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ON QUASI-ORTHODOX SEMIGROUPS WITH INVERSE TRANSVERSALS

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An inverse transversal of a regular semigroup S is an inverse subsemigroup S° that contains precisely one inverse of each element of S. Here we consider the case where S is quasi-orthodox. We give natural characterisations of such semigroups and consider various properties of congruences.

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An *inverse transversal* of a regular semigroup S is an inverse subsemigroup T with the property that $|T \cap V(x)| = 1$ for every $x \in S$, where $V(x)$ denotes the set of inverses of $x \in S$. In what follows we shall write the unique element of $T \cap V(x)$ as x°, and T as $S^\circ = \{x^\circ; x \in S\}$. Then in S° we have $(x^\circ)^{-1} = x^{\circ\circ}$, so that $x^\circ = x^{\circ\circ\circ}$ for every $x \in S$. Fundamental properties of the unary operation $x \mapsto x^\circ$ in such a semigroup are

(α) [4] $(\forall x, y \in S) \quad (xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ(xyy^\circ)^\circ = y^\circ(x^\circ xyy^\circ)^\circ x^\circ$;

(β) [2] $(\forall x, y \in S) \quad (xy^\circ)^\circ = y^{\circ\circ}x^\circ, \quad (x^\circ y)^\circ = y^\circ x^{\circ\circ}$;

(γ) [7] $\mathrm{I} = \{e \in S; e = ee^\circ\}$ and $\Lambda = \{f \in S; f = f^\circ f\}$ are sub-bands of S;

(δ) [5] S is orthodox if and only if $(\forall x, y \in S) \quad (xy)^\circ = y^\circ x^\circ$.

The sub-bands I and Λ, which are respectively left regular and right regular, are such that $\mathrm{I} \cap \Lambda$ is the semilattice $E(S^\circ)$ of idempotents of S°. Together with the inverse subsemigroup S°, they form the building bricks in the structure theorems of Saito [5].

In this paper we shall be concerned primarily with the case where S is quasi-orthodox. Yamada [8] has defined a semigroup S to be *quasi-orthodox* if there is an inverse semigroup Γ and a surjective morphism $\varphi : S \to \Gamma$ such that, for every idempotent $e \in \Gamma$, the pre-image of e under φ is a completely simple subsemigroup of S. A regular semigroup S is quasi-orthodox if and only if the subsemigroup $\langle E(S) \rangle$ is completely regular.

Saito [6] has proved that if S is regular with an inverse transversal S° then the following statements are equivalent:

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(1) S is quasi-orthodox;

(2) $(\forall x, y \in S)\quad (xy)^\circ(xy)^{\circ\circ} = y^\circ x^\circ x^{\circ\circ} y^{\circ\circ}$;

(3) $(\forall x, y \in S)\quad (xy)^{\circ\circ}(xy)^\circ = x^{\circ\circ} y^{\circ\circ} y^\circ x^\circ$.

Since Green's relations $\mathcal{L}$ and $\mathcal{R}$ on S are given by

$$(x, y) \in \mathcal{L} \Leftrightarrow x^\circ x = y^\circ y, \quad (x, y) \in \mathcal{R} \Leftrightarrow xx^\circ = yy^\circ,$$

it follows immediately that

(ϵ) S is quasi-orthodox if and only if

$$(\forall x, y \in S)\quad ((xy)^\circ, y^\circ x^\circ) \in \mathcal{H}.$$

In the same paper, Saito proved that if S is quasi-orthodox then S is orthodox if and only if S° is weakly multiplicative, in the sense that $(\Lambda \mathrm{I})^\circ \subseteq E(S^\circ)$. Now if S is orthodox it follows by (δ) and (ϵ) that S is quasi-orthodox. Since $i^\circ \in E(S^\circ)$ for every $i \in \mathrm{I}$ and $l^\circ \in E(S^\circ)$ for every $l \in \Lambda$, it is clear that if S is orthodox then S° is weakly multiplicative. Hence the following statements are equivalent:

(1) S is orthodox;

(2) S is quasi-orthodox and S° is weakly multiplicative.

Example 1. Let S be the set of real singular 2×2 matrices having a non-zero entry in the (1, 1)-position, and let M consist of S with the 2×2 zero matrix adjoined. Then, as we have shown in [1], M is a regular semigroup and relative to the definitions

$$\begin{bmatrix} a & b \\ c & a^{-1}bc \end{bmatrix}^\circ = \begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^\circ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

the set

$$M^\circ = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}; x \neq 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

is an inverse transversal of M.

