Can. J. Math., Vol. XL, No. 1, 1988, pp. 217-227

SHAPE EQUIVALENCES OF WHITNEY CONTINUA OF CURVES

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1. Introduction. By a *compactum*, we mean a compact metric space. A *continuum* is a connected compactum. A *curve* is a 1-dimensional continuum. Let X be a continuum and let $C(X)$ be the hyperspace of (nonempty) subcontinua of X. $C(X)$ is metrized with the Hausdorff metric (e.g., see [**12**] or [**18**]). One of the most convenient tools in order to study the structure of $C(X)$ is a monotone map $\omega:C(X) \to [0, \omega(X)]$ defined by H. Whitney [**25**]. A map $\omega:C(X) \to [0, \omega(X)]$ is said to be a *Whitney map* for $C(X)$ provided that

(1) $\omega(\{x\}) = 0$ for each $x \in X$, and
(2) $\omega(A) < \omega(B)$ whenever $A, B \in C(X)$, $A \subset B$ and $A \neq B$.

The continua $\{\omega^{-1}(t)\}$ $(0 < t < \omega(X))$ are called the *Whitney continua* of X. We may think of the map ω as measuring the size of a continuum. Note that $\omega^{-1}(0)$ is homeomorphic to X and $\omega^{-1}(\omega(X)) = \{X\}$. Naturally, we are interested in the structures of $\omega^{-1}(t)$ $(0 < t < \omega(X))$. In [**14**], J. Krasinkiewicz proved that if X is a circle-like continuum and ω is any Whitney map for $C(X)$, then for any $0 < t < \omega(X)$ $\omega^{-1}(t)$ is shape equivalent to X, i.e., $\mathrm{Sh}\, \omega^{-1}(t) = \mathrm{Sh}\, X$ (e.g., see [**1**] or [**17**]). In [**8**], we proved the following: If one of the conditions (i) and (ii) is satisfied, then the shape morphism

$$f_{0t}:X \to \omega^{-1}(t) \ (0 < t < \omega(X)),$$

which is defined in [**7**] and [**8**], is a shape equivalence.

(i) X is a strongly winding curve.
(ii) X is a $\theta(m)$-curve $(m < \infty)$ and each proper subcontinuum of X is tree-like.

Note that if X is a strongly winding curve, then each proper subcontinuum of X is tree-like, and note that if X is a $\theta(m)$-curve, then $\mathrm{Fd}\, \omega^{-1}(t) \leqq m - 1$ (see [**8**]). It is easily seen that if each proper subcontinuum of a continuum X is an FAR (e.g., see [**1**] or [**17**]), then $\dim X \leqq 1$. In [**8**], in order to prove the case (ii) we used the following well-known result in shape theory (e.g., see [**17**]): If $f:X \to Y$ is a cell-like

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Received September 26, 1986. This work is dedicated to Professor Masahiro Sugawara on his 60th birthday.

map between compacta and Fd X and Fd Y are finite, then f is a shape equivalence. The condition that Fd X and Fd Y are finite can not be omitted. Hence, in (ii) we needed the condition that X is a $\theta(m)$-curve. In fact, it was shown that there exist a curve X and a Whitney map ω for $C(X)$ such that Fd $\omega^{-1}(t) = \infty$ for some t $(0 < t < \omega(X))$ (see [**8**]).

In this paper, we give a better result than cases (i) and (ii). The result is best possible. Precisely, we prove the following: Let X be a curve and let ω be any Whitney map for $C(X)$. Let $0 < t < \omega(X)$. If each element of $\omega^{-1}(t)$ is a tree-like continuum, then Fd $\omega^{-1}(t) \leqq 1$. This is an affirmative answer to [**8**, (4.4)]. Hence the shape morphism $f_{0t}:X \to \omega^{-1}(t)$ is a shape equivalence, in particular,

$$\text{Sh } \omega^{-1}(t) = \text{Sh } X.$$

Conversely, if the shape morphism $f_{0t}:X \to \omega^{-1}(t)$ is a shape equivalence, then each element of $\omega^{-1}(t)$ is a tree-like continuum. As a corollary, we show that if X is a curve and calm (see [**3**]), then there is t_0 $(0 < t_0 < \omega(X))$ such that $f_{0t}:X \to \omega^{-1}(t)$ is a shape equivalence for each $0 < t \leqq t_0$.

