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Finite domination and Novikov homology over strongly $\mathbb{Z}^2$-graded rings

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Let R be a strongly $\mathbb{Z}^2$-graded ring, and let C be a bounded chain complex of finitely generated free R-modules. The complex C is $R_{(0,0)}$-finitely dominated, or of type FP over $R_{(0,0)}$, if it is chain homotopy equivalent to a bounded complex of finitely generated projective $R_{(0,0)}$-modules. We show that this happens if and only if C becomes acyclic after taking tensor product with a certain eight rings of formal power series, the graded analogues of classical Novikov rings. This extends results of Ranicki, Quinn and the first author on Laurent polynomial rings in one and two indeterminates.

Keywords: algebraic mapping torus; finite domination of chain complexes; Mather trick; Novikov homology

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Part I. Finite domination over strongly $\mathbb{Z}^2$-graded rings

1. Introduction

Let L be a unital ring, and let K be a subring of L. A bounded chain complex C of (right) L-modules is K-*finitely dominated* if C, considered as a complex of K-modules, is a retract up to homotopy of a bounded complex of finitely generated free K-modules; this happens if and only if C is homotopy equivalent, as a K-module complex, to a bounded complex of finitely generated projective K-modules [7, Proposition 3.2. (ii)]. The following result of Ranicki gives a complete homological characterization of finite domination in an important special case:

Theorem 1.1 Ranicki [8, Theorem 2]. *Let K be a unital ring, and let $K[t,t^{-1}]$ denote the* Laurent *polynomial ring in the indeterminate t. Let C be a bounded chain complex of finitely generated free $K[t,t^{-1}]$-modules. The complex C is K-finitely dominated if and only if both*

$$C \underset{K[t,t^{-1}]}{\otimes} K(\!(t^{-1})\!) \text{ and } C \underset{K[t,t^{-1}]}{\otimes} K(\!(t)\!)$$

have vanishing homology in all degrees. Here we use the notation $K(\!(t)\!) = K[\![t]\!][t^{-1}]$ *for the ring of formal* LAURENT *series in* t, *and similarly* $K(\!(t^{-1})\!) = K[\![t^{-1}]\!][t]$ *stands for the ring of formal* LAURENT *series in* t^{-1}.

The cited paper [8] also contains a discussion of the relevance of finite domination in topology. The rings $K(\!(t)\!)$ and $K(\!(t^{-1})\!)$ are known as NOVIKOV rings. The theorem can be formulated more succinctly: *The chain complex* C *is* K*-finitely dominated if and only if it has trivial* NOVIKOV *homology.*

This result was extended by QUINN and the first author to LAURENT polynomial rings in two variables; contrary to appearance, this is not a straight-forward modification of the original result, introducing additional levels of complication in homological algebra.

THEOREM 1.2 HÜTTEMANN AND QUINN [3, THEOREM I.1.2]. *Let* C *be a bounded chain complex of finitely generated free* L*-modules, where* $L = K[x, x^{-1}, y, y^{-1}]$ *is a* LAURENT *polynomial ring in two variables over the unital ring* K. *The following two statements are equivalent:*

(1) *The complex* C *is* K*-finitely dominated, i.e.,* C *is homotopy equivalent, as an* K*-module chain complex, to a bounded chain complex of finitely generated projective* K*-modules.*

(2) *The eight chain complexes listed below are acyclic (all tensor products are taken over* L*):*

$$\left.\begin{array}{ll} C \otimes K[x,\, x^{-1}](\!(y)\!)\ , & C \otimes K[x,\, x^{-1}](\!(y^{-1})\!) \\ C \otimes K[y,\, y^{-1}](\!(x)\!)\ , & C \otimes K[y,\, y^{-1}](\!(x^{-1})\!) \end{array}\right\} \tag{1.1a}$$

$$\left.\begin{array}{ll} C \otimes K(\!(x,\, y)\!)\ , & C \otimes K(\!(x^{-1},\, y^{-1})\!) \\ C \otimes K(\!(x,\, y^{-1})\!)\ , & C \otimes K(\!(x^{-1},\, y)\!) \end{array}\right\} \tag{1.1b}$$

Here $K(\!(x, y)\!) = K[\![x, y]\!][1/xy]$ *is a localization of the ring of formal power series in* x *and* y, *and the other rings are defined analogously.*

The authors of the present paper generalized theorem 1.1 in an entirely different direction, exhibiting the graded structure of LAURENT polynomial rings as the crucial property for setting up the theory.

THEOREM 1.3 HÜTTEMANN AND STEERS [4, THEOREM 1.3]. *Let* $R_*[t, t^{-1}] = \bigoplus_{k\in\mathbb{Z}} R_k$ *be a strongly* $\mathbb{Z}$*-graded ring, and let* C *be a bounded chain complex of finitely generated free* $R_*[t, t^{-1}]$*-modules. The complex* C *is* R_0*-finitely dominated if and only if both*

$$C \underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t^{-1})\!) \qquad \textit{and} \qquad C \underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t)\!)$$

have vanishing homology in all degrees. Here the rings

$$R_*(\!(t^{-1})\!) = \bigcup_{n\geqslant 0} \prod_{k\leqslant n} R_k \quad \textit{and} \quad R_*(\!(t)\!) = \bigcup_{n\geqslant 0} \prod_{k\geqslant -n} R_k$$

are used as formal analogues of the usual NOVIKOV *rings.*

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The notion of a *strongly graded ring* will be discussed in detail below. In the first instance, $R = \bigoplus_{k\in\mathbb{Z}} R_k$ is a $\mathbb{Z}$-graded ring. The (usual) LAURENT polynomial ring $R[t,t^{-1}]$ has a $\mathbb{Z}$-graded subring $R_*[t,t^{-1}]$ with kth component the set of monomials $r_k t^k$ with $r_k \in R_k$. In fact, we may identify $R_*[t,t^{-1}]$ with R itself. In a similar spirit, the NOVIKOV rings $R((t^{-1}))$ and $R((t))$ have subrings $R_*((t^{-1}))$ and $R_*((t))$ determined by the condition that the coefficient of t^k be an element of R_k, for any $k \in \mathbb{Z}$. It may be worth pointing out that these subrings do not contain the indeterminate t, which should be considered a purely notational device.

2. The main theorem

In the present paper, we take the step to strongly $\mathbb{Z}^2$-graded rings, combining ideas from both of the aforementioned publications [3] and [4]. Roughly speaking, a bounded chain complex C of finitely generated free modules over a strongly $\mathbb{Z}^2$-graded ring is finitely dominated over the degree-0 subring if and only if certain eight complexes induced from C are acyclic. Indeed, from a $\mathbb{Z}^2$-graded ring $R = \bigoplus_{\sigma\in\mathbb{Z}^2} R_\sigma$ we construct the following eight NOVIKOV-type rings:

$$\left.\begin{aligned}
R_*[x,\,x^{-1}]((y)) &= \bigcup_{n\geqslant 0}\prod_{y\geqslant -n}\bigoplus_{x\in\mathbb{Z}} R_{(x,y)}\\
R_*[x,\,x^{-1}]((y^{-1})) &= \bigcup_{n\geqslant 0}\prod_{y\geqslant -n}\bigoplus_{x\in\mathbb{Z}} R_{(x,-y)}\\
R_*[y,\,y^{-1}]((x)) &= \bigcup_{n\geqslant 0}\prod_{x\geqslant -n}\bigoplus_{y\in\mathbb{Z}} R_{(x,y)}\\
R_*[y,\,y^{-1}]((x^{-1})) &= \bigcup_{n\geqslant 0}\prod_{x\geqslant -n}\bigoplus_{y\in\mathbb{Z}} R_{(-x,y)}\\
R_*((x,\,y)) &= \bigcup_{n\geqslant 0}\prod_{x,y\geqslant -n} R_{(x,y)}\\
R_*((x,\,y^{-1})) &= \bigcup_{n\geqslant 0}\prod_{x,y\geqslant -n} R_{(x,-y)}\\
R_*((x^{-1},\,y^{-1})) &= \bigcup_{n\geqslant 0}\prod_{x,y\geqslant -n} R_{(-x,-y)}\\
R_*((x^{-1},\,y)) &= \bigcup_{n\geqslant 0}\prod_{x,y\geqslant -n} R_{(-x,y)}
\end{aligned}\right\} \tag{2.1}$$

Similar to notation used earlier, the symbols x and y do *not* stand for actual indeterminates; they are purely notational devices, emphasizing a formal similarity with LAURENT polynomial rings and their associated NOVIKOV rings. The ring $R_*[x,x^{-1}]((y))$ is the subring of $R[x,x^{-1}]((y))$ with elements $\sum_{(a,b)\in\mathbb{Z}^2} r_{a,b}x^a y^b$ such that $r_{a,b} \in R_{(a,b)}$, and similar for the other seven cases. With this notational camouflage, we obtain a perfect analogue of theorem 1.2:

THEOREM 2.1. *Let $R = \bigoplus_{k\in\mathbb{Z}^2} R_k$ be a strongly $\mathbb{Z}^2$-graded ring, and let C be a bounded chain complex of finitely generated free R-modules. The complex C is*

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$R_{(0,0)}$-finitely dominated if and only if all of the eight complexes

$$\left.\begin{array}{ll} C \underset{R}{\otimes} R_*[x,\, x^{-1}]((y)) \ , & C \underset{R}{\otimes} R_*[x,\, x^{-1}]((y^{-1})) \\ C \underset{R}{\otimes} R_*[y,\, y^{-1}]((x)) \ , & C \underset{R}{\otimes} R_*[y,\, y^{-1}]((x^{-1})) \end{array}\right\} \tag{2.2a}$$

and

$$\left.\begin{array}{ll} C \underset{R}{\otimes} R_*((x,\, y)) \ , & C \underset{R}{\otimes} R_*((x^{-1},\, y^{-1})) \\ C \underset{R}{\otimes} R_*((x,\, y^{-1})) \ , & C \underset{R}{\otimes} R_*((x^{-1},\, y)), \end{array}\right\} \tag{2.2b}$$

have vanishing homology in all degrees.

Applications in Σ-invariant theory

Let G be a group. Every character $\chi\colon G \longrightarrow \mathbb{R}$ to the additive group of the reals determines a monoid $G_\chi = \{g \in G \,|\, \chi(g) \geqslant 0\}$. Now suppose C is a non-negatively indexed chain complex of $\mathbb{Z}[G]$-modules. Then C has, by restriction of scalars, the structure of a $\mathbb{Z}[G_\chi]$-module chain complex. Following Farber *et.al.* one defines [**2**, Definition 9] the mth Σ-invariant of C as

$$\Sigma^m(C) = \{\chi \neq 0 \,|\, C \text{ has finite } m\text{-type over } \mathbb{Z}[G_\chi]\}/\mathbb{R}_+ \ ;$$

that is, $\Sigma^m(C)$ is a quotient of the set of those non-trivial χ for which there is a chain complex C' consisting of finitely generated projective $\mathbb{Z}[G_\chi]$-modules, and a $\mathbb{Z}[G_\chi]$-linear chain map $f\colon C' \longrightarrow C$ with $f_i\colon H_i(C') \longrightarrow H_i(C)$ an isomorphism for $i < m$ and an epimorphism for $i = m$. Two different characters are identified in the quotient precisely when they are positive real multiples of each other. The set $\Sigma^m(C)$ can be used to detect whether C is $\mathbb{Z}[N]$-finitely dominated:

Theorem 2.2 [**2**, Corollary 4]. *Suppose that C consists of finitely generated free $\mathbb{Z}[G]$-modules, and is such that $C_i = 0$ whenever $i > m$. Let N be a normal subgroup of G with* abel*ian quotient G/N. Then the $\mathbb{Z}[N]$-module complex C is chain homotopy equivalent to a bounded chain complex of finitely generated projective $\mathbb{Z}[N]$-modules concentrated in degrees $\leqslant m$ if and only if the set $\Sigma^m(C)$ contains the equivalence class of every non-trivial character of G that factorizes through G/N (i.e., whose kernel contains N).*

For K a unital ring and a character $\chi\colon G \longrightarrow \mathbb{R}$ we define the Novikov ring

$$\widehat{KG_\chi} = \Big\{ f\colon G \longrightarrow K \,|\, \forall t \in \mathbb{R}\colon \#\left(\operatorname{supp}(f) \cap \chi^{-1}(]-\infty,\, t]\,)\right) < \infty \Big\} \ ,$$

where $\operatorname{supp}(f) = f^{-1}(K \setminus \{0\})$, with multiplication given by the usual involution product. Note that $K[G]$ is a subring of $\widehat{KG_\chi}$. Vanishing of Novikov homology, that is, vanishing of homology with coefficients in the rings $\widehat{KG_\chi}$, characterizes whether C is finitely dominated over $K[N]$, provided G/N is free abelian of finite rank:

Theorem 2.3 Schütz [**10**, Theorem 4.7]. *Let C be a bounded chain complex of finitely generated free $K[G]$-modules. Suppose that N is a normal subgroup of G*

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with quotient $G/N \cong \mathbb{Z}^k$ a free ABEL*ian group of finite rank. The complex C is $K[N]$-finitely dominated if and only if for every non-zero character $\chi\colon G \longrightarrow \mathbb{R}$ which is trivial on N the complex $C \otimes_{K[G]} \widehat{KG_\chi}$ is acyclic.*

The non-zero characters which are trivial on N correspond bijectively to non-zero homomorphisms $\mathbb{Z}^k \cong G/N \to \mathbb{R}$, which in turn correspond bijectively to elements of $\mathbb{R}^k \setminus \{0\}$; the set of equivalence classes of such characters thus forms a sphere $S^{k-1} \cong \big(\mathbb{R}^k \setminus \{0\}\big)/\mathbb{R}_+$ of dimension $k-1$. With this identification, the Σ-invariants $\Sigma^m(C)$ and their analogues

$$\Sigma'(C) = \Big\{[\chi] \in S^{k-1} \,|\, C \underset{K[G]}{\otimes} \widehat{KG_\chi} \text{ is acyclic}\Big\}$$

are subsets of S^{k-1}. The two theorems assert that C is $K[N]$-finitely dominated, with the equivalent finite type complex concentrated in chain degrees $\leqslant m$ in case of the first theorem, if and only if $\Sigma^m(C) = S^{k-1}$ or $\Sigma'(C) = S^{k-1}$, respectively. Even for $k = 2$, computation of $\Sigma^m(C)$ and $\Sigma'(C)$ *a priori* involves infinitely many NOVIKOV rings.