Consider the subset Q of M given by

$$Q = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix}, \begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix}, \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}; x \neq 0 \right\}.$$

It is readily seen that Q is a subsemigroup of M. Since Q is clearly closed under the operation $A \mapsto A^\circ$ we have that Q is regular with

$$Q^\circ = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}; x \neq 0 \right\}$$

an inverse (in fact, a group) transversal of Q. Since $E(Q^\circ)$ is a singleton and since each of $(AB)^\circ(AB)^{\circ\circ}$ and $B^\circ A^\circ A^{\circ\circ}B^{\circ\circ}$ belong to $E(Q^\circ)$, it follows that $(AB)^\circ(AB)^{\circ\circ}=B^\circ A^\circ A^{\circ\circ}B^{\circ\circ}$ and therefore Q is quasi-orthodox. That Q is not orthodox can be seen from the observation that $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ belong to $E(Q)$ but

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\notin E(Q).$$

In what follows we shall require certain properties of congruences. For this purpose, we let $\mathrm{Con}\,S$ be the lattice of congruences on the semigroup S, and denote by $\overline{\mathrm{Con}}\,S$ the sublattice of $^\circ$-congruence, i.e., congruences ϑ with the property that $(x, y)\in\vartheta$ implies $(x^\circ, y^\circ)\in\vartheta$. In [**2**, Theorem 1] we have shown that if $X\in\{\mathrm{I}, S^\circ, \Lambda\}$ then $\mathrm{Con}\,X=\overline{\mathrm{Con}}\,X$.

Of particular interest are those congruences on $X\in\{\mathrm{I}, S^\circ, \Lambda\}$ that are *special* in the sense that they can be extended to $^\circ$-congruences on S. In [**2**, Theorem 8] we have shown that $\iota\in\mathrm{Con}\,\mathrm{I}$ is special if and only if

$$(i, j)\in\iota\Rightarrow(\forall x\in S)\quad(xi(xi)^\circ, xj(xj)^\circ)\in\iota.$$

In what follows, for $X\in\{\mathrm{I}, S^\circ, \Lambda\}$ we shall denote Green's relations on X by $\mathcal{L}_X$, $\mathcal{R}_X$, and $\mathcal{H}_X$. By a result of Hall [**3**] we have that $\mathcal{L}_X$, $\mathcal{R}_X$, $\mathcal{H}_X$ are respectively the restrictions to X of $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ on S.

Theorem 1. *If S is a regular semigroup with an inverse transversal S° then $\mathcal{L}_\mathrm{I}\in\mathrm{Con}\,\mathrm{I}$ and*

$$(i, j)\in\mathcal{L}_\mathrm{I}\Leftrightarrow i^\circ=j^\circ.$$

Moreover, $\mathcal{R}_\mathrm{I}$ reduces to equality.

Proof. Since I is left regular, $\mathcal{L}_\mathrm{I}$ is a congruence and $\mathcal{R}_\mathrm{I}$ reduces to equality. Now, since I is orthodox with $E(S^\circ)$ a semilattice transversal, we have

$$(i, j)\in\mathcal{L}_\mathrm{I}\Rightarrow i^\circ=(ij)^\circ=j^\circ i^\circ=i^\circ j^\circ=(ji)^\circ=j^\circ.$$

Conversely, let $i, j\in\mathrm{I}$ be such that $i^\circ=j^\circ$. Since $i=ii^\circ$ and $i^\circ i=i^\circ$ we have $(i, i^\circ)\in\mathcal{L}_\mathrm{I}$. Consequently, $i^\circ=j^\circ$ implies that $(i, j)\in\mathcal{L}_\mathrm{I}$. □

Corollary. *$\mathcal{L}_\mathrm{I}$ is special if and only if, for all $i, j\in\mathrm{I}$,*

$$i^\circ=j^\circ\Rightarrow(\forall x\in S)\quad(xi)^{\circ\circ}(xi)^\circ=(xj)^{\circ\circ}(xj)^\circ.$$ □

We shall denote by μ the biggest idempotent-separating congruence on S°. For every idempotent-separating congruence π on S° we let Θ_π be the relation defined on S by

$$(a, b)\in\Theta_\pi\Leftrightarrow(a^\circ, b^\circ)\in\pi.$$

Theorem 2. *If S is a regular semigroup with an inverse transversal S° then the following statements are equivalent:*

(1) *S is quasi-orthodox;*

(2) *$\mathcal{L}_I$ is special;*

(3) *$\mathcal{R}_\Lambda$ is special;*

(4) $\Theta_\mu \in \overline{\text{Con}}\, S$.

