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Specializations of Jordan Superalgebras

Dedicated to R. V. Moody on his 60th birthday.

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Abstract. In this paper we study specializations and one-sided bimodules of simple Jordan superalgebras.

Let F be a ground field of characteristic $\neq 2$. A (linear) Jordan algebra is a vector space J with a binary bilinear operation $(x, y) \to xy$ satisfying the following identities:

(J1) $xy = yx$;
(J2) $(x^2y)x = x^2(yx)$.

For an element $x \in J$ let $R(x)$ denote the right multiplication $R(x)\colon a \to ax$ in J. If $x, y, z \in J$ then by $\{x, y, z\}$ we denote their Jordan triple product $\{x, y, z\} = (xy)z + x(yz) - y(xz)$.

Examples of Jordan Algebras

(1) Let A be an associative algebra. The new operation $a \cdot b = \frac{1}{2}(ab + ba)$ defines a structure of a Jordan algebra on A. We will denote this Jordan algebra as $A^{(+)}$.
(2) Let $\star\colon A \to A$ be an involution on the algebra A, that is, $(a^\star)^\star = a$, $(ab)^\star = b^\star a^\star$. The subspace $H(A, \star)$ of symmetric elements is a subalgebra of $A^{(+)}$.
(3) Let V be a vector space over F with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle\colon V \times V \to F$. The direct sum $F1 + V$ with the product $(\alpha 1 + v)(\beta 1 + w) = (\alpha\beta + \langle v, w\rangle)1 + (\alpha w + \beta v)$ is a Jordan algebra.
(4) The algebra $H_3(\mathbb{O})$ of Hermitian 3×3 matrices over octonions with the operation $a \cdot b = \frac{1}{2}(ab + ba)$ is a Jordan algebra.

P. Jordan, J. von Neumann, E. Wigner [JNW] and A. Albert [A] showed that every simple finite dimensional Jordan algebra over an algebraically closed field is of one of the types (1)–(4).

A Jordan algebra J is called *special* if it is embeddable into an algebra of type $A^{(+)}$, where A is an associative algebra. Clearly the algebras of Examples (1)–(3) above are special. The algebra $H_3(\mathbb{O})$ is exceptional. A homomorphism $J \to A^{(+)}$ is called a *specialization* of a Jordan algebra J. N. Jacobson [J] introduced the notion of a universal associative enveloping algebra $U = U(J)$ of a Jordan algebra J and showed that

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the category of specializations of J is equivalent to the category of homomorphisms of the associative algebra $U(J)$.

Let V be a Jordan bimodule over the algebra J (see [J]). We call V a one-sided bimodule if $\{J, V, J\} = (0)$. In this case, the mapping $a \to 2R_V(a) \in \operatorname{End}_F V$ is a specialization. The category of one-sided bimodules over J is equivalent to the category of right (left) $U(J)$-modules.

N. Jacobson [J] found universal associative enveloping algebras for all special simple finite dimensional Jordan algebras.

In this paper we study specializations and one-sided bimodules of Jordan superalgebras. Let's introduce the definitions.

By a superalgebra we mean a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $A = A_{\bar{0}} + A_{\bar{1}}$. We define $|a| = 0$ if $a \in A_{\bar{0}}$ and $|a| = 1$ if $a \in A_{\bar{1}}$.

For instance, if V is a vector space of countable dimension, and $G(V) = G(V)_{\bar{0}} + G(V)_{\bar{1}}$ is the Grassmann algebra over V, that is, the quotient of the tensor algebra over the ideal generated by the symmetric tensors, then $G(V)$ is a superalgebra. Its even part is the linear span of all products of even length and the odd part is the linear span of all products of odd length.

If A is a superalgebra, its *Grassmann enveloping algebra* is the subalgebra of $A \otimes G(V)$ given by $G(A) = A_{\bar{0}} \otimes G(V)_{\bar{0}} + A_{\bar{1}} \otimes G(V)_{\bar{1}}$.

Let $\mathcal{V}$ be a homogeneous variety of algebras, that is, a class of F-algebras satisfying a certain set of homogeneous identities and all their partial linearizations (see [ZSSS]).

Definition A superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is called a $\mathcal{V}$ superalgebra if $G(A) \in \mathcal{V}$.

C. T. C. Wall [W] showed that every simple finite-dimensional associative superalgebra over an algebraically closed field F is isomorphic to the superalgebra $M_{m,n}(F) = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix} \right), A \in M_m(F), D \in M_n(F) \right\} + \left\{ \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix} \right), B \in M_{m\times n}(F), C \in M_{n\times m}(F) \right\}$ or to the superalgebra $Q(n) = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix} \right), A \in M_n(F) \right\} + \left\{ \left(\begin{smallmatrix} 0 & B \\ B & 0 \end{smallmatrix} \right), B \in M_n(F) \right\}$.

Jordan superalgebras were first studied by V. Kac [Ka2] and I. Kaplansky [Kp1], [Kp2]. In [Ka2] V. Kac (see also I. L. Kantor [K1], [K2]) classified simple finite dimensional Jordan superalgebras over an algebraically closed field of zero characteristic. In [RZ] this classification was extended to simple finite dimensional Jordan superalgebras, with semisimple even part, over characteristic $p > 2$; a few new exceptional superalgebras in characteristic 3 were added to the list. In [MZ] the remaining case of Jordan superalgebras with nonsemisimple even part was tackled.

Let's consider the examples that arise in these classifications.

If $A = A_{\bar{0}} + A_{\bar{1}}$ is an associative superalgebra then the superalgebra $A^{(+)}$, with the new product $a \cdot b = \frac{1}{2}\left(ab + (-1)^{|a||b|}ba\right)$ is Jordan. This leads to two superalgebras

(1) $M_{m,n}^{(+)}(F)$, $m \geq 1$, $n \geq 1$;
(2) $Q(n)^{(+)}$, $n \geq 2$.

If A is an associative superalgebra and $\star\colon A \to A$ is a superinvolution, that is, $(a^\star)^\star = a$, $(ab)^\star = (-1)^{|a||b|} b^\star a^\star$, then $H(A,\star) = H(A_{\bar{0}},\star) + H(A_{\bar{1}},\star)$ is a subsuperalgebra of $A^{(+)}$. The following two subalgebras of $M_{m,n}^{(+)}$ are of this type.

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(3) $\mathrm{Osp}_{m,n}(F)$ if $n = 2k$ is even. The superalgebra consists of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A^t = A \in M_m(F)$, $C = J^{-1}B^t \in M_{n\times m}(F)$, $D = J^{-1}D^tJ \in M_n(F)$, $J = \left(\begin{smallmatrix} 0 & I_k \\ -I_k & 0 \end{smallmatrix}\right)$;

(4) $P(n) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), D = A^t, B^t = B, C^t = -C \in M_n(F) \right\}$;

(5) Let $V = V_{\bar 0} + V_{\bar 1}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with a superform $\langle\ ,\ \rangle\colon V \times V \to F$ which is symmetric on $V_{\bar 0}$, skewsymmetric in $V_{\bar 1}$ and $\langle V_{\bar 0}, V_{\bar 1}\rangle = (0) = \langle V_{\bar 1}, V_{\bar 0}\rangle$.
The superalgebra $J = F1 + V = (F1 + V_{\bar 0}) + V_{\bar 1}$ is Jordan.

(6) The 3-dimensional Kaplansky superalgebra, $K_3 = Fe + (Fx + Fy)$, with the multiplication $e^2 = e$, $ex = \frac{1}{2}x$, $ey = \frac{1}{2}y$, $[x, y] = e$.

(7) The 1-parametric family of 4-dimensional superalgebras D_t is defined as $D_t = (Fe_1 + Fe_2) + (Fx + Fy)$ with the products: $e_i^2 = e_i$, $e_1e_2 = 0$, $e_ix = \frac{1}{2}x$, $e_iy = \frac{1}{2}y$, $xy = e_1 + te_2$, $i = 1, 2$.
The superalgebra D_t is simple if $t \neq 0$. In the case $t = -1$, the superalgebra D_{-1} is isomorphic to $M_{1,1}(F)$.

(8) The 10-dimensional Kac superalgebra (see [Ka2]) has been proved to be exceptional in [MeZ]. In characteristic 3 this superalgebra is not simple. It has a subalgebra of dimension 9 that is simple and exceptional (Shestakov and Vaughan Lee). There are two more examples of simple Jordan superalgebras in ch $F = 3$, both of them exceptional (see [RZ]).

