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RESEARCH ARTICLE

Generalised Dirac-Schrödinger operators and the Callias Theorem

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Abstract

We consider generalised Dirac-Schrödinger operators, consisting of a self-adjoint elliptic first-order differential operator $\mathcal{D}$ with a skew-adjoint 'potential' given by a (suitable) family of unbounded operators. The index of such an operator represents the pairing (Kasparov product) of the K-theory class of the potential with the K-homology class of $\mathcal{D}$. Our main result in this paper is a generalisation of the Callias Theorem: the index of the Dirac-Schrödinger operator can be computed on a suitable compact hypersurface. Our theorem simultaneously generalises (and is inspired by) the well-known result that the spectral flow of a path of relatively compact perturbations depends only on the endpoints.

Contents

1. Introduction

Let $\mathcal{D}_M$ be the Dirac operator on an odd-dimensional locally compact smooth spin manifold M, which determines the K-homology fundamental class $[\mathcal{D}_M] \in K_1(M)$. If $\Sigma_M \in K^1(M)$ is any element in the K-theory of M, there is a natural pairing $\langle\ ,\ \rangle \colon K^1(M) \times K_1(M) \to \mathbb{Z}$ of K-theory with K-homology, often referred to as the *index pairing*, which yields an integer

$$\big\langle \Sigma_M, [\mathcal{D}_M] \big\rangle \in \mathbb{Z}.$$

We now wish to compute this integer in terms of an index pairing on a suitable hypersurface $N \subset M$ of codimension one. Assume that $U \subset M$ is an open subset with compact closure $K = \overline{U}$ and smooth boundary $N = \partial U$. Consider the natural map $\iota_U{}^* \colon K_1(M) \to K_1(U)$ which sends $[\mathcal{D}_M]$ to the class $[\mathcal{D}_U]$ of the restriction of the Dirac operator to U. The K-homology boundary map

$$\partial \colon K_1(U) \to K_0(\partial U)$$

sends the class $[\mathcal{D}_U]$ to the class $[\mathcal{D}_{\partial U}]$ of the Dirac operator on the boundary ∂U.

Now suppose that the K-theory class Σ_M is the image of a class $\Sigma_U \in K^1(U)$ under the natural map $\iota_{U*} \colon K^1(U) \to K^1(M)$. Suppose furthermore that Σ_U is the image of a class $\Sigma_{\partial U} \in K^0(\partial U)$ under the K-theory boundary map $\partial \colon K^0(\partial U) \to K^1(U)$. Then by naturality, we have the equalities

$$\begin{aligned} \big\langle \Sigma_M, [\mathcal{D}_M] \big\rangle &= \big\langle \iota_{U*} \circ \partial(\Sigma_{\partial U}), [\mathcal{D}_M] \big\rangle = \big\langle \partial(\Sigma_{\partial U}), \iota_{U*}([\mathcal{D}_M]) \big\rangle \\ &= \big\langle \Sigma_{\partial U}, \partial \circ \iota_U{}^*([\mathcal{D}_M]) \big\rangle = \big\langle \Sigma_{\partial U}, [\mathcal{D}_{\partial U}] \big\rangle. \end{aligned}$$

To summarise the preceding argument, if there exists a class $\Sigma_{\partial U} \in K^1(\partial U)$ with $\Sigma_M = \iota_{U*} \circ \partial(\Sigma_{\partial U})$, then the index pairing on the locally compact manifold M can be computed from an index pairing on the compact hypersurface ∂U:

$$\big\langle \Sigma_M, [\mathcal{D}_M] \big\rangle = \big\langle \Sigma_{\partial U}, [\mathcal{D}_{\partial U}] \big\rangle. \tag{1.1}$$

There are two (at first sight rather different) instances in the literature where such a computation has been made. The first instance is in the case of *Dirac-Schrödinger* (or *Callias-type*) operators. These are operators of the form $\mathcal{D}_M - i\mathcal{S}$, where the ‘potential’ $\mathcal{S}$ is a self-adjoint endomorphism on some auxiliary vector bundle (of finite rank) over M. Assuming that $\mathcal{S}$ is invertible outside of $U \subset M$, the Dirac-Schrödinger operator $\mathcal{D}_M - i\mathcal{S}$ is Fredholm, and its index computes the index pairing of the K-theory class of the potential with the K-homology class of the Dirac operator:

$$\mathrm{Index}(\mathcal{D}_M - i\mathcal{S}) = \big\langle [\mathcal{S}], [\mathcal{D}_M] \big\rangle \in \mathbb{Z}.$$

The invertibility of $\mathcal{S}$ outside U ensures that $[\mathcal{S}] = \iota_{U*}([\mathcal{S}|_U])$. Moreover, since we may perturb the potential on a compact subset without changing its K-theory class, and since $\overline{U}$ is compact, we note that $j_*([\mathcal{S}|_U]) = [\mathcal{S}|_{\overline{U}}] = 0 \in K^1(\overline{U})$, where $j_* \colon K^1(U) \to K^1(\overline{U})$ is induced from the inclusion $U \hookrightarrow \overline{U}$. By exactness of the sequence

$$K^0(\partial U) \xrightarrow{\partial} K^1(U) \xrightarrow{j_*} K^1(\overline{U}),$$

we therefore know that $[\mathcal{S}|_U] = \partial(\Sigma_{\partial U})$ for some $\Sigma_{\partial U} \in K^0(\partial U)$. In fact, as we will see in Corollary 3.11, we can explicitly identify $\Sigma_{\partial U}$ as the K-theory class of the vector bundle V_+ over ∂U obtained from the positive eigenspace of the invertible self-adjoint endomorphism $\mathcal{S}|_{\partial U}$. The pairing

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$\langle [V_+], [\mathcal{D}_{\partial U}] \rangle$ can be computed as the index of the operator $(\mathcal{D}_{\partial U})_+^+$, which is obtained by twisting the chiral Dirac operator on ∂U with the vector bundle V_+. The equality (1.1) therefore yields

$$\operatorname{Index}(\mathcal{D}_M - i\mathcal{S}) = \operatorname{Index}(\mathcal{D}_{\partial U})_+^+. \tag{1.2}$$

This result is known as the *Callias Theorem*; the first version was proven by Callias [Cal78] on Euclidean space, and it has subsequently been generalised by various authors (see, for instance, [Ang90, BM92, Ang93, Råd94, Bun95, Kuc01, GW16]).

The second instance of Equation (1.1) appears in the study of *spectral flow*. Consider a 'sufficiently continuous' family of self-adjoint Fredholm operators $\{\mathcal{S}(x)\}_{x\in[0,1]}$ with invertible endpoints and with common domain on a Hilbert space $\mathcal{H}$, such that $\mathcal{S}(x)$ is a relatively compact perturbation of $\mathcal{S}(0)$ (for each $x \in [0,1]$). Then the spectral flow depends only on the endpoints and is given by (see [Les05, Theorem 3.6] and [Wah08, Proposition 2.5])

$$\operatorname{sf}\big(\{\mathcal{S}(x)\}_{x\in[0,1]}\big) = \text{rel-ind}\,\big(P_+(\mathcal{S}(1)), P_+(\mathcal{S}(0))\big). \tag{1.3}$$

Here, the left-hand side is the spectral flow of the family $\{\mathcal{S}(x)\}_{x\in[0,1]}$, and the right-hand side is given by the relative index of the pair of positive spectral projections associated to $\mathcal{S}(1)$ and $\mathcal{S}(0)$. To view this equality in the form of Equation (1.1), let $M = \mathbb{R}$, $U = (0,1)$, and $N = \partial U = \{0\} \cup \{1\}$, and extend $\mathcal{S}$ to a family on $\mathbb{R}$. By the well-known 'index = spectral flow' equality of Robbin–Salamon (see, for example, [RS95, Wah07, AW11] and [Dun19, Corollary 5.16]), the spectral flow can be described as an index pairing on $\mathbb{R}$:

$$\operatorname{sf}\big(\{\mathcal{S}(x)\}_{x\in\mathbb{R}}\big) = \operatorname{Index}\,\big(\partial_x + \mathcal{S}(\cdot)\big) = \big\langle [\mathcal{S}(\cdot)], [-i\partial_x] \big\rangle,$$

where $-i\partial_x$ is the standard Dirac operator on $\mathbb{R}$. Moreover, the assumption that $\mathcal{S}(x)$ is a relatively compact perturbation of $\mathcal{S}(0)$ (for each $x \in [0,1]$), combined with the compactness of $[0,1]$, implies that the operator $\mathcal{S}(\cdot)|_{[0,1]}$ is a relatively compact perturbation of a constant invertible family. It follows that $j_*([\mathcal{S}(\cdot)|_{(0,1)}]) = 0 \in K^1([0,1])$ (where j_* is induced from the inclusion $(0,1) \hookrightarrow [0,1]$), and therefore (by exactness, as in the case of Dirac-Schrödinger operators), $[\mathcal{S}(\cdot)|_{(0,1)}] = \partial\big(\Sigma_{\{0\}\cup\{1\}}\big)$ for some $\Sigma_{\{0\}\cup\{1\}} \in K^0(\{0\}\cup\{1\}) \simeq \mathbb{Z}\oplus\mathbb{Z}$. We will see in Corollary 3.12 that the relative index on the right-hand side of Equation (1.3) is indeed obtained from an index pairing of $\Sigma_{\{0\}\cup\{1\}}$ with the K-homology element $[\mathcal{D}_{\{0\}\cup\{1\}}] \in K_0(\{0\} \cup \{1\})$, where the latter can be identified with $(-1) \oplus 1 \in \mathbb{Z} \oplus \mathbb{Z}$.

Thus, we have seen that both the Callias Theorem (1.2) and the spectral flow result (1.3) can be viewed as a special case of the equality (1.1). Our goal in the present paper is to provide a common generalisation which unifies both these results. For this purpose, we now consider 'generalised' Dirac-Schrödinger operators, which are again operators of the form $\mathcal{D}_M - i\mathcal{S}(\cdot)$, where now the auxiliary vector bundle is of *infinite* rank, and the 'potential' $\mathcal{S}(\cdot)$ consists of a family of (unbounded) self-adjoint operators $\{\mathcal{S}(x)\}_{x\in M}$ on a fixed Hilbert space $\mathcal{H}$. (In fact, instead of $\mathcal{H}$, we will more generally consider a Hilbert C^*-module over some auxiliary C^*-algebra, but in this introduction, we limit our attention to the simpler case of a Hilbert space.) Such operators were studied in [KL13, §8] for suitably differentiable potentials, and in [Dun19] for continuous potentials. It is known that the pairing of the K-theory class of $\mathcal{S}(\cdot)$ with the K-homology class of $\mathcal{D}_M$ still equals the index of the Dirac-Schrödinger operator ([KL13, Theorem 1.2] and [Dun19, Theorem 5.15]):

$$\big\langle [\mathcal{S}(\cdot)], [\mathcal{D}_M] \big\rangle = \operatorname{Index}(\mathcal{D} - i\mathcal{S}(\cdot)).$$

Now, under the additional assumption that the potential $\mathcal{S}(\cdot)$ is given by a family of relatively compact perturbations (as in the case of the spectral flow result (1.3)), we again find that $[\mathcal{S}(\cdot)] = \iota_{U*} \circ \partial(\Sigma_{\partial U})$ for some K-theory class $\Sigma_{\partial U} \in K^0(\partial U)$. Hence, Equation (1.1) applies, and to obtain our desired generalisation of the Callias Theorem, it remains only to identify the class $\Sigma_{\partial U}$. We will see that this class can again (as in the spectral flow case) be described as a relative index of positive spectral

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projections, and our generalised Callias Theorem then provides the equality (we refer to §3.2 for the precise statement)

$$\operatorname{Index}\big(\mathcal{D} - i\mathcal{S}(\cdot)\big) = \big\langle \operatorname{rel-ind}\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big), [\mathcal{D}_N]\big\rangle.$$

Let us provide a brief summary of the contents of this paper. We start in Section 2 with some background material, describing both the classical Callias Theorem as well as the study of the spectral flow for relatively compact perturbations (in particular, we prove Equation (1.3)). In Section 3, we describe our main assumptions, definitions and the main results (the proofs are postponed to later sections). In particular, we state in §3.2 our generalisation of the classical Callias Theorem and show in §3.3 that we recover the Callias Theorem (1.2) and the spectral flow result (1.3). The remaining sections are devoted to the proofs. In Section 4, we first consider generalised Dirac-Schrödinger operators. As an important tool, we present a relative index theorem for such operators in §4.1, under fairly general assumptions. §4.2 and §4.3 then provide sufficient conditions under which we can prove that a Dirac-Schrödinger operator is Fredholm, and that its index equals the pairing between the K-theory and K-homology classes. Section 5 finally provides the proof of the generalised Callias-type theorem.

Throughout the body of this article, we will work with Hilbert C^*-modules over C^*-algebras (rather than just Hilbert spaces). An important step in our main result is to ensure that the relative index is well-defined, for which we require several operator-theoretic facts that are known for operators on Hilbert spaces, but (to the author's best knowledge) have not yet appeared in the literature for operators on Hilbert C^*-modules. We therefore include an Appendix, in which we generalise several (partly well-known) operator-theoretic results from Hilbert spaces to Hilbert C^*-modules. We mention here a few of these results, which may also be of independent interest:

- The composition of a $*$-strongly convergent sequence of adjointable operators with a compact operator yields a norm-convergent sequence (Lemma A.1).
- Let T be regular and self-adjoint. Then any relatively T-compact operator R is relatively T-bounded with arbitrarily small relative bound (Proposition A.6). Consequently, if R is symmetric, then $T + R$ is again regular and self-adjoint on $\operatorname{Dom}(T)$ (Proposition A.7).
- Let T be regular and self-adjoint, and let R be symmetric and relatively T-compact. Let $f \in C(\mathbb{R})$ be a continuous function for which the limits $\lim_{x\to\pm\infty} f(x)$ exist. Then $f(T + R) - f(T)$ is compact (Proposition A.9). Furthermore, if T and $T+R$ are both invertible, also $P_+(T+R) - P_+(T)$ is compact (where $P_+(T)$ denotes the positive spectral projection of T).

1.1. Notation

Throughout this paper, let A be a σ-unital C^*-algebra, and let E be a (possibly $\mathbb{Z}_2$-graded) countably generated Hilbert C^*-module over A (or Hilbert A-module for short) with A-valued inner product $\langle\cdot|\cdot\rangle$. (The reader unfamiliar with C^*-modules may consider the special case $A = \mathbb{C}$, so that E is simply a separable Hilbert space. For an introduction to Hilbert C^*-modules, we refer to [Lan95].) For the inner product of an element $\psi \in E$ with itself, we use the convenient short-hand notation

$$\langle\!\langle \psi \rangle\!\rangle := \langle \psi|\psi\rangle.$$

The norm of ψ is then given by $\|\psi\| = \|\langle\!\langle \psi \rangle\!\rangle\|^{\frac{1}{2}}$.

The space of adjointable linear operators $E \to E$ is denoted by $\mathcal{L}_A(E)$. For any $\psi, \eta \in E$, the rank-one operators $\theta_{\psi,\eta}$ are defined by $\theta_{\psi,\eta}\xi := \psi\langle\eta|\xi\rangle$ for $\xi \in E$. The compact operators $\mathcal{K}_A(E)$ are given by the closure of the space of finite linear combinations of rank-one operators. For two Hilbert A-modules E_1 and E_2, the adjointable linear operators $E_1 \to E_2$ are denoted by $\mathcal{L}_A(E_1, E_2)$.

A densely defined operator S is called *regular* if S is closed, the adjoint S^* is densely defined, and $1 + S^*S$ has dense range (note that on a Hilbert space, every closed operator is regular). A densely

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defined, closed, symmetric operator S is regular and self-adjoint if and only if the operators $S \pm i$ are surjective [Lan95, Lemma 9.8].

Given a densely defined, symmetric operator S on E, we can equip $\mathrm{Dom}\, S$ with the graph inner product $\langle\psi|\psi\rangle_S := \langle(S \pm i)\psi|(S \pm i)\psi\rangle = \langle\psi|\psi\rangle + \langle S\psi|S\psi\rangle$. The graph norm of S is then defined as $\|\psi\|_S := \left\|\langle\psi|\psi\rangle_S\right\|^{\frac{1}{2}} = \|(S \pm i)\psi\|$.

2. Background

As described in the Introduction, our main theorem simultaneously generalises both the classical Callias Theorem (1.2) and the spectral flow equality (1.3). In this section, we will introduce both these results.

2.1. The classical Callias Theorem

The 'classical Callias Theorem', which we aim to generalise, is a result which was first proven by Callias [Cal78] on Euclidean space, who proved that the index of a Dirac-Schrödinger operator $\mathcal{D} - i\mathcal{S}$ on $\mathbb{R}^{2n+1}$ can be computed on a sufficiently large sphere. This has become known as the Callias Theorem and has been generalised by various authors (see, for instance, [Ang90, BM92, Ang93, Råd94, Bun95, Kuc01, GW16]), replacing Euclidean space by larger classes of Riemannian manifolds and computing the index on a suitable hypersurface (which is the boundary of a compact subset). The Callias Theorem continues to be actively studied, with recent work considering, for instance, Callias-type operators on Lie manifolds [CN14], with degenerate potentials [Kot15], via cobordism invariance [BS16], on manifolds with boundary [Shi17], twisted by Hilbert C^*-bundles of finite type [Cec20], and associated to abstract spectral triples [SS23].