Proof. (1) $\Rightarrow$ (2): If (1) holds then

$$(xi)^{\circ\circ}(xi)^\circ = x^{\circ\circ}i^{\circ\circ}i^\circ x^\circ = x^{\circ\circ}i^\circ x^\circ.$$

Then (2) follows by the Corollary to Theorem 1.

(2) $\Rightarrow$ (1): For every $y \in S$, we have $(yy^\circ)^\circ = y^{\circ\circ}y^\circ = (y^{\circ\circ}y^\circ)^\circ$. If (2) holds, then it follows by the Corollary to Theorem 1 that

$$(\forall x, y \in S) \qquad (xyy^\circ)^{\circ\circ}(xyy^\circ)^\circ = (xy^{\circ\circ}y^\circ)^{\circ\circ}(xy^{\circ\circ}y^\circ)^\circ.$$

Now, on the one hand,

$$\begin{aligned}(xyy^\circ)^{\circ\circ}(xyy^\circ)^\circ &= (xyy^\circ)^{\circ\circ}(yy^\circ)^\circ(xyy^\circ(yy^\circ)^\circ)^\circ \\ &= (xyy^\circ)^{\circ\circ}y^{\circ\circ}y^\circ(xyy^\circ)^\circ \\ &= (y^\circ(xyy^\circ)^\circ)^\circ y^\circ(xyy^\circ)^\circ \\ &= (xy)^{\circ\circ}(xy)^\circ;\end{aligned}$$

and, on the other hand,

$$(xy^{\circ\circ}y^\circ)^{\circ\circ}(xy^{\circ\circ}y^\circ)^\circ = x^{\circ\circ}y^{\circ\circ}y^\circ y^{\circ\circ}y^\circ x^\circ = x^{\circ\circ}y^{\circ\circ}y^\circ x^\circ.$$

Thus $(xy)^{\circ\circ}(xy)^\circ = x^{\circ\circ}y^{\circ\circ}y^\circ x^\circ$ and so S is quasi-orthodox.

(1) $\Leftrightarrow$ (3): This is similar.

(1) $\Rightarrow$ (4): Suppose that (1) holds and that $(a, b) \in \Theta_\mu$. Then for every $e \in E(S^\circ)$ we have $a^\circ ea^{\circ\circ} = b^\circ eb^{\circ\circ}$. Since $e = e^\circ = e^{\circ\circ}$ for every $e \in E(S^\circ)$, it follows that, for every $x \in S$,

$$\begin{aligned}(ax)^\circ e(ax)^{\circ\circ} &= (ax)^\circ e^\circ e^{\circ\circ}(ax)^{\circ\circ} \\ &= (eax)^\circ(eax)^{\circ\circ} \\ &= x^\circ(ea)^\circ(ea)^{\circ\circ}x^{\circ\circ} \qquad \text{by (1)} \\ &= x^\circ a^\circ ea^{\circ\circ}x^{\circ\circ} \\ &= x^\circ b^\circ eb^{\circ\circ}x^{\circ\circ} \\ &= (bx)^\circ e(bx)^{\circ\circ}.\end{aligned}$$

Also,

$$\begin{aligned}(xa)^\circ e(xa)^{\circ\circ} &= (exa)^\circ(exa)^{\circ\circ}\\ &= a^\circ(ex)^\circ(ex)^{\circ\circ}a^{\circ\circ} && \text{by (1)}\\ &= b^\circ(ex)^\circ(ex)^{\circ\circ}b^{\circ\circ} && \text{since } (ex)^\circ(ex)^{\circ\circ} \in E(S^\circ)\\ &= (xb)^\circ e(xb)^{\circ\circ}.\end{aligned}$$

Consequently, $\Theta_\mu \in \operatorname{Con} S$. Clearly, if $(a, b) \in \Theta_\mu$ then $(a^\circ, b^\circ) \in \Theta_\mu$. Hence $\Theta_\mu \in \overline{\operatorname{Con}}\, S$.