(9) We will consider now Jordan superalgebras defined by a bracket.
If $A = A_{\bar 0} + A_{\bar 1}$ is an associative commutative superalgebra with a bracket on A, $\{\ ,\ \}\colon A\times A \to A$, the Kantor double of $(A, \{\ ,\ \})$ is the superalgebra $J = A + Ax$ with the $\mathbb{Z}/2\mathbb{Z}$ gradation $J_{\bar 0} = A_{\bar 0} + A_{\bar 1}x$, $J_{\bar 1} = A_{\bar 1} + A_{\bar 0}x$ and the multiplication in J given by: $a(bx) = (ab)x$, $(bx)a = (-1)^{|a|}(ba)x$, $(ax)(bx) = (-1)^{|b|}\{a, b\}$, and the product (in J) of two elements of A is just the product of them in A.
A bracket on A is called a *Jordan bracket* if the Kantor double $J(A, \{\ ,\ \})$ is a Jordan superalgebra. Every Poisson bracket is a Jordan bracket (see K2]).

(10) Let Z be a unital associative commutative algebra with a derivation $D\colon Z \to Z$. Consider the superalgebra $CK(Z, D) = A + M$, where $A = J_{\bar 0} = Z + \sum_{i=1}^{3} w_iZ$, $M = J_{\bar 1} = xZ + \sum_{i=1}^{3} x_iZ$ are free Z-modules of rank 4. The multiplication on A is Z-linear and $w_iw_j = 0$, $i \neq j$, $w_1^2 = w_2^2 = 1$, $w_3^2 = -1$.

Denote $x_{i\times i} = 0$, $x_{1\times 2} = -x_{2\times 1} = x_3$, $x_{1\times 3} = -x_{3\times 1} = x_2$, $-x_{2\times 3} = x_{3\times 2} = x_1$. The bimodule structure and the bracket on M are defined via the following tables:

	g	w_jg
xf	$x(fg)$	$x_j(fg^D)$
x_if	$x_i(fg)$	$x_{i\times j}(fg)$

	xg	x_jg
xf	$f^Dg - fg^D$	$-w_j(fg)$
x_if	$w_i(fg)$	0

The superalgebra $CK(Z, D)$ is simple if and only if Z does not contain proper D-invariant ideals.

In [Ka2], [K1] it was shown that simple finite dimensional Jordan superalgebras over an algebraically closed field F of zero characteristic are those of examples (1)–(8) and the Kantor double (example (9)) of the Grassmann algebra with the bracket $\{f, g\} = \sum(-1)^{|f|}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i}$.

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The examples (9), (10) are related to infinite dimensional *superconformal* Lie superalgebras (see [KL], [KMZ]). In particular, the superalgebras $CK(Z, D)$ correspond to an important superconformal algebra discovered in [CK] and [GLS].

In [MZ] it was shown that the only simple finite dimensional Jordan superalgebras over an algebraically closed field of characteristic $p > 2$ with nonsemisimple even part are superalgebras (9), (10) built on truncated polynomials.

In Section 1 we discuss reflexive superalgebras (the generic case).

In Section 2 we show that the specialization σ of the Cheng-Kac superalgebra $CK(Z, D)$ constructed in [MSZ] is universal, $U\big(CK(Z, D)\big) \simeq M_{2,2}(W)$, where W is the Weyl algebra of differential operators on Z. The restriction of σ to the superalgebra $P(2)$ is the universal specialization of $P(2)$, $U\big(P(2)\big) \simeq M_{2,2}(F[t])$.

In Section 3 we show that, for a D-simple algebra Z, the McCrimmon specialization of the Kantor double of the bracket of vector type is universal.

In Section 4 we construct the universal specialization of the superalgebra $M_{1,1}(F)$.

Finally, in Section 5, we describe all irreducible one-sided bimodules over a superalgebra $D(t)$, $t \neq -1, 0, 1$.

In what follows the ground field F is assumed to be algebraically closed.

1 Reflexive Superalgebras

Let J be a special Jordan superalgebra. A specialization $u\colon J \to U$ into an associative algebra U is said to be universal if $U = \langle u(J)\rangle$ and for an arbitrary specialization $\varphi\colon J \to A$ there exists a homomorphism of associative algebras $\chi\colon U \to A$ such that $\varphi = \chi \cdot u$. The algebra U is called the universal associative enveloping algebra of J.

Exactly in the same way as for Jordan algebras (see [J]) one can show that an arbitrary special Jordan superalgebra has a unique universal specialization $u\colon J \to U$ which is an embedding. Moreover, the algebra U is equipped with a superinvolution $*$ having all elements from $u(J)$ fixed, *i.e.*, $u(J) \subseteq H(U, *)$.

Generally speaking, an identity in U is not assumed. However, if J is a unital (super)algebra then the identity of J is automatically an identity of U (see [J]).

We call a special Jordan superalgebra reflexive if $u(J) = H(U, *)$.

Theorem 1.1 *All superalgebras of examples (1)–(4) are reflexive except the following ones:* $M^{+}_{1,1}(F)$, $\mathrm{Osp}(1, 2) \simeq D(-2)$, $P(2)$. *Hence,* $U\big(M^{(+)}_{m,n}(F)\big) \simeq M_{m,n}(F) \oplus M_{m,n}(F)$ *for* $(m, n) \neq (1, 1)$*;* $U\big(Q^{(+)}(n)\big) = Q(n) \oplus Q(n)$, $n \geq 2$*;* $U\big(\mathrm{Osp}(m, n)\big) \simeq M_{m,n}(F)$, $(m, n) \neq (1, 2)$*;* $U\big(P(n)\big) \simeq M_{n,n}(F)$, $n \geq 3$.

If A is an associative enveloping superalgebra of a special superalgebra J and a_1, a_2, a_3, a_4 are homogeneous elements from J then by a tetrad $\{a_1, a_2, a_3, a_4\} \in A$ we mean

$$\{a_1, a_2, a_3, a_4\} = a_1a_2a_3a_4 + (-1)^{\sum_{i<j} |a_i|\,|a_j|} a_4a_3a_2a_1.$$

A homogeneous element a of J is said to be a tetrad-eater if in any associative enveloping superalgebra of J any tetrad with a as one of its entries is necessarily an element of J. There exists an ideal T of the free Jordan algebra with the following property: for an arbitrary special Jordan algebra J, an arbitrary element from $T(J)$ is

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a tetrad eater (see [Z]). If J is a simple special Jordan superalgebra and $T(J_0) \neq (0)$, then every element of J is a tetrad-eater. By P. Cohn's theorem (see [SSSZ], [C], [J]) in this case J is reflexive. If B is a Jordan algebra of capacity ≥ 3 then $T(B) \neq (0)$ (see [Z]). Hence the superalgebras $P(n)$, $Q(n)$, $n \geq 3$ are reflexive.

The remaining cases of Theorem 1.1 except $Q(2)$ follow from the following lemma that was proved in [RZ]:

Lemma 1.1 (**[RZ]**) *If J is a finite-dimensional special simple Jordan superalgebra, $J_{\bar 0} = J'_{\bar 0} \oplus J''_{\bar 0}$ is semisimple and at least one of the summands is not F then J is reflexive.*

Lemma 1.2 *The superalgebra $Q(2)$ is reflexive.*

Proof The even and the odd parts of $Q(2)$ can be identified with the matrix algebra. Let $e_{ij} \in Q(2)_{\bar 0}$ and $\overline{e_{ij}} \in Q(2)_{\bar 1}$ denote the images of the unit matrix e_{ij}, $Q(2)_{\bar 0} = \sum Fe_{ij}$, $Q(2)_{\bar 1} = \sum F\overline{e_{ij}}$.

Let U be the universal associative enveloping algebra of $Q(2)$, let $\equiv$ denote the equality in U modulo $Q(2)$. We need to check that for arbitrary elements $x_i \in Q(2)$, $1 \leq i \leq 4$, the tetrad $\{x_1, x_2, x_3, x_4\} = x_1x_2x_3x_4 + (-1)^{\sum_{i<j} |x_i|\,|x_j|} x_4x_3x_2x_1$ lies in $Q(2)$ (see [C]).

We have $\{\dots, x, y, \dots\} \equiv -(-1)^{|x|\,|y|}\{\dots, y, x, \dots\}$ and $\{\dots, xy, z, \dots\} \equiv \{\dots, x, yz, \dots\} + (-1)^{|x|\,|y|}\{\dots, y, xz, \dots\}$ (see [Z]).

Now suppose that $x_1, x_2, x_3, x_4 \in \{e_{ij}, \overline{e_{ij}}, 1 \leq i, j \leq 2\}$ and $0 \not\equiv \{x_1, x_2, x_3, x_4\}$.

(i) If $x_1 = e_{11}$ or e_{22} then $x_2, x_3, x_4 \in \{e_{12}, e_{21}, \overline{e_{12}}, \overline{e_{21}}\}$.

Indeed, $\{e_{11}, x_2, x_3, x_4\} = \{e_{11}^2, x_2, x_3, x_4\} \equiv \{e_{11}, 2e_{11}x_2, x_3, x_4\}$, which implies that $x_2 = 2e_{11}x_2$.