We refer readers to [**18**] for hyperspace theory. Also, we refer readers to [**1**] and [**17**] for shape theory.

2. Preliminaries. Let X be a compactum. We say that X has *fundamental dimension* Fd $X \leqq n$ (see [**1**]) provided that for any ANR M and any map $f:X \to M$, there exist a n-dimensional polyhedron P and maps $f_1:X \to P$, $f_2:P \to M$ such that $f_2f_1 \simeq f$. A continuum X is said to be *tree-like* provided that every open cover of X can be refined by a finite open cover having nerve a tree, that is, having a simply connected 1-dimensional polyhedron.

In [**2**], J. H. Case and R. E. Chamberlin proved that a continuum X is tree-like if and only if X is 1-dimensional and every map from X to any 1-dimensional polyhedron is null-homotopic. Let X be any continuum and let ω be any Whitney map for $C(X)$. Let $0 < t < \omega(X)$. Now, we define the shape morphism $f_{0t}:X \to \omega^{-1}(t)$ as follows (see [**7**] or [**8**]). Suppose that X is lying in the Hilbert cube $Q = [0, 1]^{\infty}$. Choose a decreasing sequence $X_1 \supset X_2 \supset \dots$, of Peano continua such that for each n, X_n is a closed neighborhood of X in Q and

$$X = \bigcap_{n=1}^{\infty} X_n.$$

By Ward's result [**24**], there is a Whitney map

$$\mu:C(Q) \to [0, \mu(Q)]$$

which is an extension of ω. Set $\omega_n = \mu|C(X_n)$. Then ω_n is a Whitney map for $C(X_n)$. Note that $\omega_1^{-1}(t) \supset \omega_2^{-1}(t) \supset \dots$, and

$$\cap\ \omega_n^{-1}(t) = \omega^{-1}(t) \quad \text{for each } 0 < t < \omega(X).$$

For each n, we construct a map $f_n:X \to \omega_n^{-1}(t)$ as follows. Since X_n is a Peano continuum, there is a convex metric d_n on X_n. Define the following homotopy

$$K_n:C(X_n) \times [0, \infty) \to C(X_n)$$

by letting

$$K_n(A, s) = \{x \in X_n|\ \ d_n(x, a) \leqq s \text{ for some } a \in A\}.$$

For each n, define a map $f_n:X \to \omega_n^{-1}(t)$ by

$$f_n(x) = K_n(\{x\}, \alpha_n(\{x\})),$$

where

$$\omega_n(K_n(\{x\}, \alpha_n(\{x\}))) = t.$$

Set $f_{0t} = \{f_n\}$. Then $f_{0t}:X \to \omega^{-1}(t)$ is a shape morphism (see [7] or [8]). Note that the shape morphism f_{0t} is independent of the choices of neighborhoods X_n and convex metrics d_n on X_n.

3. Main theorem. In this section, we prove the following theorem which is our main theorem in this paper.

(3.1) THEOREM. *Let X be a curve and let ω be any Whitney map for $C(X)$. Let $0 < t < \omega(X)$. Then the following are equivalent.*