Fixing a choice of isomorphism $G/N \cong \mathbb{Z}^k$ gives a group epimorphism $\pi\colon G \to \mathbb{Z}^k$ with kernel N, which can be used to equip the group ring $K[G] = \bigoplus_{p\in\mathbb{Z}^k} K\big[\pi^{-1}(p)\big]$ with the structure of a strongly $\mathbb{Z}^k$-graded ring as indicated. The subring of degree 0 elements is precisely $K[N]$. For $k = 2$, this gives a large and natural class of strongly $\mathbb{Z}^2$-graded rings to which our main theorem 2.1 applies, providing a characterization of $K[N]$-finite domination of $K[G]$-module complexes involving eight NOVIKOV rings only. The four conditions (2.2a) occur already in theorem 2.3 (the ring $R_*[y, y^{-1}](\!(x)\!)$, for example, arising as $\widehat{KG_\chi}$ for the character ξ corresponding to the point $(1, 0) \in S^1$), but the four conditions (2.2b) are of a new type. The special case of a product group $G = N \times \mathbb{Z}^2$, with $K[G] \cong K[N][x, x^{-1}, y, y^{-1}]$ a LAURENT polynomial ring in two indeterminates, is the remit of the paper [**3**] where the reader can find additional discussion and examples.

Applications in topology

A topological space Z is finite up to homotopy if and only if (i) it is finitely dominated in the sense of being a retract up to homotopy of a finite CW complex, and (ii) a certain K-theoretical obstruction vanishes, see theorem 2.5. The obstruction is only defined if condition (i) is met. The results of [**4**] and of the present paper yield algebraic criteria to verify condition (i) for important classes of topological spaces, including regular covering spaces with deck transformation group $\mathbb{Z}$ or $\mathbb{Z}^2$.

Apart from their intrinsic interest, finiteness results of this type have manifold applications in topology themselves. For example, for a closed manifold M to fibre over S^1 it is necessary (but not sufficient) that its infinite cyclic covering $\bar{M}$ is finitely dominated, cf. [**9**, §15]. Moreover, if $\bar{M}$ is finitely dominated and M has dimension at least 6, a given 1-dimensional cohomology class $\xi \in H^1(M, \mathbb{R})$ is represented by a non-singular closed 1-form provided its LATOUR obstruction $\tau(M, \xi)$, which is defined if $\bar{M}$ is finitely dominated, vanishes, cf. [**10**].

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Homotopy finiteness and finite domination of topological spaces A connected CW-complex X is *homotopy finite* if it is homotopy equivalent to a finite CW-complex. We say that X is *finitely dominated* if it is a retract up to homotopy of a finite connected CW-complex, that is, if and only if there exist maps $s\colon X \to K$ and $r\colon K \to X$ with K a finite connected CW-complex such that $r \circ s$ is homotopic to id_X. (This is condition (i) from above.)

Any homotopy finite space is finitely dominated, and any finitely dominated space has a finitely presented fundamental group. The following two theorems, going back to the work of Wall, are fundamental (see [8, §3] for a discussion):

THEOREM 2.4. *A connected CW complex Z is finitely dominated if and only if*

- *the fundamental group $\pi = \pi_1(Z)$ is finitely presented, and*
- *the cellular chain complex $C(\tilde{Z})$ of the universal covering $\tilde{Z}$ of Z (a complex of free $\mathbb{Z}[\pi]$-modules) is $\mathbb{Z}[\pi]$-finitely dominated, that is, is chain homotopy equivalent to a bounded complex D of finitely generated projective $\mathbb{Z}[\pi]$-modules.*

THEOREM 2.5. *Let Z be a finitely dominated CW-complex, and let D be a bounded chain complex of finitely generated projective $\mathbb{Z}[\pi]$-modules homotopy equivalent to $C(\tilde{Z})$, as described in theorem 2.4. Let $[Z] = \sum_{k\in\mathbb{Z}}(-1)^k[D_k]$ denote the class of D in $\tilde{K}_0\big(\mathbb{Z}[\pi]\big)$. The class $[Z]$ does not depend on the choice of D, and is a finiteness obstruction in the sense that Z is homotopy finite if and only if $[Z] = 0$.*

This latter theorem is precisely condition (ii) from above.

Fundamental groups mapping surjectively onto $\mathbb{Z}$ Let X now be a finite connected CW-complex with fundamental group $G = \pi_1(X)$, and suppose there is a surjective group homomorphism $h\colon G \to \mathbb{Z}$ with kernel π. Then G is a semi-direct product $G = \pi \rtimes \mathbb{Z}$ since h has a section. Let $\bar{X}$ be the covering space of X determined by π; it has fundamental group π. The universal covering space of $\bar{X}$ is the universal cover $\tilde{X}$ of X. We can apply the previous theorem to $Z = \bar{X}$ and $\tilde{Z} = \tilde{X}$ to conclude that *the space $\bar{X}$ is finitely dominated if and only if π is finitely presented, and the chain complex $C(\tilde{X})$ is $\mathbb{Z}[\pi]$-finitely dominated.*

As a first remark, we note that $G = \pi \rtimes \mathbb{Z}$ is a finitely presented group in this situation since X is a finite CW-complex. If $G = \pi \times \mathbb{Z}$ is a *direct* product, then π is a retract of G in the category of groups and hence is finitely presented itself. This, however, does not hold in general for a semi-direct product $G = \pi \rtimes \mathbb{Z}$. For an explicit example, consider $\pi = \mathbb{Z}[\frac{1}{2}]$ with $\mathbb{Z}$ acting on π as ${}^k x = 2^k \cdot x$ whence

$$G = \pi \rtimes \mathbb{Z} \cong \langle y,\, z \,|\, zyz^{-1} = y^2 \rangle$$

is finitely presented, while π is not even finitely generated. This can be topologically realized by taking X to be the mapping torus of the self map $z \mapsto z^2$ of $S^1 \subset \mathbb{C}$ with $\bar{X}$ the canonical infinite cyclic cover, the dyadic solenoid [9, Remark 3.5 (ii)].

More in the spirit of the present paper is the observation that $\mathbb{Z}[G]$ is a *strongly $\mathbb{Z}$-graded* ring, with degree k component the free ABELian group with set of generators $h^{-1}(k) \subseteq G$. Thus, we can detect finite domination homologically, by combining the above discussion with the main result of [4] (Theorem 1.3) as follows.

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We write $R_*[t,t^{-1}]$ for the strongly $\mathbb{Z}$-graded ring $\mathbb{Z}[G]$, and write $H_*\big(X,\, R_*(\!(t^{\pm 1})\!)\big)$ for the NOVIKOV homology of X, the homology of the chain complex $C(\tilde{X}) \otimes_{R_*[t,t^{-1}]} R_*(\!(t^{\pm 1})\!)$.

THEOREM 2.6. *Let X be a finite connected CW-complex with fundamental group G. Suppose there exists a surjective group homomorphism $h\colon G \to \mathbb{Z}$. Let $\bar{X}$ denote the covering space of X determined by* $\ker(h)$*, and let $R_*[t,t^{-1}]$ denote the ring $\mathbb{Z}[G]$ equipped with the strong $\mathbb{Z}$-grading induced by h.*

(1) *The space $\bar{X}$ is finitely dominated if and only if*
- *the group* $\ker(h)$ *is finitely presented, and*
- *X has trivial NOVIKOV homology, $H_*\big(X,\, R_*(\!(t^{\pm 1})\!)\big) = \{0\}$.*

(2) *Suppose in addition that $G = \ker(h) \times \mathbb{Z}$, and that h is the projection map $h = \mathrm{pr}_2\colon G = \ker(h) \times \mathbb{Z} \to \mathbb{Z}$. In this case,* $\ker(h)$ *is finitely presented, and the space $\bar{X}$ is finitely dominated if and only if $H_*\big(X,\, R_*(\!(t^{\pm 1})\!)\big)$ is trivial.*

Note that the group ring $\mathbb{Z}[G]$ is isomorphic, as a $\mathbb{Z}[\ker(h)]$-ring, to a skew LAURENT polynomial ring $\mathbb{Z}[\ker(h)]_\alpha[t,t^{-1}]$, with t corresponding to $1 \in \mathbb{Z} \subseteq \ker(h) \rtimes \mathbb{Z}$ and α determined by the action of $\mathbb{Z}$ on $\ker(h)$. If $G = \ker(h) \times \mathbb{Z}$ then $\mathbb{Z}[G] = \mathbb{Z}[\ker(h)][t,t^{-1}]$ is a LAURENT polynomial ring in the usual sense.

Fundamental groups mapping surjectively onto $\mathbb{Z}^2$ Let X be a finite connected CW-complex with fundamental group $G = \pi_1(X)$, and suppose there is a surjective group homomorphism $h\colon G \to \mathbb{Z}^2$ with kernel π. If h has a section, then $G = \pi \rtimes \mathbb{Z}^2$ is a semi-direct product, but in general G may have a more complicated structure. Note that G is finitely presented but π may not be. If $G = \pi \times \mathbb{Z}^2$ is a direct product (with h being the projection map), then π is finitely presented as well. On the other hand, if G is a free group on two generators a and b, and $h\colon G \to \mathbb{Z}^2$ is the obvious surjection, $\pi = \ker(h)$ is freely generated by the (infinitely many) commutators $[a^m, b^n]$ for non-zero exponents m and n; of course G can be realized topologically by the figure-8 space $X = S^1 \vee S^1$.

Let $\bar{X}$ be the covering space of X determined by π. The universal covering space of $\bar{X}$ is the universal cover $\tilde{X}$ of X, and its cellular chain complex $C(\tilde{X})$ is a bounded complex of finitely generated free $\mathbb{Z}[G]$-modules.

The homomorphism h endows $\mathbb{Z}[G]$ with the structure of a strongly $\mathbb{Z}^2$-graded ring which we denote by $R = R_*[x,x^{-1},y,y^{-1}]$. Thus, the eight NOVIKOV rings listed in (2.1) are at our disposal. If S is any of these rings, we write $H_*(X,S)$ for the homology of the chain complex $C(\tilde{X}) \otimes_R S$. Using the main result of the present paper, theorem 2.1, we obtain the following characterization of finite domination of the covering space determined by $\ker(h)$:

THEOREM 2.7. *Let X be a finite connected CW-complex with fundamental group G. Suppose there exists a surjective group homomorphism $h\colon G \to \mathbb{Z}^2$. Let $\bar{X}$ denote the covering space of X determined by* $\ker(h)$*, and let $R = R_*[x,x^{-1},y,y^{-1}]$ denote the integral group ring $\mathbb{Z}[G]$ of G equipped with the strong $\mathbb{Z}^2$-grading induced by h.*

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(1) *The space $\bar{X}$ is finitely dominated if and only if*
- *the group* $\ker(h)$ *is finitely presented, and*
- *X has trivial* Novikov *homology*:

$$\left.\begin{aligned} H_*\big(X,\, R[x,\, x^{-1}]((y^{\pm 1}))\big) &= \{0\},\\ H_*\big(X,\, R[y,\, y^{-1}]((x^{\pm 1}))\big) &= \{0\},\\ H_*\big(X,\, R((x^{\pm 1},\, y^{\pm 1}))\big) &= \{0\}. \end{aligned}\right\} \tag{2.3}$$

(2) *Suppose in addition that $G = \ker(h) \times \mathbb{Z}^2$, and that h is the projection map $h = \mathrm{pr}_2 \colon G = \ker(h) \times \mathbb{Z}^2 \to \mathbb{Z}^2$. In this case,* $\ker(h)$ *is finitely presented, and the space $\bar{X}$ is finitely dominated if and only if X has trivial* Novikov *homology as in* (2.3).

Note that provided h has a section, the group ring $\mathbb{Z}[G]$ is isomorphic, as a $\mathbb{Z}[\ker(h)]$-ring, to a skew Laurent polynomial ring of the form $\mathbb{Z}[\ker(h)]_\alpha[x, x^{-1}, y, y^{-1}]$, with x and y corresponding to the generators of $\mathbb{Z}^2$ and α determined by the action of $\mathbb{Z}^2$ on $\ker(h)$. If $G = \ker(h) \times \mathbb{Z}^2$, then $\mathbb{Z}[G] = \mathbb{Z}[\ker(h)][x, x^{-1}, y, y^{-1}]$ is a Laurent polynomial ring in the usual sense. In general, however, $\mathbb{Z}[G]$ is not a skew Laurent polynomial ring.

An explicit algebraic example

The applications discussed so far feature strongly graded rings that are in fact group rings, and hence specific examples of crossed products which are graded rings having homogeneous units of all degrees. However, theorem 2.1 does not rely on this feature. To illustrate the extent of the generalization achieved in this paper, we describe a strongly $\mathbb{Z}^2$-graded ring $\hat{K}$ that is not a crossed product (and in particular not a Laurent polynomial ring), and a complex of finitely generated free $\hat{K}$-modules that is $\hat{K}_{(0,0)}$-finitely dominated by the above criterion.

Let $\overline{K} = K[a, b, c, d]/ab + cd - 1$; this is a strongly $\mathbb{Z}$-graded ring if we let a, c have degree 1 and b, d have degree -1. It is not a Laurent polynomial ring, in fact not a crossed product, since the only units in $\overline{K}$ are the elements of $K^\times$ in degree 0, as can be shown using ideas from Gröbner basis theory. In particular, there is no unit in $\overline{K}_1$.