This takes care of the case when all four elements x_1, x_2, x_3, x_4 are even.

(ii) $\{\overline{e_{11}}, \overline{e_{22}}, \dots\} \equiv 0$. Indeed,

$$\{\overline{e_{11}}, \overline{e_{22}}, \dots\} = \{e_{11}\overline{e_{11}}, \overline{e_{22}}, \dots\} \equiv \{e_{11}, \overline{e_{11}}\overline{e_{22}}, \dots\} + \{\overline{e_{11}}, e_{11}\overline{e_{22}}, \dots\} = 0.$$

(iii) $\{e_{12}, \overline{e_{12}}, \dots\} \equiv 0$. Indeed, $\overline{e_{12}} = 2e_{12}\overline{e_{22}}$. Hence, $\{e_{12}, \overline{e_{12}}, \dots\} = \{e_{12}, 2e_{12}\overline{e_{22}}, \dots\} \equiv \{e_{12}^2, \overline{e_{22}}, \dots\} = 0$.

This takes care of the case when x_1, x_2, x_3 are even and x_4 is odd.

Indeed, if the elements x_1, x_2, x_3 are e_{11}, e_{12}, e_{21}, then all four possibilities for x_4 are ruled out.

(iv) Fix elements $x_2, x_3, x_4 \in J$. Suppose that $\{Q(2)_0, x_2, x_3, x_4\} \equiv (0)$ and $\{\overline{e_{11}}, x_2, x_3, x_4\} \equiv 0$. Then $\{Q(2), x_2, x_3, x_4\} \equiv (0)$. Indeed, the $Q(2)_0$-bimodule $Q(2)_0$ is irreducible. Hence it is sufficient to prove that for arbitrary elements $a_1, \dots, a_k \in Q(2)_0$, we have $\{\overline{e_{11}}R(a_1)\cdots R(a_k), x_2, x_3, x_4\} \equiv 0$.

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In [Z] it was shown that for arbitrary homogenous elements x_1, x_1' we have

$$\{x_1x_1', x_2, x_3, x_4\} \equiv x_1\{x_1', x_2, x_3, x_4\} + (-1)^{|x_1|\,|x_1'|}x_1'\{x_1, x_2, x_3, x_4\}.$$

This implies the assertion.

Similarly, $\{Q(2)_0, x_2, x_3, x_4\} \equiv (0)$ and $\{\overline{e_{22}}, x_2, x_3, x_4\} \equiv 0$ imply $\{Q(2), x_2, x_3, x_4\} \equiv (0)$.

From (iv) it follows that if $\{x_1, x_2, x_3, x_4\} \not\equiv 0$, then for an arbitrary i, $1 \le i \le 4$ we can assume that x_i is even or our choice of the elements $\overline{e_{11}}, \overline{e_{22}}$. In view of (ii) this finishes the proof of the lemma.

In next section we will see that the superalgebra $P(2)$ is not reflexive.

2 The Cheng-Kac Superalgebras and $P(2)$

Let Z be an associative commutative F-algebra with a derivation $D\colon Z \to Z$. Let $CK(Z, D) = (Z + \sum_{i=1}^{3} Zw_i) + (Zx + \sum_{j=1}^{3} Zx_j)$ be the Cheng-Kac superalgebra. The subsuperalgebra of $CK(Z, D)$ spanned over F by the elements $1, w_1, w_2, w_3, x, x_1, x_2, x_3$ is isomorphic to $P(2)$.

Consider the associative Weyl algebra $W = \sum_{i\ge 0} Zt^i$ where the variable t commutes with a coefficient $a \in Z$ via $ta = D(a) + at$.

In [MSZ] we found the following embedding of $CK(Z, D)$ into the associative superalgebra $M_{2,2}(W) = \begin{pmatrix} M_2(W) & 0 \\ 0 & M_2(W)\end{pmatrix} + \begin{pmatrix} 0 & M_2(W) \\ M_2(W) & 0\end{pmatrix}$,

$$\sigma(a) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad a \in Z; \qquad \sigma(w_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma(w_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \sigma(w_3) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

$$\sigma(x) = \begin{pmatrix} 0 & 0 & 0 & 2D \\ 0 & 0 & -2D & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma(x_1) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma(x_2) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma(x_3) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 2.1 The subsuperalgebra $Z + Zx$ of $CK(Z, D)$ is a Kantor double of vector type (see [Mc]). The embedding σ above extends the embedding of Kantor doubles of vector type found by McCrimmon in [Mc].

Theorem 2.1 *The restriction of the embedding σ (see above) to $P(2)$ is a universal specialization; $U\big(P(2)\big) \simeq M_{2,2}(F[t])$, where $F[t]$ is a polynomial algebra in one variable.*

Let K and H be the subspaces of skew-symmetric and symmetric 2×2 matrices over F respectively. Let $J = P(2)$, $J_{\bar 0} = M_2(F)$, $J_{\bar 1} = \bar K + \bar H$, where $\bar K$ and $\bar H$ are isomorphic copies of K and H. The multiplication of $J_{\bar 1}$ by $J_{\bar 0}$ and the bracket on $J_{\bar 1}$ are defined via $a \cdot \bar b = \frac{1}{2}(\overline{ab + ba^t})$ and $[\bar b, \bar c] = bc - cb \in J_{\bar 0}$; $a \in M_2(F)$, $b, c \in K \cup H$.

Let $u\colon J \to U$ be the universal specialization of J. We will identify J with $u(J)$ and assume that $J \subseteq U$. The juxtaposition in the following lemma denotes multiplication in U.

Lemma 2.1

(1) $\bar H \bar H = (0)$,
(2) $\bar H \bar K \subseteq \langle J_{\bar 0} \rangle$,
(3) $\bar H \langle J_{\bar 0} \rangle = \bar 1 \langle J_{\bar 0} \rangle$.

Proof We have $[\bar H, \bar H] = (0)$. In particular, $[\overline{e_{11}}, \overline{e_{12} + e_{21}}] = 0$.

If e is an idempotent in an associative algebra R, $a, b \in R$ and $[eae, eb(1-e) + (1-e)be] = 0$, then $eaeb(1-e) = (1-e)beae = 0$, which implies $eae\big(eb(1-e) + (1-e)be\big) = \big(eb(1-e) + (1-e)be\big)\, eae = 0$.

Since the elements $\overline{e_{11}}$ and $\overline{e_{12} + e_{21}}$ lie in the corresponding Peirce components of U, we conclude that $\overline{e_{11}}(\overline{e_{12} + e_{21}}) = (\overline{e_{12} + e_{21}})\overline{e_{11}} = 0$.

To finish the proof we will need the following remark:

Remark 2.2 Let J be an arbitrary Jordan superalgebra and let A, B be two associative enveloping algebras of J. If x is an odd element of $J_{\bar 1}$ and the square of x in A lies in the center of A, then the square of x in B also lies in the center of B. Indeed, for an arbitrary element $a \in J$ we have $aR_J(x)R_J(x) = \frac{1}{2}[a, x^2]$, where $R_J(x)$ denotes the operator of right Jordan multiplication in J.

The superalgebra $J = P(2)$ has an associative enveloping algebra $M_{2,2}(F)$, where the square of $\overline{e_{11}}$ is 0.

Hence the square $\overline{e_{11}^2}$ in U lies in the center of U.

The element $\overline{e_{11}^2}$ lies in the 1-Peirce component $e_{11}Ue_{11}$ of U; the element $e_{12} + e_{21}$ lies in the $\frac{1}{2}$-Peirce component $e_{11}U(-e_{11}) + (1 - e_{11})Ue_{11}$. Hence $\overline{e_{11}^2}(e_{12} + e_{21}) = (e_{12} + e_{21})\overline{e_{11}^2}$ implies $\overline{e_{11}^2}(e_{12} + e_{21}) = 0$. But $1 = (e_{12} + e_{21})^2$. We proved that $\overline{e_{11}^2} = 0$.

Since, obviously, $\overline{e_{11}e_{22}} = 0$, we conclude that $\overline{e_{11}}\bar H = (0)$.

The Jordan $J_{\bar 0}$-bimodule $\bar H$ is generated by the element $\overline{e_{11}}$.

This implies that $\bar H \subseteq \langle J_{\bar 0} \rangle \overline{e_{11}} \langle J_{\bar 0} \rangle$, $\bar H \bar H \subseteq \langle J_{\bar 0} \rangle \overline{e_{11}} \langle J_{\bar 0} \rangle \bar H \subseteq \langle J_{\bar 0} \rangle \overline{e_{11}} \bar H \langle J_{\bar 0} \rangle = (0)$.

We proved the assertion (1).