We will cite here Anghel's version [Ang93] of the Callias Theorem. Let M be a complete odd-dimensional oriented Riemannian manifold, and let $\mathcal{D}$ be a formally self-adjoint Dirac-type operator on a hermitian Clifford bundle $\Sigma \to M$. Let $\mathcal{S} = \mathcal{S}^* \in \Gamma^\infty(\mathrm{End}\,\Sigma)$ be a hermitian bundle endomorphism such that $\mathcal{S}$ commutes with Clifford multiplication, $\mathcal{S}$ and $[\mathcal{D}, \mathcal{S}]$ are uniformly bounded, and there exists a compact subset $K \subset M$ such that $\mathcal{S}$ is uniformly invertible on the complement of K.

Without loss of generality, assume that K has a smooth compact boundary N. On $\Sigma_N := \Sigma|_N$, we have a $\mathbb{Z}_2$-grading operator Γ_N given by Clifford multiplication with the unit normal vector on N, which yields the decomposition $\Sigma_N = \Sigma_N^+ \oplus \Sigma_N^-$. Consider a 'restriction' $\mathcal{D}_N$ of $\mathcal{D}$ to Σ_N (i.e., the principal symbol of $\mathcal{D}_N$ is obtained from the principal symbol of $\mathcal{D}$ by restricting from TM to TN), which anticommutes with Γ_N. Since $\mathcal{D}_N$ is elliptic and N is compact, $\mathcal{D}_N$ has compact resolvents. In particular, $\mathcal{D}_N$ is Fredholm, and we obtain a K-homology class $[\mathcal{D}_N] \in K^0(C(N)) \equiv K_0(N)$.

Let $\mathcal{S}_N$ denote the restriction of $\mathcal{S}$ to $\Sigma_N \to N$. We define $\Sigma_{N+} := \mathrm{Ran}\, P_+(\mathcal{S}_N)$ to be the image of the positive spectral projection of $\mathcal{S}_N$, representing a K-theory class $[\Sigma_{N+}] \in K_0(C(N)) \equiv K^0(N)$.

Since $\mathcal{S}$ commutes with the Clifford multiplication, Γ_N is still a $\mathbb{Z}_2$-grading on Σ_{N+} and yields the decomposition $\Sigma_{N+} = \Sigma_{N+}^+ \oplus \Sigma_{N+}^-$. We will consider the Fredholm operator

$$(\mathcal{D}_N)_+^+ := \mathcal{D}_N|_{\Sigma_{N+}^+} : \Sigma_{N+}^+ \to \Sigma_{N+}^-.$$

Theorem 2.1 [Ang93, Theorem 1.5]. *Under the assumptions given above, we have the equalities*

$$\mathrm{Index}\left(\mathcal{D} - i\mathcal{S}\right) = \mathrm{Index}\left(\mathcal{D}_N\right)_+^+ = [\Sigma_{N+}] \otimes_{C(N)} [\mathcal{D}_N] = \int_N \hat{A}(N) \wedge \mathrm{ch}(\Sigma_{N+}).$$

We note that, while Anghel's theorem and proof focused on the first equality, the index of $\left(\mathcal{D}_N\right)_+^+$ realises the index pairing (Kasparov product) of the K-theory class $[\Sigma_{N+}]$ with the K-homology class $[\mathcal{D}_N]$, and it can be computed as $\int_N \hat{A}(N) \wedge \mathrm{ch}(\Sigma_{N+})$ by the Atiyah–Singer Index Theorem [AS63].

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2.2. Spectral flow

Next, we will describe the spectral flow equality (1.3) from the Introduction in detail (see Proposition 2.8). First, we provide the relevant definitions in the context of Hilbert C^*-modules.

An adjointable operator $F \in \mathcal{L}_A(E)$ is called *Fredholm* if there exists a *parametrix* $G \in \mathcal{L}_A(E)$ such that $GF - 1$ and $FG - 1$ are compact operators on E. If F is Fredholm, we denote by $\mathrm{Index}(F) \in K_0(A)$ the $K_0(A)$-valued index of F; for the definition of this index, we refer to [Dun19, §2.2] and references therein.

2.2.1. The relative index of projections

Consider two projections $P, Q \in \mathcal{L}_A(E)$. If the difference $P - Q$ is a *compact* operator on E, then the operator $Q\colon \mathrm{Ran}(P) \to \mathrm{Ran}(Q)$ is a Fredholm operator with parametrix $P\colon \mathrm{Ran}(Q) \to \mathrm{Ran}(P)$.

Definition 2.2. For projections $P, Q \in \mathcal{L}_A(E)$ with $P - Q \in \mathcal{K}_A(E)$, we define the *relative index of* (P, Q) by

$$\text{rel-ind}(P, Q) := \mathrm{Index}\big(Q\colon \mathrm{Ran}(P) \to \mathrm{Ran}(Q)\big) \in K_0(A).$$

For future reference, we record two important properties of the relative index:

Lemma 2.3 [Wah07, §3.2].

- *(Additivity.) If $P, Q, R \in \mathcal{L}_A(E)$ are projections with $P - Q$ and $Q - R$ compact, then*

$$\text{rel-ind}(P, R) = \text{rel-ind}(P, Q) + \text{rel-ind}(Q, R).$$

- *(Homotopy invariance.) If $\{P_t\}_{t\in[0,1]}$ and $\{Q_t\}_{t\in[0,1]}$ are strongly continuous paths of projections such that $P_t - Q_t$ is compact for each $t \in [0, 1]$, then*

$$\text{rel-ind}(P_0, Q_0) = \text{rel-ind}(P_1, Q_1).$$

Using the homotopy invariance of the relative index, we obtain the following:

Corollary 2.4. *Let $\{P_t\}_{t\in[0,1]}$ be a strongly continuous family of projections on E, such that $P_t - P_0$ is compact for each $t \in [0, 1]$. Then rel-ind(P_0, P_1) = 0.*

2.2.2. The spectral flow

The notion of spectral flow for a path of self-adjoint operators (typically parametrised by the unit interval) was first defined by Atiyah and Lusztig, and it appeared in the work of Atiyah, Patodi and Singer [APS76, §7]. Heuristically, the spectral flow of a path of self-adjoint Fredholm operators counts the net number of eigenvalues which pass through zero. An analytic definition of the spectral flow of a path of self-adjoint Fredholm operators on a Hilbert space was given by Phillips in [Phi96]. An axiomatic study of the spectral flow was given by Lesch in [Les05].

For regular self-adjoint Fredholm operators on a Hilbert C^*-module, a general definition of spectral flow was given by Wahl in [Wah07, §3], which we will largely follow here. However, we slightly adapt this definition by allowing the 'trivialising operators' (which appear in the definition of the spectral flow) to be possibly unbounded (rather than bounded, as in [Wah07, §3]). This is made possible by Proposition A.9 and Corollary A.10 (generalising [Wah07, Proposition 3.7]). For the definition and properties of relatively compact operators, we refer the reader to §A.3 in the Appendix.

Definition 2.5 (cf. [Wah07, Definition 3.4]). Let $\mathcal{D}$ be a regular self-adjoint operator on E. A *trivialising operator for* $\mathcal{D}$ is a (densely defined) symmetric operator $\mathcal{B}$ on E such that $\mathcal{B}$ is relatively $\mathcal{D}$-compact and $\mathcal{D} + \mathcal{B}$ is invertible.

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Now let $\mathcal{D}$ be a regular self-adjoint Fredholm operator on E, and let $\mathcal{B}_0$ and $\mathcal{B}_1$ be two trivialising operators for $\mathcal{D}$. By Corollary A.10, which generalises [Wah07, Proposition 3.7], the difference of spectral projections $P_+(\mathcal{D}+\mathcal{B}_1) - P_+(\mathcal{D}+B_0)$ is compact. Hence, we can define

$$\operatorname{ind}(\mathcal{D},\mathcal{B}_0,\mathcal{B}_1) := \text{rel-ind}\big(P_+(\mathcal{D}+\mathcal{B}_1), P_+(\mathcal{D}+B_0)\big). \tag{2.1}$$

Definition 2.6 (cf. [Wah07, Definition 3.9]). Let X be a compact Hausdorff space, and consider a regular operator $\mathcal{D}(\cdot) = \{\mathcal{D}(x)\}_{x\in X}$ on the Hilbert $C(X,A)$-module $C(X,E)$. A *trivialising family for* $\{\mathcal{D}(x)\}_{x\in X}$ is a family $\{\mathcal{B}(x)\}_{x\in X}$ of operators on E such that $\mathcal{B}(\cdot)$ is a trivialising operator for $\mathcal{D}(\cdot)$.

We say *there exist locally trivialising families* for $\mathcal{D}(\cdot)$ if for each $x \in X$ there exist a compact neighbourhood O_x of x and a trivialising family for $\{\mathcal{D}(y)\}_{y\in O_x}$.

We note that the existence of locally trivialising families for $\{\mathcal{D}(x)\}_{x\in X}$ then implies that $\mathcal{D}(\cdot)$ is Fredholm (using compactness of X).

Definition 2.7 (cf. [Wah07, Definition 3.10]). Let $\mathcal{D}(\cdot) = \{\mathcal{D}(t)\}_{t\in[0,1]}$ be a regular self-adjoint operator on the Hilbert $C([0,1],A)$-module $C([0,1],E)$, for which locally trivialising families exist. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be such that there is a trivialising family $\{\mathcal{B}^i(t)\}_{t\in[t_i,t_{i+1}]}$ of $\{\mathcal{D}(t)\}_{t\in[t_i,t_{i+1}]}$ for each $i = 0,\ldots,n-1$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be trivialising operators of $\mathcal{D}(0)$ and $\mathcal{D}(1)$. Then we define

$$\begin{aligned}
&\operatorname{sf}\big(\{\mathcal{D}(t)\}_{t\in[0,1]};\mathcal{A}_0,\mathcal{A}_1\big)\\
&\quad:= \operatorname{ind}\big(\mathcal{D}(0),\mathcal{A}_0,\mathcal{B}^0(0)\big) + \sum_{i=1}^{n-1}\operatorname{ind}\big(\mathcal{D}(t_i),\mathcal{B}^{i-1}(t_i),\mathcal{B}^i(t_i)\big) + \operatorname{ind}\big(\mathcal{D}(1),\mathcal{B}^{n-1}(1),\mathcal{A}_1\big) \in K_0(A),
\end{aligned}$$

where ind is defined in Equation (2.1). If we assume furthermore that the endpoints $\mathcal{D}(0)$ and $\mathcal{D}(1)$ are invertible, then the *spectral flow of* $\{\mathcal{D}(t)\}_{t\in[0,1]}$ is defined by

$$\begin{aligned}
\operatorname{sf}\big(\{\mathcal{D}(t)\}_{t\in[0,1]}\big) &:= \operatorname{sf}\big(\{\mathcal{D}(t)\}_{t\in[0,1]};0,0\big)\\
&= \operatorname{ind}\big(\mathcal{D}(0),0,\mathcal{B}^0(0)\big) + \sum_{i=1}^{n-1}\operatorname{ind}\big(\mathcal{D}(t_i),\mathcal{B}^{i-1}(t_i),\mathcal{B}^i(t_i)\big) + \operatorname{ind}\big(\mathcal{D}(1),\mathcal{B}^{n-1}(1),0\big).
\end{aligned}$$

As in [Wah07], the definition of the spectral flow is independent of the choice of subdivision and the choice of trivialising families $\{\mathcal{B}^i(t)\}_{t\in[t_i,t_{i+1}]}$. In particular, using [Wah07, Lemma 3.5], we may choose the trivialising families to be *bounded*, and thus, we recover the definition of the spectral flow given in [Wah07, Definition 3.10].

2.2.3. Spectral flow for relatively compact perturbations

For our attempt to generalise the Callias Theorem to the case of *infinite-rank* bundles, we take some inspiration from the study of spectral flow. In particular, the following result motivates the idea that a Callias-type theorem should hold for generalised Dirac-Schrödinger operators whenever the ‘potential’ is given by a family of relatively compact perturbations.

Proposition 2.8 (cf. [Wah07, Example in §3.4]). *Let $\mathcal{T}(\cdot) = \{\mathcal{T}(t)\}_{t\in[0,1]}$ be a regular self-adjoint operator on the Hilbert $C([0,1],A)$-module $C([0,1],E)$, such that*

- *the endpoints $\mathcal{T}(0)$ and $\mathcal{T}(1)$ are invertible;*
- *$\mathcal{T}(t)\colon \operatorname{Dom}\mathcal{T}(0) \to E$ depends norm-continuously on t; and*
- *$\mathcal{T}(t) - \mathcal{T}(0)$ is relatively $\mathcal{T}(0)$-compact for each $t \in [0,1]$.*

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Then the following statements hold:

1. *There exists a trivialising family for* $\{\mathcal{T}(t)\}_{t\in[0,1]}$.
2. *We have the equality*

$$sf\big(\{\mathcal{T}(t)\}_{t\in[0,1]}\big) = \text{rel-ind}\,\big(P_+(\mathcal{T}(1)), P_+(\mathcal{T}(0))\big). \tag{2.2}$$

Proof.

1. We observe that the family of operators $\mathcal{B}(t) := \mathcal{T}(0) - \mathcal{T}(t)$ ($t \in [0,1]$) yields a densely defined symmetric operator $\mathcal{B}(\cdot)$ on $C([0,1],E)$, such that $\mathcal{T}(\cdot)+\mathcal{B}(\cdot)$ is invertible. Moreover, $\mathcal{B}(t)\big(\mathcal{T}(t)\pm i\big)^{-1}$ is compact for each $t \in [0,1]$ (where we use that $\operatorname{Dom}\mathcal{T}(t) = \operatorname{Dom}\mathcal{T}(0)$ by Proposition A.7). Since $\big(\mathcal{T}(t)\pm i\big)\big(\mathcal{T}(0)\pm i\big)^{-1}$ is norm-continuous in t, also the family of inverses $\big(\mathcal{T}(0)\pm i\big)\big(\mathcal{T}(t)\pm i\big)^{-1}$ is norm-continuous, and therefore, $\mathcal{B}(t)\big(\mathcal{T}(t)\pm i\big)^{-1}$ is norm-continuous in t. This shows that $\mathcal{B}(\cdot)$ is relatively $\mathcal{T}(\cdot)$-compact. Thus, $\mathcal{B}(\cdot)$ is a trivialising operator for $\mathcal{T}(\cdot)$.
2. We can insert the trivialising family $\{\mathcal{B}(t)\}_{t\in[0,1]}$ from the first statement into Definition 2.7 to obtain

$$\begin{aligned}\text{sf}\big(\{\mathcal{T}(t)\}_{t\in[0,1]}\big) &= \text{ind}\,\big(\mathcal{T}(0), 0, \mathcal{B}(0)\big) + \text{ind}\,\big(\mathcal{T}(1), \mathcal{B}(1), 0\big)\\ &= \text{rel-ind}\,\big(P_+(\mathcal{T}(0)+\mathcal{B}(0)), P_+(\mathcal{T}(0))\big) + \text{rel-ind}\,\big(P_+(\mathcal{T}(1)), P_+(\mathcal{T}(1)+\mathcal{B}(1))\big)\\ &= \text{rel-ind}\,\big(P_+(\mathcal{T}(1)), P_+(\mathcal{T}(0))\big),\end{aligned}$$

where we used that $\mathcal{B}(0) = 0$ and $\mathcal{T}(1)+\mathcal{B}(1) = \mathcal{T}(0)$. □

We remark that for paths of operators on Hilbert *spaces* (rather than Hilbert modules), the identity (2.2) has been shown to hold even under more general continuity assumptions; see [Les05, Theorem 3.6] and [Wah08, Proposition 2.5].

3. A generalised Callias-type theorem

3.1. Generalised Dirac-Schrödinger operators

Throughout this section, we will consider the following setting.

Assumption (A). Let A be a σ-unital C^*-algebra, and let E be a countably generated Hilbert A-module. Let M be a connected Riemannian manifold (typically non-compact), and let $\mathcal{D}$ be an essentially self-adjoint elliptic first-order differential operator on a hermitian vector bundle $\mathtt{F} \to M$. Let $\{\mathcal{S}(x)\}_{x\in M}$ be a family of regular self-adjoint operators on E satisfying the following assumptions:

(A1) The domain $W := \operatorname{Dom}\mathcal{S}(x)$ is independent of $x \in M$, and the inclusion $W \hookrightarrow E$ is compact (where W is viewed as a Hilbert A-module equipped with the graph norm of $\mathcal{S}(x_0)$, for some $x_0 \in M$).
(A2) The map $\mathcal{S}\colon M \to \mathcal{L}_A(W,E)$ is norm-continuous.
(A3) There is a compact subset $K \subset M$ such that $\mathcal{S}(x)$ is uniformly invertible on $M\setminus K$.

Given the family of operators $\{\mathcal{S}(x)\}_{x\in M}$ on E, we obtain a closed symmetric operator $\mathcal{S}(\cdot)$ on $C_0(M,E)$, which is defined as the closure of the operator $\big(\mathcal{S}(\cdot)\psi\big)(x) := \mathcal{S}(x)\psi(x)$ on the initial dense domain $C_c(M,W)$. By [Dun19, Proposition 3.4], the operator $\mathcal{S}(\cdot)$ on the Hilbert $C_0(M,A)$-module $C_0(M,E)$ is regular self-adjoint and Fredholm. Consequently, we obtain from [Dun19, Proposition 2.14] a well-defined K-theory class

$$[\mathcal{S}(\cdot)] \in KK^1(\mathbb{C}, C_0(M,A)) \simeq K_1(C_0(M,A)).$$

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Furthermore, since $\mathcal{D}$ is an essentially self-adjoint first-order differential operator, and since the ellipticity of $\mathcal{D}$ ensures that $\mathcal{D}$ also has locally compact resolvents [HR00, Proposition 10.5.2], we know that $(C_0^1(M), L^2(M,\mathtt{F}), \mathcal{D})$ is an (odd) spectral triple, which represents a K-homology class

$$[\mathcal{D}] \in KK^1(C_0(M),\mathbb{C}) \simeq K^1(C_0(M)) \equiv K_1(M).$$

We consider the balanced tensor product $L^2(M, E\otimes\mathtt{F}) := C_0(M,E)\otimes_{C_0(M)} L^2(M,\mathtt{F})$. The operator $\mathcal{S}(\cdot)\otimes 1$ is well-defined on $\operatorname{Dom}\mathcal{S}(\cdot)\otimes_{C_0(M)} L^2(M,\mathtt{F}) \subset L^2(M,E\otimes\mathtt{F})$, and is denoted simply by $\mathcal{S}(\cdot)$ as well. By [Lan95, Proposition 9.10], $\mathcal{S}(\cdot)$ is regular self-adjoint on $L^2(M,E\otimes\mathtt{F})$.