(4) ⇒ (1): If $\Theta_\mu \in \overline{\operatorname{Con}}\, S$ then, observing that $(x, x^{\circ\circ}) \in \Theta_\mu$ for every $x \in S$, we have that $(xy, x^{\circ\circ}y^{\circ\circ}) \in \Theta_\mu$ for all $x, y \in S$ and therefore

$$(xy)^\circ x^{\circ\circ}x^\circ(xy)^{\circ\circ} = (x^{\circ\circ}y^{\circ\circ})^\circ x^{\circ\circ}x^\circ(x^{\circ\circ}y^{\circ\circ})^{\circ\circ}.$$

Clearly, the right hand side reduces to $y^\circ x^\circ x^{\circ\circ}y^{\circ\circ}$. As for the left hand side, this can be written $(x^\circ xy)^\circ x^\circ(xy)^{\circ\circ} = (xy)^\circ(xy)^{\circ\circ}$. Hence we see that S is quasi-orthodox. □

Theorem 3. *Let S be a regular semigroup with an inverse transversal S° and let π be an idempotent-separating congruence on S°. Then the following statements are equivalent:*

(1) $\Theta_\pi \in \overline{\operatorname{Con}}\, S$;

(2) $(\forall x, y \in S) \quad ((xy)^\circ, y^\circ x^\circ) \in \pi$;

(3) $(\forall i \in \mathrm{I})(\forall l \in \Lambda) \quad ((li)^\circ, l^\circ i^\circ) \in \pi$.

Proof. (1) ⇒ (2): Clearly, for every $x \in S$, we have $(x, x^{\circ\circ}) \in \Theta_\pi$ so if (1) holds we have $(xy, x^{\circ\circ}y^{\circ\circ}) \in \Theta_\pi$ whence $((xy)^\circ, y^\circ x^\circ) \in \pi$.

(2) ⇒ (3): This is clear.

(3) ⇒ (1): If (3) holds, we observe that, for $i, j \in \mathrm{I}$ and $l, m \in \Lambda$,

$$i^\circ = j^\circ, l^\circ = m^\circ \Rightarrow ((li)^\circ, (mj)^\circ) \in \pi.$$

Hence, if $(x, y) \in \Theta_\pi$ we have $(x^\circ x^{\circ\circ}, y^\circ y^{\circ\circ}) \in \pi$ and therefore, since π is idempotent-separating, $x^\circ x^{\circ\circ} = y^\circ y^{\circ\circ}$. Applying the above observation we deduce that, for every $a \in S$,

$$((x^\circ xaa^\circ)^\circ, (y^\circ yaa^\circ)^\circ) \in \pi.$$

Since $(x^\circ, y^\circ) \in \pi$ it follows that

$$((xa)^\circ, (ya)^\circ) = (a^\circ(x^\circ xaa^\circ)^\circ x^\circ, a^\circ(y^\circ yaa^\circ)^\circ y^\circ) \in \pi$$

and therefore $(xa, ya) \in \Theta_\pi$. Similarly, we can show that $(ax, ay) \in \Theta_\pi$, and therefore $\Theta_\pi \in \operatorname{Con} S$. It now follows from the definition of Θ_π that $\Theta_\pi \in \overline{\operatorname{Con}} S$. □

Corollary. *S is quasi-orthodox if and only if μ is such that*

$$(\forall x, y \in S) \qquad ((xy)^\circ, y^\circ x^\circ) \in \mu.$$

Proof. This follows immediately by Theorem 2. □

In order to proceed, we require some general facts concerning °-congruences. In [2] we have established the general form of such congruences. Specifically, we define a triple $(\iota, \pi, \lambda) \in \operatorname{Con} \mathrm{I} \times \overline{\operatorname{Con}} S^\circ \times \operatorname{Con} \Lambda$ to be

(a) *balanced* if $\iota|_{E(S^\circ)} = \pi|_{E(S^\circ)} = \lambda|_{E(S^\circ)}$;

(b) *linked* if for all $i_1, i_2 \in \mathrm{I}$, all $x_1, x_2 \in S^\circ$, and all $l_1, l_2 \in \Lambda$,

$$(i_1, i_2) \in \iota, (l_1, l_2) \in \lambda \Rightarrow \begin{cases} (l_1 i_1 (l_1 i_1)^\circ, l_2 i_2 (l_2 i_2)^\circ) \in \iota & [\alpha] \\ ((l_1 i_1)^\circ, (l_2 i_2)^\circ) \in \pi & [\beta] \\ ((l_1 i_1)^\circ l_1 i_1, (l_2 i_2)^\circ l_2 i_2) \in \lambda & [\gamma] \end{cases}$$

$$(i_1, i_2) \in \iota, (x_1, x_2) \in \pi \Rightarrow (x_1 i_1 x_1^\circ, x_2 i_2 x_2^\circ) \in \iota \qquad [\delta]$$

$$(l_1, l_2) \in \lambda, (x_1, x_2) \in \pi \Rightarrow (x_1^\circ l_1 x_1, x_2^\circ l_2 x_2) \in \lambda \qquad [\epsilon]$$