Let $x = \overline{e_{12} - e_{21}}$. If $a \in J_{\bar 0} = M_2(F)$ and $\operatorname{tr}(a) = 0$, then $a \cdot x = 0$. In particular, if $a \in H$ and $\operatorname{tr}(a) = 0$ then $ax + xa = 0$. Now choose an arbitrary element $h \in H$ and consider $(a \cdot \bar h)x$. Clearly, $[a \cdot \bar h, x] \in J_{\bar 0}$.

Furthermore, $(a\bar h + \bar h a)x + x(a\bar h + \bar h a) - \bar h(ax + xa) - (ax + xa)\bar h = a[\bar h, x] - [\bar h, x]a \in \langle J_{\bar 0} \rangle$. Hence $(a \cdot \bar h)x \in \langle J_{\bar 0} \rangle$.

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Denote $H^0 = \{a \in H \mid \mathrm{tr}(a) = 0\}$. We proved that $(H^0 \cdot \bar{H})x \subseteq \langle J_{\bar{0}} \rangle$. Now, notice that $\bar{h} = h \cdot \bar{1}$ for $h \in H$ and $\bar{1} = (e_{11} - e_{22}) \cdot \overline{e_{11} - e_{22}}$. Hence $\bar{H} = H^0 \cdot \bar{H}$. This finishes the proof of (2).

Clearly $\bar{H}\langle J_{\bar{0}} \rangle = \overline{e_{11}}\langle J_{\bar{0}} \rangle + \overline{e_{22}}\langle J_{\bar{0}} \rangle + (\overline{e_{12} + e_{21}})\langle J_{\bar{0}} \rangle$. But $\overline{e_{ii}} = \bar{1}e_{ii}$ since $\bar{1} = \overline{e_{11}} + \overline{e_{22}}$ is the Peirce decomposition of $\bar{1}$ with respect to the idempotents e_{11}, e_{22}. Hence $\overline{e_{11}}\langle J_{\bar{0}} \rangle, \overline{e_{22}}\langle J_{\bar{0}} \rangle \subseteq \bar{1}\langle J_{\bar{0}} \rangle$.

Denote $s = e_{12} + e_{21}$. Then $\bar{s} = 2(\overline{e_{11}} \cdot s) = s\overline{e_{11}} + \overline{e_{11}}s$.

Since $s^2 = 1$ it follows that $s\overline{e_{11}} = s\overline{e_{11}}ss = \overline{e_{22}}s$.

Now we have $\bar{s} = s\overline{e_{11}} + \overline{e_{11}}s = \overline{e_{22}}s + \overline{e_{11}}s = \bar{1}s$. Lemma is proved.

Corollary 2.1 $U = \sum_{i \geq 0} \langle J_{\bar{0}} \rangle x^i + \bar{1}\langle J_{\bar{0}} \rangle$.

Proof In an arbitrary product involving elements from $J_{\bar{0}}$, $\bar{H}$, x we can use $J_{\bar{0}}\bar{H} \subseteq \bar{H}J_{\bar{0}} + \bar{H}$, $x\bar{H} \subseteq \bar{H}x + J_{\bar{0}}$ to move all factors from $\bar{H}$ to the left end.

If $a \in J_{\bar{0}}$, $\mathrm{tr}(a) = 0$, then $ax + xa^t = 0$. Hence in a product involving only elements from $J_{\bar{0}}$ and x we can move all x's together. Now the result follows from Lemma 2.1.

Lemma 2.2 $\bar{H}x\langle J_{\bar{0}} \rangle \triangleleft \langle J_{\bar{0}} \rangle$.

Proof We need to show that $\bar{H}x\langle J_{\bar{0}} \rangle$ is a left ideal in $\langle J_{\bar{0}} \rangle$. Choose arbitrary elements $a \in J_{\bar{0}}$, $h \in H$. Then $a\bar{h}x = (a\bar{h} + \bar{h}a)x - \bar{h}(ax + xa) + \bar{h}xa \in \bar{H}x\langle J_{\bar{0}} \rangle$. Lemma is proved.

Lemma 2.3 *The subalgebra of $M_{2,2}(W)$ generated by $\sigma(J)$ is $M_{2,2}(F[t])$.*

Proof

Step 1 $\langle \sigma(w_1), \sigma(w_2), \sigma(w_3) \rangle = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in M_2(F) \right\}$. Indeed, $M_2(F)$ is generated by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence $\langle \sigma(w_1), \sigma(w_2) \rangle = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in M_2(F) \right\}$.

It implies that

$$\sigma(w_3) + \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \in \langle \sigma(w_1), \sigma(w_2), \sigma(w_3) \rangle.$$

Now

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; a \in M_2(F) \right\} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}; b \in M_2(F) \right\}$$
$$\subseteq \langle \sigma(w_1), \sigma(w_2), \sigma(w_3) \rangle,$$

which implies the result.

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Step 2 $\langle \sigma(w_1), \sigma(w_2), \sigma(w_3), \sigma(x_1), \sigma(x_2), \sigma(x_3)\rangle = \left\{ \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) ; a, b, c \in M_2(F) \right\}$.

It suffices to notice that $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \sigma(x_3) = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$.

Step 3 $\langle \sigma(J)\rangle \supseteq M_{2,2}(F)$.

We have

$$\sigma(x)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_{21} & -a_{22} & 0 & 0 \\ a_{11} & a_{12} & 0 & 0 \end{pmatrix},$$

which implies that $\left\{ \left(\begin{smallmatrix} 0 & 0 \\ d & 0 \end{smallmatrix}\right) ; d \in M_2(F) \right\} \subseteq \langle \sigma(J)\rangle$.

Step 4 We have $\frac{1}{2} e_{11}\sigma(x)e_{11} = e_{11}(t) \in \langle \sigma(J)\rangle$. Hence $M_{2,2}(F[t]) = \langle \overline{M_{2,2}(F)}, e_{11}(t)\rangle = \langle \sigma(J)\rangle$. Lemma is proved.

By the universal property of $u \colon J \to U$, there exists a unique homomorphism $\chi \colon U \to M_{2,2}(F[t])$ of associative superalgebras such that $\sigma = \chi \cdot u$.

Lemma 2.4 *The restriction of χ to $\bar{1}\langle J_{\bar{0}}\rangle$ is an embedding.*

Proof We have already proved that $\bar{H}x\langle J_{\bar{0}}\rangle$ is an ideal of $\langle J_{\bar{0}}\rangle$. Furthermore, this ideal is proper. Indeed, it is nonzero, since $\sigma(\bar{H})\sigma(x) \neq (0)$ in $M_4(F[t])$,

$$\sigma(x_1)\sigma(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$$

Let's assume that the ideal $\bar{H}x\langle J_{\bar{0}}\rangle = \langle J_{\bar{0}}\rangle$. Then $\bar{H}\cdot\bar{H} = (0)$ implies $\bar{H}(\bar{H}x\langle J_{\bar{0}}\rangle) = (0)$ and therefore $\bar{H} = (0)$, the contradiction.

The dimension of the subalgebra $\langle J_{\bar{0}}\rangle$ of U is ≤ 8. By Step 1 of the proof of Lemma 2.3 we have $\langle J_{\bar{0}}\rangle \cong M_2(F) \oplus M_2(F)$. Hence $\bar{H}x\langle J_{\bar{0}}\rangle$ is a direct summand of $\langle J_{\bar{0}}\rangle$ of dimension 4. Let $\langle J_{\bar{0}}\rangle = \bar{H}x\langle J_{\bar{0}}\rangle \oplus L$, where $L \cong M_2(F)$.

Since $\bar{1}\langle J_{\bar{0}}\rangle = \bar{1}\bar{H}x\langle J_{\bar{0}}\rangle + \bar{1}L$ and $\bar{1}\bar{H} = (0)$ by Lemma 2.1 (1), it follows that $\dim_F \bar{1}\langle J_{\bar{0}}\rangle \leq 4$. Now it remains to notice that $\chi(\bar{1}\langle J_{\bar{0}}\rangle) = \sigma(\bar{1})\langle \sigma(J_{\bar{0}})\rangle = \left\{ \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) ; a \in M_2(F) \right\}$ has dimension 4. Lemma is proved.

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We have

$$\sigma(x)=\begin{pmatrix}0&0&0&2t\\0&0&-2t&0\\0&-1&0&0\\1&0&0&0\end{pmatrix},\quad \sigma(x)^{2k}=2^k\begin{pmatrix}t^k&0&0&0\\0&t^k&0&0\\0&0&t^k&0\\0&0&0&t^k\end{pmatrix},$$

$$\sigma(x)^{2k+1}=2^k\begin{pmatrix}0&0&0&2t^{k+1}\\0&0&-2t^{k+1}&0\\0&-t^k&0&0\\t^k&0&0&0\end{pmatrix}.$$

Now we are ready to finish the proof of Theorem 2.1. Let $a=\sum u_i x^i+\overline{e_{11}}v+\overline{e_{22}}w\in\ker\chi$; $u_i,v,w\in\langle J_{\bar 0}\rangle$. Let $\chi(u_i)=\left(\begin{smallmatrix}a_i'&0\\0&a_i''\end{smallmatrix}\right)$, $\chi(v)=\left(\begin{smallmatrix}b'&0\\0&b''\end{smallmatrix}\right)$, $\chi(w)=\left(\begin{smallmatrix}c'&0\\0&c''\end{smallmatrix}\right)$, where $a_i',a_i'',b',b'',c',c''\in M_2(F)$.