The $\mathbb{Z}^2$-graded ring $\hat{K} = \overline{K} \otimes_{\overline{K}_0} \overline{K}$, where $\hat{K}_{(k,\ell)} = \overline{K}_k \otimes_{\overline{K}_0} \overline{K}_\ell$, can be seen to be strongly graded as the necessary partitions of unity are present as in point (3) of proposition 4.5. It is not a crossed product since, for example, there is no unit in $\hat{K}_{(1,0)} = \overline{K}_1$.

We will define a fourfold chain complex V concentrated in degrees $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \mathbb{Z}^4$ with $\epsilon_j = 0, 1$. The entries are $\hat{K}$, a free $\hat{K}$-module of rank 1; the non-trivial differentials are multiplication by

$$1 - a \otimes 1 \text{ in 1-direction,}$$

$$1 - 1 \otimes c \text{ in 2-direction,}$$

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$$1 - d \otimes 1 \text{ in 3-direction,}$$
$$1 - 1 \otimes bcd \text{ in 4-direction.}$$

That is, the map from position $(1, \epsilon_2, \epsilon_3, \epsilon_4)$ to position $(0, \epsilon_2, \epsilon_3, \epsilon_4)$ is multiplication by $1 - a \otimes 1$, the map from position $(\epsilon_1, 1, \epsilon_3, \epsilon_4)$ to position $(\epsilon_1, 0, \epsilon_3, \epsilon_4)$ is multiplication by $1 - 1 \otimes c$, and so on.

The totalization C of V can be obtained by taking 'iterated mapping cones', and up to isomorphism it does not matter in which order we choose the different directions. Now the map $1 - a \otimes 1$ is an isomorphism with inverse

$$(1 - a \otimes 1)^{-1} = 1 + a \otimes 1 + a^2 \otimes 1 + a^3 \otimes 1 + \dots$$

over the rings $\hat{K}_*[y, y^{-1}]((x))$ and $\hat{K}_*((x, y^{\pm 1}))$, by the usual telescoping sum argument familiar from the geometric series. Note that the series defines an element in each of these rings since $a^k \otimes 1 = (a \otimes 1)^k$ has degree $(k, 0)$. Thus, after tensoring V with one of these rings, the mapping cones in 1-direction will be acyclic, hence the tensor product of C with one of these rings will be acyclic.

Similarly, the map $1 - 1 \otimes bcd$ is an isomorphism over $\hat{K}_*[x, x^{-1}]((y^{-1}))$ and $\hat{K}_*((x^{\pm 1}, y^{-1}))$, with inverse

$$(1 - 1 \otimes bcd)^{-1} = 1 + 1 \otimes bcd + 1 \otimes (bcd)^2 + 1 \otimes (bcd)^3 + \dots \ ;$$

this is an element of the rings in question since the degree of $1 \otimes (bcd)^k$ is $(0, -k)$. Consequently, after tensoring V with one of these rings the mapping cones in 4-direction will be acyclic, hence the tensor product of C with one of these rings will be acyclic. The cases of $1 - 1 \otimes c$ and $1 - d \otimes 1$ are similar, involving the rings $\hat{K}_*[x, x^{-1}]((y))$ and $\hat{K}_*((x^{\pm 1}, y))$ in the former case, the rings $\hat{K}_*[y, y^{-1}]((x^{-1}))$ and $\hat{K}_*((x^{-1}, y^{\pm 1}))$ in the latter.

Structure of the paper

The paper can be seen as an amalgamation of the two publications [**3**] and [**4**], combining the graded viewpoint of the latter with the elaborate homological algebra of the former. Replacing central indeterminates by inherently non-commutative structures is a non-trivial task, and it is rather surprising that with the right set-up the overall pattern of proof remains virtually unchanged.

We start with recalling basic concepts from the theory of strongly graded rings. The 'if' implication of the main theorem is verified in part II, building on [**4**] and [**3**]; the main point is to relate the given chain complex with a complex of diagrams which are the analogues of well-known line bundles on the scheme $\mathbb{P}^1 \times \mathbb{P}^1$. Part III focuses on the 'only if' implication. The main technical device is the algebraic torus (a 'two-dimensional' version of the algebraic mapping torus), and the Mather trick which allows to replace the complex C with an algebraic torus of a complex D consisting of finitely generated projective $R_{(0,0)}$-modules.

3. Graded rings and Novikov rings

Definition 3.1. *A $\mathbb{Z}^2$-graded ring is a (unital) ring L equipped with a direct sum decomposition into additive subgroups $L = \bigoplus_{\sigma \in \mathbb{Z}^2} L_\sigma$ such that $L_\sigma L_\tau \subseteq L_{\sigma+\tau}$ for*

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all $\sigma, \tau \in \mathbb{Z}^2$, where $L_\sigma L_\tau$ consists of the finite sums of ring products xy with $x \in L_\sigma$ and $y \in L_\tau$. The summands L_σ are called the (homogeneous) components of L; elements of L_σ are called homogeneous of degree σ.

Relevant examples are the polynomial ring $K[x, y]$ and the LAURENT polynomial ring $R = K[x, x^{-1}, y, y^{-1}]$ in two variables, equipped with the usual $\mathbb{Z}^2$-grading by the exponents of x and y, respectively; here K may be any unital ring. The main difference between these examples, from the point of view taken in this paper, is that the latter is strongly $\mathbb{Z}^2$-graded (definition 4.4) but the former is not.

From a unital ring R, for the moment not equipped with a grading, we can construct various 'NOVIKOV rings' of formal LAURENT series as follows:

$$\begin{aligned}
R[x,\, x^{-1}]((y)) &= R[x,\, x^{-1}][[y]][y^{-1}] \\
R[y,\, y^{-1}]((x)) &= R[y,\, y^{-1}][[x]][x^{-1}] \\
R[x,\, x^{-1}]((y^{-1})) &= R[x,\, x^{-1}][[y^{-1}]][y] \\
R[y,\, y^{-1}]((x^{-1})) &= R[y,\, y^{-1}][[x^{-1}]][x] \\
R((x,\, y)) &= R[[x,\, y]][(xy)^{-1}] \\
R((x^{-1},\, y^{-1})) &= R[[x^{-1},\, y^{-1}]][xy] \\
R((x,\, y^{-1})) &= R[[x,\, y^{-1}]][(xy^{-1})^{-1}] \\
R((x^{-1},\, y)) &= R[[x^{-1},\, y]][(x^{-1}y)^{-1}]
\end{aligned}$$

Here x and y are central indeterminates which commute with each other, and with the elements of R. The slightly informal notation $S[[x]][x^{-1}]$ is meant to denote the ring of formal power series in the indeterminate x with coefficients in S, localized at the multiplicative system $\{x^k \mid k \geqslant 0\}$.

Returning to the case of a $\mathbb{Z}^2$-graded ring R, we can introduce subrings of rings of formal power or LAURENT series, denoted by adding the decoration '$*$' as a subscript to the ring R, consisting of those elements such that the coefficient of $x^s y^t$ is an element of $R_{(s,t)}$. The resulting unital subrings will not contain the elements x and y as, for example, $x = 1 \cdot x$ has coefficient $1 \in R_{(0,0)}$, whereas the monomial x has degree $(1, 0)$. At this point, we may treat the 'variables' as a purely notational device keeping track of degrees of various elements. As an illustration, we put forward the ring $R_*[x, x^{-1}, y, y^{-1}]$ which has elements the finite sums of terms $r_{s,t} x^s y^t$, with $s, t \in \mathbb{Z}$ and $r_{s,t} \in R_{(s,t)}$. We may in fact identify this ring with the (graded) ring R itself. When applied to the NOVIKOV rings listed above, we arrive at the identifications listed in (2.1).

4. Partitions of unity and strongly graded rings

DEFINITION 4.1. *Let $S = \bigoplus_{\sigma \in \mathbb{Z}^2} S_\sigma$ be a $\mathbb{Z}^2$-graded unital ring. For $\rho \in \mathbb{Z}^2$, an finite sum expression of the form $1 = \sum_j u_j v_j$ with $u_j \in S_\rho$ and $v_j \in S_{-\rho}$ is called a partition of unity of type $(\rho, -\rho)$.*

By a straightforward computation, one can show:

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LEMMA 4.2. *For $\rho, \rho' \in \mathbb{Z}^2$ let $1 = \sum_j u_j v_j$ and $1 = \sum_k u'_k v'_k$ be partitions of unity of type ρ and ρ', respectively. Then*

$$1 = \sum_k \sum_j (u_j u'_j)(v'_j v_j)$$

is a partition of unity of type $\rho + \rho'$.

If $S_\rho S_{-\rho} = S_{(0,0)}$ then, since $1 \in S_{(0,0)}$, there exists a partition of unity of type $(\rho, -\rho)$. Conversely, if a partition of unity of type $(\rho, -\rho)$ exists, then $1 \in S_\rho S_{-\rho}$ and thus $S_{(0,0)} \subseteq S_\rho S_{-\rho}$ so that $S_{(0,0)} = S_\rho S_{-\rho}$; this implies moreover

$$S_\sigma = S_\sigma S_{(0,0)} = S_\sigma S_\rho S_{-\rho} \subseteq S_{\sigma+\rho} S_{-\rho} \subseteq S_\sigma$$

so that $S_\sigma = S_{\sigma+\rho} S_{-\rho}$ for any $\sigma \in \mathbb{Z}^2$, and similarly $S_\sigma = S_\rho S_{\sigma-\rho}$.

PROPOSITION 4.3. *Let R be a $\mathbb{Z}^2$-graded ring. Let $\lambda, \rho \in \mathbb{Z}^2$ be such that there exists a partition of unity $1 = \sum_j \alpha_j \beta_j$ of type $(\lambda, -\lambda)$. The multiplication map*

$$\pi_{\lambda,\rho} \colon R_\lambda \underset{R_{(0,0)}}{\otimes} R_\rho \longrightarrow R_{\lambda+\rho}, \quad x \otimes y \mapsto xy$$

is an isomorphism of $R_{(0,0)}$-bimodules; its inverse can be written as

$$\mu_{\lambda,\rho} = \pi^{-1}_{\lambda,\rho} \colon R_{\lambda+\rho} \longrightarrow R_\lambda \underset{R_{(0,0)}}{\otimes} R_\rho \ , \quad z \mapsto \sum_j \alpha_j \otimes \beta_j z.$$

The map $\mu_{\lambda,\rho}$ does not depend on the choice of partition of unity.

Proof. The map $\pi_{\lambda,\rho}$ is an $R_{(0,0)}$-balanced, thus well-defined, $R_{(0,0)}$-bimodule homomorphism. Hence, it is enough to show that $\mu_{\lambda,\rho}$, considered as a homomorphism of right $R_{(0,0)}$-modules, is its inverse. For $z \in R_{\lambda+\rho}$, we calculate

$$\pi_{\lambda,\rho} \circ \mu_{\lambda,\rho}(z) = \sum_j \alpha_j \beta_j z = z$$

so that $\pi_{\lambda,\rho} \circ \mu_{\lambda,\rho}$ is the identity map. Similarly, for $x \in R_\lambda$ and $y \in R_\rho$, we have

$$\mu_{\lambda,\rho} \circ \pi_{\lambda,\rho}(x \otimes y) = \sum_j \alpha_j \otimes (\beta_j x) y = \sum_j \alpha_j \beta_j x \otimes y = x \otimes y$$

(since $\beta_j x \in R_{(0,0)}$) so that $\mu_{\lambda,\rho} \circ \pi_{\lambda,\rho}$ is the identity map. □

DEFINITION 4.4 DADE [1, §1]:. *A $\mathbb{Z}^2$-graded ring $L = \bigoplus_{\rho \in \mathbb{Z}^2} L_\rho$ is called a strongly $\mathbb{Z}^2$-graded ring if $L_\kappa L_\lambda = L_{\kappa+\lambda}$ for all $\kappa, \lambda \in \mathbb{Z}^2$.*

The establishing example is the LAURENT polynomial ring in two variables, $R = K[x, x^{-1}, y, y^{-1}]$. This is a strongly $\mathbb{Z}^2$-graded ring when x and y are given degrees $(1, 0)$ and $(0, 1)$, respectively.

Using [1, Proposition 1.6] and lemma 4.2, one obtains the following characterization of strongly $\mathbb{Z}^2$-graded rings:

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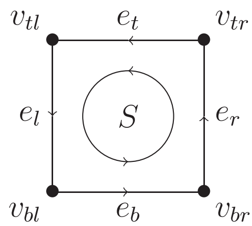

Figure 1. Labelling and orientation of the faces of the square S.

PROPOSITION 4.5. *Let R be a $\mathbb{Z}^2$-graded ring. The following statements are equivalent:*

(1) *The ring R is strongly graded.*

(2) *For every $\rho \in \mathbb{Z}^2$, there is at least one partition of unity of type $(\rho, -\rho)$.*

(3) *There is at least one partition of unity of each of the types*
- $((1,0),(-1,0))$ *and* $((-1,0),(1,0))$,
- $((0,1),(0,-1))$ *and* $((0,-1),(0,1))$.

(4) *For every $\rho \in \mathbb{Z}^2$, we have $L_\rho L_{-\rho} = L_{(0,0)}$.*

COROLLARY 4.6. *If $L = \bigoplus_{\kappa \in \mathbb{Z}^2} L_\kappa$ is strongly graded, then all L_κ are invertible $L_{(0,0)}$-bimodules. In particular, the L_κ are finitely generated projective as right and as left $L_{(0,0)}$-modules.*

Part II. Contractibility of Novikov homology implies finite domination

5. Rings and modules associated to faces of a square

We denote by the symbol $\mathfrak{S}$ the set of non-empty faces of the polytope $S = [-1,1] \times [-1,1] \subset \mathbb{R}^2$, partially ordered by inclusion. In figure 1, we show our chosen labelling and orientation of the faces (the orientations will be used later to define incidence numbers).