(a) *The shape morphism $f_{0t}:X \to \omega^{-1}(t)$ is a shape equivalence.*

(b) *Each element of $\omega^{-1}(t)$ is a tree-like continuum.*

Proof. First, we shall show that (b) implies (a). Now we shall prove that Fd $\omega^{-1}(t) \leqq 1$. Let M be an ANR and let $\alpha:\omega^{-1}(t) \to M$ be any map. Then we must show that there are maps $\beta:\omega^{-1}(t) \to P$ and $\gamma:P \to M$ such that $\gamma\beta \simeq \alpha$, where P is a 1-dimensional polyhedron. Since X is a curve, there is an inverse sequence $\{X_n, p_{nn+1}\}$ of compact 1-dimensional connected polyhedra X_n such that

$$X = \text{invlim}\{X_n, p_{nn+1}\}.$$

By [8], there are sequences $t_1 > t_2 > \dots$, and $t'_1 < t'_2 < \dots$, of positive numbers such that

$$\lim t_n = t, \quad \lim t'_n = t \quad \text{and}$$

$$\omega^{-1}(t) = \text{invlim}\{\omega_n^{-1}([t'_n, t_n]), p^*_{nn+1}|\omega^{-1}([t'_{n+1}, t_{n+1}])\},$$

where

$$p^*_{nn+1}:C(X_{n+1}) \to C(X_n)$$

is defined by

$$p^*_{nn+1}(A) = \{p_{nn+1}(a) \,|a \in A\},$$

ω_n is a Whitney map for $C(X_n)$ ($n = 1, 2, \dots,$). Take a natural number n and a map

$$\alpha' : \omega_n^{-1}([t'_n, t_n]) \to M$$

such that

$$\alpha'(p_n^*|\omega^{-1}(t)) \simeq \alpha,$$

where $p_n : X \to X_n$ is the projection (e.g., see [**17**, p. 63, Theorem 8]).

Consider the universal covering $q : \tilde{X}_n \to X_n$. Since X_n is a polyhedron, we may assume that $\tilde{X}_n$ is a polyhedral tree and q is a simplicial map (e.g., see [**23**, p. 144]). Let $G(\tilde{X}_n|X_n)$ be the group of covering transformations of q. Let $\tilde{d}_n$ be the metric on $\tilde{X}_n$ such that $\tilde{d}_n$ is the path length metric and

$$\tilde{d}_n(x, y) = |s - s'| \quad \text{for } x, y \in \langle V_0, V_1 \rangle \quad \text{and}$$

$$x = sV_0 + (1 - s)V_1, \quad y = s'V_0 + (1 - s')V_1,$$

where $\langle V_0, V_1 \rangle$ is an edge of $\tilde{X}_n$. Note that if $h \in G(\tilde{X}_n|X_n)$, then h is a simplicial homeomorphism and

$$\tilde{d}_n(x, y) = \tilde{d}_n(h(x), h(y)) \quad \text{for each } x, y \in \tilde{X}_n.$$

Let $A \in \omega^{-1}(t)$. Since A is a tree-like continuum, $p_n|A : A \to X_n$ is null-homotopic. Since X_n is an ANR, there is a closed neighborhood $U(A)$ of A in X such that

$$p_n|U(A) : U(A) \to X_n$$

is null-homotopic. Set

$$U(A)* = \{B \in \omega^{-1}(t) \,|B \subset U(A)\}.$$

Then $U(A)*$ is closed in $\omega^{-1}(t)$. Since $\omega^{-1}(t)$ is compact, we have a finite family $\{U(A_1)*, U(A_2)*, \dots, U(A_m)*\}$ such that

$$\omega^{-1}(t) = \cup\{U(A_i)*|1 \leqq i \leqq m\}.$$

Set

$$L(A) = \{g|g : A \to \tilde{X}_n \text{ is a lifting of } p_n|A, \text{ i.e., } qg = p_n|A\}.$$