Then

$$\sum_i 2^i\begin{pmatrix}a_{2i}'&0\\0&a_{2i}''\end{pmatrix}t^i+\sum 2^i\begin{pmatrix}a_{2i+1}'&0\\0&a_{2i+1}''\end{pmatrix}\begin{pmatrix}0&0&0&2t^{i+1}\\0&0&-2t^{i+1}&0\\0&-t^i&0&0\\t^i&0&0&0\end{pmatrix}$$
$$+\begin{pmatrix}0&e_{11}\\0&0\end{pmatrix}\begin{pmatrix}b'&0\\0&b''\end{pmatrix}+\begin{pmatrix}0&e_{22}\\0&0\end{pmatrix}\begin{pmatrix}c'&0\\0&c''\end{pmatrix}=0,$$

which implies that $a_{2i}'=a_{2i}''=a_{2i+1}'=a_{2i+1}''=0$.

Hence $a=\overline{e_{11}}v+\overline{e_{22}}w\in\bar H\langle J_{\bar 0}\rangle=\bar 1\langle J_{\bar 0}\rangle$.

By Lemma 2.4, $a=0$. Hence χ is an isomorphism. Theorem 2.1 is proved.

Theorem 2.2 *The embedding σ is universal, that is, $U\big(CK(Z,D)\big)\cong M_{2,2}(W)$.*

As above we will identify the Jordan superalgebra $J=CK(Z,D)$ with $u(J)$, *i.e.*, we assume that $J=CK(Z,D)\subseteq U(J)=U$. The superalgebra J is generated by Z and by the superalgebra $\langle w_i,x,x_j;1\le i,j\le 3\rangle\cong P(2)$. The multiplication in U will be denoted by juxtaposition.

By the universal property of u there exists a homomorphism $\chi\colon U\to M_{2,2}(W)$ of associative superalgebras such that $\sigma=\chi\cdot u$. By Theorem 2.2 the subalgebra generated by $P(2)$ in U is the universal associative enveloping algebra of $P(2)$ and $\chi\colon\langle P(2)\rangle\to M_{2,2}(F[t])$ is an isomorphism.

We have $\langle w_1,w_2,w_3,\bar H\rangle=\chi^{-1}\big\{\left(\begin{smallmatrix}a&c\\0&b\end{smallmatrix}\right);a,b,c\in M_2(F)\big\}$ and

$$\left\langle w_1,w_2,w_3,\bar H,\chi^{-1}\begin{pmatrix}0&0\\0&I\end{pmatrix}x\chi^{-1}\begin{pmatrix}I&0\\0&0\end{pmatrix}\right\rangle=\chi^{-1}\big(M_{2,2}(F)\big).$$

Lemma 2.5 *Z commutes with $\chi^{-1}\big(M_{2,2}(F)\big)$ in U.*

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Proof We only need to show that Z commutes with all generators of $\chi^{-1}\big(M_{2,2}(F)\big)$.

Choose an arbitrary element $\alpha \in Z$. Then

$$\pm[\alpha, w_i] = [(\alpha w_j)\cdot w_j, w_i] = [w_j, \alpha w_j \cdot w_i] + [\alpha w_j, w_j \cdot w_i] = 0 \quad \text{for } i \neq j;$$
$$\pm[\alpha, x_i] = [\alpha w_i \cdot w_i, x_i] = [w_i, \alpha w_i \cdot x_i] + [\alpha w_i, w_i \cdot x_i] = 0.$$

Finally, denote $E_2 = \chi^{-1}\big(\left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)\big)$, $E_1 = \chi^{-1}\big(\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)\big)$. By what we proved above, α commutes with E_1, E_2. We have $[\alpha, E_2 x E_1] = E_2[\alpha, x]E_1$ and $[\alpha, x] = [\alpha w_1 \cdot w_1, x] = [w_1, \alpha w_1 \cdot x] + [\alpha w_1, w_1 \cdot x]$, where $w_1 \cdot x = 0, \alpha w_1 \cdot x = x_1 D(\alpha)$.

Since $\chi(w_1) = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$, from Theorem 2.1, it follows that w_1 commutes with E_1, E_2. Hence $E_2[w_1, x_1 \cdot D(\alpha)]E_1 = \big[w_1, E_2\big(x_1 \cdot D(\alpha)\big)E_1\big]$. The element $D(\alpha)$ lies in Z, hence commutes with E_1, E_2. Therefore $E_2\big(x_1 \cdot D(\alpha)\big)E_1 = E_2 x_1 E_1 \cdot D(\alpha)$.

We have $\chi(x_1) = \left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$. Hence, by Theorem 2.1, $E_2 x_1 = 0$, and the lemma is proved.

Lemma 2.6 *Arbitrary elements from Z commute in U.*

Proof Let $\alpha, \beta \in Z$, $1 \le i \le 3$. Let us show that $[\alpha \cdot w_i, \beta] = 0$. Indeed, for $j \neq i$ we have $\pm[\alpha w_i, \beta] = [\alpha w_i, (\beta w_j) w_j] = [(\alpha w_i)\cdot(\beta w_j), w_j] + [\alpha w_i \cdot w_j, \beta w_j] = 0$.

Now $\alpha = \pm(\alpha w_i)\cdot w_i$. If β commutes with αw_i and with w_i then it commutes with α, and the lemma is proved.

Proof of Theorem 2.2 The algebra U is generated by $P(2)$ and Z. By Theorem 2.2, the subalgebra $\langle P(2)\rangle$ of U is generated by $\chi^{-1}\big(M_{2,2}(F)\big)$ and by x^2. We have $\big[Z, \chi^{-1}\big(M_{2,2}(F)\big)\big] = (0)$, $[Z, x^2] \subseteq Z$ and $\big[\chi^{-1}\big(M_{2,2}(F)\big), x^2\big] = (0)$. Hence $U = \sum_{i \ge 0} \chi^{-1}\big(M_{2,2}(F)\big) Z (x^2)^i$, which easily implies that $\operatorname{Ker}\chi = (0)$. Theorem is proved.

3 Specializations of Kantor Doubles

Let $\Gamma = \Gamma_{\bar{0}} + \Gamma_{\bar{1}}$ be an arbitrary associative commutative superalgebra with a Jordan bracket $\{\ ,\ \}$. Then $D(a) = \{a, 1\}$ is a derivation of Γ. The bracket is said to be of vector type if $\{a, b\} = D(a)b - aD(b)$.

In [Mc] it was proved that the Kantor double of a bracket of vector type is a special superalgebra. Furthermore, in [Mc], [K-Mc2] two important examples of classical and Grassmann Poisson brackets were analysed and it was shown that in both cases the Kantor doubles are exceptional.

The following proposition from [MSZ] completely determines which "superconformal" Kantor doubles (see [KMZ]) and which simple finite dimensional Kantor doubles (see [MZ]) are special.

Proposition 3.1 **(see [MSZ])** *Let $\Gamma = \Gamma_{\bar{0}} + \Gamma_{\bar{1}}$ be a finitely generated associative commutative superalgebra with a Jordan bracket $\{\ ,\ \}$ such that the superalgebra $J = J(\Gamma, \{\ ,\ \})$ does not contain nonzero nilpotent ideals.*

(1) *If $\Gamma_{\bar{1}}\Gamma_{\bar{1}} \neq (0)$, then the superalgebra J is exceptional.*

(2) *Suppose that either $\Gamma_{\bar{1}} = (0)$ or $\Gamma_{\bar{1}}$ contains an element ξ such that $\Gamma_{\bar{1}} = \Gamma_{\bar{0}}\xi$ and $\{\Gamma_0, \xi\} = (0)$, $\{\xi, \xi\} = -1$. Then the superalgebra $J(\Gamma, \{\ ,\ \})$ is special if and only if the restriction of $\{\ ,\ \}$ on Γ_0 is of vector type.*

Let $1 \in Z$ be an associative commutative algebra with a derivation $D\colon Z \to Z$ and the bracket of vector type $\{a, b\} = D(a)b - aD(b)$. The Kantor double $J(Z, \{\ ,\ \})$ is simple if and only if Z does not contain proper D-invariant ideals (see [K-Mc], [MZ]). Let $W = \sum_{i=0}^{\infty} Zt^i$, $ta = D(a) + at$, $a \in Z$ be the Weyl algebra. We recall the McCrimmon specialization $m\colon J(Z, \{\ ,\ \}) \to M_2(W)$,

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \in Z; \qquad m(x) = \begin{pmatrix} 0 & 2t \\ -1 & 0 \end{pmatrix}.$$

Theorem 3.1 *Suppose that the algebra Z does not contain proper D-invariant ideals. Then the McCrimmon specialization is universal, that is, $U\big(J(Z, \{\ ,\ \})\big) = M_{1,1}(W)$.*

Remark 3.1 The assumption that Z does not contain proper D-invariant ideals is essential. Indeed, let $Z = F[t_1, t_2]$ be the algebra of polynomials in two variables, $D = 0$. Let $u\colon Z \to U$ be the universal specialization of the Jordan algebra $Z^{(+)}$. The algebra U is not commutative (see [Jac]). Let J be the Kantor double of Z corresponding to the zero bracket, $J = Z + Zx$. Then the mapping $f\colon J \to M_{1,1}(U)$, $f(a) = \begin{pmatrix} u(a) & 0 \\ 0 & u(a) \end{pmatrix}$, $a \in Z$, $f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, is a specialization such that the images of t_1, t_2 do not commute.