With each face $F \in \mathfrak{S}$, we associate its *barrier cone*

$$T_F = \operatorname{cone}\{x - y \,|\, x \in S,\ \ y \in F\} \subseteq \mathbb{R}^2 \ ,$$

where cone (X) denotes the set of finite linear combinations of elements of X with non-negative real coefficients. For example, $T_{v_{bl}}$ is the first quadrant and T_{e_r} is the left half-plane. Given a $\mathbb{Z}^2$-graded ring $R = \bigoplus_{\rho \in \mathbb{Z}^2} R_\rho$, we let $A_F = R_*[T_F]$ denote the subring of $R = R_*[x, x^{-1}, y, y^{-1}]$ consisting of elements with support in T_F; explicitly,

$$A_F = R_*[T_F] = \bigoplus_{\rho \in T_F \cap \mathbb{Z}^2} R_\rho \ .$$

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Thus, for example,

$$\begin{array}{lll} A_{v_{br}} & = R_*[T_{v_{br}}] & = R_*[x^{-1}, y], \\ A_{e_l} & = R_*[T_{e_l}] & = R_*[x, y, y^{-1}], \\ A_S & = R_*[T_S] & = R_*[x, x^{-1}, y, y^{-1}] = R. \end{array}$$

For $k \in \mathbb{Z}$, we introduce the $R_*[T_F]$-bimodules

$$R_*[kF + T_F] = \bigoplus_{\sigma \in (kF + T_F) \cap \mathbb{Z}^2} R_\sigma ,$$

where $kF + T_F = \{kx + y \,|\, x \in F \text{ and } y \in T_F\}$. In different notation,

$$\begin{array}{lll} R_*[kv_{br} + T_{v_{br}}] & = x^{-k}y^{-k}R_*[x^{-1}, y] & = x^{-k}y^{-k}A_{v_{br}}, \\ R_*[ke_l + T_{e_l}] & = x^{-k}R_*[x, y, y^{-1}] & = x^{-k}A_{e_l}, \\ R_*[kS + T_S] & = R_*[x, x^{-1}, y, y^{-1}] & = R. \end{array}$$

As a final bit of notation for now, we let $p_F \in F \cap \mathbb{Z}^2$ denote the barycentre of F. For example, $p_{e_t} = (0, 1)$ and $p_S = (0, 0)$. Then $kF + T_F = kp_F + T_F$, and every element σ of $kF + T_F$ can in fact be written in the form $kp_F + \tau$ for some $\tau \in T_F$. Also, given faces $F \subseteq G$, the barrier cones satisfy $T_G = T_F + \mathbb{R} \cdot (p_G - p_F)$ so that, in particular, $k(p_G - p_F) \in T_G$.

Lemma 5.1. *Let R be a strongly $\mathbb{Z}^2$-graded ring. For all $k \in \mathbb{Z}$ and for all $F, G \in \mathfrak{S}$ with $F \subseteq G$, the $R_*[T_F]$-linear inclusion map $\alpha_{F,G} \colon R_*[kF + T_F] \longrightarrow R_*[kG + T_G]$ is such that its adjoint map*

$$\alpha^\sharp_{F,G} \colon R_*[kF + T_F] \underset{R_*[T_F]}{\otimes} R_*[T_G] \longrightarrow R_*[kG + T_G] \ , \qquad r \otimes s \mapsto rs$$

is an isomorphism.

Proof. Choose a partition of unity of type $(kp_F, -kp_F)$, say $1 = \sum_j u_j v_j$ with $u_j \in R_{kp_F}$ and $v_j \in R_{-kp_F}$. The map

$$\beta_{F,G} \colon R_*[kG + T_G] \longrightarrow R_*[kF + T_F] \underset{R_*[T_F]}{\otimes} R_*[T_G] \ , \qquad x \mapsto \sum_j u_j \otimes v_j x$$

is well-defined. First, u_j has degree $kp_F \in kF \subseteq kF + T_F$ whence $u_j \in R_*[kF + T_F]$. Second, every element σ of $(kG + T_G) \cap \mathbb{Z}^2$ can be written in the form $kp_G + \tau$, with $\tau \in T_G \cap \mathbb{Z}^2$. Thus,

$$\sigma - kp_F = kp_G + \tau - kp_F = \tau + k(p_G - p_F) \in T_G \ .$$

It follows that for $x \in R_*[kG + T_G]$, the product $v_j x$ is an element of $R_*[T_G]$. The map $\beta_{F,G}$ is $R_*[T_G]$-linear and satisfies $\alpha^\sharp_{F,G} \circ \beta_{F,G} = \mathrm{id}$ by direct calculation, using $1 = \sum_j u_j v_j$. We also have

$$\beta_{F,G} \circ \alpha^\sharp_{F,G}(r \otimes s) = \beta_{F,G}(rs) = \sum_j u_j \otimes v_j rs \underset{(*)}{=} \sum_j u_j v_j r \otimes s = r \otimes s$$

where the equality labelled $(*)$ is true since $v_j r \in R_*[T_F]$ for any $r \in R_*[kF + T_F]$, whence $\beta_{F,G} \circ \alpha^\sharp_{F,G} = \mathrm{id}$. $\square$

6. Čech complexes

Incidence numbers and Čech complexes

Suppose P is a poset equipped with a strictly increasing 'rank' function

$$\mathrm{rk}\colon P \longrightarrow \mathbb{N} = \{0,\, 1,\, 2,\, \cdots\}$$

and with a notion of 'incidence numbers'

$$[\cdot,\, \cdot]\colon P \times P \longrightarrow \mathbb{Z}$$

satisfying the following conditions:

(DI1) $[x\colon y] = 0$ unless $x > y$ and $\mathrm{rk}(y) = \mathrm{rk}(x) - 1$.

(DI2) For all $z < x$ with $\mathrm{rk}(z) = \mathrm{rk}(x) - 2,$ the set

$$P(z < x) = \{y \in P \,|\, z < y < x\}$$

is finite, and

$$\sum_{y \in P(z<x)} [x\colon y] \cdot [y\colon z] = 0.$$

(DI3) For $x \in P$ with $\mathrm{rk}(x) = 1$, the set

$$P(<x) = \{y \in P \,|\, y < x\}$$

is finite, and

$$\sum_{y \in P(<x)} [x\colon y] = 0.$$

We can then associate a 'Čech complex' with a diagram of K-modules (K a unital ring) indexed by P:

Definition 6.1. *Given diagram* $\Phi\colon P \longrightarrow K\text{-Mod}$ *with structure maps* $\varphi_{x,y}\colon \Phi_y \longrightarrow \Phi_x,$ *define its* Čech *complex* $\Gamma(\Phi) = \Gamma_P(\Phi)$ *to be the chain complex concentrated in non-positive degrees with chain modules*

$$\Gamma(\Phi)_{-n} = \bigoplus_{x \in \mathrm{rk}^{-1}(n)} \Phi_x \qquad (n \geqslant 0)$$

with differential induced by $[x : y]\varphi_{x,y}$ *for* $x > y$.

It is easy to check that by virtue of (DI1) and (DI2), this is indeed a chain complex. Condition (DI3) ensures that a cone $M \xrightarrow{\cdot} \Phi$ on $\Phi,$ that is, a collection of K-module maps $\kappa_x\colon M \longrightarrow \Phi_x$ satisfying the compatibility condition $\kappa_x = \varphi_{x,y} \circ \kappa_y,$ yields a chain complex

$$\ldots \leftarrow \Gamma_P(\Phi)_{-1} \leftarrow \Gamma_P(\Phi)_0 \xleftarrow{\kappa} M$$

with co-augmentation κ induced by the maps κ_x for $x \in \mathrm{rk}^{-1}(0)$.

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Čech complexes of diagrams of chain complexes

The assignment $\Phi \mapsto \Gamma_P(\Phi)$ is in fact an exact functor from the category of P-indexed diagrams of K-modules to the category of chain complexes of K-modules. Hence, it can be prolongated, by levelwise application, to a functor Γ_P from the category of P-indexed diagrams of K-module chain complexes to the category of double chain complexes of K-modules. The resulting double complexes have rows $\Gamma_P(C_t)$, that is, $\Gamma_P(C)_{s,t} = \Gamma_P(C_t)_s$, and commuting differentials. By construction, $\Gamma_P(C)_{s,t} = 0$ for $s > 0$. Passage to Čech complexes is homotopy invariant in the following sense:

Lemma 6.2. *Let $\chi\colon \Phi \longrightarrow \Psi$ be a natural transformation of functors $\Phi, \Psi\colon P \longrightarrow$ $\mathit{Ch}\,(K\text{-Mod})$. Suppose that for each $p \in P$ the component $\chi_p\colon \Phi_p \longrightarrow \Psi_p$ is a quasi-isomorphism. Suppose also that the rank function is bounded above. Then the induced map of chain complexes*

$$\mathrm{Tot}\Gamma(\chi)\colon \mathrm{Tot}\Gamma(\Phi) \longrightarrow \mathrm{Tot}\Gamma(\Psi)$$

is also a quasi-isomorphism.

Proof. This is a standard result, we include an elementary proof for convenience. For $p \geqslant 0$ let Γ^p, denote the horizontal truncation at $-p$ of Γ. That is, $\Gamma^p(\Phi)$ is defined to be the double chain complex that agrees with $\Gamma(\Phi)$ in columns $-p, -p+1, \cdots, 0,$ and is trivial otherwise. As the rank function is bounded above, $\Gamma^p = \Gamma$ for sufficiently large p.

The obvious surjection $\Gamma^{p+1}(\Phi) \longrightarrow \Gamma^p(\Phi)$, given by identity maps in horizontal degrees $-p$ and higher, has kernel the vertical chain complex

$$K_{p+1}(\Phi) = \bigoplus_{x \in \mathrm{rk}^{-1}(p+1)} \Phi_x,$$

considered as a double chain complex concentrated in column $-(p+1)$. Its totalization is then the shift suspension $K_{p+1}(\Phi)[-p-1]$ of the chain complex K. By induction on $p \geqslant 0$, using the five lemma for ladder diagrams of long exact homology sequences induced by the diagram of short exact sequences below,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{p+1}(\Phi)[-p-1] & \longrightarrow & \mathrm{Tot}\Gamma^{p+1}(\Phi) & \longrightarrow & \mathrm{Tot}\Gamma^p(\Phi) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{p+1}(\Psi)[-p-1] & \longrightarrow & \mathrm{Tot}\Gamma^{p+1}(\Psi) & \longrightarrow & \mathrm{Tot}\Gamma^p(\Psi) & \longrightarrow & 0 \end{array}$$

the induced maps $\mathrm{Tot}\Gamma^p(\chi)\colon \mathrm{Tot}\Gamma^p(\Phi) \longrightarrow \mathrm{Tot}\Gamma^p(\Psi)$ are seen to be quasi-isomorphisms for all $p \in \mathbb{N}$. □

7. Quasi-coherent diagrams

As before, we denote by the symbol $\mathfrak{S}$ the poset of non-empty faces of the square $S = [-1,1] \times [-1,1]$. For the purpose of taking Čech complexes, we equip $\mathfrak{S}$ with the rank function $\mathrm{rk}(F) = \dim(F)$, and the usual incidence numbers coming from

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$$\begin{tikzcd}
M^{v_{tl}} \arrow[r, "\alpha^{v_{tl},e_t}"] \arrow[d, "\alpha^{v_{tl},e_l}"'] & M^{e_t} \arrow[d, "\alpha^{e_t,S}"'] & M^{v_{tr}} \arrow[l, "\alpha^{v_{tr},e_t}"'] \arrow[d, "\alpha^{v_{tr},e_r}"'] \\
M^{e_l} \arrow[r, "\alpha^{e_l,S}"] & M^{S} & M^{e_r} \arrow[l, "\alpha^{e_r,S}"'] \\
M^{v_{bl}} \arrow[r, "\alpha^{v_{bl},e_b}"] \arrow[u, "\alpha^{v_{bl},e_l}"] & M^{e_b} \arrow[u, "\alpha^{e_b,S}"] & M^{v_{br}} \arrow[l, "\alpha^{v_{br},e_b}"'] \arrow[u, "\alpha^{v_{br},e_r}"]
\end{tikzcd}$$

Figure 2. Quasi-coherent diagram.

the orientations indicated in figure 1. For example, $[e_r : v_{tr}] = 1$ and $[e_r : v_{br}] = -1$, and $[S : e_?] = 1$ for any decoration $? \in \{t, l, b, r\}$. Let R be a $\mathbb{Z}^2$-graded ring.

Definition 7.1. *A quasi-coherent diagram of modules is a functor*

$$M \colon \mathfrak{S} \longrightarrow R_{(0,0)}\text{-Mod} , \quad F \mapsto M^F$$

as depicted in figure 2. In addition, for each F, the entry M^F is to be equipped with a specified structure of an $R_*[T_F]$-module, extending the given $R_{(0,0)}$-module structure. For an inclusion of faces $F \subseteq G$, we require the structure map $\alpha^{F,G} \colon M^F \longrightarrow M^G$ to be $R_*[T_F]$-linear, such that the adjoint map

$$\alpha_\sharp^{F,G} \colon M^F \underset{R_*[T_F]}{\otimes} R_*[T_G] \longrightarrow M^G , \quad m \otimes x \mapsto \alpha^{F,G}(m) \cdot x$$

is an isomorphism of $R_*[T_G]$-modules. A quasi-coherent diagram of chain complexes is a chain complex of quasi-coherent diagrams of modules.

We remark that a quasi-coherent diagram of chain complexes can be considered as a functor defined on $\mathfrak{S}$ with values in the category of chain complexes of $R_{(0,0)}$-modules, subject to conditions as above specified levelwise. Moreover, any quasi-coherent diagram of modules can be considered as a quasi-coherent diagram of complexes concentrated in chain degree 0.

In case R is a Laurent polynomial ring in two variables with coefficients in a commutative ring K, a quasi-coherent diagram of modules is nothing but a quasi-coherent sheaf of modules on the product $\mathbb{P}^1_K \times \mathbb{P}^1_K$ of the projective line over R_K with itself.