The operator $1\otimes\mathcal{D}$ is not well-defined on $L^2(M,E\otimes\mathtt{F})$. Instead, using the canonical isomorphism $L^2(M,E\otimes\mathtt{F}) \simeq E\otimes L^2(M,\mathtt{F})$, we consider the operator $1\otimes\mathcal{D}$ on $E\otimes L^2(M,\mathtt{F})$ with domain $E\otimes\operatorname{Dom}\mathcal{D}$. Alternatively, we can extend the exterior derivative on $C_0^1(M)$ to an operator

$$d\colon C_0^1(M,E) \xrightarrow{\simeq} E\otimes C_0^1(M) \xrightarrow{1\otimes d} E\otimes\Gamma_0(T^*M) \xrightarrow{\simeq} \Gamma_0(E\otimes T^*M).$$

Denoting by σ the principal symbol of $\mathcal{D}$, we can define an operator $1\otimes_d\mathcal{D}$ on the Hilbert space $C_0(M,E)\otimes_{C_0(M)} L^2(M,\mathtt{F})$ by setting

$$(1\otimes_d\mathcal{D})(\xi\otimes\psi) := \xi\otimes\mathcal{D}\psi + (1\otimes\sigma)(d\xi)\psi.$$

Under the isomorphism $C_0(M,E)\otimes_{C_0(M)} L^2(M,\mathtt{F}) \simeq E\otimes L^2(M,\mathtt{F})$, the operator $1\otimes\mathcal{D}$ on $E\otimes L^2(M,\mathtt{F})$ agrees with $1\otimes_d\mathcal{D}$ on $C_0(M,E)\otimes_{C_0(M)} L^2(M,\mathtt{F})$. We will denote this operator on $L^2(M,E\otimes\mathtt{F})$ simply as $\mathcal{D}$. The operator $\mathcal{D}$ is regular self-adjoint on $L^2(M,E\otimes\mathtt{F})$ (see also [KL13, Theorem 5.4]).

Definition 3.1. Consider M, $\mathcal{D}$ and $\mathcal{S}(\cdot)$ satisfying assumption (A). We define the operator

$$\mathcal{D}_{\mathcal{S}} := \mathcal{D} - i\mathcal{S}(\cdot)$$

on the initial domain $C_c^1(M,W)\otimes_{C_0^1(M)}\operatorname{Dom}\mathcal{D}$. Since $\mathcal{D}+i\mathcal{S}(\cdot) \subset \big(\mathcal{D}-i\mathcal{S}(\cdot)\big)^*$ is densely defined (on the same domain), $\mathcal{D}-i\mathcal{S}(\cdot)$ is closable, and (with slight abuse of notation) we denote its closure simply by $\mathcal{D}_{\mathcal{S}}$ as well.

The operator $\mathcal{D}_{\mathcal{S}}$ is called a *generalised Dirac-Schrödinger operator* if $\mathcal{D}_{\mathcal{S}}$ is regular and Fredholm, and $\mathcal{D}_{\mathcal{S}}^* = \mathcal{D}_{-\mathcal{S}}$. In this case, we obtain a well-defined $K_0(A)$-valued index

$$\operatorname{Index}\mathcal{D}_{\mathcal{S}} \in K_0(A).$$

For the definition of this index, we refer to [Dun19, §2.2] and references therein.

We note that, despite our use of the term 'Dirac-Schrödinger' operator, we do not assume that the operator $\mathcal{D}$ is of Dirac-type (although a Dirac-type operator is of course the typical example, as described in the Introduction). Furthermore, we note that regularity, the Fredholm property and the adjoint relation of $\mathcal{D}_{\mathcal{S}}$ do not follow automatically from assumption (A).

In order to prove the Fredholm property of $\mathcal{D}_{\mathcal{S}}$, we consider in addition to assumption (A) also the following assumption:

Assumption (B). We assume the following conditions are satisfied:

(B1) the map $\mathcal{S}\colon M\to\mathcal{L}_A(W,E)$ is weakly differentiable (i.e., for each $\psi\in W$ and $\eta\in E$, the map $x\mapsto\langle\mathcal{S}(x)\psi|\eta\rangle$ is differentiable), and the weak derivative $d\mathcal{S}(x)\colon W\to E\otimes T_x^*(M)$ is bounded for all $x\in M$.

(B2) the operator $\big[\mathcal{D},\mathcal{S}(\cdot)\big]\big(\mathcal{S}(\cdot)\pm i\big)^{-1}$ is well-defined and bounded (in the sense of [KL12, Assumption 7.1] and [Dun19, Definition 5.5]): there exists a core $\mathcal{E}\subset\operatorname{Dom}\mathcal{D}$ for $\mathcal{D}$ such that for all $\xi\in\mathcal{E}$ and for all $\mu\in(0,\infty)$, we have the inclusions

$$\big(\mathcal{S}(\cdot)\pm i\mu\big)^{-1}\xi \in \operatorname{Dom}\mathcal{S}(\cdot)\cap\operatorname{Dom}\mathcal{D} \quad\text{and}\quad \mathcal{D}\big(\mathcal{S}(\cdot)\pm i\mu\big)^{-1}\xi \in \operatorname{Dom}\mathcal{S}(\cdot),$$

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and the map $[\mathcal{D}, \mathcal{S}(\cdot)](\mathcal{S}(\cdot) \pm i\mu)^{-1} : \mathcal{E} \to L^2(M, E \otimes \mathtt{F})$ extends to a bounded operator for all $\mu \in (0, \infty)$.

Remark 3.2.

1. Assumption (B) requires the potential $\mathcal{S}(\cdot)$ to be differentiable (in a suitable sense). Alternatively, it is also possible to deal with continuous potentials, as is done in [Dun19].
2. As described in [KL13, Remark 8.4], assumption (B1) already implies assumption (A2).
3. If, in addition to (B1), we assume that $\mathcal{D}$ has bounded propagation speed, that $\operatorname{Dom} \mathcal{S}(\cdot) \subset C_0(M, W)$ (i.e., that there exists $C > 0$ such that for all $x \in M$, we have $\|\cdot\|_W \le C\|\cdot\|_{\mathcal{S}(x)}$), and that the weak derivative $d\mathcal{S}(\cdot)$ is *uniformly* bounded, then the boundedness of $[\mathcal{D}, \mathcal{S}(\cdot)](\mathcal{S}(\cdot) \pm i)^{-1}$ in (B2) already follows. Indeed, as in [KL13, Lemma 8.5 & Theorem 8.6], we can then write

$$[\mathcal{D}, \mathcal{S}(\cdot)](\mathcal{S}(\cdot) \pm i)^{-1} = \sigma_{\mathcal{D}} \circ d\mathcal{S}(\cdot) \circ (\mathcal{S}(\cdot) \pm i)^{-1} :$$

$$L^2(M, E \otimes \mathtt{F}) \xrightarrow{(\mathcal{S}(\cdot)\pm i)^{-1}} \operatorname{Dom} \mathcal{S}(\cdot) \otimes_{C_0(M)} L^2(M, \mathtt{F}) \hookrightarrow L^2(M, W \otimes \mathtt{F})$$
$$\xrightarrow{d\mathcal{S}(\cdot)} L^2(M, E \otimes T^*M \otimes \mathtt{F}) \xrightarrow{\sigma_{\mathcal{D}}} L^2(M, E \otimes \mathtt{F})$$

as a composition of bounded operators.

Thanks to assumption (B), we have the following:

Proposition 3.3 [KL12, Theorem 7.10]. *The operators $\mathcal{D}_{\pm\mathcal{S}}$ are regular on the domain* $\operatorname{Dom} \mathcal{D}_{\mathcal{S}}$ *and satisfy $\mathcal{D}_{\pm\mathcal{S}}^* = \mathcal{D}_{\mp\mathcal{S}}$.*

The following theorem will be proven as the first statement of Theorem 4.4 below. It states that the operator $\mathcal{D}_{\mathcal{S}}$ is Fredholm, provided that (if necessary) the potential $\mathcal{S}(\cdot)$ is rescaled by a sufficiently large $\lambda > 0$.

Theorem 3.4. *There exists $\lambda_0 > 0$ such that for any $\lambda \ge \lambda_0$, the operator $\mathcal{D}_{\lambda\mathcal{S}}$ is Fredholm and thus a generalised Dirac-Schrödinger operator.*

Our next theorem then describes the Fredholm index of a Dirac-Schrödinger operator in terms of the *index pairing* between the K-theory class of the potential $\mathcal{S}(\cdot)$ and the K-homology class of the elliptic operator $\mathcal{D}$. Results of this form were previously given by Bunke [Bun95] (see also [Kuc01]) in the classical case and in [KL13, Dun19] for 'generalised' Dirac-Schrödinger operators.

Theorem 3.5. *Let M be a connected Riemannian manifold, and let $\{\mathcal{S}(x)\}_{x\in M}$ and $\mathcal{D}$ satisfy assumptions (A) and (B). Then there exists $\lambda_0 > 0$ such that for any $\lambda \ge \lambda_0$, the $K_0(A)$-valued index of $\mathcal{D}_{\lambda\mathcal{S}}$ equals the pairing of $[\mathcal{S}(\cdot)] \in K_1(C_0(M, A))$ with $[\mathcal{D}] \in K^1(C_0(M))$.*

The proof is given in §4.3. It relies on identifying the classes as elements in Kasparov's KK-theory via the isomorphisms $K_1(C_0(M, A)) \simeq KK^1(\mathbb{C}, C_0(M, A))$, $K^1(C_0(M)) \simeq KK^1(C_0(M), \mathbb{C})$ and $K_0(A) \simeq KK^0(\mathbb{C}, A)$, and then computing the index pairing using the description of the unbounded Kasparov product given in [KL13].

3.2. Generalised Callias-type operators

Let M, $\mathcal{D}$ and $\mathcal{S}(\cdot)$ satisfy assumptions (A) and (B) such that $\mathcal{D}_{\lambda\mathcal{S}}$ is Fredholm (and hence a generalised Dirac-Schrödinger operator) for $\lambda \ge \lambda_0 > 0$. In the remainder of this section, we furthermore assume the following:

Assumption (C). Without loss of generality, assume that the compact subset K from assumption (A3) has a smooth compact boundary N. We assume furthermore that the following conditions are satisfied:

(C1) The operator $\mathcal{D}$ is of 'product form' near N in the following sense. There exists a collar neighbourhood $C \simeq (-2\varepsilon, 2\varepsilon) \times N$ of N (with $(-2\varepsilon, 0) \times N$ in the interior of K), where we can

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identify $\mathtt{F}|_C$ with the pullback of $\mathtt{F}_N := \mathtt{F}|_N \to N$ to $C \simeq (-2\varepsilon, 2\varepsilon) \times N$, so that $\Gamma^\infty(\mathtt{F}|_C) \simeq C^\infty\big((-2\varepsilon, 2\varepsilon)\big) \otimes \Gamma^\infty(\mathtt{F}_N)$. On this collar neighbourhood, we have $\mathcal{D}|_C \simeq -i\partial_r \otimes \Gamma_N + 1 \otimes \mathcal{D}_N$, where $\mathcal{D}_N$ is an essentially self-adjoint elliptic first-order differential operator on $\mathtt{F}_N \to N$, and where $\Gamma_N \in \Gamma^\infty(\operatorname{End} \mathtt{F}_N)$ is a self-adjoint unitary satisfying $\Gamma_N \mathcal{D}_N = -\mathcal{D}_N \Gamma_N$.

(C2) For any $x, y \in K$, $\mathcal{S}(x) - \mathcal{S}(y)$ is relatively $\mathcal{S}(x)$-compact.

Moreover, we fix an (arbitrary) invertible regular self-adjoint operator $\mathcal{T}$ on E with domain $\operatorname{Dom} \mathcal{T} = W$, such that $\mathcal{S}(x) - \mathcal{T}$ is relatively $\mathcal{T}$-compact for some (and hence, by (C2), for every) $x \in K$.

Remark 3.6.

1. For the definition and properties of relatively compact operators, we refer the reader to §A.3 in the Appendix.
2. The product form of $\mathcal{D}$ in assumption (C1) is typical of Dirac operators corresponding to a product metric on the collar neighbourhood C of N, where Γ_N is given by Clifford multiplication with the unit normal vector ∂_r to N (actually, one might often write $\mathcal{D}'_C = -i(1 \otimes \Gamma_N)(\partial_r \otimes 1 + 1 \otimes \mathcal{D}_N)$, but these two product forms are in fact unitarily equivalent). However, in this paper, we do not insist that $\mathcal{D}$ is of Dirac-type. One can view assumption (C1) as requiring precisely those properties of Dirac operators which we need below (in particular, to prove Lemma 4.3).
3. We remind the reader that assumption (C2) is motivated by the spectral flow result from Proposition 2.8.
4. We note that, for the operator $\mathcal{T}$, we can for instance choose $\mathcal{T} = \mathcal{S}(x_0)$ for some $x_0 \in K$, but it can be useful to allow for arbitrary relatively compact perturbations.

Definition 3.7. If assumptions (A), (B) and (C) are satisfied, then the generalised Dirac-Schrödinger operator $\mathcal{D}_{\lambda\mathcal{S}}$ is called a *generalised Callias-type operator*.
(We always implicitly assume that $\lambda \geq \lambda_0 > 0$ such that $\mathcal{D}_{\lambda\mathcal{S}}$ is Fredholm.)

We consider the invertible regular self-adjoint operator $\mathcal{T}(\cdot)$ on $C(N, E)$ corresponding to the constant family $\mathcal{T}(y) := \mathcal{T}$ (for $y \in N$). The restriction of the potential $\mathcal{S}(\cdot)$ to the hypersurface N also yields an invertible regular self-adjoint operator $\mathcal{S}_N(\cdot) = \{\mathcal{S}(y)\}_{y \in N}$ on $C(N, E)$. We recall that $\mathcal{S}(y) - \mathcal{T}$ is relatively $\mathcal{T}$-compact for each $y \in N$. Furthermore, $\mathcal{S}(y)\big(\mathcal{T} \pm i\big)^{-1}$ depends norm-continuously on y by assumption (A2). Hence, $\mathcal{S}_N(\cdot) - \mathcal{T}(\cdot)$ is relatively $\mathcal{T}(\cdot)$-compact. We then know from Corollary A.10 that the difference of positive spectral projections $P_+(\mathcal{S}_N(\cdot)) - P_+(\mathcal{T}(\cdot))$ is compact, so that the relative index $\operatorname{rel-ind}\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big)$ is well-defined in $K_1(C(N, A))$ (see Definition 2.2). We are now ready to state our generalisation of the Callias Theorem.

Theorem 3.8 (Generalised Callias Theorem). *Let $\mathcal{D}_{\lambda\mathcal{S}}$ be a generalised Callias-type operator. Then we have the equality*

$$\operatorname{Index}\big(\mathcal{D}_{\lambda\mathcal{S}}\big) = \operatorname{rel-ind}\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big) \otimes_{C(N)} [\mathcal{D}_N] \in K_0(A),$$

where $\otimes_{C(N)}$ denotes the pairing $K_1(C(N, A)) \times K^1(C(N)) \to K_0(A)$.

Remark 3.9. Although the relative index depends explicitly on the choice of $\mathcal{T}$, the theorem in particular shows that the pairing $\operatorname{rel-ind}\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big) \otimes_{C(N)} [\mathcal{D}_N]$ on the right-hand side is in fact independent of this choice. This independence of $\mathcal{T}$ can be understood as a consequence of the cobordism invariance of the index (since N is the boundary of K, the index of $\mathcal{D}_N$ vanishes). In fact, one can also turn this around and prove the cobordism invariance of the index as a consequence of the Callias Theorem (by considering the trivial rank-one bundle $M \times \mathbb{C}$ with the potential $\mathcal{S}(\cdot) = 1$, and the operator $\mathcal{T} = -1$ on $E = A = \mathbb{C}$).

We observe next that our assumption (C2) ensures that the class $[\mathcal{S}(\cdot)]$ of the potential depends only on the hypersurface N. This is the crucial observation which enables one to obtain the index of the Callias-type operator from a computation on the hypersurface N, as in Equation (1.1) in the Introduction.

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Consider the open subset $U := K \cup C \subset M$ with compact closure $\overline{U}$ and boundary $\partial U \simeq N$. We have the short exact sequence

$$0 \to C_0(U, A) \xrightarrow{j} C(\overline{U}, A) \to C(N, A) \to 0 \tag{3.1}$$

and the corresponding cyclic six-term exact sequences in K-theory and K-homology.

Proposition 3.10. *The K-theory class $[\mathcal{S}(\cdot)] \in K_1(C_0(M, A))$ is uniquely determined by an element $\Sigma_N \in K_0(C(N, A))$. More explicitly, we have*

$$[\mathcal{S}(\cdot)] = \iota_{U*} \circ \partial(\Sigma_N),$$

where $\partial\colon K_0(C(N, A)) \to K_1(C_0(U, A))$ denotes the exponential map in the cyclic six-term exact sequence in K-theory corresponding to the short exact sequence (3.1), and where $\iota_{U*}\colon K_1(C_0(U, A)) \to K_1(C_0(M, A))$ is induced by the inclusion $\iota_U\colon C_0(U, A) \hookrightarrow C_0(M, A)$.

