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ON HOMOTOPY DOMINATION

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ABSTRACT. A short proof of the following result of Bernstein and Ganea is given:

"Let X be a topological space which is homotopy dominated by a closed connected n-dimensional manifold M. If $H^n(X; Z_2) \neq 0$ then X has the homotopy type of M".

It is also shown that the manifold in this theorem can be replaced by a Poincaré complex.

In this note we give a short proof of the Berstein–Ganea Theorem about spaces which are homotopy dominated by manifolds. In fact our proof will enable us to extend the Berstein–Ganea result to spaces which are homotopy dominated by Poincaré complexes.

Let X be a topological space which is homotopy dominated by a closed (compact without boundary) connected topological n-dimensional manifold M. This means that there exist maps $f: X \to M$, $g: M \to X$ such that gf is homotopic to id_X ($gf \simeq \mathrm{id}_X$). In [1] Berstein and Ganea proved the following:

THEOREM 1. *Let X be a topological space which is homotopy dominated by a closed connected topological n-dimensional manifold M. If $H^n(X; Z_2) \neq 0$ then X has the homotopy type of M.*

The proof of this nice geometric result (see [3] for its applications, compare also [4]) given in [1] is rather long and really uses the geometry of M (compare the proof of Lemma 2 in [1]). In our proof we will only use Poincaré duality which will enable us to extend Theorem 1 to spaces which are homotopy dominated by Poincaré complexes.

First we give the proof of Theorem 1 and next show how to extend this result. Note (see Remark 4(b)) that the class of Poincaré complexes is essentially larger then the class of topological manifolds.

Proof of Theorem 1. Being homotopy dominated by M, X has the homotopy type of some CW complex (see [6]) so we can assume X is a CW complex. Suppose M is orientable. From the Universal Coefficient Theorem we infer $H_n(X: Z) \neq 0$ and because $gf \simeq \mathrm{id}_X$ and $H_n(M; Z) \approx Z$ we have

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$H_n(X: Z) \approx Z$. Now observe that $f_\#: \Pi_1(X) \to \Pi_1(M)$ is an isomorphism. The homomorphism $f_\#$ is injective by the homotopy domination condition, the surjectivity of $f_\#$ follows by the same argument as in the classical Hopf Theorem (comp. [1]). Denote by $\Lambda, \bar{\Lambda}$ the group rings $Z\Pi_1(M)$ and $Z\Pi_1(X)$ respectively. Let $[M] \in H_n(M; Z)$ be the fundamental class of M and let $[X] \in H_n(X; Z)$ be given by $f_n([X]) = [M]$. We have the Poincaré duality isomorphism with local coefficients (see [9]) given by $[M] \cap : H^q(M; \Lambda) \to H_{n-q}(M; \Lambda)$. Let $P: H_{n-q}(M; \Lambda) \to H^q(M; \Lambda)$ be its inverse. The map $f: X \to M$ induces the homomorphism $f_*: H_*(X; f^*(\Lambda)) \to H_*(M; \Lambda)$, where $f^*(\Lambda)$ is the local system induced by f. Since $f_\#: \Pi_1(X) \to \Pi_1(M)$ is the isomorphism then we can write $f_*: H_*(X; \bar{\Lambda}) \to H_*(M; \Lambda)$ and $f^*: H^*(M; \Lambda) \to H^*(X; \Lambda)$ in cohomology.

Consider the map $f'_*: H_*(M; \Lambda) \to H_*(X; \bar{\Lambda})$ given by $f'(y) = [X] \cap f^*P(y)$, $y \in H_{n-q}(M; \Lambda)$. Observe that $f_*f'_*: H_*(M; \Lambda) \to H_*(M; \Lambda)$ is the identity homomorphism. Indeed $f_*f'_*(y) = f_*([X] \cap f^*P(y)) = [M] \cap P(y) = y$. This implies the surjectivity of $f_*: H_*(X; \bar{\Lambda}) \to H_*(M; \Lambda)$ which together with the injectivity of f_* which follows from the homotopy domination condition gives us the isomorphism $f_*: H_*(X; \bar{\Lambda}) \to H_*(M; \Lambda)$. After the identification of $H_*(X; \bar{\Lambda})$, $H_*(M; \Lambda)$ with the homology groups of the universal coverings i.e. with $H_*(\tilde{X}; Z)$, $H_*(\tilde{M}; Z)$ respectively, from the Whitehead Theorem we obtain that X has the homotopy type of M.