The set $\mathrm{BLT}(S)$ of balanced linked triples forms a lattice that is isomorphic to $\overline{\operatorname{Con}} S$. Every $\vartheta \in \overline{\operatorname{Con}} S$ is of the form $\Psi(\iota, \pi, \lambda)$ where $(\iota, \pi, \lambda) \in \mathrm{BLT}(S)$ and

$$(a, b) \in \Psi(\iota, \pi, \lambda) \Leftrightarrow (aa^\circ, bb^\circ) \in \iota, \quad (a^\circ, b^\circ) \in \pi, \quad (a^\circ a, b^\circ b) \in \lambda.$$

Concerning the special congruences on $X \in \{\mathrm{I}, S^\circ, \Lambda\}$ we have also established in [2] the following results:

(A) if $\iota \in \operatorname{Con} \mathrm{I}$ is special then the biggest extension of ι to a °-congruence on S is the congruence $\hat{\iota} \in \overline{\operatorname{Con}} S$ given by

$$(a, b) \in \hat{\iota} \Leftrightarrow (\forall i \in \mathrm{I}) \quad (ai(ai)^\circ, bi(bi)^\circ) \in \iota;$$

(B) a dual result if $\lambda \in \operatorname{Con} \Lambda$ is special;

(C) $\pi \in \operatorname{Con} S^\circ$ is special if and only if

$$(x, y) \in \pi \Rightarrow (\forall i \in \mathrm{I})(\forall l \in \Lambda) \quad ((lxi)^\circ, (lyi)^\circ) \in \pi.$$

In this case the biggest extension of $\pi \in \operatorname{Con} S^\circ$ to a °-congruence on S is the congruence $\hat{\pi} \in \overline{\operatorname{Con}} S$ given by

$$(a, b) \in \hat{\pi} \Leftrightarrow (\forall i \in \mathrm{I})(\forall l \in \Lambda) \quad ((lai)^\circ, (lbi)^\circ) \in \pi.$$

In view of Theorem 2, when S is quasi-orthodox there are balanced linked triples of the form $(\mathcal{L}_\mathrm{I}, -, -)$ and $(-, -, \mathcal{R}_\Lambda)$. Since, by Theorem 1, $\mathcal{L}$ and $\mathcal{R}$ reduce to equality on $E(S^\circ)$, the middle components of any such triples must be idempotent-separating congruences on S°. Conversely, any balanced linked triple of the form (ι, π, λ) in which π is idempotent-separating is such that $\iota \subseteq \mathcal{L}_\mathrm{I}$ and $\lambda \subseteq \mathcal{R}_\Lambda$. In fact, since in such a triple $\iota|_{E(S^\circ)}$ is equality we have, by Theorem 1,

$$(i_1, i_2) \in \iota \Rightarrow i_1^\circ = i_2^\circ \Rightarrow (i_1, i_2) \in \mathcal{L}_\mathrm{I},$$

so that $\iota \subseteq \mathcal{L}_\mathrm{I}$, and similarly $\lambda \subseteq \mathcal{R}_\Lambda$.

In order to establish the existence of balanced linked triples of the form $(\mathcal{L}_\mathrm{I}, -, \mathcal{R}_\Lambda)$ or, equivalently, the existence of $^\circ$-congruences on S that simultaneously extend $\mathcal{L}_\mathrm{I}$ and $\mathcal{R}_\Lambda$, we require the following general result.