Proof. The invertibility of the potential $\mathcal{S}(\cdot)$ outside of the compact subset K ensures that the class $[\mathcal{S}(\cdot)]$ depends only on the restriction of $\mathcal{S}(\cdot)$ to U. Indeed, we have from [Dun19, Lemma 3.8] the equality $[\mathcal{S}(\cdot)] = \iota_{U*}\big([\mathcal{S}(\cdot)|_U]\big)$.

We may assume, without loss of generality, that assumption (C2) holds for all $x \in \overline{U}$ (see Lemma 5.1 below for an explicit computation). Using compactness of $\overline{U}$, it follows that the operator $\mathcal{S}(\cdot)|_{\overline{U}}$ is a relatively compact perturbation of the invertible operator $\mathcal{T}(\cdot)_{\overline{U}} = \{\mathcal{T}\}_{x\in\overline{U}}$, and therefore, $j_*\big(\big[\mathcal{S}(\cdot)|_U\big]\big) = 0 \in K_1(C(\overline{U}, A))$. From the cyclic six-term exact sequence in K-theory, we conclude that $\big[\mathcal{S}(\cdot)|_U\big]$ lies in the image of the exponential map $\partial\colon K_0(C(N, A)) \to K_1(C_0(U, A))$, so there exists a class $\Sigma_N \in K_0(C(N, A))$ such that $\partial(\Sigma_N) = [\mathcal{S}(\cdot)|_U]$. □

The above proposition ensures that we can apply Equation (1.1), and combined with Theorem 3.5, we obtain the equality

$$\operatorname{Index}\big(\mathcal{D}_{\lambda\mathcal{S}}\big) = \Sigma_N \otimes_{C(N)} [\mathcal{D}_N].$$

Thus, in order to prove Theorem 3.8, it remains to explicitly identify the K-theory class $\Sigma_N \in K_0(C(N, A))$ as the relative index of the positive spectral projections $P_+(\mathcal{S}_N(\cdot))$ and $P_+(\mathcal{T}(\cdot))$. We will obtain this identification in Section 5 by first reducing the general statement to the special case of a cylindrical manifold $\mathbb{R} \times N$ (see Theorem 5.4). The main advantage of considering the cylindrical manifold is, roughly speaking, that we can then invert the boundary map in order to explicitly compute a solution Σ_N of the equation $[\mathcal{S}(\cdot)|_U] = \partial(\Sigma_N)$.

3.3. Special cases

In this subsection, we reconsider the two well-known special cases of our generalised Callias Theorem, described in Section 2.

First, in the special case when E is a finite-dimensional Hilbert space, we recover the classical Callias Theorem 2.1 (though only for globally trivial bundles). In fact, we find that the statement of the classical Callias Theorem continues to hold if E is a finitely generated projective module over a *unital* C^*-algebra A.

Corollary 3.11. *Let $\mathcal{D}_{\lambda\mathcal{S}}$ be a generalised Callias-type operator. Suppose furthermore that A is unital and that E is finitely generated and projective over A. Then*

$$\operatorname{Index}\big(\mathcal{D}_{\lambda\mathcal{S}}\big) = \big[\operatorname{Ran} P_+(\mathcal{S}_N(\cdot))\big] \otimes_{C(N)} [\mathcal{D}_N] \in K_0(A).$$

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Proof. The assumptions on A and E ensure that all operators on E are compact. In particular, the operator $\mathcal{T} := -1$ is a relatively compact perturbation of each $\mathcal{S}_N(y)$. With $P_+(\mathcal{T}(\cdot)) = 0$, we therefore obtain

$$\text{rel-ind}\left(P_+(\mathcal{S}_N(\cdot)), 0\right) = \text{Index}\left(0\colon \text{ Ran}\, P_+(\mathcal{S}_N(\cdot)) \to \{0\}\right) = \left[\text{ Ran}\, P_+(\mathcal{S}_N(\cdot))\right].$$

Thus, from Theorem 3.8, we find that

$$\text{Index}\left(\mathcal{D}_{\lambda\mathcal{S}}\right) = \text{rel-ind}\left(P_+(\mathcal{S}_N(\cdot)), 0\right) \otimes_{C(N)} [\mathcal{D}_N] = \left[\text{ Ran}\, P_+(\mathcal{S}_N(\cdot))\right] \otimes_{C(N)} [\mathcal{D}_N]. \qquad \square$$

Second, in the special case where $M = \mathbb{R}$, we recover the equality between the spectral flow and the relative index of spectral projections of the end-points from Proposition 2.8.

Corollary 3.12. *Consider the operator $\mathcal{D} = -i\partial_t$ on the manifold $M = \mathbb{R}$ and a potential $\mathcal{S}(\cdot) = \{\mathcal{S}(t)\}_{t\in\mathbb{R}}$ satisfying assumptions (A), (B) and (C). Suppose for simplicity that the compact subset from assumption (A3) is given by the unit interval $K = [0, 1]$. Then we have the equality*

$$sf\left(\{\mathcal{S}(t)\}_{t\in[0,1]}\right) = \text{rel-ind}\left(P_+(\mathcal{S}(1)), P_+(\mathcal{S}(0))\right).$$

Proof. From Theorem 3.8, we obtain

$$\text{Index}\left(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\right) = \text{rel-ind}\left(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\right) \otimes_{C(N)} [\mathcal{D}_N],$$

where the 'hypersurface' $N = \{0, 1\}$ consists of the endpoints of the unit interval, and $\mathcal{T}$ is any relatively compact perturbation of $\mathcal{S}(0)$. We will examine both the left-hand side and the right-hand side of the above equation.

First, the left-hand side is given by

$$\text{Index}\left(-i\partial_t - i\lambda\mathcal{S}(\cdot)\right) = [\mathcal{S}(\cdot)] \otimes_{C_0(\mathbb{R})} [-i\partial_t] = \text{sf}\left(\{\mathcal{S}(t)\}_{t\in[0,1]}\right),$$

where the first equality is obtained from Theorem 3.5, and the second from [Dun19, Proposition 2.21] (using that trivialising families exist by Proposition 2.8.(1)).

For the right-hand side, we examine the product form of $-i\partial_t$ near $N = \{0, 1\}$. The operator $\mathcal{D}_N$ is just the zero operator on $\mathtt{F}_N = \mathtt{F}_{\{0\}} \oplus \mathtt{F}_{\{1\}} \simeq \mathbb{C} \oplus \mathbb{C}$. We note that the coordinate r increases in the outward direction, so we have $t = -r$ near 0 and $t = 1 + r$ near 1. Thus, on a collar neighbourhood of N, we can write $-i\partial_t \simeq i\partial_r \oplus (-i\partial_r) = -i\partial_r \otimes \Gamma_N$, where the operator Γ_N is given by $(-1) \oplus 1$ on $\mathtt{F}_{\{0\}} \oplus \mathtt{F}_{\{1\}}$. Thus, $\mathtt{F}_{\{0\}} = \mathtt{F}^-_{\{0\}} = \mathbb{C}$ and $\mathtt{F}_{\{1\}} = \mathtt{F}^+_{\{1\}} = \mathbb{C}$, and we can identify $[\mathcal{D}_N] \in KK^0(\mathbb{C}^2, \mathbb{C}) \simeq K^0(\mathbb{C}^2)$ with $(-1) \oplus 1 \in \mathbb{Z} \oplus \mathbb{Z}$. Then the Kasparov product over $C(N) = \mathbb{C}^2$ can be calculated as follows:

$$\begin{aligned}
&\text{rel-ind}\left(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\right) \otimes_{\mathbb{C}^2} [\mathcal{D}_N] \\
&\quad = \text{rel-ind}\left(P_+(\mathcal{S}(0)), P_+(\mathcal{T})\right) \otimes (-1) + \text{rel-ind}\left(P_+(\mathcal{S}(1)), P_+(\mathcal{T})\right) \otimes 1 \\
&\quad = \text{rel-ind}\left(P_+(\mathcal{T}), P_+(\mathcal{S}(0))\right) + \text{rel-ind}\left(P_+(\mathcal{S}(1)), P_+(\mathcal{T})\right) \\
&\quad = \text{rel-ind}\left(P_+(\mathcal{S}(1)), P_+(\mathcal{S}(0))\right). \qquad \square
\end{aligned}$$

4. Generalised Dirac-Schrödinger operators

Consider M, $\mathcal{D}$ and $\mathcal{S}(\cdot)$ satisfying assumption (A). We have defined in Definition 3.1 the operator $\mathcal{D}_{\mathcal{S}} := \mathcal{D} - i\mathcal{S}(\cdot)$ on the initial domain $C^1_c(M, W) \otimes_{C^1_0(M)} \text{Dom}\,\mathcal{D}$. We now also define the operators

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$$\widetilde{\mathcal{D}} := \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{pmatrix}, \qquad \widetilde{\mathcal{S}}(\cdot) := \begin{pmatrix} 0 & +i\mathcal{S}(\cdot) \\ -i\mathcal{S}(\cdot) & 0 \end{pmatrix},$$
$$\widetilde{\mathcal{D}}_{\mathcal{S}} := \widetilde{\mathcal{D}} + \widetilde{\mathcal{S}}(\cdot) = \begin{pmatrix} 0 & \mathcal{D} + i\mathcal{S}(\cdot) \\ \mathcal{D} - i\mathcal{S}(\cdot) & 0 \end{pmatrix},$$

on the initial domain $\left(C_c^1(M,W) \otimes_{C_0^1(M)} \operatorname{Dom}\mathcal{D}\right)^{\oplus 2}$. The operator $\widetilde{\mathcal{D}}_{\mathcal{S}}$ is odd with respect to the $\mathbb{Z}_2$-grading $\Gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We recall that $\mathcal{D}_{\mathcal{S}}$ is called a *generalised Dirac-Schrödinger operator* if $\widetilde{\mathcal{D}}_{\mathcal{S}}$ is regular, self-adjoint and Fredholm. In this case, the operator $\widetilde{\mathcal{D}}_{\mathcal{S}}$ yields a class

$$[\widetilde{\mathcal{D}}_{\mathcal{S}}] \in KK^0(\mathbb{C}, A),$$

corresponding to the $K_0(A)$-valued index of $\mathcal{D}_{\mathcal{S}}$ under the isomorphism $KK^0(\mathbb{C}, A) \simeq K_0(A)$. For the construction of this class in Kasparov's KK-theory and its relation to the Fredholm index, we refer to [Dun19, §2.2].

4.1. Relative index theorem

An important tool for our index computations is the relative index theorem [Dun19, Theorem 4.7], which is an adaptation of a theorem by Bunke [Bun95, Theorem 1.14]. Here, we shall adapt [Dun19, Theorem 4.7] in order to allow for more general situations (in particular, we avoid the assumption (A4) from [Dun19, §3.2]).

We consider the following setting. For $j = 1, 2$, let $\mathrm{F}^j \to M^j$, $\mathcal{D}^j$ and $\mathcal{S}^j(\cdot)$ be as in assumption (A), and assume that the operators $\{\mathcal{S}^j(x)\}_{x \in M^j}$ act on the same Hilbert A-module E. Suppose we have partitions $M^j = \overline{U}^j \cup_{N^j} \overline{V}^j$, where N^j are smooth compact hypersurfaces. Let C^j be open tubular neighbourhoods of N^j, and assume that there exists an isometry $\phi \colon C^1 \to C^2$ (with $\phi(N^1) = N^2$) covered by an isomorphism $\Phi \colon \mathrm{F}^1|_{C^1} \to \mathrm{F}^2|_{C^2}$, such that $\mathcal{D}^1|_{C^1}\Phi^* = \Phi^*\mathcal{D}^2|_{C^2}$ and $\mathcal{S}^2(\phi(x)) = \mathcal{S}^1(x)$ for all $x \in C^1$.

We will identify C^1 with C^2 (as well as N^1 with N^2) via ϕ, and we simply write C (and N). Define two new Riemannian manifolds

$$M^3 := \overline{U}^1 \cup_N \overline{V}^2, \qquad\qquad M^4 := \overline{U}^2 \cup_N \overline{V}^1.$$

Moreover, we glue the bundles using Φ to obtain hermitian vector bundles $\mathrm{F}^3 \to M^3$ and $\mathrm{F}^4 \to M^4$. For $j = 3, 4$, we then obtain corresponding operators $\mathcal{D}^j$ and $\mathcal{S}^j(\cdot)$ satisfying assumption (A).

Theorem 4.1 (Relative index theorem). *Assume that $\widetilde{\mathcal{D}}_{\mathcal{S}}^j$ (for $j = 1, 2$) are regular self-adjoint Fredholm operators with locally compact resolvents. Then $\widetilde{\mathcal{D}}_{\mathcal{S}}^3$ and $\widetilde{\mathcal{D}}_{\mathcal{S}}^4$ are also regular self-adjoint Fredholm operators with locally compact resolvents. Moreover, we have the equality*

$$\operatorname{Index}\left(\mathcal{D}^1 - i\mathcal{S}^1(\cdot)\right) + \operatorname{Index}\left(\mathcal{D}^2 - i\mathcal{S}^2(\cdot)\right) = \operatorname{Index}\left(\mathcal{D}^3 - i\mathcal{S}^3(\cdot)\right) + \operatorname{Index}\left(\mathcal{D}^4 - i\mathcal{S}^4(\cdot)\right) \in K_0(A).$$

Proof. First, we need to check that $\widetilde{\mathcal{D}}_{\mathcal{S}}^3$ and $\widetilde{\mathcal{D}}_{\mathcal{S}}^4$ are also regular self-adjoint and Fredholm. We give the proof only for $\widetilde{\mathcal{D}}_{\mathcal{S}}^3$. We choose smooth functions χ_1 and χ_2 such that

$$\operatorname{supp}\chi_1 \subset U^1 \cup C, \qquad \operatorname{supp}\chi_2 \subset V^2 \cup C, \qquad \chi_1^2 + \chi_2^2 = 1.$$

For $\lambda > 0$, we define

$$R_\pm(\lambda) := \chi_1\left(\widetilde{\mathcal{D}}_{\mathcal{S}}^1 \pm i\lambda\right)^{-1}\chi_1 + \chi_2\left(\widetilde{\mathcal{D}}_{\mathcal{S}}^2 \pm i\lambda\right)^{-1}\chi_2.$$

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Then

$$\big(\widetilde{\mathcal{D}}^3_{\mathcal{S}} \pm i\lambda\big)R_\pm(\lambda) = 1 + [\widetilde{\mathcal{D}}^1, \chi_1]\big(\widetilde{\mathcal{D}}^1_{\mathcal{S}} \pm i\lambda\big)^{-1}\chi_1 + [\widetilde{\mathcal{D}}^2, \chi_2]\big(\widetilde{\mathcal{D}}^2_{\mathcal{S}} \pm i\lambda\big)^{-1}\chi_2 =: 1 + \mathcal{K}_\pm(\lambda).$$

We can pick λ sufficiently large, such that the norm of $\mathcal{K}_\pm(\lambda)$ is less than one. Then $1 + \mathcal{K}_\pm(\lambda)$ is invertible, and $R_\pm(\lambda)(1 + \mathcal{K}_\pm(\lambda))^{-1}$ is a right inverse of $\widetilde{\mathcal{D}}^3_{\mathcal{S}} \pm i\lambda$. Similarly, we can also obtain a left inverse, which proves that $\widetilde{\mathcal{D}}^3_{\mathcal{S}}$ is regular self-adjoint. Moreover, since $R_\pm(\lambda)$ is locally compact, we see that $\widetilde{\mathcal{D}}^3_{\mathcal{S}}$ has locally compact resolvents.

Next, given parametrices Q_1 and Q_2 for $\widetilde{\mathcal{D}}^1_{\mathcal{S}}$ and $\widetilde{\mathcal{D}}^2_{\mathcal{S}}$, respectively, we define

$$Q_3 := \chi_1 Q_1 \chi_1 + \chi_2 Q_2 \chi_2.$$

Then

$$\widetilde{\mathcal{D}}^3_{\mathcal{S}} Q_3 - 1 = \chi_1\big(\widetilde{\mathcal{D}}^1_{\mathcal{S}} Q_1 - 1\big)\chi_1 + [\widetilde{\mathcal{D}}^1, \chi_1]Q_1\chi_1 + \chi_2\big(\widetilde{\mathcal{D}}^2_{\mathcal{S}} Q_2 - 1\big)\chi_2 + [\widetilde{\mathcal{D}}^2, \chi_2]Q_2\chi_2.$$

The terms $\chi_j\big(\widetilde{\mathcal{D}}^j_{\mathcal{S}} Q_j - 1\big)\chi_j$ are compact because Q_j are parametrices. Furthermore, the terms $[\widetilde{\mathcal{D}}^j, \chi_j]Q_j\chi_j$ are compact because $[\widetilde{\mathcal{D}}^j, \chi_j]$ are compactly supported and $\widetilde{\mathcal{D}}^j_{\mathcal{S}}$ have locally compact resolvents. Hence, Q_3 is a right parametrix for $\widetilde{\mathcal{D}}^3_{\mathcal{S}}$. A similar calculation shows that Q_3 is also a left parametrix, and therefore, $\widetilde{\mathcal{D}}^3_{\mathcal{S}}$ is Fredholm. Similarly, the operator $\widetilde{\mathcal{D}}^4_{\mathcal{S}}$ is also regular self-adjoint and Fredholm. The proof of the equality $\operatorname{Index}\big(\mathcal{D}^1 - i\mathcal{S}^1(\cdot)\big) + \operatorname{Index}\big(\mathcal{D}^2 - i\mathcal{S}^2(\cdot)\big) = \operatorname{Index}\big(\mathcal{D}^3 - i\mathcal{S}^3(\cdot)\big) + \operatorname{Index}\big(\mathcal{D}^4 - i\mathcal{S}^4(\cdot)\big) \in K_0(A)$is then exactly as in [Dun19, Theorem 4.7]. □

4.2. The Fredholm index

From here on, we consider $M, \mathcal{D}$ and $\mathcal{S}(\cdot)$ satisfying assumptions (A) and (B). Our aim in this subsection is to prove Theorem 3.4 (see Theorem 4.4 below). We first observe that, thanks to assumption (B), the operator $\widetilde{\mathcal{D}}_{\mathcal{S}}$ has locally compact resolvents.