Given a $\mathbb{Z}^2$-graded ring R and $k \in \mathbb{Z}$, we denote by $D(k)$ the diagram depicted in figure 3.

The arrows are inclusion maps. *For a strongly $\mathbb{Z}^2$-graded ring, this diagram is quasi-coherent* as the adjoint maps are isomorphisms in light of lemma 5.1. These diagrams will play a central role later on. They are the analogues of certain line

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$$\begin{tikzcd}
\begin{array}{c} R_*[kv_{tl} + T_{v_{tl}}] \\ x^{-k}y^k R_*[x, y^{-1}] \end{array} \arrow[r] \arrow[d] & \begin{array}{c} R_*[ke_t + T_{e_t}] \\ y^k R_*[x, x^{-1}, y^{-1}] \end{array} \arrow[d] & \begin{array}{c} R_*[kv_{tr} + T_{v_{tr}}] \\ x^k y^k R_*[x^{-1}, y^{-1}] \end{array} \arrow[l] \arrow[d] \\
\begin{array}{c} R_*[ke_l + T_{e_l}] \\ x^{-k} R_*[x, y, y^{-1}] \end{array} \arrow[r] & \begin{array}{c} R_*[kS + T_S] \\ R_*[x, x^{-1}, y, y^{-1}] \end{array} & \begin{array}{c} R_*[ke_r + T_{e_r}] \\ x^k R_*[x^{-1}, y, y^{-1}] \end{array} \arrow[l] \\
\begin{array}{c} R_*[kv_{bl} + T_{v_{bl}}] \\ x^{-k}y^{-k} R_*[x, y] \end{array} \arrow[r] \arrow[u] & \begin{array}{c} R_*[ke_b + T_{e_b}] \\ y^{-k} R_*[x, x^{-1}, y] \end{array} \arrow[u] & \begin{array}{c} R_*[kv_{br} + T_{v_{br}}] \\ x^k y^{-k} R_*[x^{-1}, y] \end{array} \arrow[l] \arrow[u]
\end{tikzcd}$$

Figure 3. The quasi-coherent diagram $D(k)$.

bundles on $\mathbb{P}^1_{R_{(0,0)}} \times \mathbb{P}^1_{R_{(0,0)}}$, *viz.*, the external tensor square of $\mathcal{O}(k)$. We thus are led to expect the following calculation of its 'global sections' and (trivial) 'higher cohomology' (note $H_{-n}\Gamma_{\mathfrak{S}} \cong \lim^n$ as shown, e.g., in [**5**, Corollary 2.19]):

Proposition 7.2. *For $k \geqslant 0$, the complex*

$$\Gamma_{\mathfrak{S}}\big(D(k)\big) \xleftarrow{\iota} \bigoplus_{x,y=-k}^{k} R_{(x,y)} \leftarrow 0$$

is exact, where ι is the diagonal embedding, induced from the inclusions of its source into $D(k)^v$ with v a vertex of S.

More explicitly, the sequence

$$0 \leftarrow \Gamma_{\mathfrak{S}}\big(D(k)\big)_{-2} \xleftarrow{d_{-1}} \Gamma_{\mathfrak{S}}\big(D(k)\big)_{-1} \xleftarrow{d_0} \Gamma_{\mathfrak{S}}\big(D(k)\big)_0 \xleftarrow{\iota} \bigoplus_{x,y=-k}^{k} R_{(x,y)} \leftarrow 0 \qquad (7.1)$$

is exact.

Proof. The complex consists of $\mathbb{Z}^2$-graded $R_{(0,0)}$-modules and degree-preserving maps, so exactness can be checked in each degree separately. In degree $(x, y) \in [-k, k]^2$, the complex is the dual of the augmented cellular chain complex of the square (equipped with its obvious cellular structure), which is contractible, tensored with $R_{(x,y)}$. For all other degrees (x, y), we see the dual of the cellular complex associated to the complement of a visibility subcomplex of the boundary of S. In either case, the resulting complex is acyclic. More details on the computation can be found in the appendix of §2.5 in [**6**], for example. $\square$

8. Extending chain complexes of modules to quasi-coherent diagrams of chain complexes

Suppose that M is a finitely generated free module over the $\mathbb{Z}^2$-graded ring R. Then there exists a quasi-coherent diagram which has M as its middle entry. More precisely, we fix an isomorphism between M and R^n, for some $n \geqslant 0$, and have $(\bigoplus_n D(k))^S \cong M$ for any $k \in \mathbb{Z}$. This shows that finitely generated free R-modules can be extended to quasi-coherent diagrams. We need the following chain complex version of this fact:

Proposition 8.1. *Let R be a $\mathbb{Z}^2$-graded ring, and let C be a bounded above chain complex of finitely generated free R-modules. Then there exists a complex $\mathcal{Y}$ of diagrams of the form $\bigoplus_r D(k)$, for various $r \geqslant 0$, such that $C \cong \mathcal{Y}^S$ as R-module complexes. More precisely, suppose that C is concentrated in degrees $\leqslant t$. There are numbers*

$$0 \leqslant k_t \leqslant k_{t-1} \leqslant k_{t-2} \leqslant \dots$$

and numbers $r_n \geqslant 0$, with $r_n = 0$ whenever $C_n = 0$, such that we can choose $\mathcal{Y}_n = \bigoplus_{r_n} D(k_n)$. If C is bounded, then so is $\mathcal{Y}$. If the ring R is strongly $\mathbb{Z}^2$-graded, then $\mathcal{Y}$ consists of quasi-coherent diagrams.

Proof. We identify the chain module C_n with R^{r_n}, for suitable numbers $r_n \geqslant 0$, where $r_n = 0$ if and only if $C_n = 0$; this amounts to choosing basis elements for the free modules C_n. The action of the differential $C_n \longrightarrow C_{n-1}$ is then given by left multiplication by a matrix D_n of size $r_{n-1} \times r_n$, with entries in R.

Given a homogeneous non-zero element $x \in R_{(s,t)}$, we write $a(x) = \max(|s|, |t|)$; we also agree $a(0) = 0$. For a general element $x \in R$, let $a(x)$ denote the maximum of the values $a(x_i)$, where the x_i are the homogeneous components of x. We denote by a_n the maximum of the values $a(x)$ where x varies over the entries of the matrix D_n, with the convention that $a_n = 0$ if D_n is the empty matrix (i.e., if $r_n = 0$ or $r_{n-1} = 0$).

Suppose that C is concentrated in chain degrees $\leqslant t$. We set $k_n = 0$ for $n \geqslant t$, and

$$k_n = \sum_{j=n+1}^{t} a_j$$

for $k < t$. We then define the chain complex $\mathcal{Y}$ of quasi-coherent diagrams by setting $\mathcal{Y}_n = \bigoplus_{r_n} D(k_n)$; the differentials $d_n \colon \mathcal{Y}_n \longrightarrow \mathcal{Y}_{n-1}$ are given by left multiplication by the matrix D_n in each component, so that

$$d_n^F \colon \mathcal{Y}_n^F = \Big(\bigoplus_{r_n} D(k_n)\Big)^F \xrightarrow{D_n} \Big(\bigoplus_{r_{n-1}} D(k_{n-1})\Big)^F = \mathcal{Y}_{n-1}^F.$$

(This is well defined: The numbers k_n are chosen precisely to ensure that d_n^F maps $\mathcal{Y}_n^F$ to $\mathcal{Y}_{n-1}^F$.) As $D_{n-1} \cdot D_n = 0$, we have $d_{n-1} \circ d_n = 0$, and $C = \mathcal{Y}^S$ by construction.

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It was observed before that the diagrams $D(k)$ are quasi-coherent if R is strongly $\mathbb{Z}^2$-graded (this follows from lemma 5.1); hence $\mathcal{Y}$ consists of quasi-coherent diagrams in this case. □

COROLLARY 8.2. *Let R, C, $\mathcal{Y}$, r_n and k_n be as in proposition 8.1, with C bounded. Then $D = \lim \mathcal{Y}$ is a bounded complex of $R_{(0,0)}$-modules with chain modules*

$$D_n = \Big(\bigoplus_{x,y=-k_n}^{k_n} R_{(x,y)} \Big)^{r_n} .$$

If R is strongly $\mathbb{Z}^2$-graded, these $R_{(0,0)}$-modules are finitely generated projective. The natural map $D \longrightarrow \operatorname{Tot} \Gamma_{\mathfrak{S}}(\mathcal{Y})$ is a homotopy equivalence of $R_{(0,0)}$-module complexes so that $\operatorname{Tot} \Gamma_{\mathfrak{S}}(\mathcal{Y})$ is $R_{(0,0)}$-finitely dominated.

Proof. Consider the double complex concentrated in columns -2, -1, 0 and 1 with nth row given by

$$\Gamma_{\mathfrak{S}}(\mathcal{Y}_n) \xleftarrow{\iota} \Big(\bigoplus_{x,y=-k_n}^{k_n} R_{(x,y)} \Big)^{r_n} .$$

Column 1 is the chain complex $\lim \mathcal{Y}$, see proposition 7.2, which consists of finitely generated projective $R_{(0,0)}$-modules if R is strongly graded by corollary 4.6. As the rows are exact (by proposition 7.2 again), standard homological algebra asserts that the natural map $\lim \mathcal{Y} \longrightarrow \operatorname{Tot} \Gamma(\mathcal{Y})$ is a quasi-isomorphism. In the strongly graded case, this is a homotopy equivalence (all complexes are bounded below and consist of projective $R_{(0,0)}$-modules) so that $\operatorname{Tot} \Gamma(\mathcal{Y})$ is $R_{(0,0)}$-finitely dominated. □

9. Flags and stars, and their associated rings

DEFINITION 9.1. *For $F \in \mathfrak{S}$, define the star of F, denoted $\operatorname{st}(F)$, as the set of faces of S that contain F; this is a sub-poset of $\mathfrak{S}$. Let $\mathcal{N}_F$ be the set of flags in $\operatorname{st}(F)$; an element $\tau = \langle P_0, P_1, \cdots, P_k \rangle$ of $\mathcal{N}_F$ is a strictly increasing sequence $P_0 \subset P_1 \subset \ldots \subset P_k$ of faces of S with $F \subseteq P_0$. We consider $\mathcal{N}_F$ as a poset with order given by inclusion (or refinement) of flags, the smaller flag in the partial order having fewer elements.*

For each $F \in \mathfrak{S}$, we equip $\mathcal{N}_F$ with the rank function

$$\operatorname{rk}\big(\langle P_0, P_1, \cdots, P_k \rangle\big) = k$$

and with standard simplicial incidence numbers: $[\tau : \sigma] = 0$ unless the flag σ is obtained from $\tau = \langle P_0, P_1, \cdots, P_k \rangle$ by omitting the entry P_j, in which case $[\tau : \sigma] = (-1)^j$. Whenever we form a ČECH complex of a diagram indexed by $\mathcal{N}_F$, we will use these data.

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We now attach a ring $A\langle P_0, P_1, \cdots, P_k\rangle$ to each flag $\langle P_0, P_1, \cdots, P_k\rangle$ of faces in $\mathfrak{S}$, as follows:

$$\begin{array}{lcl} A\langle v_{bl}\rangle & = & R_*[[x, y]] \\ A\langle e_b\rangle & = & R_*[x, x^{-1}][[y]] \\ A\langle S\rangle & = & R_*[x, x^{-1}, y, y^{-1}] = R \\ A\langle v_{bl}, e_b\rangle & = & R_*((x))[[y]] \\ A\langle v_{bl}, S\rangle & = & R_*((x, y)) \\ A\langle e_b, S\rangle & = & R_*[x, x^{-1}]((y)) \\ A\langle v_{bl}, e_b, S\rangle & = & R_*((x))((y)) \end{array}$$

The effect of replacing 'left' by 'right' (*i.e.*, replacing the subscript 'l' by 'r' throughout) is to replace x by x^{-1}, while replacing 'bottom' by 'top' (*i.e.*, replacing the subscript 'b' by 't' throughout) means replacing y by y^{-1}. Swapping 'bottom' and 'left' amounts to swapping x with y, and swapping 'top' and 'right' results in x^{-1} and y^{-1} swapping places. Thus, for example,

$$A\langle v_{tl}, e_l\rangle = R_*((y^{-1}))[[x]] \quad \text{and} \quad A\langle v_{tr}, e_r, S\rangle = R_*((y^{-1}))((x^{-1})) \ .$$

By direct inspection, this collection of rings is seen to have the following properties:

(1) If $\sigma \subseteq \tau$, then $A\langle\sigma\rangle \subseteq A\langle\tau\rangle$. In particular, if $S \in \tau$, then $R = R_*[x, x^{-1}, y, y^{-1}] = A\langle S\rangle$ is a subring of $A\langle\tau\rangle$.

(2) If $\tau \in \mathcal{N}_F$ (*i.e.*, if all faces in τ contain F), then $A_F \subseteq A\langle\tau\rangle$.

Fix $F \in \mathfrak{S}$. The rings $A\langle\tau\rangle$ defined above fit into a commutative diagram

$$E_F \colon \mathcal{N}_F \longrightarrow A_F\text{-Mod} \ , \quad \tau \mapsto A\langle\tau\rangle$$

of A_F-modules with structure maps given by inclusions. For $F = S$, the diagram E_S has a single entry indexed by the flag $\langle S\rangle$, and takes value $A\langle S\rangle = R$ there. For $F = e_l$, the left-hand edge of S, the diagram E_{e_l} looks like this:

$$\begin{array}{ccccc} R_*[y, y^{-1}][[x]] & \longrightarrow & R_*[y, y^{-1}]((x)) & \longleftarrow & R_*[x, x^{-1}, y, y^{-1}] \\ A\langle e_l\rangle & & A\langle e_l, S\rangle & & A\langle S\rangle \end{array} \tag{9.1}$$

And finally, for $F = v_{bl}$ the bottom left vertex of S, the diagram $E_{v_{bl}}$ is depicted in figure 4.