Theorem 4. *Let S be a regular semigroup with an inverse transversal S°. Given special congruences $\iota \in \operatorname{Con}\mathrm{I}$ and $\lambda \in \operatorname{Con}\Lambda$, there exists a balanced linked triple of the form $(\iota, -, \lambda)$ if and only if $\iota \subseteq \hat{\lambda}|_\mathrm{I}$ and $\lambda \subseteq \hat{\iota}|_\Lambda$. In this case, the biggest such triple has middle component $\hat{\iota}|_{S^\circ} \cap \hat{\lambda}|_{S^\circ}$.*

Proof. Suppose that $\iota \in \operatorname{Con}\mathrm{I}$ and $\lambda \in \operatorname{Con}\Lambda$ are such that there is a balanced linked triple of the form $(\iota, -, \lambda)$. Observe that, by [α], if $(l_1, l_2) \in \lambda$ then for every $i \in \mathrm{I}$, $(l_1 i(l_1 i)^\circ, l_2 i(l_2 i)^\circ) \in \iota$ and therefore $(l_1, l_2) \in \hat{\iota}|_\Lambda$. Thus $\lambda \subseteq \hat{\iota}|_\Lambda$, and similarly $\iota \subseteq \hat{\lambda}|_\mathrm{I}$.

Conversely, $\hat{\iota}$ corresponds to the balanced linked triple $(\iota, \hat{\iota}|_{S^\circ}, \hat{\iota}|_\Lambda)$, and $\hat{\lambda}$ corresponds to the balanced linked triple $(\hat{\lambda}|_\mathrm{I}, \hat{\lambda}|_{S^\circ}, \lambda)$. Therefore, if the stated conditions hold, $\hat{\iota} \cap \hat{\lambda}$ corresponds to the balanced linked triple $(\iota, \hat{\iota}|_{S^\circ} \cap \hat{\lambda}|_{S^\circ}, \lambda)$.

Finally, for any balanced linked triple (ι, π, λ), it follows by [δ] that, for $x_1, x_2 \in S^\circ$,

$$\begin{aligned}(x_1, x_2) \in \pi &\Rightarrow (\forall i \in \mathrm{I}) \quad (x_1 i(x_1 i)^\circ, x_2 i(x_2 i)^\circ) = (x_1 i x_1^\circ, x_2 i x_2^\circ) \in \iota \\ &\Rightarrow (x_1, x_2) \in \hat{\iota}|_{S^\circ}\end{aligned}$$

and so $\pi \subseteq \hat{\iota}|_{S^\circ}$. Similarly, $\pi \subseteq \hat{\lambda}|_{S^\circ}$ and we conclude that the biggest balanced linked triple of the form $(\iota, -, \lambda)$ has $\hat{\iota}|_{S^\circ} \cap \hat{\lambda}|_{S^\circ}$ as its middle component. □

To see that, when S is quasi-orthodox, there exist balanced linked triples of the form $(\mathcal{L}_\mathrm{I}, -, \mathcal{R}_\Lambda)$ we may use Theorem 4 as follows. By Theorem 2 we can consider $\widehat{\mathcal{L}_\mathrm{I}}$ and $\widehat{\mathcal{R}_\Lambda}$. Now, by (A) and Theorem 1, we have

$$\begin{aligned}(a, b) \in \widehat{\mathcal{L}_\mathrm{I}} &\Leftrightarrow (\forall i \in \mathrm{I}) \quad (ai(ai)^\circ, bi(bi)^\circ) \in \mathcal{L}_\mathrm{I} \\ &\Leftrightarrow (\forall i \in \mathrm{I}) \quad (ai)^{\circ\circ}(ai)^\circ = (bi)^{\circ\circ}(bi)^\circ \\ &\Leftrightarrow (\forall i \in \mathrm{I}) \quad a^{\circ\circ} i^\circ a^\circ = b^{\circ\circ} i^\circ b^\circ,\end{aligned}$$

and similarly,

$$(a, b) \in \widehat{\mathcal{R}_\Lambda} \Leftrightarrow (\forall l \in \Lambda) \quad a^\circ l^\circ a^{\circ\circ} = b^\circ l^\circ b^{\circ\circ}.$$

Since $E(S^\circ) = \{i^\circ; i \in \mathrm{I}\} = \{l^\circ; l \in \Lambda\}$, we deduce from these expressions that

$$(a, b) \in \widehat{\mathcal{L}_\mathrm{I}} \Leftrightarrow (a, b) \in \widehat{\mathcal{R}_\Lambda}.$$

It follows that $\widehat{\mathcal{L}_\mathrm{I}} = \widehat{\mathcal{R}_\Lambda}$. Consequently $\mathcal{L}_\mathrm{I} = \widehat{\mathcal{L}_\mathrm{I}}|_\mathrm{I} = \widehat{\mathcal{R}_\Lambda}|_\mathrm{I}$ and similarly $\mathcal{R}_\Lambda = \widehat{\mathcal{L}_\mathrm{I}}|_\Lambda$. Thus, by Theorem 4, balanced linked triples of the form $(\mathcal{L}_\mathrm{I}, -, \mathcal{R}_\Lambda)$, and hence $^\circ$-congruences on S that simultaneously extend $\mathcal{L}_\mathrm{I}$ and $\mathcal{R}_\Lambda$, exist.