Proposition 4.2 [KL13, Theorem 6.7]. *The operator $\phi(\widetilde{\mathcal{D}}_{\mathcal{S}} \pm i)^{-1}$ on $L^2(M, E \otimes \mathtt{F})^{\oplus 2}$ is compact for any $\phi \in C_0(M)$. Moreover, if $(\mathcal{S}(\cdot) \pm i)^{-1}$ is compact on $C_0(M, E)$, then $(\widetilde{\mathcal{D}}_{\mathcal{S}} \pm i)^{-1}$ is also compact.*

In order to prove the Fredholm property of $\widetilde{\mathcal{D}}_{\mathcal{S}}$, we need to rescale the potential $\mathcal{S}(\cdot)$ by a sufficiently large $\lambda > 0$. First, we need the following pointwise estimate.

Lemma 4.3. *There exist $\lambda_0 > 0$ and $\epsilon > 0$ such that for any $\lambda \geq \lambda_0$, there exists a compactly supported smooth function $f \in C_c^\infty(M)$ such that for all $x \in M$ and $\psi(x) \in (W \otimes \mathtt{F})^{\oplus 2}$, we have the inequality*

$$\langle\!\langle \lambda\widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle + \big\langle \{\widetilde{\mathcal{D}}, \lambda\widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle + \langle\!\langle f(x)\psi(x)\rangle\!\rangle \geq \epsilon\langle\!\langle \psi(x)\rangle\!\rangle.$$

Proof. We roughly follow the proof of [Dun19, Lemma 5.8], but with somewhat different estimates.

First, since $\lambda\mathcal{S}$ also satisfies assumption (B), we know from Propositions 3.3 and 4.2 that $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}$ is regular self-adjoint and has locally compact resolvents. For any $\alpha \in (0, \infty), x \in M$ and $\psi(x) \in (W \otimes \mathtt{F})^{\oplus 2}$, we have (using the same arguments as in the proof of [KL12, Lemma 7.5])

$$\pm 2\big\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \leq \alpha^2\langle\!\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x)\rangle\!\rangle + \alpha^{-2}\langle\!\langle \psi(x)\rangle\!\rangle,$$

where $\{\cdot,\cdot\}$ denotes the anti-commutator. Using that $\delta_x := \big\|\big[\mathcal{D}, \mathcal{S}(\cdot)\big](x)\big(\mathcal{S}(x) \pm i\big)^{-1}\big\|$ is bounded, we obtain

$$\pm 2\big\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \leq \alpha^2\delta_x^2\langle\!\langle (\widetilde{\mathcal{S}}(x) \pm i)\psi(x)\rangle\!\rangle + \alpha^{-2}\langle\!\langle \psi(x)\rangle\!\rangle. \quad (4.1)$$

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We distinguish between the cases $x \in M \setminus K$ and $x \in K$:

$x \in M \setminus K$: Let $c \equiv c_{M\setminus K} := \inf_{x\in M\setminus K} \|\mathcal{S}(x)^{-1}\|^{-1}$. Then combining (4.1) with the norm inequality $\|(\widetilde{\mathcal{S}}(x) \pm i)\widetilde{\mathcal{S}}(x)^{-1}\| \leq 1 + c^{-1}$, we obtain

$$\pm 2\big\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \leq \alpha^2 \delta_x^2 (1 + c^{-1})^2 \langle\!\langle \widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle + \alpha^{-2}\langle\!\langle \psi(x)\rangle\!\rangle.$$

Now setting $\alpha = \lambda^{1/2}\delta_x^{-1}(1 + c^{-1})^{-1}$ yields

$$\pm 2\big\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \leq \lambda \langle\!\langle \widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle + \lambda^{-1}\delta_x^2(1 + c^{-1})^2\langle\!\langle \psi(x)\rangle\!\rangle$$

and in particular

$$\big\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \geq -\frac{1}{2}\lambda \langle\!\langle \widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle - \frac{1}{2}\lambda^{-1}\delta_x^2(1 + c^{-1})^2\langle\!\langle \psi(x)\rangle\!\rangle.$$

Thus, we have

$$\begin{aligned}\langle\!\langle \lambda\widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle + \big\langle \{\widetilde{\mathcal{D}}, \lambda\widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle &\geq \frac{1}{2}\langle\!\langle \lambda\widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle - \frac{1}{2}\delta_x^2(1 + c^{-1})^2\langle\!\langle \psi(x)\rangle\!\rangle \\ &\geq \frac{1}{2}\big(\lambda^2 c^2 - \delta_x^2(1 + c^{-1})^2\big)\langle\!\langle \psi(x)\rangle\!\rangle.\end{aligned}$$

Now set $\delta_{M\setminus K} := \sup_{x\in M\setminus K} \delta_x = \sup_{x\in M\setminus K} \big\|\big[\mathcal{D}, \mathcal{S}(\cdot)\big](x)\big(\mathcal{S}(x) \pm i\big)^{-1}\big\|$, and pick $\lambda_0 > 0$ large enough such that $\epsilon := \frac{1}{2}\big(\lambda_0^2 c^2 - \delta_{M\setminus K}^2(1 + c^{-1})^2\big) > 0$. Then we have shown that for all $x \in M \setminus K$ and all $\lambda \geq \lambda_0$, we have

$$\langle\!\langle \lambda\widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle + \big\langle \{\widetilde{\mathcal{D}}, \lambda\widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \geq \epsilon\langle\!\langle \psi(x)\rangle\!\rangle. \tag{4.2}$$

$x \in K$: Set $\delta_K := \sup_{x\in K} \delta_x = \sup_{x\in K} \big\|\big[\mathcal{D}, \mathcal{S}(\cdot)\big](x)\big(\mathcal{S}(x) \pm i\big)^{-1}\big\|$, fix $\lambda \geq \lambda_0$ and pick a compactly supported smooth function $f \in C_c^\infty(M)$ such that $f(x)^2 \geq \epsilon + \frac{1}{2}(\lambda^2 + \delta_K^2)$ for all $x \in K$. Inserting $\alpha = \lambda^{1/2}\delta_K^{-1}$ into (4.1), we see that for any $x \in K$, we have

$$\pm 2\big\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \leq \lambda \langle\!\langle \widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle + (\lambda + \lambda^{-1}\delta_K^2)\langle\!\langle \psi(x)\rangle\!\rangle$$

and in particular

$$\big\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle \geq -\frac{1}{2}\lambda \langle\!\langle \widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle - \frac{1}{2}(\lambda + \lambda^{-1}\delta_K^2)\langle\!\langle \psi(x)\rangle\!\rangle.$$

Thus, for any $x \in K$, we have

$$\begin{aligned}&\langle\!\langle \lambda\widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle + \big\langle \{\widetilde{\mathcal{D}}, \lambda\widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\big|\, \psi(x)\big\rangle + \langle\!\langle f(x)\psi(x)\rangle\!\rangle \\ &\quad\geq \frac{1}{2}\langle\!\langle \lambda\widetilde{\mathcal{S}}(x)\psi(x)\rangle\!\rangle - \frac{1}{2}(\lambda^2 + \delta_K^2)\langle\!\langle \psi(x)\rangle\!\rangle + f(x)^2\langle\!\langle \psi(x)\rangle\!\rangle \geq \epsilon\langle\!\langle \psi(x)\rangle\!\rangle.\end{aligned} \tag{4.3}$$

Combining Equations (4.2) and (4.3), we have thus shown the desired inequality for *any* $x \in M$. □

Theorem 4.4.

1. *There exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, the operator $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}$ is Fredholm, and thus, $\mathcal{D}_{\lambda\mathcal{S}}$ is a generalised Dirac-Schrödinger operator.*

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2. *Suppose there exists a compact subset $\hat{K} \supset K$ such that $\hat{\delta} < \frac{\hat{c}^2}{\hat{c}+1}$, where*

$$\hat{\delta} := \sup_{x \in M \setminus \hat{K}} \left\| \left[\mathcal{D}, \mathcal{S}(\cdot)\right](x) \left(\mathcal{S}(x) \pm i\right)^{-1} \right\|, \qquad \hat{c} := \inf_{x \in M \setminus \hat{K}} \|\mathcal{S}(x)^{-1}\|^{-1}.$$

Then the first statement holds with $\lambda_0 = 1$. In particular, $\mathcal{D}_{\mathcal{S}}$ is a generalised Dirac-Schrödinger operator.

Proof. Let $\lambda \geq \lambda_0$, $\epsilon > 0$ and $f \in C_c^\infty(M)$ be given by Lemma 4.3. For any $\psi \in \mathrm{Dom}(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2)$, we then compute

$$\begin{aligned}
\left\langle \psi \,\middle|\, (\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2)\psi \right\rangle &= \left\langle \widetilde{\mathcal{D}}_{\lambda\mathcal{S}}\psi \,\middle|\, \widetilde{\mathcal{D}}_{\lambda\mathcal{S}}\psi \right\rangle + \left\langle \psi \,\middle|\, f^2\psi \right\rangle \\
&= \langle\!\langle \widetilde{\mathcal{D}}\psi \rangle\!\rangle + \langle\!\langle \lambda\widetilde{\mathcal{S}}(\cdot)\psi \rangle\!\rangle + \left\langle \widetilde{\mathcal{D}}\psi \,\middle|\, \lambda\widetilde{\mathcal{S}}(\cdot)\psi \right\rangle + \left\langle \lambda\widetilde{\mathcal{S}}(\cdot)\psi \,\middle|\, \widetilde{\mathcal{D}}\psi \right\rangle + \langle\!\langle f\psi \rangle\!\rangle \\
&\geq \langle\!\langle \lambda\widetilde{\mathcal{S}}(\cdot)\psi \rangle\!\rangle + \left\langle \{\widetilde{\mathcal{D}}, \lambda\widetilde{\mathcal{S}}(\cdot)\}\psi \,\middle|\, \psi \right\rangle + \langle\!\langle f\psi \rangle\!\rangle \\
&= \int_M \Big(\langle\!\langle \lambda\widetilde{\mathcal{S}}(x)\psi(x) \rangle\!\rangle + \left\langle \{\widetilde{\mathcal{D}}, \lambda\widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\middle|\, \psi(x) \right\rangle + \langle\!\langle f(x)\psi(x) \rangle\!\rangle \Big) \mathrm{dvol}(x) \\
&\geq \epsilon \int_M \langle\!\langle \psi(x) \rangle\!\rangle \mathrm{dvol}(x) = \epsilon \langle\!\langle \psi \rangle\!\rangle,
\end{aligned} \tag{4.4}$$

where the inequality on the last line is given by Lemma 4.3. Hence, we have shown that the spectrum of $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2$ is contained in $[\epsilon, \infty)$, and therefore, we have a well-defined inverse $\left(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2\right)^{-1} \in \mathcal{L}_A\left(L^2(M, E \otimes \mathrm{F})^{\oplus 2}\right)$.

We can then construct a parametrix for $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}$ as follows. Pick a smooth function $\chi \in C_c^\infty(M)$ such that $0 \leq \chi \leq 1$, and $\chi(x) = 1$ for all $x \in \operatorname{supp} f$. Write $\chi' := \sqrt{1 - \chi^2}$. Using that $f\chi' = 0$, we calculate that

$$\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}\chi'\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}\left(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2\right)^{-1}\chi' = [\widetilde{\mathcal{D}}, \chi']\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}\left(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2\right)^{-1}\chi' + (\chi')^2.$$

Define the operator

$$Q := \chi\left(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}} - i\right)^{-1}\chi + \chi'\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}\left(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2\right)^{-1}\chi'.$$

We then compute

$$\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}Q - 1 = \left[\widetilde{\mathcal{D}}, \chi\right](\widetilde{\mathcal{D}}_{\lambda\mathcal{S}} - i)^{-1}\chi + i\chi(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}} - i)^{-1}\chi + \left[\widetilde{\mathcal{D}}, \chi'\right]\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}\left(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2\right)^{-1}\chi'.$$

The operators $[\widetilde{\mathcal{D}}, \chi]$ and $[\widetilde{\mathcal{D}}, \chi']$ are smooth and compactly supported, and therefore bounded. Since $(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}} - i)\left(\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2 + f^2\right)^{-\frac{1}{2}}$ is also bounded, it follows from Proposition 4.2 that $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}Q - 1$ is compact. Hence, Q is a right parametrix for $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}$. A similar calculation shows that Q is also a left parametrix, and therefore, $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}$ is Fredholm. We have thus proven the first statement.

For the second statement, we note that we may replace K by the larger compact set $\hat{K}$. Using the inequality $\hat{\delta} < \frac{\hat{c}^2}{\hat{c}+1}$, the proof of Lemma 4.3 (picking $\lambda_0 = 1$) shows that for all $x \in M \setminus \hat{K}$, we have

$$\langle\!\langle \widetilde{\mathcal{S}}(x)\psi(x) \rangle\!\rangle + \left\langle \{\widetilde{\mathcal{D}}, \widetilde{\mathcal{S}}(\cdot)\}(x)\psi(x) \,\middle|\, \psi(x) \right\rangle \geq \epsilon \langle\!\langle \psi(x) \rangle\!\rangle,$$

for $\epsilon := \frac{1}{2}\left(\hat{c}^2 - \hat{\delta}^2(1 + \hat{c}^{-1})^2\right) > 0$. Thus, in this case, the first statement holds with $\lambda_0 = 1$. □

Proposition 4.5. *Suppose that $\{\mathcal{S}(x)\}_{x \in M}$ is uniformly invertible on all of M. Then there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, the generalised Dirac-Schrödinger operator $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}$ is also invertible.*

Proof. Since $\mathcal{S}(\cdot)$ is uniformly invertible, Equation (4.2) now holds for all $x \in M$ (for $\lambda \geq \lambda_0 > 0$), and therefore, Equation (4.4) holds with $f \equiv 0$, which shows that $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}^2$ (and hence $\widetilde{\mathcal{D}}_{\lambda\mathcal{S}}$) is invertible. □

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4.3. The index pairing

In this subsection, we will prove Theorem 3.5. Similarly to [Dun19, Proposition 5.14], we show first that we can replace M by a manifold with cylindrical ends, without affecting the index of the generalised Dirac-Schrödinger operator.

Proposition 4.6. *There exist a precompact open subset U of M and a generalised Dirac-Schrödinger operator $\mathcal{D}'_{\lambda\mathcal{S}}$ on $M' := \overline{U} \cup_{\partial U} (\partial U \times [0,\infty))$ satisfying assumptions (A) and (B), such that*

1. *the operators $\mathcal{D}'$ and $\mathcal{S}'(\cdot)$ on M' agree with $\mathcal{D}$ and $\mathcal{S}(\cdot)$ on M when restricted to U;*
2. *the metric and the operators $\mathcal{D}'$ and $\mathcal{S}'(\cdot)$ on M' are of product form on $\partial U \times [1,\infty)$;*
3. *we have, for λ sufficiently large, the equality* $\operatorname{Index}\big(\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)\big) = \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big) \in K_0(A)$.

In particular, M' is complete and $\mathcal{D}'$ has bounded propagation speed.

Proof. The proof is similar to the proof of [Dun19, Proposition 5.14] but requires some minor adaptations. For completeness, we include the details here.

Let U be a precompact open neighbourhood of K, with smooth compact boundary ∂U. Consider the manifold $M' := \overline{U} \cup_{\partial U} \big(\partial U \times [0,\infty)\big)$ with cylindrical ends. For some $0 < \epsilon < 1$, let $C \simeq \partial U \times (-\epsilon,\epsilon)$ be a tubular neighbourhood of ∂U, such that there exists a diffeomorphism $\phi\colon U \cup C \to \overline{U} \cup_{\partial U} \big(\partial U \times [0,\epsilon)\big) \subset M'$ (which preserves the subset U). Equip M' with a Riemannian metric which is of product form on $\partial U \times [1,\infty)$ (ensuring that M' is complete), and which agrees with $g|_U$ on U. Let $\mathrm{F}' \to M'$ be a hermitian vector bundle which agrees with $\mathrm{F}|_U$ on U. Let $\mathcal{D}'$ be a symmetric elliptic first-order differential operator on $\mathrm{F}' \to M'$, which is of product form on $\partial U \times [1,\infty)$, and which agrees with $\mathcal{D}|_{U\cup C}$ on $U \cup C$. Then $\mathcal{D}'$ has bounded propagation speed and is essentially self-adjoint by [HR00, Proposition 10.2.11].

Let $0 < \delta < \epsilon$ and let $\chi \in C^\infty(\mathbb{R})$ be such that $0 \le \chi(r) \le 1$ for all $r \in \mathbb{R}$, $\chi(r) = 1$ for all r in a neighbourhood of 0, and $\chi(r) = 0$ for all $|r| > \delta$. Consider the family $\{\mathcal{S}'(x)\}_{x\in M'}$ given by

$$\mathcal{S}'(x) := \begin{cases} \mathcal{S}(x), & x \in U, \\ \chi(r)\mathcal{S}(x) + (1-\chi(r))\mathcal{S}(y), & x = (r,y) \in [0,\infty) \times \partial U. \end{cases}$$

We choose δ small enough such that $\chi(r)\mathcal{S}(y) + (1-\chi(r))\mathcal{S}(x)$ is invertible for all $x \in [0,\delta] \times \partial U$. Then the family $\{\mathcal{S}'(x)\}_{x\in M'}$ also satisfies assumptions (A) and (B). Thus, we have constructed a Dirac-Schrödinger operator $\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)$ on M', satisfying the desired properties 1 and 2. It remains to prove the equality $\operatorname{Index}\big(\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)\big) = \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big)$, for which we invoke the relative index theorem.