We let $\Gamma_{\mathcal{N}_F}(E_F)$ denote the Čech complex of E_F. Considering A_F as a chain complex concentrated in chain level 0, we have a chain map

$$\xi^F \colon A_F \longrightarrow \Gamma_{\mathcal{N}_F}(E_F)$$

given by the diagonal embedding $A_F \longrightarrow \bigoplus_{G \supseteq F} A\langle G\rangle$. This is indeed a chain map thanks to property (DI3) of incidence numbers.

Lemma 9.2. *The map ξ^F is a quasi-isomorphism of chain complexes of A_F-modules.*

$$\begin{tikzcd}
\begin{array}{c} A\langle e_l\rangle \\ R_*[y,y^{-1}][[x]] \end{array} \arrow[rr] \arrow[dd] & & \begin{array}{c} A\langle e_l,S\rangle \\ R_*[y,y^{-1}]((x)) \end{array} \arrow[dl] & & \begin{array}{c} A\langle S\rangle \\ R_*[x,x^{-1},y,y^{-1}] \end{array} \arrow[ll] \arrow[ddll] \arrow[dd] \\
& \begin{array}{c} A\langle v_{bl},e_l,S\rangle \\ R_*((y))((x)) \end{array} & & & \\
\begin{array}{c} A\langle v_{bl},e_l\rangle \\ R_*((y))[[x]] \end{array} \arrow[ur] & & \begin{array}{c} A\langle v_{bl},S\rangle \\ R_*((x,y)) \end{array} \arrow[ul] \arrow[dr] & & \begin{array}{c} A\langle e_b,S\rangle \\ R_*[x,x^{-1}]((y)) \end{array} \arrow[dl] \\
& & & \begin{array}{c} A\langle v_{bl},e_b,S\rangle \\ R_*((x))((y)) \end{array} & \\
\begin{array}{c} A\langle v_{bl}\rangle \\ R_*[[x,y]] \end{array} \arrow[uu] \arrow[uurr] \arrow[rr] & & \begin{array}{c} A\langle v_{bl},e_b\rangle \\ R_*((x))[[y]] \end{array} \arrow[ur] & & \begin{array}{c} A\langle e_b\rangle \\ R_*[x,x^{-1}][[y]] \end{array} \arrow[ll] \arrow[uu]
\end{tikzcd}$$

Figure 4. The diagram $E_{v_{bl}}$.

Proof. Note that for $F = S$, there is actually nothing to show as the category $\mathcal{N}_S$ has a single object; for $F = e_l$, the claim is equivalent to saying that the sequence

$$0 \longrightarrow R_*[x,\, y,\, y^{-1}] \xrightarrow{\binom{1}{1}} R_*[y,\, y^{-1}][[x]] \oplus R_*[x,\, x^{-1},\, y,\, y^{-1}]$$
$$\xrightarrow{(1\ -1)} R_*[y,\, y^{-1}]((x)) \longrightarrow 0$$

is exact, and it is not too hard to see that it is actually split exactly as a sequence of $R_{(0,0)}$-modules. The case of $F = v_?$ a vertex is somewhat more demanding in terms of book-keeping. Full details are worked out in the proof of Lemma 4.5.1 in [**3**], writing $R_*[x,x^{-1}]((y))$ in place of $R[x,x^{-1}]((y))$ (and similar for the other rings); the proof carries over *mutatis mutandis* since the claim can be checked in each degree separately. □

For faces $G \supsetneqq F$, let E_F^G denote the $\mathcal{N}_F$-indexed diagram which agrees with E_G on $\mathcal{N}_G$, and is zero everywhere else. The obvious map of diagrams $E_F \longrightarrow E_F^G$, given by identities where possible, results in a map of chain complexes

$$\Gamma_{\mathcal{N}_F}(E_F) \longrightarrow \Gamma_{\mathcal{N}_F}(E_F^G) = \Gamma_{\mathcal{N}_G}(E_G)\ ,$$

given by projecting away from all those summands in the source which involve flags in $\mathcal{N}_F \setminus \mathcal{N}_G$. These maps are functorial on the poset $\mathfrak{S}$ so that there results a commutative diagram

$$E\colon \mathfrak{S} \longrightarrow R_{(0,0)}\text{-Mod}\ , \quad F \mapsto \Gamma_{\mathcal{N}_F}(E_F)\ .$$

Proposition 9.3. *The maps ξ^F constructed above assemble to a natural transformation*

$$\xi\colon D(0) \longrightarrow E.$$

Its components $\xi^F\colon A_F \longrightarrow \Gamma_{\mathcal{N}_F}(E_F)$ are quasi-isomorphisms of complexes of A_F-modules.

Proof. This is the content of lemma 9.2, together with the observation that the maps ξ^F are natural in F with respect to the structure maps in E. □

Given a flat A_F-module M_F, we thus obtain a quasi-isomorphism

$$\chi^F\colon M_F \xrightarrow{\cong} M_F \underset{A_F}{\otimes} A_F \xrightarrow{\mathrm{id}\otimes\xi^F} M_F \underset{A_F}{\otimes} \Gamma_{\mathcal{N}_F}(E_F) \cong \Gamma_{\mathcal{N}_F}(M_F \underset{A_F}{\otimes} E_F)\ ;$$

here $M_F \otimes_{A_F} E_F$ stands for the pointwise tensor product of the module M_F with the entries of the diagram E_F. In other words, the sequence

$$\begin{aligned} 0 \longrightarrow M_F \longrightarrow \Gamma_{\mathcal{N}_F}(M_F \underset{A_F}{\otimes} E_F)_0 \longrightarrow \Gamma_{\mathcal{N}_F}(M_F \underset{A_F}{\otimes} E_F)_{-1} \\ \longrightarrow \Gamma_{\mathcal{N}_F}(M_F \underset{A_F}{\otimes} E_F)_{-2} \longrightarrow 0 \end{aligned} \tag{9.2}$$

is exact.

For M an arbitrary quasi-coherent diagram, we obtain a natural transformation

$$\chi\colon M \xrightarrow{\cong} M \underset{D(0)}{\otimes} D(0) \longrightarrow M \underset{D(0)}{\otimes} E \xrightarrow{\cong} M' \tag{9.3}$$

where the target M' is the diagram of chain complexes

$$M'\colon F \mapsto \Gamma_{\mathcal{N}_F}(M_F \underset{A_F}{\otimes} E_F),$$

$M \otimes_{D(0)} D(0)$ denotes the pointwise tensor product

$$M \underset{D(0)}{\otimes} D(0)\colon F \mapsto M_F \underset{A_F}{\otimes} D(0)^F = M_F \underset{A_F}{\otimes} A_F\ ,$$

and $M \otimes_{D(0)} E$ stands similarly for the pointwise tensor product of the diagrams M and E. If moreover M is such that the entry M_F is a flat A_F-module, the components of χ are quasi-isomorphisms.

10. From trivial Novikov homology to finite domination

We will now prove the 'if' implication of theorem 2.1. So let R be a strongly $\mathbb{Z}^2$-graded ring. Suppose that C is a bounded complex of finitely generated free R-modules, and further that the complexes listed in (2.2a) and (2.2b) are acyclic.

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In view of the assumed freeness of C, these eight complexes are then actually contractible. They are of the form

$$C \underset{R}{\otimes} A\langle e, S\rangle \qquad \text{and} \qquad C \underset{R}{\otimes} A\langle v, S\rangle$$

where e and v denote an edge and a vertex of S, respectively. As tensor products preserve contractions, it follows that for all eight maximal flags $\langle v, e, S\rangle$, for v a vertex of S and e an edge incident to v, the complex

$$C \underset{R}{\otimes} A\langle v, e, S\rangle \cong C \underset{R}{\otimes} A\langle v, S\rangle \underset{A\langle v,S\rangle}{\otimes} A\langle v, e, S\rangle \simeq 0 \tag{10.1}$$

is contractible (and hence acyclic) as well.

Extend the complex C to a complex of sheaves $\mathcal{Y}\colon F \mapsto \mathcal{Y}_F$, which can be done by proposition 8.1. In more detail, this means that we find a complex of sheaves such that C is identified with $\mathcal{Y}^S$, and such that there is a quasi-isomorphism $\lim \mathcal{Y} \longrightarrow \operatorname{Tot}\Gamma_{\mathfrak{S}}(\mathcal{Y})$ with $R_{(0,0)}$-finitely dominated source (corollary 8.2).

Let, for the moment, $F \neq \emptyset$ denote a fixed face of S. We observe that if $F \neq S$,

$$\mathcal{Y}_F \underset{A_F}{\otimes} A\langle F, S\rangle \cong \mathcal{Y}_F \underset{A_F}{\otimes} A_S \underset{A_S}{\otimes} A\langle F, S\rangle \underset{(\dagger)}{\cong} C \underset{R}{\otimes} A\langle F, S\rangle \simeq 0\ . \tag{10.2}$$

The isomorphism labelled (†) combines two facts: first, the entries of the complex $\mathcal{Y}$ are quasi-coherent diagrams (proposition 8.1) so that $\mathcal{Y}_F \otimes_{A_F} A_S \cong \mathcal{Y}^S$; second, by construction of $\mathcal{Y}$ there is an identification of $\mathcal{Y}^S$ with C.

If $F = v$ is a vertex and $e \supseteq v$ an edge incident to e, we make use of (10.1) to conclude similarly

$$\mathcal{Y}_v \underset{A_v}{\otimes} A\langle v, e, S\rangle \cong \mathcal{Y}_v \underset{A_v}{\otimes} A_S \underset{A_S}{\otimes} A\langle v, e, S\rangle \cong C \underset{R}{\otimes} A\langle v, e, S\rangle \simeq 0\ . \tag{10.3}$$

For arbitrary $F \neq \emptyset$, exactness of the sequence (9.2) implies that the double complex

$$\mathcal{Y}_F \longrightarrow \Gamma_{\mathcal{N}_F}(\mathcal{Y}_F \underset{A_F}{\otimes} E_F)$$

has exact rows (using the fact that $\mathcal{Y}_F$ consists of projective A_F-modules), so there results a quasi-isomorphism

$$\operatorname{Tot}(\chi^F)\colon \mathcal{Y}_F \longrightarrow \operatorname{Tot}\Gamma_{\mathcal{N}_F}(\mathcal{Y}_F \underset{A_F}{\otimes} E_F).$$

We will now make use of the fact that some entries of the diagram $\mathcal{Y}_F \otimes_{A_F} E_F$ are known to be acyclic. Let Z_F denote the $\mathcal{N}_F$-indexed diagram which agrees with $\mathcal{Y}_F \otimes_{A_F} E_F$ on those flags not containing S, and is zero elsewhere. Let K_F denote the diagram which takes the value C at $\{S\}$, and is zero otherwise. The obvious surjective map $\mathcal{Y}_F \otimes_{A_F} E_F \longrightarrow Z_F \oplus K_F$ is a pointwise quasi-isomorphism,

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by (10.2) and (10.3), and hence induces a quasi-isomorphism

$$\operatorname{Tot}\Gamma_{\mathcal{N}_F}(\mathcal{Y}_F \underset{A_F}{\otimes} E_F) \xrightarrow{\simeq} \operatorname{Tot}\Gamma_{\mathcal{N}_F}(Z_F \oplus K_F).$$

We compute further that the target of this map is

$$\operatorname{Tot}\Gamma_{\mathcal{N}_F}(Z_F) \oplus \operatorname{Tot}\Gamma_{\mathcal{N}_F}(K_F) = \operatorname{Tot}\Gamma_{\mathcal{N}_F}(Z_F) \oplus C.$$

In total, this yields a quasi-isomorphism

$$\Xi^F \colon \mathcal{Y}_F \longrightarrow \operatorname{Tot}\Gamma_{\mathcal{N}_F}(Z_F) \oplus C.$$

Allowing F to vary again, and making use of the naturality of the constructions above, we see that we obtain a natural quasi-isomorphism of diagrams

$$\Xi \colon \mathcal{Y} \longrightarrow \operatorname{Tot}\Gamma_{\mathcal{N}_{(-)}}(Z_{(-)}) \oplus \operatorname{con}(C),$$

with the $\mathfrak{S}$-indexed diagram

$$\Gamma_{\mathcal{N}_{(-)}}(Z_{(-)}) \colon F \mapsto \Gamma_{\mathcal{N}_F}(Z_F)$$

and the constant diagram

$$\operatorname{con}(C) \colon F \mapsto C\ .$$

It follows that $\operatorname{Tot}\Gamma_{\mathfrak{S}}(\mathcal{Y})$, which is an $R_{(0,0)}$-finitely dominated by corollary 8.2, is quasi-isomorphic to

$$\operatorname{Tot}\Gamma_{\mathfrak{S}}\big(\operatorname{Tot}\Gamma_{\mathcal{N}_{(-)}}(Z_{(-)})\big) \oplus \operatorname{Tot}\Gamma_{\mathfrak{S}}\big(\operatorname{con}(C)\big)\ .$$

The second summand is, in turn, quasi-isomorphic to C. (In effect, this is true since the nerve of $\mathfrak{S}$ is contractible; for a more explicit argument, observe that the double complex

$$\Gamma_{\mathfrak{S}}\big(\operatorname{con}(C)\big) \longleftarrow C$$

concentrated in columns -2, -1, 0 and 1 has acyclic rows so that its totalization is acyclic. But the totalization is, up to isomorphism, the mapping cone of the map $C \longrightarrow \operatorname{Tot}\Gamma_{\mathfrak{S}}\big(\operatorname{con}(C)\big)$ which is thus a weak equivalence. See also [**3**, Lemma 4.6.4].)

Thus in the derived category of the ring $R_{(0,0)}$, the complex C is a retract of $\operatorname{Tot}\Gamma_{\mathfrak{S}}(\mathcal{Y})$, and as both complexes are bounded and consist of projective $R_{(0,0)}$-modules, this makes C a retract up to homotopy of $\operatorname{Tot}\Gamma_{\mathfrak{S}}(\mathcal{Y})$. As the latter is $R_{(0,0)}$-finitely dominated so is C, as was to be shown.