Since $p_n|U(A_1)$ is null-homotopic, there is a lifting

$$g_1 : U(A_1) \to \tilde{X}_n$$

of $p_n|U(A_1)$. Now we define $\varphi(A, g_1|A)$ for each $A \in U(A_1)*$ as follows: Let $* \in \tilde{X}_n$. Let $A \in U(A_1)*$. Note that $g_1(A)$ is a compact tree. Then $\varphi(A, g_1|A)$ is the unique point of $\tilde{X}_n$ such that

$$\varphi(A, g_1|A) \in g_1(A) \quad \text{and}$$

$$[*, \varphi(A, g_1|A)] \cap g_1(A) = \{\varphi(A, g_1|A)\},$$

where $[a, b]$ denotes the arc from a to b in $\widetilde{X}_n$ for $a, b \in \widetilde{X}_n$. Note that the function

$$\varphi_1' : U(A_1)* \to \widetilde{X}_n$$

defined by

$$\varphi_1'(A) = \varphi(A, g_1|A) \quad \text{for } A \in U(A_1)*$$

is continuous. Let $g \in L(A)$. We define $\varphi(A, g)$ by

$$\varphi(A, g) = h(\varphi_1'(A)),$$

where $h \in G(\widetilde{X}_n|X_n)$ such that $hg_1 = g$. Assume that $\varphi(A, g)$ is defined for

$$A \in \cup\{U(A_j)*|1 \leqq j \leqq i\} \quad \text{and} \quad g \in L(A).$$

Let $A \in U(A_{i+1})*$ and $g \in L(A)$. We shall define $\varphi(A, g)$ as follows: Take a lifting

$$g_{i+1} : U(A_{i+1}) \to \widetilde{X}_n$$

of $p_n|U(A_{i+1})$. Consider the following function

$$\psi : U(A_{i+1})* \cap (\cup\{U(A_j)*|1 \leqq j \leqq i\}) \to \widetilde{X}_n,$$

which is defined by $\psi(A) = \varphi(A, g_{i+1}|A)$.

We shall show that ψ is continuous. Let

$$A_k \in U(A_{i+1})* \cap (\cup\{U(A_j)*|1 \leqq j \leqq i\}) \quad (k = 1, 2, \dots,)$$

and $\lim A_k = A$. Without loss of generality, we may assume that A, $A_k \in U(A_j)*$ $(j \leqq i)$ for all k. Take $h \in G(\widetilde{X}_n|X_n)$ such that

$$h(g_j|A) = g_{i+1}|A.$$

Let $a \in A$. Take open sets $V(g_j(a))$ and $V(g_{i+1}(a))$ of $\widetilde{X}_n$ such that

$$g_j(a) \in V(g_j(a)), \quad g_{i+1}(a) \in V(g_{i+1}(a)),$$

$q|V(g_j(a))$ and $q|V(g_{I+1}(a))$ are homeomorphisms and

$$h(V(g_j(a)) = V(g_{i+1}(a)).$$

Take a sequence $\{a_k\}$ of points such that

$$a_k \in A_k \quad (k = 1, 2, \dots,) \quad \text{and} \quad \lim a_k = a.$$

Then

$$g_j(a_k) \in V(g_j(a)) \quad \text{and} \quad g_{i+1}(a_k) \in V(g_{i+1}(a))$$

for almost all k. Thus $h(g_j(a_k)) = g_{i+1}(a_k)$, which implies that

$$h(g_j|A_k) = g_{i+1}|A_k$$

(because A_k is connected). Note that

$$\varphi_j' : U(A_j)^* \to \widetilde{X}_n$$

defined by

$$\varphi_j'(B) = \varphi(B, g_j|B) \quad \text{for each } B \in U(A_j)^*$$

is continuous. Then

$$\begin{aligned} \lim \psi(A_k) &= \lim \varphi(A_k, g_{i+1}|A_k) \\ &= \lim h(\varphi(A_k, g_j|A_k)) \\ &= h(\varphi(A, g_j|A) = \varphi(A, g_{i+1}|A) = \psi(A). \end{aligned}$$