In what follows $J = J(Z, \{\ ,\ \})$, $U = U(J)$, juxtaposition denotes the multiplication in U. We will identify elements from J with their images in U.

Lemma 3.1 *Z is generated by $D(Z)$.*

Proof Suppose that $D^2 \neq 0$. The ideal $Z\big(D^2(Z)\big)$ is D-invariant, hence $Z = Z\big(D^2(Z)\big) \subseteq D\big(ZD(Z)\big) + D(Z)D(Z) \subseteq D(Z) + D(Z)D(Z)$.

Now suppose that $D^2 = 0$. Then for arbitrary elements $a, b \in Z$ we have $D^2(ab) = D^2(a)b + aD^2(b) + 2D(a)D(b)$ which implies that $D(Z)D(Z) = (0)$. Now, $Z = ZD(Z)$, the contradiction.

Lemma 3.2 *For arbitrary elements $a, b \in Z$ the commutator $[a, b]$ lies in the center of U.*

Proof For an arbitrary element $c \in J$ we have $\big[c, [a, b]\big] = 4cD(a, b) = 0$. Hence the commutator $[a, b]$ commutes with an arbitrary element from J. Now it suffices to note that the algebra U is generated by J.

Lemma 3.3 $[Z, Z] = (0)$.

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Proof Let S denote the linear span of all elements $\big[[x^2,a],b\big]$; $a,b \in Z$. By Lemma 3.2 $\big[[x^2,a],b\big] = \big[[x^2,b],a\big]$. Hence S is spanned by elements $\big[[x^2,a],a\big]$, $a \in Z$.

Let us show that for an arbitrary element $c \in Z$, $Sc \subseteq S$. Indeed, $S \subseteq [Z,Z]$. By Lemma 3.2 S lies in the center of U. Hence $\big[[x^2,a],a\big]c = \big[[x^2,a],a\big] \cdot c = \big[[x^2,a],a\cdot c\big] - \big[[x^2,a],c\big] \cdot a$. Now $\big[[x^2,a],c\big] \cdot a = \big[[x^2,c],a\big] \cdot a = \frac{1}{2}\big[[x^2,c],a^2\big] \in S$.

Now let us show that $SD(Z) = (0)$. For an arbitrary element $c \in Z$ we have $D(c) = \{c,1\} = \{c \cdot x, x\} = \frac{1}{2}[c,x^2]$.

If $s \in S$ then $s[c,x^2] = [sc,x^2] - [s,x^2]c = 0$, since the elements s and sc both lie in the center of U.

By Lemma 3.1 the identity 1 of the algebra U can be expressed as a linear combination of products of elements from $D(Z)$. Hence $S \cdot 1 = (0)$ and $S = (0)$.

We proved that Z commutes with $[x^2,Z] = D(Z)$. By Lemma 3.1, $[Z,Z] = (0)$, and the lemma is proved.

Lemma 3.4 $[Z,x][Z,x] = (0)$.

Proof Choose an arbitrary element $a \in Z$. We have $\big[[x^2,a],a\big] = 2\big[[x,a],a\big] \cdot x + 2[a,x]^2 = 0$, which implies that $[a,x]^2 = 0$. Hence for arbitrary elements $a,b \in Z$, $[a,x] \cdot [b,x] = 0$.

Let us show that for arbitrary elements $a,b \in Z$, $\Big[\big[[b,x],x\big],a\Big] = 0$.

Indeed, $\big[[b,x],x\big] = 4b \cdot x^2 - (b \cdot x) \cdot x$. Now, $[(b \cdot x) \cdot x, a] = [b \cdot x, a \cdot x] + [x, a \cdot (b \cdot x)] = \{b,a\} + [x,(ab) \cdot x] = D(b)a - bD(a) - D(ab) = -2D(a)b$; and $[b \cdot x^2, a] = b \cdot [x^2,a] + x^2 \cdot [b,a] = -2D(a)b$.

Finally, $0 = \Big[\big[[a,b],x\big],x\Big] = \Big[\big[[a,x],x\big],b\Big] + 2\big[[a,x],[b,x]\big] + \Big[a,\big[[b,x],x\big]\Big]$ which implies $\big[[a,x],[b,x]\big] = (0)$. This finishes the proof of the lemma.

Lemma 3.5 *If $a,b \in Z$ and $aD(b) = 0$ then $a[b,x] = 0$.*

Proof Denote $s = a[b,x]$. We have $sx = a[b,x]x = a([b,x^2] - x[b,x]) = -ax[b,x] = -xa[b,x] - [a,x][b,x] = -xs$.

Hence, $[s,x^2] = 0$.

For an arbitrary $c \in Z$ the element $sc = (ac)[b,x]$ is of the same type as s, hence $[sc,x^2] = 0$.

Now, $s[c,x^2] = [sc,x^2] - [s,x^2]c = 0$. We proved that $sD(Z) = (0)$. In the same way as in the proof of Lemma 3.3 this implies that $s = 0$, and the lemma is proved.

By the universal property of the associative superalgebra U there exists a homomorphism $\chi\colon U \to M_2(W)$ such that $m = \chi \cdot u$. Recall that we identify J with $u(J) \subseteq U$ and therefore assume that $u(a) = a$, $a \in J$.

By Lemmas 3.3 and 3.4 an arbitrary element $\omega \in U_{\bar{0}}$ can be represented as

$$\omega = \sum_i a_i x^{2i} + \sum_j x^{2j} b_j x [c_j, x],$$

where $a_i, b_j, c_j \in Z$. We have

$$\chi(\omega) = \sum_i (-2)^i \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} t^i & 0 \\ 0 & t^i \end{pmatrix} + \sum_j (-2)^j \begin{pmatrix} t^j & 0 \\ 0 & t^j \end{pmatrix} \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2D(c_j) \end{pmatrix}$$
$$= \begin{pmatrix} \sum(-2)^i a_i t^i & 0 \\ 0 & \sum(-2)^i a_i t^i + \sum(-2)^{j+1} t^j b_j D(c_j) \end{pmatrix}.$$

If $\chi(\omega) = 0$, then $a_i = b_j D(c_j) = 0$ for all i, j. By Lemma 3.5 this implies that $\omega = 0$.

An arbitrary element $\omega \in U_1$ can be represented as

$$\omega = \sum_i x^{2i+1} a_i + \sum_j x^{2j} b_j [c_j, x].$$

We have

$$\chi(\omega) = \begin{pmatrix} 0 & \sum(-1)^i 2^{i+1} t^{i+1} a_i + \sum(-2)^{j+1} t^j b_j D(c_j) \\ \sum(-1)^{i+1} 2^i t^i a_i & 0 \end{pmatrix}.$$

Again if $\chi(\omega) = 0$ then $a_i = b_j D(c_j) = 0$ which implies $\omega = 0$.

It is easy to check that the image of m generates the whole algebra $M_2(W)$. Hence χ is an isomorphism. Theorem 3.1 is proved.

Now let us examine the case when $\Gamma_{\bar 0} = Z$ is an associative commutative algebra with a derivation $D\colon Z \to Z$; $\Gamma_{\bar 1} = Z\xi$, $\{a, b\} = D(a)b - aD(b)$ for $a, b \in Z$, $\{Z, \xi\} = (0)$, $\{\xi, \xi\} = -1$. Then the Kantor double $J = J(\Gamma, \{\ ,\ \})$ can be identified with the subsuperalgebra of $CK(Z, D)$ generated by Z, ω_1, x. If the algebra Z does not contain proper D-invariant ideals, then this subsuperalgebra is $J = Z + Z\omega_1 + Zx_1 + Zx$.