Part III. Finite domination implies triviality of Novikov homology

11. Algebraic tori

DEFINITION 11.1. *Let $R = \bigoplus_{\sigma\in\mathbb{Z}^2} R_\sigma$ be a strongly $\mathbb{Z}^2$-graded unital ring. Given an R-module M and an element $\rho \in \mathbb{Z}^2$, we define the map of right R-modules*

$$\chi_\rho\colon M \underset{R_{(0,0)}}{\otimes} R \longrightarrow M \underset{R_{(0,0)}}{\otimes} R\ , \qquad m\otimes r \mapsto \sum_j m u_j \otimes v_j r\ ,$$

where $1 = \sum_j u_j v_j$ is any partition of unity of type $(-\rho, \rho)$.

The map χ_ρ is $R_{(0,0)}$-balanced and independent of the choice of partition of unity as its restriction to $M \otimes_{R_{(0,0)}} R_\sigma$ can be re-written as

$$M \underset{R_{(0,0)}}{\otimes} R_\sigma \xrightarrow[\mu_{-\rho,\sigma+\rho}]{\cong} M \underset{R_{(0,0)}}{\otimes} R_{-\rho} \underset{R_{(0,0)}}{\otimes} R_{\sigma+\rho} \xrightarrow[\omega]{} M \underset{R_{(0,0)}}{\otimes} R$$

with $\mu_{-\rho,\sigma+\rho}$ the $R_{(0,0)}$-bimodule isomorphism from proposition 4.3, and with $\omega(m\otimes x\otimes y) = (mx)\otimes y$. In case $M = R$, the map χ_ρ is an R-bimodule homomorphism.

Of course $\chi_{(0,0)} = \mathrm{id}$. As χ_ρ does not depend on the choice of partition of unity, lemma 4.2 implies:

COROLLARY 11.2. *For all $\rho, \sigma \in \mathbb{Z}^2$, there are equalities of maps $\chi_\rho\chi_\sigma = \chi_{\rho+\sigma} = \chi_\sigma\chi_\rho$. The maps χ_ρ are isomorphisms of R-modules with $\chi_\rho^{-1} = \chi_{-\rho}$.*

We can consider

$$M \underset{R_{(0,0)}}{\otimes} R = \bigoplus_{\rho\in\mathbb{Z}^2} M \underset{R_{(0,0)}}{\otimes} R_\rho$$

as a $\mathbb{Z}^2$-graded R-module. With respect to this grading, we observe:

LEMMA 11.3. *The map χ_ρ maps homogeneous elements of degree σ to homogeneous elements of degree $\rho+\sigma$. For the homogeneous element $m\otimes r$ of degree $-\rho$, the formula $\chi_\rho(m\otimes r) = mr\otimes 1$ holds.*

Let C be a chain complex of R-modules. Then χ_ρ, applied in each chain level, defines a chain map

$$\chi_\rho\colon C \underset{R_{(0,0)}}{\otimes} R \longrightarrow C \underset{R_{(0,0)}}{\otimes} R.$$

Let D be an additional $R_{(0,0)}$-chain complex, let $\alpha\colon C \longrightarrow D$ and $\beta\colon D \longrightarrow C$ be $R_{(0,0)}$-linear chain maps, and let $H\colon \mathrm{id}_C \simeq \beta\alpha$ be a homotopy from $\beta\alpha$ to id such

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that $dH + Hd = \beta\alpha - \mathrm{id}_C$. Then the outer square in the diagram $T(\alpha, \beta; H)$

$$\begin{tikzcd}
D \underset{R_{(0,0)}}{\otimes} R \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"] \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"'] \arrow[dr, "{\alpha^*\chi_{e_1}H^*\chi_{e_2}\beta^* - \alpha^*\chi_{e_2}H^*\chi_{e_1}\beta^*}"] & D \underset{R_{(0,0)}}{\otimes} R \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"] \\
D \underset{R_{(0,0)}}{\otimes} R \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"'] & D \underset{R_{(0,0)}}{\otimes} R
\end{tikzcd} \qquad T(\alpha, \beta; H)$$

(writing $f^* = f \otimes \mathrm{id}$, for any map f) is homotopy commutative, with the diagonal arrow recording a preferred homotopy of the two possible compositions. Note that if $\beta\alpha = \mathrm{id}_C$, we can choose $H = 0$ and in this case the outer squares of $T(\alpha, \beta; 0)$ commute by corollary 11.2.

Definition 11.4. *The algebraic torus $\mathfrak{T}(\alpha, \beta; H)$ is defined as the totalization of the homotopy commutative diagram $T(\alpha, \beta; H)$. That is, $\mathfrak{T}(\alpha, \beta; H)$ is the complex with chain modules*

$$\begin{aligned} &\mathfrak{T}(\alpha, \beta; H)_n \\ &\quad = (D_{n-2} \underset{R_{(0,0)}}{\otimes} R) \oplus (D_{n-1} \underset{R_{(0,0)}}{\otimes} R) \oplus (D_{n-1} \underset{R_{(0,0)}}{\otimes} R) \oplus (D_n \underset{R_{(0,0)}}{\otimes} R) \end{aligned}$$

and boundary $d_{\mathfrak{T}(\alpha,\beta;H)}$ given by the following matrix:

$$\begin{pmatrix} d^* & 0 & 0 & 0 \\ \mathrm{id}^* - \alpha^*\chi_{e_1}\beta^* & -d^* & 0 & 0 \\ \mathrm{id}^* - \alpha^*\chi_{e_2}\beta^* & 0 & -d^* & 0 \\ \alpha^*(\chi_{e_1}H^*\chi_{e_2} - \chi_{e_2}H^*\chi_{e_1})\beta^* & \mathrm{id}^* - \alpha^*\chi_{e_2}\beta^* & -\mathrm{id}^* + \alpha^*\chi_{e_1}\beta^* & d^* \end{pmatrix}$$

12. Canonical resolutions

Let C be a chain complex of R-modules. The commutative diagram

$$\begin{tikzcd}
C \underset{R_{(0,0)}}{\otimes} R \arrow[r, "\mathrm{id}^* - \chi_{e_1}"] \arrow[d, "\mathrm{id}^* - \chi_{e_2}"'] & C \underset{R_{(0,0)}}{\otimes} R \arrow[d, "\mathrm{id}^* - \chi_{e_2}"] \\
C \underset{R_{(0,0)}}{\otimes} R \arrow[r, "\mathrm{id}^* - \chi_{e_1}"] & C \underset{R_{(0,0)}}{\otimes} R
\end{tikzcd} \qquad T(\mathrm{id}_C, \mathrm{id}_C; 0)$$

gives rise, *via* totalization, to the algebraic torus $\mathfrak{T}(\mathrm{id}_C, \mathrm{id}_C; 0)$. The R-module structure map $\gamma\colon (x, r) \mapsto xr$ induces a map of R-module chain complexes

$$\kappa\colon \mathfrak{T}(\mathrm{id}_C, \mathrm{id}_C; 0) \longrightarrow C\ .$$

PROPOSITION 12.1 Canonical resolution. *The map κ is a quasi-isomorphism. If C is a bounded below complex of projective R-modules, then κ is a chain homotopy equivalence.*

In preparation of the proof, we note that $\mathfrak{T}(\mathrm{id}_C, \mathrm{id}_C; 0)$ can be described as the totalization (in the usual sense) of a double complex of the form

$$C \underset{R_{(0,0)}}{\otimes} R \longrightarrow \Big(C \underset{R_{(0,0)}}{\otimes} R\Big)^2 \longrightarrow C \underset{R_{(0,0)}}{\otimes} R\,. \tag{12.1}$$

The R-module structure map $\gamma\colon (x,r) \mapsto xr$ of C can be used to augment this to the R-module double complex

$$0 \to C \underset{R_{(0,0)}}{\otimes} R \to \Big(C \underset{R_{(0,0)}}{\otimes} R\Big)^2 \to C \underset{R_{(0,0)}}{\otimes} R \xrightarrow{\gamma} C \to 0\,. \tag{12.2}$$

We will consider a single row of this double complex, that is, a sequence of the form

$$0 \longrightarrow M \underset{R_{(0,0)}}{\otimes} R \xrightarrow{\alpha} \Big(M \underset{R_{(0,0)}}{\otimes} R\Big)^2 \xrightarrow{\beta} M \underset{R_{(0,0)}}{\otimes} R \xrightarrow{\gamma} M \longrightarrow 0 \tag{12.3}$$

where M is a right R-module. Here the maps α and β are given by the matrices

$$\alpha = \begin{pmatrix} \mathrm{id} - \chi_{e_1} \\ \mathrm{id} - \chi_{e_2} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathrm{id} - \chi_{e_2} & -(\mathrm{id} - \chi_{e_1}) \end{pmatrix}\,.$$

LEMMA 12.2. *The complex (12.3) is exact.*

Proof. We allow M to be an arbitrary $R_{(0,0)}$-module initially. We have the direct sum decomposition $M \otimes_{R_{(0,0)}} R = \bigoplus_{(i,j)} M \otimes_{R_{(0,0)}} R_{(i,j)}$; in fact, $M \otimes_{R_{(0,0)}} R$ is a $\mathbb{Z}^2$-graded R-module in this way. Any element $z \in M \otimes_{R_{(0,0)}} R$ can be expressed uniquely in the form

$$z = \sum_{i,j \in \mathbb{Z}} m_{i,j} \quad \text{where } m_{i,j} \in M \underset{R_{(0,0)}}{\otimes} R_{(i,j)}\,,$$

such that $m_{i,j} = 0$ for almost all pairs (i,j). We say that z *has x-amplitude in the interval* $[a,b]$ if $m_{i,j} = 0$ for $i \notin [a,b]$. The *support of* z is the (finite) set of all pairs (i,j) with $m_{i,j} \neq 0$.

After these initial comments, we proceed to verify exactness of the sequence. We remark first that γ is surjective since $\gamma(p \otimes 1) = p$. As a matter of fact, $\sigma(p) = p \otimes 1$ defines an $R_{(0,0)}$-linear section σ of γ.

Next, we show that α is injective. Let $z = \sum_{i,j\in\mathbb{Z}} m_{i,j}$ be an element of $\ker(\alpha)$. Since $\chi_{e_2}(m_{i,j}) \in M \otimes_{R_{(0,0)}} R_{(i,j+1)}$, the equality

$$\alpha(z) = \begin{pmatrix} \displaystyle\sum_{i,j\in\mathbb{Z}} \big(m_{i,j} - \chi_{e_1}(m_{i,j})\big) \\ \displaystyle\sum_{i,j\in\mathbb{Z}} \big(m_{i,j} - \chi_{e_2}(m_{i,j})\big) \end{pmatrix} = 0$$

implies, by considering the homogeneous component of degree (i,j) of the second entry, that $m_{i,j} - \chi_{e_2}(m_{i,j-1}) = 0$. If $z \neq 0$, there exists (i,j) in the support of z

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such that $m_{i,j-1} = 0$. For these indices, we have $m_{i,j} = \chi_{e_2}(m_{i,j-1}) = \chi_{e_2}(0) = 0$, a contradiction. This enforces $z = 0$ whence α is injective.

To show that $\operatorname{im}\alpha = \ker\beta$, we will take an element (z_1, z_2) of $\ker\beta$ and show that it can be reduced to 0 by subtracting a sequence of elements of $\operatorname{im}\alpha \subseteq \ker\beta$. This clearly implies that $(z_1, z_2) \in \operatorname{im}\alpha$ as required.

So let $(z_1, z_2) \in \ker\beta$, where

$$z_1 = \sum_{i,j\in\mathbb{Z}} m_{i,j} \quad \text{and} \quad z_2 = \sum_{i,j\in\mathbb{Z}} n_{i,j}\ , \quad \text{with } m_{i,j},\, n_{i,j} \in M \underset{R_{(0,0)}}{\otimes} R_{(i,j)}.$$

Choose integers a, b and k such that z_1 has x-amplitude in $[a, k]$ and z_2 has x-amplitude in $[a, b]$. If $k > a$, we define

$$u = \sum_{j\in\mathbb{Z}} \chi_{-e_1}(m_{k,j}) \in M \underset{R_{(0,0)}}{\otimes} R\ , \quad \text{with } \chi_{-e_1}(m_{k,j}) \in M \underset{R_{(0,0)}}{\otimes} R_{(k-1,j)},$$

and set $(z'_1, z'_2) = (z_1, z_2) - \alpha(u)$. The (k, j)-homogeneous component of z'_1 vanishes by construction of u and corollary 11.2, so that z'_1 has x-amplitude in $[a, k-1]$ while the x-amplitude of z_2 remains in $[a, b]$. The element (z'_1, z'_2) will be in $\operatorname{im}\alpha$ if and only if $(z_1, z_2) \in \operatorname{im}\alpha$.

By iteration, we may thus assume that our initial pair (z_1, z_2) is such that the x-amplitude of z_1 is in $\{a\} = [a, a]$. This actually necessitates $z_2 = 0$. To see this, assume $z_2 \neq 0$. We can then choose (i, j) in the support of z_2 (so that in particular $i \geqslant a$) such that $(i + 1, j)$ is not in the support of z_2, i.e., such that $n_{i+1,j} = 0$. Since $\beta(z_1, z_2) = 0$ we have, by considering the homogeneous component of degree $(i + 1, j)$, the equality

$$m_{i+1,j} - \chi_{e_2}(m_{i+1,j-1}) - n_{i+1,j} + \chi_{e_1}(n_{i,j}) = 0\ .$$

But the first three terms vanish, by choice of i and our hypothesis on z_1, so that $\chi_{e_1}(n_{i,j}) = 0$ whence $n_{i,j} = 0$, contradicting the choice of i.