Hence ψ is continuous. Since $\widetilde{X}_n$ is a tree, there is a continuous extension

$$\widetilde{\psi} : U(A_{i+1})^* \to \widetilde{X}_n$$

of ψ. Let $A \in U(A_{i+1})^*$. We can choose the unique point

$$\varphi_{i+1}'(A, g_{i+1}|A)$$

of $\widetilde{X}_n$ such that

$$\varphi_{i+1}'(A, g_{i+1}|A) \in g_{i+1}(A)$$

and

$$[\varphi_{i+1}'(A, g_{i+1}|A), \widetilde{\psi}(A)] \cap g_{i+1}(A) = \{\varphi_{i+1}'(A, g_{i+1}|A)\}.$$

Then

$$\varphi_{i+1}' : U(A_{i+1})^* \to \widetilde{X}_n$$

is continuous and

$$\varphi_{i+1}'(A) = \varphi(A, g_{i+1}|A)$$

for each

$$A \in U(A_{i+1})^* \cap (\cup\{U(A_j)^* | \quad 1 \leqq j \leqq i\}).$$

For each $A \in U(A_{i+1})^*$ and $g \in L(A)$, set

$$\varphi(A, g) = h(\varphi_{i+1}'(A)),$$

where $h \in G(\widetilde{X}_n|X_n)$ such that

$$h(g_{i+1}|A) = g.$$

By induction, we can conclude that for each $A \in \omega^{-1}(t)$ and $g \in L(A)$, $\varphi(A, g)$ is well defined. Note that $\varphi(A, g) \in g(A)$ and

$$q(\varphi(A, g)) = q(\varphi(A, g')) \quad \text{for any } g, g' \in L(A).$$

Also, the function

$$\varphi_i' : U(A_i)^* \rightarrow \widetilde{X}_n$$

defined by

$$\varphi_i'(A) = \varphi(A, g_i|A)$$

is continuous ($1 \leqq i \leqq m$).

Define a function $f : \omega^{-1}(t) \rightarrow X_n$ by

$$f(A) = q(\varphi(A, g)),$$

where $g \in L(A)$. Then f is well defined. Since

$$\varphi_i' : U(A_i)^* \rightarrow \widetilde{X}_n$$

is continuous ($1 \leqq i \leqq m$), we see that f is continuous. Define a homotopy

$$H : \omega^{-1}(t) \times I \rightarrow \omega^{-1}([0, t_n])$$

by

$$H(A, t) = q(H_{(A, g)}(g(A), t)),$$

where $g \in L(A)$ and

$$H_{(A, g)} : g(A) \times I \rightarrow g(A)$$

denotes the homotopy defined by

$$H_{(A, g)}(x, t) \in [x, \varphi(A, g)] \quad \text{and}$$

$$\widetilde{d}_n(x, H_{(A, g)}(x, t)) = t\widetilde{d}_n(x, \varphi(A, g))$$

for $x \in g(A)$ and $t \in I$. Then H is well defined and continuous. Note that $H(A, 0) = p_n^*(A)$ and $H(A, 1) = f(A)$ for each $A \in \omega^{-1}(t)$. Since X_n is locally connected, there is a retraction

$$R : \omega_n^{-1}([0, t_n]) \rightarrow \omega_n^{-1}([t_n', t_n]).$$

Then we have the following commutative diagram in homotopy category:

$$\begin{array}{ccc}
\omega^{-1}(t) & \xrightarrow{\;p_n^*\;} & \omega_n^{-1}([t_n', t_n]) \\
{\scriptstyle f}\downarrow & & \uparrow{\scriptstyle R} \\
X_n & \xrightarrow[\;i\;]{} & \omega_n^{-1}([0, t_n])
\end{array}$$