Let $M^1 := M$ and $M^2 := \partial U \times \mathbb{R}$. Let $C' = \phi(C)$ be the collar neighbourhood of ∂U in M'. We equip M^2 with a complete Riemannian metric which agrees with the metric of M' on $C' \cup \big(\partial U \times (0,\infty)\big)$, and which is of product form on $(-\infty,-1] \times \partial U$. We extend the vector bundle $\mathrm{F}'|_{C'\cup(\partial U\times(0,\infty))}$ to a bundle $\mathrm{F}^2 \to M^2$, and we pick an essentially self-adjoint elliptic first-order differential operator $\mathcal{D}^2$ on F^2 such that $\mathcal{D}^2|_{C'\cup(\partial U\times(0,\infty))} = \mathcal{D}'|_{C'\cup(\partial U\times(0,\infty))}$ (for instance, we can take $\mathcal{D}^2$ to be of product form on $(-\infty,-1) \times \partial U$). We define a family $\{\mathcal{S}^2(x)\}_{x\in M^2}$ by $\mathcal{S}^2(y,r) := \chi(r)\mathcal{S}(y,r) + (1-\chi(r))\mathcal{S}(y)$ for all $y \in \partial U$ and $r \in \mathbb{R}$. Then $\mathrm{F}^2 \to M^2, \mathcal{D}^2$ and $\mathcal{S}^2(\cdot)$ satisfy assumptions (A) and (B). By cutting and pasting along ∂U, we obtain manifolds $M^3 = M'$ and $M^4 = \big(\partial U \times (-\infty,0]\big) \cup_{\partial U} (M \setminus U)$, with corresponding operators $\mathcal{D}^3$, $\mathcal{S}^3(\cdot)$, $\mathcal{D}^4$, and $\mathcal{S}^4(\cdot)$. By Theorem 4.1, we have $\operatorname{Index}\big(\mathcal{D}^1 - i\lambda\mathcal{S}^1(\cdot)\big) + \operatorname{Index}\big(\mathcal{D}^2 - i\lambda\mathcal{S}^2(\cdot)\big) = \operatorname{Index}\big(\mathcal{D}^3 - i\lambda\mathcal{S}^3(\cdot)\big) + \operatorname{Index}\big(\mathcal{D}^4 - i\lambda\mathcal{S}^4(\cdot)\big)$. The potentials $\mathcal{S}^2(\cdot)$ and $\mathcal{S}^4(\cdot)$ are both uniformly invertible, so by Proposition 4.5, we have $\operatorname{Index}\big(\mathcal{D}^2 - i\lambda\mathcal{S}^2(\cdot)\big) = \operatorname{Index}\big(\mathcal{D}^4 - i\lambda\mathcal{S}^4(\cdot)\big) = 0$ (for λ sufficiently large). Since $M^1 = M$ and $M^3 = M'$, we conclude that $\operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big) = \operatorname{Index}\big(\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)\big)$. □

Proposition 4.7 (cf. [Dun19, Proposition 5.10]). *Let M', $\mathcal{D}'$ and $\mathcal{S}'(\cdot)$ be as in Proposition 4.6. Then we have the equality* $[\mathcal{S}'(\cdot)] \otimes_{C_0(M')} [\mathcal{D}'] = \operatorname{Index}\big(\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)\big)$.

Proof. This follows basically from [Dun19, Proposition 5.10]; however, there it was assumed that the graph norms of $\mathcal{S}'(x)$ are *uniformly* equivalent and that the weak derivative $d\mathcal{S}'(\cdot)$ is *uniformly* bounded.

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Together with the bounded propagation speed of $\mathcal{D}'$, this implies the boundedness of $\big[\mathcal{D}', \mathcal{S}'(\cdot)\big]\big(\mathcal{S}'(\cdot) \pm i\big)^{-1}$, as explained in Remark 3.2.3. The proofs given in [Dun19, §5.1] actually only rely on this boundedness of $\big[\mathcal{D}', \mathcal{S}'(\cdot)\big]\big(\mathcal{S}'(\cdot) \pm i\big)^{-1}$. As the latter is required by our assumption (B2), the proof of [Dun19, Proposition 5.10] follows through and the statement follows. □

We are now ready to prove:

Theorem 3.5. *Let M be a connected Riemannian manifold, and let $\{\mathcal{S}(x)\}_{x\in M}$ and $\mathcal{D}$ satisfy assumptions (A) and (B). Then there exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$, the $K_0(A)$-valued index of $\mathcal{D}_{\lambda\mathcal{S}}$ equals the pairing of $[\mathcal{S}(\cdot)] \in K_1(C_0(M, A))$ with $[\mathcal{D}] \in K^1(C_0(M))$.*

Proof. From Proposition 4.6, we obtain a complete manifold M' and a generalised Dirac-Schrödinger operator satisfying assumptions (A) and (B), such that $\mathcal{D}'$ has bounded propagation speed, and such that $\operatorname{Index}\big(\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)\big) = \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big) \in K_0(A)$ (for λ sufficiently large). As in the proof of [Dun19, Theorem 5.15], we have

$$[\mathcal{S}(\cdot)] \otimes_{C_0(M)} [\mathcal{D}] = [\mathcal{S}(\cdot)|_U] \otimes_{C_0(U)} [\mathcal{D}|_U] = [\mathcal{S}'(\cdot)] \otimes_{C_0(M')} [\mathcal{D}'].$$

Moreover, we know from Proposition 4.7 that $[\mathcal{S}'(\cdot)] \otimes_{C_0(M')} [\mathcal{D}'] = \operatorname{Index}\big(\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)\big)$. Altogether, we conclude that

$$[\mathcal{S}(\cdot)] \otimes_{C_0(M)} [\mathcal{D}] = [\mathcal{S}'(\cdot)] \otimes_{C_0(M')} [\mathcal{D}'] = \operatorname{Index}\big(\mathcal{D}' - i\lambda\mathcal{S}'(\cdot)\big) = \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big). \qquad □$$

5. Proof of the main theorem

Let M, $\mathcal{D}$ and $\mathcal{S}(\cdot)$ satisfy assumptions (A), (B) and (C), and consider the generalised Callias-type operator $\mathcal{D}_{\lambda\mathcal{S}}$. We will show that we can replace the manifold M by a cylindrical manifold $\mathbb{R}\times N$ without changing the index of $\mathcal{D}_{\lambda\mathcal{S}}$. Thus, we can reduce the proof of our generalised Callias Theorem (Theorem 3.8) from the general statement to the case of a cylindrical manifold. This reduction is made possible by the relative index theorem (Theorem 4.1).

Lemma 5.1. *We may replace the collar neighbourhood C by a smaller collar neighbourhood $C' \simeq (-2\varepsilon', 2\varepsilon') \times N$ (with $0 < \varepsilon' < \varepsilon$) and the potential $\mathcal{S}(\cdot)$ by a potential $\mathcal{S}'(\cdot)$ satisfying the following:*

- *for all $x \in K \setminus C'$: $\mathcal{S}'(x) = \mathcal{T}$;*
- *for all $x = (r, y) \in C'$: $\mathcal{S}'(x) = \varrho(r)\mathcal{T} + \big(1 - \varrho(r)\big)\mathcal{S}(y)$, for some function $\varrho \in C^\infty(\mathbb{R})$ such that $0 \leq \varrho(r) \leq 1$ for all $r \in \mathbb{R}$, $\varrho(r) = 1$ for all r in a neighbourhood of $(-\infty, -\varepsilon']$, and $\varrho(r) = 0$ for all r in a neighbourhood of $[0, \infty)$,*

such that $[\mathcal{S}(\cdot)] = [\mathcal{S}'(\cdot)] \in K_1(C_0(M, A))$ and (for λ sufficiently large) $\operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big) = \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}'(\cdot)\big) \in K_0(A)$.

Proof.

1. In a first step, we replace $\mathcal{S}(\cdot)$ by a potential $\mathcal{S}''(\cdot)$ which is of 'product form' near N. Let $0 < \varepsilon' < \varepsilon/2$ and let $\chi \in C^\infty(\mathbb{R})$ be such that $0 \leq \chi(r) \leq 1$ for all $r \in \mathbb{R}$, $\chi(r) = 1$ for all $|r| \leq 2\varepsilon'$, and $\chi(r) = 0$ for all $|r| > 3\varepsilon'$. Consider the family $\{\mathcal{S}''(x)\}_{x\in M}$ given by

$$\mathcal{S}''(x) := \begin{cases} \mathcal{S}(x), & x \in M \setminus C, \\ \chi(r)\mathcal{S}(y) + (1 - \chi(r))\mathcal{S}(x), & x = (r, y) \in C \simeq (-2\varepsilon, 2\varepsilon) \times N. \end{cases}$$

   We choose ε' small enough such that $\chi(r)\mathcal{S}(y) + (1 - \chi(r))\mathcal{S}(x)$ is invertible for all $x \in [-3\varepsilon', 3\varepsilon'] \times N$. Then $\mathcal{S}''(\cdot)$ satisfies $\mathcal{S}''(r, y) = \mathcal{S}(y) = \mathcal{S}''(0, y)$ for all $x = (r, y)$ in the collar

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neighbourhood $C' \simeq (-2\varepsilon', 2\varepsilon') \times N$ of N. Connecting $\mathcal{S}(\cdot)$ and $\mathcal{S}''(\cdot)$ via a straight-line homotopy, we obtain $[\mathcal{S}''(\cdot)] = [\mathcal{S}(\cdot)]$. Moreover, since $\mathcal{S}''(\cdot)$ again satisfies assumptions (A) and (B), we can apply Theorem 3.5: there exist λ_0, λ_0'' such that for all $\lambda \geq \max\{\lambda_0, \lambda_0''\}$, we have

$$\operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big) \overset{\lambda\geq\lambda_0}{=} [\mathcal{S}(\cdot)] \otimes_{C_0(M)} [\mathcal{D}] = [\mathcal{S}''(\cdot)] \otimes_{C_0(M)} [\mathcal{D}] \overset{\lambda\geq\lambda_0''}{=} \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}''(\cdot)\big).$$

2. Picking ϱ as in the statement with ε' from step 1, we consider the potential $\mathcal{S}'(\cdot)$ given by

$$\mathcal{S}'(x) := \begin{cases} \mathcal{T}, & x \in K \setminus C', \\ \varrho(r)\mathcal{T} + \big(1 - \varrho(r)\big)\mathcal{S}(y), & x = (r, y) \in C', \\ \mathcal{S}''(x), & x \in M \setminus (K \cup C'). \end{cases} \tag{5.1}$$

We note that $\mathcal{S}'(\cdot)$ again satisfies assumption (A) and therefore also defines a class $[\mathcal{S}'(\cdot)] \in K_1(C_0(M, A))$. The difference $\mathcal{S}'(x) - \mathcal{S}''(x)$ is relatively $\mathcal{S}''(x)$-compact (by choice of $\mathcal{T}$) and vanishes outside of K. Moreover, by assumption (A2), we know that $\big(\mathcal{S}''(x) \pm i\big)\big(\mathcal{T} \pm i\big)^{-1}$ is norm-continuous in x, and consequently, the family of inverses $\big(\mathcal{T} \pm i\big)\big(\mathcal{S}''(x) \pm i\big)^{-1}$ is also norm-continuous in x. Therefore, $\big(\mathcal{S}'(x) - \mathcal{S}''(x)\big)\big(\mathcal{S}''(x) \pm i\big)^{-1}$ is norm-continuous in x. Hence, the family $\mathcal{S}'(\cdot) - \mathcal{S}''(\cdot)$ is relatively $\mathcal{S}''(\cdot)$-compact, and it follows from Proposition A.11 that $[\mathcal{S}'(\cdot)] = [\mathcal{S}''(\cdot)] \in KK^1(\mathbb{C}, C_0(M, A)) \simeq K_1(C_0(M, A))$.

Next, since $\mathcal{S}'(\cdot)$ again satisfies assumption (B), we can again apply Theorem 3.5, and as in step 1, we obtain (for λ sufficiently large) that $\operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}'(\cdot)\big) = \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}''(\cdot)\big)$. □

Definition 5.2. Consider the cylindrical manifold $\mathbb{R} \times N$, along with the pullback vector bundle $\mathtt{F}_{\mathbb{R}\times N}$ obtained from $\mathtt{F}_N \to N$. We identify $\Gamma_c^\infty(\mathtt{F}_{\mathbb{R}\times N}) \simeq C_c^\infty(\mathbb{R}) \otimes \Gamma^\infty(\mathtt{F}_N)$ and consider the essentially self-adjoint elliptic first-order differential operator $\mathcal{D}_{\mathbb{R}\times N}$ on $\mathtt{F}_{\mathbb{R}\times N}$ given by

$$\mathcal{D}_{\mathbb{R}\times N} := -i\partial_r \otimes \Gamma_N + 1 \otimes \mathcal{D}_N.$$

Let $\varrho \in C^\infty(\mathbb{R})$ be as in Lemma 5.1. We define the family $\{\mathcal{S}_{\mathbb{R}\times N}(r, y)\}_{(r,y)\in\mathbb{R}\times N}$ on E given by

$$\mathcal{S}_{\mathbb{R}\times N}(r, y) := \varrho(r)\mathcal{T} + \big(1 - \varrho(r)\big)\mathcal{S}(y).$$

The operator Γ_N from assumption (C1) provides a $\mathbb{Z}_2$-grading on $\mathtt{F}_N$, yielding the decomposition $\mathtt{F}_N = \mathtt{F}_N^+ \oplus \mathtt{F}_N^-$. By assumption, the essentially self-adjoint elliptic first-order differential operator $\mathcal{D}_N$ is odd with respect to this $\mathbb{Z}_2$-grading, and thus, $\mathcal{D}_N$ defines an even K-homology class $[\mathcal{D}_N] \in K^0(C(N)) \equiv K_0(N)$. Similarly, the ungraded operator $\mathcal{D}_{\mathbb{R}\times N}$ yields an odd K-homology class $[\mathcal{D}_{\mathbb{R}\times N}] \in K^1(C_0(\mathbb{R} \times N)) \equiv K_1(\mathbb{R} \times N)$. Furthermore, the operator $-i\partial_r$ on $L^2(\mathbb{R})$ yields an odd K-homology class $[-i\partial_r] \in K^1(C_0(\mathbb{R})) \equiv K_1(\mathbb{R})$.

Lemma 5.3. *The external product of* $[-i\partial_r] \in K^1(C_0(\mathbb{R}))$ *with* $[\mathcal{D}_N] \in K^0(C(N))$ *equals* $[\mathcal{D}_{\mathbb{R}\times N}] \in K^1(C_0(\mathbb{R} \times N))$.

Proof. The statement follows from the description of the odd-even (internal) Kasparov product given in [BMS16, Example 2.38] (noting that the argument remains valid in the simpler case of an external Kasparov product). □

Theorem 5.4. *Consider the cylindrical manifold* $\mathbb{R} \times N$ *with the operators* $\mathcal{D}_{\mathbb{R}\times N}$ *and* $\mathcal{S}_{\mathbb{R}\times N}(\cdot)$ *from Definition 4.2. Then, for* λ *sufficiently large,*

$$\operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big) = \operatorname{Index}\big(\mathcal{D}_{\mathbb{R}\times N} - i\lambda\mathcal{S}_{\mathbb{R}\times N}(\cdot)\big).$$

Proof. Using Lemma 5.1, we may assume that the potential $\mathcal{S}(\cdot)$ agrees with the potential $\mathcal{S}_{\mathbb{R}\times N}(\cdot)$ on the collar neighbourhood C of N and that $\mathcal{S}(x) = \mathcal{T}$ for all $x \in K \setminus C$. We will then apply the relative index theorem (Theorem 4.1) twice.

First, define $V := M \setminus K$, and consider the manifolds

$$M^1 \equiv M = K \cup_N \overline{V}, \qquad M^2 := \mathbb{R} \times N = \big((-\infty, 0] \times N\big) \cup_{\{0\}\times N} \big([0, \infty) \times N\big).$$

On M^2, we consider the operator $\mathcal{D}^2 := \mathcal{D}_{\mathbb{R}\times N}$ and the potential $\mathcal{S}^2(r, y) := \mathcal{S}(y)$, satisfying assumptions (A) and (B). We identify N in M with $\{0\}\times N$ in M^2. Cutting and pasting then gives us two new manifolds $M^3 = K \cup_N \big([0, \infty) \times N\big)$ and $M^4 = \big((-\infty, 0] \times N\big) \cup_N \overline{V}$. Using the relative index theorem, we know that $\mathrm{Index}\,\big(\mathcal{D}^1 - i\lambda\mathcal{S}^1(\cdot)\big) + \mathrm{Index}\,\big(\mathcal{D}^2 - i\lambda\mathcal{S}^2(\cdot)\big) = \mathrm{Index}\,\big(\mathcal{D}^3 - i\lambda\mathcal{S}^3(\cdot)\big) + \mathrm{Index}\,\big(\mathcal{D}^4 - i\lambda\mathcal{S}^4(\cdot)\big) \in K_0(A)$ (for λ sufficiently large). Since $\mathcal{S}^2(\cdot)$ and $\mathcal{S}^4(\cdot)$ are invertible, we know from Proposition 4.5 that also $\mathcal{D}^2 - i\lambda\mathcal{S}^2(\cdot)$ and $\mathcal{D}^4 - i\lambda\mathcal{S}^4(\cdot)$ are invertible (for λ sufficiently large), and it follows that $\mathrm{Index}\,\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big) \equiv \mathrm{Index}\,\big(\mathcal{D}^1 - i\lambda\mathcal{S}^1(\cdot)\big) = \mathrm{Index}\,\big(\mathcal{D}^3 - i\lambda\mathcal{S}^3(\cdot)\big) \in K_0(A)$. Thus, we have replaced the subset V by the half cylinder $(0, \infty) \times N$, with the potential $\mathcal{S}_{\mathbb{R}\times N}(r, y) = \mathcal{S}(y)$ for all $r \in (0, \infty)$.