Thus, $z_2 = 0$. This in turn implies that $\beta(z_1, 0) = 0$ so that $m_{a,\ell} - \chi_{e_2}(m_{a,\ell-1}) = 0$ for any ℓ. If z_1 is non-zero, we let ℓ be minimal with $m_{a,\ell} \neq 0$. Then $m_{a,\ell} = \chi_{e_2}(m_{a,\ell-1}) = \chi_{e_2}(0) = 0$, a contradiction. We conclude that $(z_1, z_2) = 0 \in \operatorname{im}\alpha$ as required.

To show $\operatorname{im}\beta = \ker\gamma$, it is enough to produce an $R_{(0,0)}$-linear map

$$\varphi\colon M \underset{R_{(0,0)}}{\otimes} R \longrightarrow \Big(M \underset{R_{(0,0)}}{\otimes} R\Big)^2$$

such that $\beta\varphi + \sigma\gamma = \mathrm{id}$, where σ is the section of γ given by $\sigma(p) = p \otimes 1$. It is enough to define φ on homogeneous primitive tensors $x = m \otimes r$ with $r \in R_\rho$, for the various $\rho \in \mathbb{Z}^2$.

- If $a, b \geqslant 0$:

$$\varphi(x) = \Big(-\sum_{k=1}^{b} \chi_{(0,-k)}(x) \qquad \sum_{\ell=1}^{a} \chi_{(-\ell,-b)}(x) \Big)$$

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- If $a, b < 0$:

$$\varphi(x) = \Big(\sum_{k=0}^{|b|-1} \chi_{(0,k)}(x) \qquad - \sum_{\ell=0}^{|a|-1} \chi_{(\ell,|b|)}(x) \Big)$$

- If $a < 0$ and $b \geqslant 0$:

$$\varphi(x) = \Big(- \sum_{k=1}^{b} \chi_{(0,-k)}(x) \qquad - \sum_{\ell=0}^{|a|-1} \chi_{(\ell,|b|)}(x) \Big)$$

- If $a \geqslant 0$ and $b < 0$:

$$\varphi(x) = \Big(\sum_{k=0}^{|b|-1} \chi_{(0,k)}(x) \qquad \sum_{\ell=1}^{a} \chi_{(-\ell,-b)}(x) \Big)$$

In view of corollary 11.2, computing $\beta\varphi(m \otimes r)$, for $r \in R$ a homogeneous element of degree ρ, results in a telescoping sum simplifying to $m \otimes r - \chi_{-\rho}(m \otimes r)$. But $\chi_{-\rho}(m \otimes r) = mr \otimes 1$ by lemma 11.3 whence $\chi_{-\rho}(m \otimes r) = \sigma\gamma(m \otimes r)$ so $\beta\varphi + \sigma\gamma = \mathrm{id}$ as required. $\square$

Proof of proposition 12.1. The map κ is a quasi-isomorphism if and only if its mapping cone is acyclic. But this mapping cone is precisely the totalization of the double complex (12.2). As it is concentrated in a finite vertical strip, and as its rows are acyclic by lemma 12.2, the totalization is acyclic by standard results for double complexes. $\square$

13. The Mather trick

Let C be a chain complex of R-modules, and let D be a chain complex of $R_{(0,0)}$-modules. Suppose we have $R_{(0,0)}$-linear chain maps $\alpha\colon C \longrightarrow D$ and $\beta\colon D \longrightarrow C$ and a chain homotopy $H\colon \mathrm{id}_C \simeq \beta\alpha$ such that $dH + Hd = \beta\alpha - \mathrm{id}_C$. A calculation shows:

Lemma 13.1. *The matrix*

$$\begin{pmatrix} \alpha^* & 0 & 0 & 0 \\ -\alpha^*\mu_1 H^* & \alpha^* & 0 & 0 \\ -\alpha^*\mu_2 H^* & 0 & \alpha^* & 0 \\ K & \alpha^*\mu_2 H^* & -\alpha^*\mu_1 H^* & \alpha^* \end{pmatrix},$$

where we abbreviate

$$K = \alpha^*(\mu_1 H^* \mu_2 - \mu_2 H^* \mu_1) H^* \colon C_n \otimes R \longrightarrow D_{n+2} \otimes R \ ,$$

defines a chain map of algebraic tori

$$\lambda\colon \mathfrak{T}(\mathrm{id}_C, \mathrm{id}_C; 0) \longrightarrow \mathfrak{T}(\alpha, \beta; H) \ . \qquad \square$$

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THEOREM 13.2 MATHER trick. *If the map α is a quasi-isomorphism, then C is quasi-isomorphic to $\mathfrak{T}(\alpha, \beta; H)$. More precisely, the maps*

$$C \xleftarrow{\kappa} \mathfrak{T}(\mathrm{id}_C, \mathrm{id}_C; 0) \xrightarrow{\lambda} \mathfrak{T}(\alpha, \beta; H)$$

are quasi-isomorphism.

Proof. The map κ is a quasi-isomorphism by lemma 12.1. The map λ, defined in lemma 13.1, is a quasi-isomorphism if α is since the representing matrix of λ is lower triangular with diagonal terms α^*. Note that $\alpha^* = \alpha \otimes \mathrm{id}_R$ is a quasi-isomorphism since R is strongly graded and hence is a projective $R_{(0,0)}$-module. □

14. Novikov homology

Let K be a $\mathbb{Z}^2$-graded R-module, with R a $\mathbb{Z}^2$-graded ring as usual. In analogy to (2.1), we can define

$$K_*[x, x^{-1}]((y)) = \bigcup_{n \geqslant 0} \prod_{y \geqslant -n} \bigoplus_{x \in \mathbb{Z}} K_{(x,y)} \ ,$$

which is an $R_*[x, x^{-1}]((y))$-module in a natural way; similarly, we can define

$$K_*((x, y)) = \bigcup_{n \geqslant 0} \prod_{x,y \geqslant -n} K_{(x,y)} \ ,$$

which is an $R_*((x, y))$-module in a natural way; and so on.

For R-modules of the form $K = M \otimes_{R_{(0,0)}} R$, with M an $R_{(0,0)}$-module, these constructions are close to the usual induction functors; for example:

PROPOSITION 14.1. *Let M be a right $R_{(0,0)}$-module. There is a natural map*

$$\Psi_M \colon M \underset{R_{(0,0)}}{\otimes} R_*((x, y)) \longrightarrow (M \underset{R_{(0,0)}}{\otimes} R)_*((x, y)) \ ,$$
$$m \otimes \sum_{x,y} r_{x,y} \mapsto \sum_{x,y} m \otimes r_{x,y}$$

which is an isomorphism if M is finitely presented.

Proof. This can be verified by standard techniques: establish the result for $M = R_{(0,0)}$ first (in which case it is trivial), extend to the case $M = R^n_{(0,0)}$ and then treat the general case using a finite presentation of M. □

From now on, assume that C is a bounded below complex of finitely generated projective R-modules which is $R_{(0,0)}$-finitely dominated. Thus, we choose, once and for all, a bounded complex of finitely generated projective $R_{(0,0)}$-modules D and mutually inverse chain homotopy equivalences $\alpha \colon C \longrightarrow D$ and $\beta \colon D \longrightarrow C$, and a chain homotopy $H \colon \mathrm{id}_C \simeq \beta\alpha$ so that $dH + Hd = \beta\alpha - \mathrm{id}_C$.

The MATHER trick 13.2 guarantees that C and $\mathfrak{T}(\alpha, \beta; H)$ are quasi-isomorphic; as both R-module complexes are bounded below and consist of projective R-modules, they are in fact chain homotopy equivalent.

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LEMMA 14.2. *The chain complex $C \otimes_R R_*(\!(x,y)\!)$ is contractible.*

Proof. By the previous remarks, it is enough to show that the chain complex $\mathfrak{T}(\alpha,\beta;H) \otimes_R R_*(\!(x,y)\!)$ is acyclic. Recall that $\mathfrak{T}(\alpha,\beta;H)$ is obtained from the diagram $T(\alpha,\beta,H)$ by a totalization process, which commutes with induction. Thus, we can realize $\mathfrak{T}(\alpha,\beta;H) \otimes_R R_*(\!(x,y)\!)$ by totalizing the following diagram:

$$\begin{tikzcd}
D \underset{R_{(0,0)}}{\otimes} R_*(\!(x,y)\!) \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"] \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"'] \arrow[dr, "\alpha^*\chi_{e_1}H^*\chi_{e_2}\beta^* - \alpha^*\chi_{e_2}H^*\chi_{e_1}\beta^*" description] & D \underset{R_{(0,0)}}{\otimes} R_*(\!(x,y)\!) \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"] \\
D \underset{R_{(0,0)}}{\otimes} R_*(\!(x,y)\!) \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"'] & D \underset{R_{(0,0)}}{\otimes} R_*(\!(x,y)\!)
\end{tikzcd}$$

(We have used implicitly that the two functors $- \otimes_{R_{(0,0)}} R_*(\!(x,y)\!)$ and $- \otimes_{R_{(0,0)}} R \otimes_R R_*(\!(x,y)\!)$ are naturally isomorphic.) By lemma 14.1, we can re-write this further as the totalization of the diagram

$$\begin{tikzcd}
D_*(\!(x,y)\!) \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"] \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"'] \arrow[dr, "\alpha^*\chi_{e_1}H^*\chi_{e_2}\beta^* - \alpha^*\chi_{e_2}H^*\chi_{e_1}\beta^*" description] & D_*(\!(x,y)\!) \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"] \\
D_*(\!(x,y)\!) \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"'] & D_*(\!(x,y)\!)
\end{tikzcd} \tag{14.1}$$

with maps suitably interpreted. To wit, an element z of $D_*(\!(x,y)\!)$ (in some fixed chain level) has the form $z = \sum_{x,y \geqslant a} z_{x,y}$ for some $a \in \mathbb{Z}$ and certain $z_{x,y} \in D \otimes_{R_{(0,0)}} R_{x,y}$, and

$$(\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*)(z) = \sum_{x,y \geqslant a} z_{x,y} - \alpha^*\chi_{e_1}\beta^*(z_{x-1,y}) \ ,$$

see lemma 11.3; similar formulæ can be written out for the other maps.

The point is that *the self-map* $\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*$ *of* $D_*(\!(x,y)\!)$ *is an isomorphism with inverse given by the 'geometric series'*

$$\begin{aligned}
P(z) &= \sum_{k \geqslant 0} \big(\alpha^*\chi_{e_1}\beta^*\big)^k(z) \\
&= z + \alpha^*\chi_{e_1}\beta^*(z) + \alpha^*\chi_{e_1}\beta^*\alpha^*\chi_{e_1}\beta^*(z) + \dots .
\end{aligned}$$

This follows immediately from the usual telescoping sum argument, once it is understood that the series actually defines a well-defined self-map of $D_*(\!(x,y)\!)$. But this is the case because $\alpha^*\chi_{e_1}\beta^*$ maps an element of degree (x,y) to an element of degree $(x+1,y)$.

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It is now a matter of computation to verify that the matrix

$$p = \begin{pmatrix} 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14.2}$$

almost defines a contraction of the totalization of (14.1), in the sense that $dp + pd$ is an automorphism q (in each chain level)—in fact, q is given by a triagonal matrix with identity diagonal entries—where d denotes the boundary map of the totalization. It follows that $q^{-1}p$ is a chain contraction as required. □

LEMMA 14.3. *The complex $C \otimes_R R_*[y, y^{-1}]((x))$ is contractible.*

Proof. This is shown just like the previous lemma. It is enough to demonstrate that the chain complex $\mathfrak{T}(\alpha, \beta; H) \otimes_R R_*[y, y^{-1}]((x))$ is acyclic; this complex can be written, using a variant of lemma 14.1, as the totalization of the following diagram:

$$\begin{tikzcd} D_*[y,y^{-1}]((x)) \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"] \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"'] \arrow[dr, "\alpha^*\chi_{e_1}H^*\chi_{e_2}\beta^* - \alpha^*\chi_{e_2}H^*\chi_{e_1}\beta^*" description] & D_*[y,y^{-1}]((x)) \arrow[d, "\mathrm{id}^* - \alpha^*\chi_{e_2}\beta^*"] \\ D_*[y,y^{-1}]((x)) \arrow[r, "\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*"'] & D_*[y,y^{-1}]((x)) \end{tikzcd} \tag{14.3}$$

The self-map $\mathrm{id}^* - \alpha^*\chi_{e_1}\beta^*$ of $D_*[y, y^{-1}]((x))$ is an isomorphism with inverse $P(z) = \sum_{k \geqslant 0} \left(\alpha^*\chi_{e_1}\beta^*\right)^k(z)$; the matrix p from (14.2) is such that $dp + pd$ is an automorphism q (in each chain level), where d denotes the boundary map of the totalization. Consequently, $q^{-1}p$ is a chain contraction. □

Proof of only if in theorem 2.1. By lemma 14.2, the complex $C \otimes_R R_*((x, y))$ is acyclic. Replacing R by the strongly $\mathbb{Z}^2$-graded ring $\bar{R}$ with $\bar{R}_{(x,y)} = R_{(-x,y)}$, effectively substituting x^{-1} for x, gives that $C \otimes_{\bar{R}} \bar{R}_*((x, y)) = C \otimes_R R_*((x^{-1}, y))$ is acyclic. The remaining cases of (1.1b) are dealt with by similar re-indexing.

By lemma 14.3, the complex $C \otimes_R R_*[y, y^{-1}]((x))$ is acyclic. Replacing R by the strongly $\mathbb{Z}^2$-graded ring $\bar{R}$ with $\bar{R}_{(x,y)} = R_{(-x,y)}$ gives that $C \otimes_{\bar{R}} \bar{R}_*[y, y^{-1}]((x)) = C \otimes_R R_*[y, y^{-1}]((x^{-1}))$ is acyclic. The other cases of (1.1a) are dealt with by using $\bar{R}_{(x,y)} = R_{(y,x)}$ and $\bar{R}_{(x,y)} = R_{(-y,x)}$, respectively. □

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