We now determine precisely the nature of middle components in such triples.

Theorem 5. *Let S be a quasi-orthodox semigroup with an inverse transversal S°. Then $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$ if and only if π is idempotent-separating on S° and $\Theta_\pi \in \overline{\mathrm{Con}}\, S$.*

Proof. $\Rightarrow$: If $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$ then, by the above observations, π is idempotent-separating and $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda)$ is the biggest balanced linked triple of the form $(-, \pi, -)$. Consequently, $\hat{\pi} = \Psi(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda)$ and therefore

$$\begin{aligned}(x, y) \in \hat{\pi} &\Leftrightarrow x^\circ x^{\circ\circ} = y^\circ y^{\circ\circ}, (x^\circ, y^\circ) \in \pi, x^{\circ\circ}x^\circ = y^{\circ\circ}y^\circ \\ &\Leftrightarrow (x^\circ, y^\circ) \in \pi\end{aligned}$$

and so $\hat{\pi} = \Theta_\pi$. Hence $\Theta_\pi \in \overline{\mathrm{Con}}\, S$.

$\Leftarrow$: If π is idempotent-separating on S° and $\Theta_\pi \in \overline{\mathrm{Con}}\, S$ then it is readily seen that $\Theta_\pi|_\mathrm{I} = \mathcal{L}_\mathrm{I}, \Theta_\pi|_{S^\circ} = \pi$, and $\Theta_\pi|_\Lambda = \mathcal{R}_\Lambda$. Consequently $\Theta_\pi = \Psi(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda)$. □

Corollary. *Let S be a quasi-orthodox semigroup with an inverse transversal S°. Then*

(1) *the biggest idempotent-separating congruence μ on S° is special with $(\mathcal{L}_\mathrm{I}, \mu, \mathcal{R}_\Lambda)$ the biggest balanced linked triple of the form $(\mathcal{L}_\mathrm{I}, -, \mathcal{R}_\Lambda)$;*

(2) *the biggest extension of μ in $\overline{\mathrm{Con}}\, S$ is $\Psi(\mathcal{L}_\mathrm{I}, \mu, \mathcal{R}_\Lambda) = \Theta_\mu$.*

Proof. This follows from Theorems 2, 4 and 5 on observing that $\widehat{\mathcal{L}_\mathrm{I}}|_{S^\circ} = \widehat{\mathcal{R}_\Lambda}|_{S^\circ} = \mu$. □

In what follows we shall denote by ζ the $^\circ$-congruence on S generated by the set $\{(a, a^{\circ\circ}); a \in S\}$.

Theorem 6. *Let S be a quasi-orthodox semigroup with an inverse transversal S°. Then the smallest balanced linked triple of the form $(\mathcal{L}_\mathrm{I}, -, \mathcal{R}_\Lambda)$ has middle component $\zeta|_{S^\circ}$.*

Proof. Let T be the set of idempotent-separating congruences π on S° such that $\Theta_\pi \in \overline{\mathrm{Con}}\, S$. Then for every $a \in S$ we have $(a, a^{\circ\circ}) \in \bigcap_{\pi \in T} \Theta_\pi$. It follows that $\zeta \subseteq \bigcap_{\pi \in T} \Theta_\pi$. For every $\pi \in T$ we then have $\zeta \subseteq \Theta_\pi$ and so $\zeta|_{S^\circ} \subseteq \Theta_\pi|_{S^\circ} = \pi$. Hence $\zeta|_{S^\circ}$ is idempotent-separating. Now

$$(a, b) \in \zeta \Leftrightarrow (a^{\circ\circ}, b^{\circ\circ}) \in \zeta \Leftrightarrow (a^\circ, b^\circ) \in \zeta \Leftrightarrow (a, b) \in \Theta_{\zeta|_{S^\circ}}.$$

Consequently, $\zeta = \Theta_{\zeta|_{S^\circ}}$ and therefore $\zeta|_{S^\circ} \in T$. It follows that $\zeta|_{S^\circ} = \min T$ whence we obtain the result by Theorem 5. □

We can now describe the balanced linked triples of the form $(\mathcal{L}_\mathrm{I}, -, \mathcal{R}_\Lambda)$.