Second, we can similarly apply the relative index theorem again to replace the subset $K\setminus\big([-\varepsilon, 0]\times N\big)$ by the half cylinder $(-\infty, -\varepsilon) \times N$, equipped with the constant potential $\mathcal{S}_{\mathbb{R}\times N}(r, y) = \mathcal{T}$ for all $r \in (-\infty, -\varepsilon)$, $y \in N$. This completes the proof. □

We are now ready to prove our main theorem.

Theorem 3.8 (Generalised Callias Theorem). *Let $\mathcal{D}_{\lambda\mathcal{S}}$ be a generalised Callias-type operator. Then we have the equality*

$$\mathrm{Index}\,\big(\mathcal{D}_{\lambda\mathcal{S}}\big) = \text{rel-ind}\,\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big) \otimes_{C(N)} [\mathcal{D}_N] \in K_0(A),$$

where $\otimes_{C(N)}$ denotes the pairing $K_1(C(N, A)) \times K^1(C(N)) \to K_0(A)$.

Proof. Consider the cylindrical manifold $\mathbb{R} \times N$ with the operators $\mathcal{D}_{\mathbb{R}\times N}$ and $\mathcal{S}_{\mathbb{R}\times N}(\cdot)$ from Definition 4.2. From Proposition 2.8, we have the equality

$$\text{rel-ind}\,\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big) = \mathrm{sf}\big(\{\mathcal{S}_{\mathbb{R}\times N}(r)\}_{r\in[-\epsilon,0]}\big).$$

Moreover, by [Dun19, Proposition 2.21], the spectral flow of the family $\{\mathcal{S}_{\mathbb{R}\times N}(r)\}_{r\in[-\epsilon,0]}$ equals $[\mathcal{S}_{\mathbb{R}\times N}(\cdot)] \otimes_{C_0(\mathbb{R})} [-i\partial_r]$, so we obtain

$$\text{rel-ind}\,\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big) = [\mathcal{S}_{\mathbb{R}\times N}(\cdot)] \otimes_{C_0(\mathbb{R})} [-i\partial_r].$$

For C^*-algebras A, B and C, recall the map $\tau_C\colon KK(A, B) \to KK(A\otimes C, B\otimes C)$ given by the external Kasparov product with the identity element $1_C \in KK(C, C)$. Applying this with $C = C(N)$, we then have the equalities

$$\begin{aligned}
\text{rel-ind}\,\big(P_+(\mathcal{S}_N(\cdot)), P_+(\mathcal{T}(\cdot))\big) \otimes_{C(N)} [\mathcal{D}_N] &= \Big([\mathcal{S}_{\mathbb{R}\times N}(\cdot)] \otimes_{C_0(\mathbb{R})} [-i\partial_r]\Big) \otimes_{C(N)} [\mathcal{D}_N] \\
&= [\mathcal{S}_{\mathbb{R}\times N}(\cdot)] \otimes_{C_0(\mathbb{R}\times N)} \tau_{C(N)}([-i\partial_r]) \otimes_{C(N)} [\mathcal{D}_N] \\
&= [\mathcal{S}_{\mathbb{R}\times N}(\cdot)] \otimes_{C_0(\mathbb{R}\times N)} \Big([-i\partial_r] \otimes [\mathcal{D}_N]\Big) \\
&= [\mathcal{S}_{\mathbb{R}\times N}(\cdot)] \otimes_{C_0(\mathbb{R}\times N)} [\mathcal{D}_{\mathbb{R}\times N}],
\end{aligned}$$

where the second and third equalities follow from the properties of the Kasparov product, and the fourth equality is given by Lemma 5.3.

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Since the operators $\mathcal{S}_{\mathbb{R}\times N}(\cdot)$ and $\mathcal{D}_{\mathbb{R}\times N}$ on the manifold $\mathbb{R}\times N$ satisfy the assumptions (A) and (B), we may apply Theorem 3.5 to compute the Kasparov product on the manifold $\mathbb{R}\times N$ and obtain

$$[\mathcal{S}_{\mathbb{R}\times N}(\cdot)] \otimes_{C_0(\mathbb{R}\times N)} [\mathcal{D}_{\mathbb{R}\times N}] = \operatorname{Index}\big(\mathcal{D}_{\mathbb{R}\times N} - i\lambda\mathcal{S}_{\mathbb{R}\times N}(\cdot)\big).$$

Finally, from Theorem 5.4, we know that

$$\operatorname{Index}\big(\mathcal{D}_{\mathbb{R}\times N} - i\lambda\mathcal{S}_{\mathbb{R}\times N}(\cdot)\big) = \operatorname{Index}\big(\mathcal{D} - i\lambda\mathcal{S}(\cdot)\big). \qquad \square$$

A. Appendix

In this Appendix, we collect several statements regarding (mostly unbounded) operators on Hilbert C^*-modules. Many of these statements are well-known for operators on Hilbert spaces, but they have not yet appeared (to the author's best knowledge) in the literature for operators on Hilbert C^*-modules. While some of the proofs of these statements are similar to proofs in the Hilbert space context, we have for completeness included detailed proofs in our Hilbert C^*-module context.

Throughout this Appendix, we consider a C^*-algebra A and a Hilbert A-module E. We start with a basic lemma (well-known in the Hilbert space setting), whose proof given in, for example, [HS12, Theorem 9.19], remains valid for adjointable operators on Hilbert C^*-modules.

Lemma A.1. *Let $T_n \xrightarrow{*s} T \in \mathcal{L}_A(E)$ be a $*$-strongly convergent sequence of adjointable operators on E. Then for any compact operator $K \in \mathcal{K}_A(E)$, we have norm-convergence $KT_n \to KT$ and $T_nK \to TK$.*

Proof. For any $\epsilon > 0$, there exists a finite-rank operator $F_\epsilon = \sum_{j=1}^N \theta_{\psi_j,\varphi_j}$ such that $\|K - F_\epsilon\| < \epsilon$. For each $\xi \in E$, we can estimate

$$\|(T_n - T)F_\epsilon\xi\| = \|(T_n - T)\sum_{j=1}^N \psi_j\langle\varphi_j|\xi\rangle\| \leq \sum_{j=1}^N \|(T_n - T)\psi_j\|\,\|\varphi_j\|\,\|\xi\|.$$

Since $T_n\psi_j$ converges to $T\psi_j$ for each j, we obtain for n large enough that $\|(T_n - T)F_\epsilon\| < \epsilon$. Furthermore, since T_n converges strongly to T, the uniform boundedness principle implies that there exists $M \in (0,\infty)$ such that $\|T_n\| \leq M$ for all n. Thus, for n large enough, we obtain

$$\|T_nK - TK\| \leq \|(T_n - T)(K - F_\epsilon)\| + \|(T_n - T)F_\epsilon\| \leq \epsilon\|T_n - T\| + \epsilon \leq \epsilon(M + \|T\| + 1).$$

As $\epsilon > 0$ was arbitrary, this proves T_nK converges to TK in norm.

Next, using that T_n, T are adjointable, we can also estimate, for any $\xi \in E$,

$$\|F_\epsilon(T_n - T)\xi\| = \|\sum_{j=1}^N \psi_j\langle(T_n^* - T^*)\varphi_j|\xi\rangle\| \leq \sum_{j=1}^N \|\psi_j\|\,\|(T_n^* - T^*)\varphi_j\|\,\|\xi\|.$$

By assumption, T_n^* also converges strongly to T^*, so for n large enough, we obtain that $\|F_\epsilon(T_n - T)\| < \epsilon$. Then, proceeding as above, also KT_n converges to KT in norm. $\square$

A.1. Interpolation

The following results are based on [Les05, Proposition A.1]. We follow the adaptation to the case of operators on Hilbert C^*-modules as given in the proof of [LM19, Lemma 7.7].

Proposition A.2. *Let T be an invertible positive regular self-adjoint operator on E. Let S be a densely defined symmetric operator on E with* $\operatorname{Dom} S \supset \operatorname{Dom} T$. *Then the following statements hold:*

1. *ST^{-1} is bounded and adjointable, and $T^{-1}S$ is densely defined and bounded and extends to an adjointable operator $\overline{T^{-1}S}$ with $(\overline{T^{-1}S})^* = ST^{-1}$.*
2. *The operator $T^{-1}ST$ is densely defined, and its adjoint equals $(T^{-1}ST)^* = TST^{-1}$.*
3. *If $T^{-1}ST$ or TST^{-1} is bounded and extends to an adjointable operator, then in fact both $T^{-1}ST$ and TST^{-1} are bounded and extend to adjointable operators, and $\|\overline{TST^{-1}}\| = \|\overline{T^{-1}ST}\|$.*
4. *The operator $T^{-\frac{1}{2}}ST^{-\frac{1}{2}}$ is bounded and extends to an adjointable operator, and its norm satisfies the inequality $\|T^{-\frac{1}{2}}ST^{-\frac{1}{2}}\| \leq \|ST^{-1}\|$.*

Proof.

1. Since S is closable and $\operatorname{Ran} T^{-1} \subset \operatorname{Dom} S$, it is a consequence of the closed graph theorem that ST^{-1} is bounded. For all $\psi \in \operatorname{Dom} S$ and $\xi \in E$, we have $\langle T^{-1}S\psi|\xi\rangle = \langle \psi|ST^{-1}\xi\rangle$, which shows that $T^{-1}S$ has a densely defined adjoint and is therefore closable. Moreover, on the dense subset $\operatorname{Dom} S$, $T^{-1}S$ agrees with the adjoint $(ST^{-1})^*$. Thus, ST^{-1} is bounded and has a densely defined adjoint, which implies that ST^{-1} is in fact adjointable with $(ST^{-1})^* = \overline{T^{-1}S}$.
2. Since T is regular and self-adjoint, $\operatorname{Dom} T^2$ is dense in E, so $\operatorname{Dom}(T^{-1}ST) = \operatorname{Dom}(ST) \supset \operatorname{Dom}(T^2)$ is also dense. Let $\xi \in \operatorname{Dom}(TST^{-1})$ and $\eta \in \operatorname{Dom}(T^{-1}ST)$. Then $T^{-1}\xi \in \operatorname{Dom} S$ with $ST^{-1}\xi \in \operatorname{Dom} T$, and $\eta \in \operatorname{Dom} T$ with $T\eta \in \operatorname{Dom} S$. Consequently,
$$\langle TST^{-1}\xi|\eta\rangle = \langle ST^{-1}\xi|T\eta\rangle = \langle T^{-1}\xi|ST\eta\rangle = \langle \xi|T^{-1}ST\eta\rangle,$$
so $\xi \in \operatorname{Dom}(T^{-1}ST)^*$ and $TST^{-1} \subset (T^{-1}ST)^*$. For the converse, consider $\xi \in \operatorname{Dom}(T^{-1}ST)^*$ and $\eta \in \operatorname{Dom}(T^2) \subset \operatorname{Dom}(T^{-1}ST)$. Then
$$\langle (T^{-1}ST)^*\xi|\eta\rangle = \langle \xi|T^{-1}ST\eta\rangle = \langle T^{-1}\xi|ST\eta\rangle = \langle ST^{-1}\xi|T\eta\rangle.$$
Since $\operatorname{Dom} T^2$ is a core for T, the above equality continues to hold for all $\eta \in \operatorname{Dom} T$. Thus, $ST^{-1}\xi \in \operatorname{Dom} T^* = \operatorname{Dom} T$ and $TST^{-1}\xi = T^*ST^{-1}\xi = (T^{-1}ST)^*\xi$, which shows $(T^{-1}ST)^* \subset TST^{-1}$.
3. Assuming $\overline{T^{-1}ST}$ is adjointable, it follows from 2 that $TST^{-1} = (\overline{T^{-1}ST})^*$ is also bounded and adjointable. Similarly, assuming $\overline{TST^{-1}}$ is adjointable, it follows from 2 that $\overline{T^{-1}ST} = (\overline{TST^{-1}})^*$ is also bounded and adjointable.
4. For simplicity, we assume that $\|T^{-1}\| \leq 1$. For $\xi, \eta \in \operatorname{Dom} T$ and $0 \leq \operatorname{Re} z \leq 1$, consider the operator $P_z := T^{-z}ST^{-1+z}$ and the function
$$f(z) := \langle P_z\xi|\eta\rangle = \langle T^{-z}ST^{-1+z}\xi|\eta\rangle.$$
f is weakly holomorphic on the strip $0 < \operatorname{Re} z < 1$. Moreover, from the estimate
$$\langle\!\langle P_z\xi\rangle\!\rangle \leq \langle\!\langle ST^{-1+z}\xi\rangle\!\rangle \leq \|ST^{-1}\|^2\langle\!\langle T^z\xi\rangle\!\rangle \leq \|ST^{-1}\|^2\langle\!\langle T\xi\rangle\!\rangle,$$
we obtain that $\|f(z)\| \leq \|ST^{-1}\|\,\|T\xi\|\,\|\eta\|$, so f is a bounded function.
Now consider a bounded linear functional $\varphi\colon A \to \mathbb{C}$ with $\|\varphi\| \leq 1$. Since the function $\varphi \circ f$ is holomorphic and bounded on the strip $0 \leq \operatorname{Re} z \leq 1$, it follows from the Hadamard 3-line Theorem that $\varphi \circ f$ is bounded by its suprema on the boundary $\operatorname{Re} z \in \{0, 1\}$. On this boundary $\operatorname{Re} z \in \{0, 1\}$, we have $\|P_z\| = \|P_0\| = \|ST^{-1}\|$, so from the Hadamard 3-line Theorem, we obtain for all $0 \leq \operatorname{Re} z \leq 1$ that
$$\begin{aligned}\left|\varphi\big(f(z)\big)\right| = \left|\varphi\big(\langle P_z\xi|\eta\rangle\big)\right| &\leq \sup_{w\in\mathbb{C}:\operatorname{Re} w=0,1} \left|\varphi\big(\langle P_w\xi|\eta\rangle\big)\right| \\ &\leq \sup_{w\in\mathbb{C}:\operatorname{Re} w=0,1} \left\|\langle P_w\xi|\eta\rangle\right\| \leq \|ST^{-1}\|\,\|\xi\|\,\|\eta\|.\end{aligned}$$

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Since there exists a bounded linear functional φ with $|\varphi(f(z))| = \|f(z)\|$, it follows that also $\|f(z)\| \leq \|ST^{-1}\| \, \|\xi\| \, \|\eta\|$. Taking the supremum over all ξ and η with $\|\xi\| = \|\eta\| = 1$, we conclude that P_z is bounded and extends to an adjointable operator $\overline{P_z}$ satisfying $\|\overline{P_z}\| \leq \|ST^{-1}\|$. □

Corollary A.3. *Let T be an invertible positive regular self-adjoint operator on E. Let $F = F^* \in \mathcal{L}_A(E)$ and assume that the operator $T^{-\frac{1}{2}}FT^{\frac{1}{2}}$ is bounded and extends to an adjointable operator. Then*

$$\|F\| \leq \|\overline{T^{-\frac{1}{2}}FT^{\frac{1}{2}}}\| = \|\overline{T^{\frac{1}{2}}FT^{-\frac{1}{2}}}\|.$$

Proof. We note that $T^{\frac{1}{2}}$ is also an invertible positive regular selfadjoint operator. Certainly, $\mathrm{Dom}\, F = E \supset \mathrm{Dom}\, T^{\frac{1}{2}}$, and since $T^{-\frac{1}{2}}FT^{\frac{1}{2}}$ is bounded (and extends to an adjointable operator), we know from Proposition A.2.3 that also $T^{\frac{1}{2}}FT^{-\frac{1}{2}}$ is bounded (and extends to an adjointable operator) and $\|T^{\frac{1}{2}}FT^{-\frac{1}{2}}\| = \|T^{-\frac{1}{2}}FT^{\frac{1}{2}}\|$.

Now consider the symmetric operator $S := T^{\frac{1}{2}}FT^{\frac{1}{2}}$. We have just seen that $ST^{-1} = T^{\frac{1}{2}}FT^{-\frac{1}{2}}$ is bounded (i.e., $\mathrm{Dom}\, T \subset \mathrm{Dom}\, S$). Hence, by Proposition A.2.4, we find that

$$\|F\| = \|T^{-\frac{1}{2}}ST^{-\frac{1}{2}}\| \overset{A.2.4}{\leq} \|ST^{-1}\| = \|T^{\frac{1}{2}}FT^{-\frac{1}{2}}\| \overset{A.2.3}{=} \|T^{-\frac{1}{2}}FT^{\frac{1}{2}}\|.$$

□

A.2. Convergence of unbounded operators

The following result generalises one of the statements in [Les05, Proposition 2.2], regarding convergence of unbounded operators with respect to certain topologies, to the context of regular operators on Hilbert C^*-modules.