Theorem 3.2 *Suppose that the algebra Z does not contain proper D-invariant ideals. Then, the restriction of the embedding $\sigma\colon CK(Z, D) \to M_{2,2}(W)$ to the superalgebra $J = Z + Z\omega_1 + Zx_1 + Zx$ is a universal specialization of J; $U(J) \simeq M_{1,1}(W) \oplus M_{1,1}(W)$.*

As always we identify the superalgebra J with its image in the universal associative enveloping superalgebra U.

Let $\langle Z, x\rangle$ denote the subsuperalgebra of U generated by Z, x.

Lemma 3.6 $U = \langle Z, x\rangle + \langle Z, x\rangle\omega_1$.

Proof For an arbitrary element $a \in Z$ we have $x(\omega_1 a) = x_1 D(a)$. Since $1 \in D(Z)Z$ it follows that x_1 lies in the subalgebra generated by Z, ω_1, x. The element ω_1 commutes with Z in U and anticommutes with x. This implies the lemma.

Let $\langle \sigma(Z), \sigma(x)\rangle$ be the subalgebra of $M_4(W)$ generated by $\sigma(Z), \sigma(x)$.

Lemma 3.7 *If $A, B \in \langle \sigma(Z), \sigma(x)\rangle$ and $A + B\sigma(\omega_1) = 0$, then $A = B = 0$.*

Proof We have $\sigma(\omega_1) = \left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right)$, where I is the identity matrix in $M_2(W)$. It is easy to see that $\langle \sigma(Z), \sigma(x)\rangle \subseteq \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right); a, b \in M_2(W)\right\}$. Now

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} a-c & b+d \\ -b+d & a+c \end{pmatrix} = 0$$

implies that $a = b = c = d = 0$. Lemma is proved.

Now we can finish the proof of Theorem 3.2. Indeed, the homomorphism $\sigma\colon \langle Z, x\rangle \to \langle \sigma(Z), \sigma(x)\rangle$ is an isomorphism, because $\langle Z, x\rangle \simeq M_2(W)$ is a simple algebra. This implies that $\sigma\colon U \to M_4(W)$ is an embedding.

$$\langle \sigma(Z), \sigma(x)\rangle = \left\{ \begin{pmatrix} \beta_1 & 0 & 0 & \beta_3 \\ 0 & \beta_2 & \beta_4 & 0 \\ 0 & -\beta_3 & \beta_1 & 0 \\ -\beta_4 & 0 & 0 & \beta_2 \end{pmatrix} \,\middle|\, \beta_i \in W \right\}, \quad \sigma(\omega_1) = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

So

$$\langle \sigma(Z), \sigma(x), \sigma(\omega_1)\rangle = \left\{ \begin{pmatrix} \beta_1 & 0 & 0 & \beta_5 \\ 0 & \beta_2 & \beta_6 & 0 \\ 0 & \beta_7 & \beta_3 & 0 \\ \beta_8 & 0 & 0 & \beta_4 \end{pmatrix} \,\middle|\, \beta_i \in W, 1 \le i \le 8 \right\}.$$

This superalgebra is isomorphic to $M_2(W) \oplus M_2(W)$, and the theorem is proved.

4 Specializations of $M_{1,1}(F)$

Denote $J = M_{1,1}(F)$, $v = e_{22} - e_{11} \in J_{\bar 0}$, $x = e_{12}$, $y = e_{21} \in J_{\bar 1}$. The universal associative enveloping superalgebra U of J can be presented by generators v, x, y and relators $v^2 - 1 = 0$, $xv + vx = 0$, $yv + vy = 0$, $yx - xy - v = 0$. Let $v < x < y$ and consider the lexicographic order on the set of words in v, x, y. Then the system of relators above is closed with respect to compositions (see [Be], [Bo]). Hence the system of irreducible words $x^i y^j$, $vx^i y^j$; $i, j \ge 0$ is a Groebner-Shirshov basis of U.

By Remark 2.2, the squares x^2, y^2 lie in the center of U. The algebra U is a free module over the central subalgebra $F[x^2, y^2]$ with free generators 1, x, y, xy, v, vx, vy, vxy.

Consider the ring of polynomials and the field of rational functions in two variables, $F[z_1, z_2] \subseteq F(z_1, z_2)$. Let K be the quadratic extension of $F(z_1, z_2)$ generated by a root of the equation $a^2 + a - z_1 z_2 = 0$. Consider the subring $A = F[z_1, z_2] + F[z_1, z_2]a$ and the subspaces $M_{12} = F[z_1, z_2] + F[z_1, z_2]a^{-1}z_2$, $M_{21} = F[z_1, z_2]z_1 + F[z_1, z_2]a$ of K. Then $\left(\begin{smallmatrix} A & M_{12} \\ M_{21} & A \end{smallmatrix}\right)$ is a subring of $M_2(K)$.

Let's consider the mapping $u\colon M_{1,1}(F) \to \left(\begin{smallmatrix} A & M_{12} \\ M_{21} & A \end{smallmatrix}\right)$,

$$u\left(\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\right) = \begin{pmatrix} \alpha_{11} & \alpha_{12} + \alpha_{21} a^{-1} z_2 \\ \alpha_{12} z_1 + \alpha_{21} a & \alpha_{22} \end{pmatrix}.$$

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A straightforward verification shows that u is a specialization of the Jordan superalgebra $J = M_{1,1}(F)$. Hence, it extends to a homomorphism $\chi\colon U \to \begin{pmatrix} A & M_{12} \\ M_{21} & A \end{pmatrix}$. Clearly, $\chi(x^2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_1 \end{pmatrix}$, $\chi(y^2) = \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix}$. Again the straightforward computation shows that the elements 1, $\chi(x)$, $\chi(y)$, $\chi(xy)$, $\chi(v)$, $\chi(vx)$, $\chi(vy)$, $\chi(vxy)$ are free generators of the $F[z_1, z_2]$-module $\begin{pmatrix} A & M_{12} \\ M_{21} & A \end{pmatrix}$, which implies the following:

Theorem 4.1 *$U\big(M_{1,1}(F)\big) \simeq \begin{pmatrix} A & M_{12} \\ M_{21} & A \end{pmatrix}$. The mapping*

$$u\colon \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha_{11} & \alpha_{12} + \alpha_{21} a^{-1} z_2 \\ \alpha_{12} z_1 + \alpha_{21} a & \alpha_{22} \end{pmatrix}$$

is a universal specialization.

Remark 4.1 One sided finite dimensional Jordan bimodules over $M_{1,1}(F)$ are not necessarily completely reducible. Indeed, if I is an ideal of $F[z_1, z_2]$ then $\begin{pmatrix} I+Ia & I+Ia^{-1}z_2 \\ Iz_1+Ia & I+Ia \end{pmatrix}$ is an ideal of $\begin{pmatrix} A & M_{12} \\ M_{21} & A \end{pmatrix}$. If the quotient $F[z_1, z_2]/I$ is finite-dimensional and not semisimple, then so is the quotient $\begin{pmatrix} A & M_{12} \\ M_{21} & A \end{pmatrix} / \begin{pmatrix} I+IA & I+Ia^{-1}z_2 \\ Iz_1+Ia & I+Ia \end{pmatrix}$.

5 Specializations of Superalgebras $D(t)$

Let $t \in F$. Consider the 4-dimensional superalgebra $D(t)$, $D(t)_{\bar 0} = Fe_1 + Fe_2$, $D(t)_{\bar 1} = Fx + Fy$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = 0$, $e_i x = \frac{1}{2}x$, $e_i y = \frac{1}{2}y$, $1 \le i \le 2$, $[x, y] = e_1 + te_2$.

Clearly, $D(-1) \cong M_{1,1}(F)$, $D(0) \cong K_3 \oplus F1$, $D(1)$ is a Jordan superalgebra of a superform.

We will start with the superalgebra K_3. Let $\mathrm{osp}(1,2)$ denote the Lie subsuperalgebra of $M_{1,2}(F)$ which consists of skewsymmetric elements with respect to the orthosympletic superinvolution. Let x, y be the standard basis of the odd part of $\mathrm{osp}(1,2)$.

As always $U\big(\mathrm{osp}(1,2)\big)$ denotes the universal associative enveloping algebra of the Lie superalgebra $\mathrm{osp}(1,2)$. Let $U^*\big(\mathrm{osp}(1,2)\big)$ be the ideal (of codimension one) of $U\big(\mathrm{osp}(1,2)\big)$ generated by $\mathrm{osp}(1,2)$.

Theorem 5.1 **(I. Shestakov [S1])** *The universal enveloping algebra of K_3 is isomorphic to $U^*\big(\mathrm{osp}(1,2)\big) / \mathrm{id}([x,y]^2 - [x,y])$, where $\mathrm{id}([x,y]^2 - [x,y])$ is the ideal of $U\big(\mathrm{osp}(1,2)\big)$ generated by $[x,y]^2 - [x,y]$.*

Remark 5.1 The ideal U^* above appeared because we do not assume an identity in the enveloping algebra $U(K_3)$ of the Jordan superalgebra. The unital hull of $U(K_3)$ is, of course, isomorphic to $U\big(\mathrm{osp}(1,2)\big) / \mathrm{id}([x,y]^2 - [x,y])$.