Theorem 7. *Let S be a quasi-orthodox semigroup with an inverse transversal S°. Then $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$ if and only if π belongs to the interval $[\zeta|_{S^\circ}, \mu]$ of* Con S°.

Proof. $\Rightarrow$: If $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$ then, by Theorem 5, π is idempotent-separating, so $\pi \subseteq \mu$. By Theorem 6, $\zeta|_{S^\circ} \subseteq \pi$.
$\Leftarrow$: If $\pi \in [\zeta|_{S^\circ}, \mu]$ then π is necessarily idempotent-separating and therefore the triple $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda)$ is balanced. Now since, by the Corollary to Theorem 5, $(\mathcal{L}_\mathrm{I}, \mu, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$ the triple $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda)$ satisfies the conditions [α], [γ], [δ], [ϵ] the last two of which follow from the fact that $\pi \subseteq \mu$. Since $(\mathcal{L}_\mathrm{I}, \zeta|_{S^\circ}, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$ and $\zeta|_{S^\circ} \subseteq \pi$, the triple $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda)$ also satisfies [β]. Hence $(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$. □

Corollary 1. *Every $\pi \in [\zeta|_{S^\circ}, \mu]$ is special.* □

Corollary 2. *The °-congruences on S that simultaneously extend $\mathcal{L}_\mathrm{I}$ and $\mathcal{R}_\Lambda$ are precisely those of the form $\Psi(\mathcal{L}_\mathrm{I}, \pi, \mathcal{R}_\Lambda)$ where $\pi \in [\zeta|_{S^\circ}, \mu]$.* □

Example 2. Concerning the semigroup Q of Example 1, we can describe the congruence ζ as follows. For every $X \in Q$ let x_{11} be the entry in the (1, 1)-position. Define a relation ρ on Q by

$$(A, B) \in \rho \Leftrightarrow (\exists n \in \mathbb{Z}) \quad a_{11} = 2^n b_{11}.$$

It is easily seen that ρ is a °-congruence on Q that identifies A and $A^{\circ\circ}$ for every $A \in Q$. Consequently, $\zeta \subseteq \rho$. Observe now that since the congruence ζ identifies the matrices

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

we see, on pre-multiplying by $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, that ζ identifies the matrices

$$\begin{bmatrix} 2x & 2x \\ 2x & 2x \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

and, by recursion, identifies the matrices

$$\begin{bmatrix} 2^n x & 2^n x \\ 2^n x & 2^n x \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

If, therefore, $(A, B) \in \rho$ we have $a_{11} = 2^n b_{11}$ where we can assume that the integer n is non-negative, and consequently

$$A \overset{\zeta}{\equiv} \begin{bmatrix} 2^n b_{11} & 0 \\ 0 & 0 \end{bmatrix} \overset{\zeta}{\equiv} \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \end{bmatrix} \overset{\zeta}{\equiv} B.$$

Hence $\rho \subseteq \zeta$ and therefore $\zeta = \rho$.

We shall denote by ω_{S° the relation of equality on S°. As the following result shows, the $^\circ$-congruence ζ can be used to provide a measure of the distinction between quasi-orthodox and orthodox.

Theorem 8. *Let S be a quasi-orthodox semigroup with an inverse transversal S°. Then S is orthodox if and only if $\zeta|_{S^\circ} = \omega_{S^\circ}$.*

Proof. $\Rightarrow$: If S is orthodox then we have the identity $(xy)^\circ = y^\circ x^\circ$ and so it follows by Theorem 3 that $\omega_{S^\circ} \in T$. Since $\zeta|_{S^\circ} = \min T$ we deduce that $\zeta|_{S^\circ} = \omega_{S^\circ}$.

$\Leftarrow$: If $\zeta|_{S^\circ} = \omega_{S^\circ}$ then $\Theta_{\omega_{S^\circ}} \in \overline{\mathrm{Con}}\, S$ and Theorem 3 gives the identity $(li)^\circ = l^\circ i^\circ$. Thus $(\Lambda I)^\circ \subseteq E(S^\circ)$ and so S° is weakly multiplicative. Consequently, S is orthodox. □

Corollary. *If S is orthodox then $(\mathcal{L}_I, \pi, \mathcal{R}_\Lambda) \in \mathrm{BLT}(S)$ if and only if π is idempotent-separating.*

Proof. This follows by Corollary 1 of Theorem 7. □

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