Proposition A.4. *Let $\mathcal{D}$ be a regular selfadjoint operator on E. We view the domain $W := \mathrm{Dom}\, \mathcal{D} \subset E$ as a Hilbert A-module with the graph norm. Let T and T_n (for all $n \in \mathbb{N}$) be regular selfadjoint operators on E with $\mathrm{Dom}\, T = \mathrm{Dom}\, \mathcal{D}$ and $\mathrm{Dom}\, T_n = \mathrm{Dom}\, \mathcal{D}$ for all $n \in \mathbb{N}$.*

If $(T_n - T)(\mathcal{D} + i)^{-1}$ converges in norm to 0 as $n \to \infty$, then also $F_{T_n} - F_T$ converges in norm to 0 as $n \to \infty$.

Proof. The proof is similar to the Hilbert space proof given in [Les05, Proposition 2.2]. For completeness, we include the details here.

We pick $0 < \epsilon < \frac{1}{2}$. Since $\mathrm{Dom}\, \mathcal{D} = \mathrm{Dom}\, T$, we know that $(\mathcal{D} + i)(T + i)^{-1}$ is bounded, so there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, we have

$$\left\|(T - T_n)(T + i)^{-1}\right\| \leq \epsilon \quad \text{and} \quad \left\|(T + i)^{-1}(T - T_n)\right\| \leq \epsilon.$$

Consequently, the operator $(T+i)^{-1}(T_n+i) = 1-(T+i)^{-1}(T-T_n)$ is invertible, and from the Neumann series, we obtain the norm bound

$$\left\|(T_n + i)^{-1}(T + i)\right\| \leq \sum_{k=0}^{\infty} \epsilon^k = \frac{1}{1-\epsilon} < 2.$$

Thus, for all $\psi \in E$, we have (in the C^*-algebra A) the inequality

$$\langle\!\langle (T_n + i)^{-1}\psi \rangle\!\rangle \leq \frac{1}{(1-\epsilon)^2} \langle\!\langle (T + i)^{-1}\psi \rangle\!\rangle,$$

or equivalently, we have the operator inequality

$$(1 + T_n^2)^{-1} \leq \frac{1}{(1-\epsilon)^2}(1 + T^2)^{-1}.$$

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Furthermore, from the estimate $\|(T+i)^{-1}(T_n+i)\| \le 1+\epsilon$, we similarly obtain the operator inequality

$$(1+T^2)^{-1} \le (1+\epsilon)^2(1+T_n^2)^{-1}.$$

Thus, taking square roots, we have

$$\frac{1}{1+\epsilon}(1+T^2)^{-\frac{1}{2}} \le (1+T_n^2)^{-\frac{1}{2}} \le \frac{1}{1-\epsilon}(1+T^2)^{-\frac{1}{2}}.$$

Subtracting $(1+T^2)^{-\frac{1}{2}}$ yields

$$-\frac{\epsilon}{1+\epsilon}(1+T^2)^{-\frac{1}{2}} \le (1+T_n^2)^{-\frac{1}{2}} - (1+T^2)^{-\frac{1}{2}} \le \frac{\epsilon}{1-\epsilon}(1+T^2)^{-\frac{1}{2}},$$

from which we obtain the norm estimate

$$\left\|(1+T^2)^{\frac{1}{4}}(1+T_n^2)^{-\frac{1}{2}}(1+T^2)^{\frac{1}{4}} - 1\right\| \le \frac{\epsilon}{1-\epsilon}.$$

In particular, $(1+T^2)^{\frac{1}{4}}(1+T_n^2)^{-\frac{1}{2}}(1+T^2)^{\frac{1}{4}}$ is bounded. Since also $(1+T^2)^{-\frac{1}{4}}T_n(1+T^2)^{-\frac{1}{4}}$ is bounded by Proposition A.2.4, the estimate

$$\left\|(1+T^2)^{-\frac{1}{4}}F_{T_n}(1+T^2)^{\frac{1}{4}}\right\| \le \left\|(1+T^2)^{-\frac{1}{4}}T_n(1+T^2)^{-\frac{1}{4}}\right\| \left\|(1+T^2)^{\frac{1}{4}}(1+T_n^2)^{-\frac{1}{2}}(1+T^2)^{\frac{1}{4}}\right\|$$

shows that $(1+T^2)^{-\frac{1}{4}}F_{T_n}(1+T^2)^{\frac{1}{4}}$ is bounded. Thus, we can use Corollary A.3 to estimate the difference of the bounded transforms of T and T_n (for all $n \ge N_0$):

$$\begin{aligned}
\left\|F_T - F_{T_n}\right\| &\overset{A.3}{\le} \left\|(1+T^2)^{-\frac{1}{4}}\big(F_T - F_{T_n}\big)(1+T^2)^{\frac{1}{4}}\right\| \\
&\le \left\|(1+T^2)^{-\frac{1}{4}}\big(T - T_n\big)(1+T^2)^{-\frac{1}{4}}\right\| \\
&\quad + \left\|(1+T^2)^{-\frac{1}{4}}T_n\big((1+T^2)^{-\frac{1}{2}} - (1+T_n^2)^{-\frac{1}{2}}\big)(1+T^2)^{\frac{1}{4}}\right\| \\
&\overset{A.2.4}{\le} \left\|\big(T - T_n\big)(1+T^2)^{-\frac{1}{2}}\right\| \\
&\quad + \left\|(1+T^2)^{-\frac{1}{4}}T_n(1+T^2)^{-\frac{1}{4}}\right\| \left\|1 - (1+T^2)^{\frac{1}{4}}(1+T_n^2)^{-\frac{1}{2}}(1+T^2)^{\frac{1}{4}}\right\| \\
&\overset{A.2.4}{\le} \epsilon + \left\|T_n(1+T^2)^{-\frac{1}{2}}\right\|\frac{\epsilon}{1-\epsilon} \\
&\le \epsilon + (1+\epsilon)\frac{\epsilon}{1-\epsilon} = \epsilon\left(1 + \frac{1+\epsilon}{1-\epsilon}\right) \le 4\epsilon,
\end{aligned}$$

where we used that $\epsilon < \frac{1}{2}$. We note that this inequality still holds for all $n \ge N_0$. Since $0 < \epsilon < \frac{1}{2}$ was arbitrary, this proves that $\left\|F_T - F_{T_n}\right\| \to 0$ as $n \to \infty$. □

A.3. Relatively compact perturbations

In this subsection, we study relatively compact perturbations of regular self-adjoint operators. Propositions A.6 and A.7 below are well-known facts for operators on Hilbert spaces but appear not to be present in the literature on Hilbert C^*-modules.

Definition A.5. Let T be a regular self-adjoint operator on E. A densely defined operator R on E is called *relatively T-compact* if $\mathrm{Dom}(T) \subset \mathrm{Dom}(R)$ and $R(T \pm i)^{-1}$ is compact.

The assumption $\mathrm{Dom}(T) \subset \mathrm{Dom}(R)$ implies that R is also relatively T-bounded. In fact, relative T-compactness implies that the relative T-bound can be chosen to be arbitrarily small. This is a well-known

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fact for operators on Hilbert spaces; we show next that this fact remains true on Hilbert C^*-modules, by adapting the proof of [HS12, Theorem 14.2].

Proposition A.6. *Let T be a regular self-adjoint operator on E, and let R be relatively T-compact. Then for all $\epsilon > 0$, there exists $C_\epsilon \geq 0$ such that for all $\psi \in \mathrm{Dom}(T)$, we have*

$$\|R\psi\| \leq \epsilon\|T\psi\| + C_\epsilon\|\psi\|.$$

Proof. We note that the operators $(T-i)(T-in)^{-1}$ converge $*$-strongly to 0 as $n \to \infty$ (this follows, for example, from [KL12, Lemma 7.2]). We can write

$$R(T-in)^{-1} = R(T-i)^{-1}(T-i)(T-in)^{-1}.$$

Since $R(T-i)^{-1}$ is compact and $(T-i)(T-in)^{-1} \xrightarrow{*s} 0$, the operator $R(T-in)^{-1}$ converges to 0 in norm by Lemma A.1. Thus, given any $\epsilon > 0$, we can choose n large enough such that $\|R(T-in)^{-1}\| < \epsilon$. Then for any $\psi \in \mathrm{Dom}(T)$, we have

$$\|R\psi\| \leq \|R(T-in)^{-1}\|\,\|(T-in)\psi\| \leq \epsilon\|(T-in)\psi\| \leq \epsilon\|T\psi\| + \epsilon n\|\psi\|,$$

where $C_\epsilon := \epsilon n$ is independent of ψ. □

Proposition A.7. *Let T be a regular self-adjoint operator on E, and let R be a symmetric operator on E which is relatively T-compact. Then $T+R$ is also regular and self-adjoint on* $\mathrm{Dom}(T+R) = \mathrm{Dom}(T)$.

Proof. By Proposition A.6, we have for any $0 < a < 1$ that $\|R\psi\| \leq a\|T\psi\| + C_a\|\psi\|$ for all $\psi \in \mathrm{Dom}(T)$. It then follows from the Kato-Rellich Theorem on Hilbert C^*-modules ([KL12, Theorem 4.5]) that $T+R$ is also regular and self-adjoint with $\mathrm{Dom}(T+R) = \mathrm{Dom}(T)$. □

The following result generalises [Les05, Proposition 3.4] to the context of regular operators on Hilbert C^*-modules.

Proposition A.8. *Let T be a regular self-adjoint operator on E, and let R be a symmetric operator on E which is relatively T-compact. Then $F_{T+R} - F_T$ is compact.*

Proof. The proof is similar to the Hilbert space proof given in [Les05, Proposition 3.4] but requires minor adaptations. For completeness, we include the details here.

1. We first prove a special case: assume that R is compact and $\mathrm{Ran}\,R \subset \mathrm{Dom}\,T$. In this case, it suffices to show the compactness of $F_{T+R} - F_T - R\big(1+(T+R)^2\big)^{-\frac12} = T\big(1+(T+R)^2\big)^{-\frac12} - T\big(1+T^2\big)^{-\frac12}$. We note that $(T+R)^2 - T^2 = TR + R(T+R)$ is well-defined on $\mathrm{Dom}\,T = \mathrm{Dom}(T+R)$. We can then use the integral formula $(1+T^2)^{-\frac12} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac12}(1+\lambda+T^2)^{-1}\,\mathrm{d}\lambda$ (and similarly for $T+R$) along with the resolvent identity to rewrite
$$\begin{aligned} &T\big(1+(T+R)^2\big)^{-\frac12} - T\big(1+T^2\big)^{-\frac12} \\ &\quad= \frac{1}{\pi}\int_0^\infty \lambda^{-\frac12} T\big(1+\lambda+T^2\big)^{-1}\big(T^2-(T+R)^2\big)\big(1+\lambda+(T+R)^2\big)^{-1}\,\mathrm{d}\lambda \\ &\quad= -\frac{1}{\pi}\int_0^\infty \lambda^{-\frac12} T\big(1+\lambda+T^2\big)^{-1}\big(TR+R(T+R)\big)\big(1+\lambda+(T+R)^2\big)^{-1}\,\mathrm{d}\lambda. \end{aligned}$$
Observing that the integrand is compact and of order $\mathcal{O}(\lambda^{-\frac32})$, we see that the integral converges in norm to a compact operator, and we conclude that $F_{T+R} - F_T$ is compact.
2. We now prove the general case by reducing to the special case. For $n \in \mathbb{N}$, consider compactly supported continuous functions $\phi_n \in C_c(\mathbb{R})$ satisfying $0 \leq \phi_n(x) \leq 1$ for all $x \in \mathbb{R}$, $\phi_n(x) = 1$ if $|x| \leq n$, and $\phi_n(x) = 0$ if $|x| \geq n+1$. Using continuous functional calculus, we construct the

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operators $\phi_n(T) \in \mathcal{L}_A(E)$, and we note that $\operatorname{Ran}\phi_n(T) \subset \operatorname{Dom} T$ (since ϕ_n is compactly supported). We now consider the operators

$$R_n := \phi_n(T)R\phi_n(T) = \underbrace{\phi_n(T)}_{\text{bounded}}\ \underbrace{R(T-i)^{-1}}_{\text{compact}}\ \underbrace{(T-i)\phi_n(T)}_{\text{bounded}},$$

and we note that R_n is compact with $\operatorname{Ran} R_n \subset \operatorname{Dom} T$. Thus, the special case applies to R_n.

As $n \to \infty$, the operators $\phi_n(T)$ converge strongly (hence, by self-adjointness, $*$-strongly) to the identity (see, for example, [KL12, Lemma 7.2]). Since $R(T-i)^{-1}$ is compact, it follows from Lemma A.1 that $R_n(T-i)^{-1}$ converges in norm to $R(T-i)^{-1}$. From Proposition A.4, we therefore obtain that F_{T+R_n} converges in norm to F_{T+R}. Since $F_{T+R_n} - F_T$ is compact by the special case, we conclude that also $F_{T+R} - F_T$ is compact. □

Proposition A.9. *Let T be a regular self-adjoint operator on E, and let R be a symmetric operator on E which is relatively T-compact. Let $f \in C(\mathbb{R})$ be a continuous function for which the limits $\lim_{x\to\pm\infty} f(x)$ exist. Then $f(T+R) - f(T)$ is compact.*

Proof. The statement clearly holds for constant functions, and by Proposition A.8 also for the 'bounded transform function' $b \in C(\mathbb{R})$ given by $b(x) := x(1+x^2)^{-\frac{1}{2}}$. It remains to prove the statement for functions $f \in C_0(\mathbb{R})$ vanishing at infinity, and for this, it suffices to consider $f(x) = (x \pm i)^{-1}$. But for the latter, the statement follows immediately from the resolvent identity and compactness of $R(T \pm i)^{-1}$:

$$(T + R \pm i)^{-1} - (T \pm i)^{-1} = -(T + R \pm i)^{-1} R (T \pm i)^{-1}.$$ □

The following result partly generalises [Les05, Corollary 3.5] to the context of regular operators on Hilbert C^*-modules, under somewhat stronger assumptions.

Corollary A.10. *Let T be a regular self-adjoint operator on E, and let R be a symmetric operator on E which is relatively T-compact. Assume that T and T + R are both invertible. Then the difference of positive spectral projections $P_+(T+R) - P_+(T)$ is compact.*

Proof. Since T and $T+R$ are invertible, there exists an $\epsilon > 0$ such that $(-\epsilon, \epsilon)$ does not intersect with the union $\operatorname{spec}(T) \cup \operatorname{spec}(T+R)$ of the spectra of T and $T+R$. Then the positive spectral projections can be defined via continuous functional calculus (i.e., we can take $\chi \in C(\mathbb{R})$ with $\chi|_{(-\infty,-\epsilon]} \equiv 0$ and $\chi|_{[\epsilon,\infty)} \equiv 1$ and see that $P_+(T) = \chi(T)$ and $P_+(T+R) = \chi(T+R)$). The statement then follows from Proposition A.9. □

A regular operator T on E is called *Fredholm* if there exists a *parametrix* Q such that (the closure of) $QT - 1$ and $TQ - 1$ are compact operators on E. We recall that an odd resp. even regular self-adjoint Fredholm operator T on a possibly $\mathbb{Z}_2$-graded Hilbert A-module E yields a well-defined class $[T]$ in $KK^0(\mathbb{C}, A) \simeq K_0(A)$ resp. $KK^1(\mathbb{C}, A) \simeq K_1(A)$; for details of the construction, we refer to [Dun19, §2.2].

Our last result shows that this K-theory class is stable under relatively compact perturbations.

Proposition A.11. *Let T be a regular self-adjoint Fredholm operator on E, and let R be a symmetric operator on E which is relatively T-compact. Then*

1. *$T + R$ is also regular, selfadjoint and Fredholm, and any parametrix for T is also a parametrix for $T + R$.*
2. *$[T+R] = [T] \in KK^p(\mathbb{C}, A) \simeq K_p(A)$ (where $p = 0$ if R, T are odd, and $p = 1$ otherwise).*

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Proof.

1. We first note that $T + R$ is also regular and selfadjoint by Proposition A.7. If $Q \in \mathcal{L}_A(E)$ is a parametrix for T, then it is also a parametrix for $T + R$ since
$$(T + R)Q - 1 = (TQ - 1) + R(T - i)^{-1}(T - i)Q$$
is compact. Similarly, also $Q(T + R) - 1$ is compact.
2. Let $T_t := T + tR$, and consider the operator $T_\bullet = \{T_t\}_{t\in[0,1]}$ on the Hilbert $C([0, 1], A)$-module $C([0, 1], E)$. Since $t \mapsto T_t\psi$ is continuous for each $\psi \in \mathrm{Dom}(T)$, we know that $T_\bullet$ is regular and self-adjoint ([DM20, Lemma 1.15]).

   If $Q \in \mathcal{L}_A(E)$ is a parametrix for T, then by 1, it is also a parametrix for T_t for each $t \in [0, 1]$ since tR is relatively T-compact. Consequently, noting that $t \mapsto tRQ$ is norm-continuous, the constant family $Q_\bullet = \{Q\}_{t\in[0,1]}$ is a parametrix for $T_\bullet$. Hence, $T_\bullet$ is a regular self-adjoint Fredholm operator on the Hilbert $C([0, 1], A)$-module $C([0, 1], E)$ and therefore a *homotopy* between T and $T + R$ (in the sense of [Dun19, Definition 2.13]). Thus, $[T] = [T + R]$ by [Dun19, Proposition 2.14]. □

In the $\mathbb{Z}_2$-graded case, where we have a decomposition $E = E_+ \oplus E_-$ and T is odd (i.e., maps $E_\pm \to E_\mp$), the class $[T] \in KK^0(\mathbb{C}, A)$ corresponds to the $K_0(A)$-valued index of $T_+ := T|_{E_+} : E_+ \to E_-$ under the isomorphism $KK^0(\mathbb{C}, A) \simeq K_0(A)$. Thus, in this case, the above result translates into the stability of the $K_0(A)$-valued index under relatively compact perturbations.

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