Clearly, if $\mathrm{ch}\, F = 0$ then K_3 does not have nonzero specializations that are finite dimensional algebras. If $\mathrm{ch}\, F = p > 0$ then K_3 has such specializations. For example, $K_3 \subseteq CK(F[a \mid a^p = 0], d/da)$.

Theorem 5.2 **(I. Shestakov [S1])** *Let $t \neq -1, 1$. Then the universal enveloping algebra of $D(t)$ is isomorphic to*

$$U\big(\operatorname{osp}(1,2)\big) / \operatorname{id}\big([x,y]^2 - (1+t)[x,y] + t\big).$$

Corollary 5.1 *If $\operatorname{ch} F = 0$ then all finite dimensional one-sided bimodules over $D(t)$, $t \neq -1, 1$, are completely reducible.*

Indeed, it is known (see [Ka1]) that finite dimensional representations of the Lie superalgebra $\operatorname{osp}(1,2)$ are completely reducible.

From now on in this section we will assume that $t \neq -1, 0, 1$ and $\operatorname{ch} F = 0$.

We will classify irreducible finite-dimensional one-sided bimodules over $D(t)$.

Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a finite dimensional irreducible right module over the associative superalgebra $U\big(D(t)\big)$. We will identify elements from $D(t)$ with their right multiplications on V, *i.e.*, $D(t) \subseteq \operatorname{End}_F V$.

Let us notice that $V_{\bar{0}} \neq (0)$. Otherwise, $Vx = Vy = (0)$, which implies that $VD(t) = (0)$, the contradiction.

Let $E = \frac{1}{1+t}x^2$, $F = -\frac{1}{1+t}y^2$, $H = -\frac{1}{1+t}(xy + yx)$.

It is easy to check that $[E,F] = H$, $[E,H] = -2E$, $[F,H] = 2F$, *i.e.*, the elements E, F, H span the Lie algebra sl_2. The subspace $V_{\bar{0}}e_i$ is invariant under the sl_2.

Suppose that $V_{\bar{0}}e_1 \neq (0)$. In the sl_2-module $V_{\bar{0}}e_1$ choose a highest weight element $v \neq 0$, *i.e.*, $vH = \lambda v$, $vF = 0$.

Now we will consider an infinite dimensional Verma type module $\tilde{V} = \tilde{v}U\big(D(t)\big)$, whose homomorphic image is V. The module $\tilde{V}$ is defined by one generator $\tilde{v}$ and the relations: $\tilde{v}H = \lambda\tilde{v}$, $\tilde{v}e_1 = \tilde{v}$, $\tilde{v}y^2 = 0$.

From $\tilde{v}H = \lambda\tilde{v}$ it follows that $\tilde{v}(xy + yx) = -(t+1)\lambda\tilde{v}$. Taking into account that $xy = yx + e_1 + te_2$ we get $\tilde{v}yx = \alpha\tilde{v}$, where $\alpha = -\frac{1}{2}\big(1 + (1+t)\lambda\big)$. Now $0 = (\tilde{v}yx - \alpha\tilde{v})y - \tilde{v}y(xy - yx - e_1 - te_2) = (t - \alpha)\tilde{v}y$. Hence $\alpha = t$ or $\tilde{v}y = 0$.

Suppose that $\alpha = t$ or equivalently, $\lambda = \frac{-1-2t}{1+t}$. Then the system of relators of $\tilde{V}$: $\tilde{v}e_1 - \tilde{v} = 0$, $\tilde{v}y^2 = 0$, $\tilde{v}yx - t\tilde{v} = 0$ together with the system of relators of $D(t)$: $e_1^2 - e_1 = 0$, $xe_1 + e_1x - x = 0$, $ye_1 + e_1y - y = 0$, $xy - yx - t - (1-t)e_1 = 0$ and the lexicographic order $e_1 < y < x < v$ is closed with respect to compositions (see [Be], [Bo]). Hence the irreducible elements $\tilde{v}$, $\tilde{v}y$, $\tilde{v}x^i$, $i \geq 1$ form a basis of the module $\tilde{V}$. We will denote this module as $\tilde{V}_1(t)$.

If $\tilde{v}y = 0$ then $\tilde{v}yx = \alpha\tilde{v}$ implies that $\alpha = 0$, *i.e.*, $\lambda = -\frac{1}{1+t}$. In this case the system of relators of $\tilde{V}$ is: $\tilde{v}e_1 - \tilde{v} = 0$, $\tilde{v}y = 0$. As above, this system, together with the system of relators of $D(t)$ (see above) and the lexicographic order, is closed with respect to compositions. Hence, the irreducible elements $\tilde{v}$, $\tilde{v}x^i$, $i \geq 1$ form a basis of $\tilde{V}$. We will refer to this module as $\tilde{V}_2(t)$.

Changing parity we get two new bimodules $\tilde{V}_1(t)^{\mathrm{op}}$ and $\tilde{V}_2(t)^{\mathrm{op}}$.

Each of these bimodules has a unique irreducible homomorphic image $V_1(t)$ or $V_2(t)$ or $V_1(t)^{\mathrm{op}}$ or $V_2(t)^{\mathrm{op}}$.

Coming back to the irreducible finite dimensional module V, if $V_{\bar{0}} = V_{\bar{0}}e_1$ and for a highest weight element v we have $vy \neq 0$ then $V \cong V_1(t)$. If $vy = 0$, then $V \cong V_2(t)$. In case that $V_{\bar{0}} = V_{\bar{0}}e_2$ and for a highest weight element v we have $vy \neq 0$, then $V \cong V_2(t)^{\mathrm{op}}$. If $vy = 0$, then $V \cong V_1(t)^{\mathrm{op}}$.

From the representation theory of sl_2 it follows that $\dim_F V_1(t) < \infty$ if and only if $\lambda = m$, a nonnegative integer. Then $t = \frac{-1-m}{2+m}$, $\dim_F V_1(t)_{\bar{0}} = m+1$, $\dim_F V_1(t)_{\bar{1}} = m+2$. Similarly, $\dim_F V_2(t) < \infty$ if and only if $\lambda = m$ a positive integer. Then $t = \frac{-1-m}{m}$, $\dim_F V_2(t)_{\bar{0}} = m+1$, $\dim_F V_2(t)_{\bar{1}} = m$.

For other values of t the module $\tilde{V}_i(t)$ is irreducible and the superalgebra $D(t)$ does not have nonzero finite dimensional specializations.

Theorem 5.3 *If $t = -\frac{m}{m+1}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $V_1(t)$ and $V_1(t)^{\mathrm{op}}$.*

If $t = -\frac{m+1}{m}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $V_2(t)$ and $V_2(t)^{\mathrm{op}}$.

If t can not be represented as $-\frac{m}{m+1}$ or $-\frac{m+1}{m}$, where m is a positive integer, then $D(t)$ does not have nonzero finite dimensional specializations.

Remark 5.2 If $\mathrm{ch}\, F = p > 2$ then for an arbitrary t the superalgebra $D(t)$ can be embedded into a finite dimensional associative superalgebra. It suffices to notice that $D(t) \subseteq CK(F[a \mid a^p = 0], d/da)$.

6 The Jordan Superalgebra of a Superform

Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space, $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = 2n$; let $\langle\, , \rangle \colon V \times V \to F$ be a supersymmetric bilinear form on V. The universal associative enveloping algebra of the Jordan algebra $F1 + V_{\bar{0}}$ is the Clifford algebra $\mathrm{Cl}(m) = \langle 1, e_1, \ldots, e_m \mid e_ie_j + e_je_i = 0, i \neq j, e_i^2 = 1\rangle$ (see [J]). Assuming the generators $e_1, \ldots, e_m$ to be odd, we get a $\mathbb{Z}/2\mathbb{Z}$-gradation on $\mathrm{Cl}(m)$.

In $V_{\bar{1}}$ we can find a basis $v_1, w_1, \ldots, v_n, w_n$ such that $\langle v_i, w_j\rangle = \delta_{ij}$, $\langle v_i, v_j\rangle = \langle w_i, w_j\rangle = 0$. Consider the Weyl algebra $W_n = \langle 1, x_i, y_i, 1 \leq i \leq n \mid [x_i, y_j] = \delta_{ij}, [x_i, x_j] = [y_i, y_j] = 0\rangle$. Assuming x_i, y_i, $1 \leq i \leq n$ to be odd, we make W_n a superalgebra. The universal associative enveloping algebra of $F1 + V$ is isomorphic to the (super)tensor product $\mathrm{Cl}(m) \otimes_F W_n$.

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