubic (1) can be expressed in any particular form, then the roots of the cubic (7) can be obtained from the ~~previous~~ expressions by replacing α by α'. Thus if the roots of (1) can be expressed in series of positive powers of $\frac{\alpha}{s}$, then the roots of (7) can be expressed in series of positive powers of $\frac{\alpha'}{s}$.

Now $\alpha' = \alpha \cos(2p_1 - p_2)$ and we have inferred that when q_2 can be expressed in a series of positive powers of $\frac{\alpha}{s}$, p_1 and p_2 can be expressed in a similar form; thus $\cos(2p_1 - p_2)$ can be expressed in a series of positive powers of $\frac{\alpha}{s}$, so that a series of positive powers of $\frac{\alpha'}{s}$ is equivalent to a series of positive powers of $\frac{\alpha}{s}$.

We have, therefore, obtained another proof of the theorem already proved in Part II., that the value of q_2 can be expressed in the form of a series of positive powers of $\frac{\alpha}{s}$ so long as the roots of the cubic (1) are expressible in series of positive powers of $\frac{\alpha}{s}$; the terms of the series for q_2 will contain factors which are trigonometric functions depending on the time.

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2. It has been shown by Whittaker * that for any dynamical system in which the motion is of a type not far removed from a steady motion or an equilibrium-configuration, the equations of motion may be expressed in a general form applicable to all such cases. It has also been shown that the same general form may be applied to motion which is not of this character and in particular to motion such as that of the planets round the sun, or the moon round the earth.†

This general form may be stated as follows:—

The equations of motion are in the Hamiltonian form

$$\frac{dq_r}{dt}=\frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt}=-\frac{\partial H}{\partial q_r}, \ (r=1, 2, \ldots, n)$$

where

$$H=a_{0,0,\ldots,0}+\Sigma a_{n_1,n_2,\ldots,n_n}\cos(n_1p_1+n_2p_2+\ldots+n_np_n), \quad \ldots(8)$$

and the coefficients a are functions of $q_1, q_2, \ldots, q_n$ only; moreover the periodic part of H is small compared with the non-periodic part $a_{0,0,\ldots,0}$; a term which has for argument $(n_1p_1+n_2p_2+\ldots+n_np_n)$ has its coefficient $a_{n_1,n_2,\ldots,n_n}$, at least of order

$$\tfrac{1}{2}\{\,|n_1|+|n_2|+\ldots+|n_n|\,\}$$

in the small quantities $q_1, q_2, \ldots q_n$; and the expansion of $a_{0,0,\ldots,0}$ begins with the term $(s_1q_1+s_2q_2+\ldots+s_nq_n)$.

Whittaker has also shown how these equations may be integrated, the coordinates p_r and q_r being expressed in the form of trigonometric series; the method consists in the repeated application of contact transformations, thereby removing periodic terms from H and ultimately reducing the problem to the equilibrium problem; it is essentially the method employed in Part I. of this work.

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* *Proc. Lond. Math Soc.*, 34 (1902), p. 206; or "Analytical Dynamics," §§ 182–186.

† See Delauney, *Théorie de la Lune*, and Tisseraud, *Annales de l'Obs. de Paris, Mémoires*, 18 (1885).

In order to integrate the system we must be able to find n independent particular integrals, expressing relations between the q's and the p's. One of these particular integrals will be the integral of energy

$$\mathrm{H} = \text{constant.} \qquad \text{.............................(9)}$$

Let us suppose that the remaining $(n-1)$ integrals are such that each of them involves some of the q's, *i.e.* that they are not merely relations between the p's alone, and further let us assume that the p's only occur in the arguments of trigonometric functions. These conditions will be satisfied in general in practical problems.

We may then use these $(n-1)$ equations to express all the q's in terms of one of them, say q_1, and certain trigonometric functions of the p's. When we substitute these values for the q's in the integral of energy (9), we shall obtain an equation in q_1, involving also certain trigonometric functions of the p's. This equation may then be rationalised so that it becomes an equation involving positive integral powers of q_1, and trigonometric functions of the p's; let it be

$$\mathrm{F}(q_1) = 0, \qquad \text{............................(10)}$$

where F is a polynomial in q_1, whose coefficients may involve trigonometric functions of the p's. The degree of F will be at least that of the greatest of the expressions

$$\tfrac{1}{2}\{\, |n_1| + |n_2| + \dots + |n_n| \,\}$$

arising from the expression for H (eqn. 9). For any particular values of the p's the roots of equation (10) will give the corresponding values of q_1.

If it is possible to find such values of the p's that all the trigonometric functions in equation (10) have their maximum values (these being supposed finite), we shall get an equation corresponding to the cubic equation (1) of the particular case previously considered. Equation (10) corresponds precisely to the generalized form (7) of the cubic.

Now if the roots of equation (10) can be expressed in power series in any particular form, for any particular initial conditions, then the coordinate q_1 can, for the same initial conditions, be expressed as a power series of the same form, whose terms will have coefficients involving trigonometric functions depending on the time.

Conversely, if a solution for q_1 is known, consisting of a power series whose terms have coefficients involving trigonometric functions depending on the time, and if it is desired to ascertain for what initial conditions this trigonometric series is convergent, it is sufficient to ascertain for what initial conditions the roots of equation (10) may be expressed as series of the same form for all possible values of the p's; the p's occur in the series for the roots of equation (10) in the arguments of certain trigonometric functions.

This provides a method of testing the convergence of the trigonometric series which express the solution of the general problem of dynamics, and may be used, as in Part II of this work, to determine the range of initial conditions for which these trigonometric series are convergent.

From the discussion given in Part III. it seems legitimate to infer that the divergence of the series solution in the general case does not necessarily imply any discontinuity in the dynamical system, but may denote merely the failure of the series solution to represent the coordinates of the system throughout the whole range of initial conditions for which a real solution exists.