Now let M be nonorientable. Using the Poincaré duality with Z_2 coefficients strictly analogous as previous we obtain mutually inverse cohomology isomorphisms

$$f^*: H^*(M; Z_2) \to H^*(X; Z_2), \qquad g^*: H^*(X; Z_2) \to H^*(M; Z_2).$$

Let $p: \tilde{M} \to M$ be a 2-fold orientable covering of M. Consider the following diagram of induced 2-fold coverings

$$\text{(1)} \qquad \begin{array}{ccccc} M' & \xrightarrow{\tilde{g}} & X' & \xrightarrow{\tilde{f}} & \tilde{M} \\ \downarrow{\scriptstyle \bar{p}'} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{g} & X & \xrightarrow{f} & M \end{array}$$

Denote by ξ, ξ' the canonical line bundles associated with $p: \tilde{M} \to M$, and $\bar{p}': M' \to M$ respectively (see [7] p. 145). Observe that $\xi \approx \xi'$. To see it we show that the classifying maps $f_\xi: M \to RP^\infty$, $f_{\xi'}: M \to RP^\infty$ for ξ and ξ' are homotopic. Denote by $[M, RP^\infty]$ the set of homotopy classes of maps from M to RP^∞. There is the bijection between $[M, RP^\infty]$ and $H^1(M: Z_2)$ given by $h \to h^1(w) \in H^1(M; Z_2)$, where $h \in [M, RP^\infty]$ and $w \in H^1(RP^\infty; Z_2)$ is the canonical generator. Since the bundle ξ' is the induced bundle $(fg)^*(\xi)$ then $f_{\xi'} = f_\xi fg: M \to RP^\infty$ and $(f_{\xi'})^1(w) = g^1 f^1 f_\xi^1(w) = f_\xi^1(w)$, hence $f_\xi \simeq f_{\xi'}$, and $\xi \approx \xi'$.

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From the isomorphism $\xi \approx \xi'$ we infer $M' = \tilde{M}$ and we can write the diagram (1) as follows:

$$\begin{tikzcd} X' \arrow[r, shift left, "\tilde{f}"] \arrow[d, "p'"] & \tilde{M} \arrow[l, shift left, "\tilde{g}"] \arrow[d, "p"] \\ X \arrow[r, shift left, "f"] & M \arrow[l, shift left, "g"] \end{tikzcd}$$

Hence we have $\tilde{g}\tilde{f} \simeq \mathrm{id}_{X'}$. From the Gysin exact sequence (see [7] p. 145)

$$\cdots \to H^n(X: Z_2) \to H^n(X'; Z_2) \to H^n(X; Z_2) \to H^{n+1}(X; Z_2) = 0$$

we have $H^n(X'; Z_2) \neq 0$. By the previous arguments we know that X' has the homotopy type of $\tilde{M}$. Our proof will be complete if we show that $f_{\#}: \Pi_1(X) \to \Pi_1(M)$ is the isomorphism. This we obtain by the following five-lemma argument:

$$\begin{tikzcd} 0 \arrow[r] & p_{\#}(\Pi_1(\tilde{M})) \arrow[r] & \Pi_1(M) \arrow[r] & \Pi_1(M)/p_{\#}(\Pi_1(\tilde{M}) \arrow[r] & 0 \\ 0 \arrow[r] & p_{\#}(\Pi_1(X')) \arrow[r] \arrow[u, "\approx"] & \Pi_1(X) \arrow[r] \arrow[u, "f_{\#}"] & \Pi_1(X)/p_{\#}(\Pi_1(X') \arrow[r] \arrow[u, "\approx"] & 0 \end{tikzcd}$$

Now we consider the more general situation.

Definition 2. A connected CW complex Y (not necessarily finite) is called a Poincaré complex of formal dimension n if there exists a class $[Y] \in H_n(Y; Z)$ such that for all integers r the cap product with $[Y]$ induces an isomorphism

$$[Y] \cap : H^r(Y; \Lambda) \to H_{n-r}(Y; \Lambda).$$

Theorem 3. *Let X be a topological space which is homotopy dominated by a Poincaré complex Y of the formal dimension n. If $H^n(X; Z_2) \neq 0$ then X has the homotopy type of Y.*

Proof. By Lemma 1.1 in [9] we may assume that for any integer r and left Λ-module B $[Y] \cap : H^r(Y: B) \to H_{n-r}(Y; B)$ is an isomorphism. In particular we have the Poincaré duality with respect to the ordinary integer coefficients. This implies $H_{n-1}(X; Z)$ is torsion free and as in the previous case using the Universal Coefficient Theorem we obtain $H_n(X; Z) \approx Z$. Now we proceed strictly analogous as in the case of manifold. If $f: X \to Y$, $g: Y \to X$ are maps such that $gf \simeq \mathrm{id}_X$ then by the easy modification (analogous as in the case of manifold) of Proposition 1.2 in [2] we obtain an isomorphism $f_{\#}: \Pi_1(X) \to \Pi_1(Y)$. The rest of the proof is evident.

Remark 4. (a) Our definition of a Poincaré complex corresponds to the definition of the orientable Poincaré complex. But it is not difficult to see that Theoerem 3 remains true in the case when the Poincaré complex Y is

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nonorientable. Namely, we can use the 2-fold orientable covering of Y and proceed analogously as in the case of a manifold.

(b) The class of Poincaré complexes (even if we will consider only finite CW complexes) is essentially larger from the homotopy point of view than the class of topological manifolds. By Kirby–Siebenmann (see [5]) every closed connected topological n-dimensional manifold has the homotopy type of a finite Poincaré complex of the formal dimension n. On the other hand by Corollary 5.4.1 in [9] for any prime number p there exists a finite Poincaré complex Y with the signature $\sigma(\tilde{Y}) \neq p\sigma(Y)$. Every such Y does not have the homotopy type of a closed connected topological manifold (comp. Theorem 8 in [8]).

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