Hence

https://doi.org/10.4153/CJM-1988-009-5 Published online by Cambridge University Press

$$\alpha \simeq \alpha'(p_n^*|\omega^{-1}(t)) \simeq \alpha'(Ri)f.$$

Since $\dim X_n \leqq 1$, we conclude that

$$\mathrm{Fd}\ \omega^{-1}(t) \leqq 1.$$

By the proof of [**8**, (4.2)], we see that the shape morphism

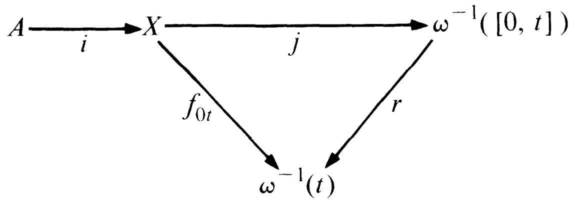
$$f_{0t}:X \to \omega^{-1}(t)$$

is a shape equivalence.

Next, we shall prove that (a) implies (b). Let $A \in \omega^{-1}(t)$ and let $f:A \to M$ be any map, where M is an ANR. We must show that f is null-homotopic. Since $\dim X \leqq 1$, there is an extension $\tilde{f}:X \to M$ of f. By [**7**], there is a shape deformation retract

$$r:\omega^{-1}([0, t]) \to \omega^{-1}(t)$$

such that $r|X = f_{0t}$. Hence we have the following commutative diagram in shape category:



where i and j are the inclusion maps, respectively. Note that $C(A)$ has trivial shape (see [**19**]) and

$$ji(A) \subset C(A) \subset \omega^{-1}([0, t]),$$

which implies that $ji = c$ in shape category, where c is a constant map. Since f_{0t} is a shape equivalence, $i = c'$ in shape category, where c' is a constant map. Hence $f = \tilde{f}i$ is null-homotopic (note that M is an ANR). Since $\dim A \leqq 1$, A is a tree-like continuum. This completes the proof.

A continuum X is said to be *calm* [**3**] provided that for any ANR M containing X, there is a neighborhood U of X in M such that for every neighborhood V of X there is a neighborhood $W \subset V \cap U$ of X in M such that if P is any polyhedron and $f, g:P \to W$ are any maps which are homotopic in U, then f is homotopic to g in V.

As a corollary of (3.1), we have

(3.2) COROLLARY. *Let X be a curve and let ω be any Whitney map for $C(X)$. If X is calm (see [**3**]), then there is t_0 $(0 < t_0 < \omega(X))$ such that the shape morphism $f_{0t}:X \to \omega^{-1}(t)$ is a shape equivalence for $0 < t \leqq t_0$, in particular,* $\mathrm{Sh}\ \omega^{-1}(t) = \mathrm{Sh}\ X$.

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Proof. Let $\underline{X} = \{X_n, p_{nn+1}\}$ be an inverse sequence of 1-dimensional connected polyhedra X_n such that

$$X = \text{invlim } \underline{X}.$$

Since X is calm, we may assume that for each $n \geqq 1$, there is $i(n) \geqq n$ such that if $f, g:P \to X_{i(n)}$ are any maps such that if

$$p_{1i(n)}f \simeq p_{1i(n)}g,$$

then

$$p_{ni(n)}f \simeq p_{ni(n)}g,$$

where P is any polyhedron. Now, we shall show that there is t_0 ($0 < t_0 < \omega(X)$) such that each element of $\omega^{-1}(t_0)$ is a tree-like continuum. Suppose, on the contrary, that there is a sequence $A_1, A_2, \ldots,$ of non-degenerate subcontinua of X such that each A_k is not tree-like and

$$\lim \text{diam } A_k = 0.$$

Let $p_n:X \to X_n$ be the projection. Then there is k such that

$$p_1|A_k:A_k \to X_1$$

is null-homotopic. Since A_k is not tree-like, there is n such that

$$p_n|A_k:A_k \to X_n$$

is not null-homotopic (note that $p_i(A_k)$ is an ANR which is a retract of X_i for each i). Then

$$p_1|A_k = p_{1i(n)}(p_{i(n)}|A_k).$$

Since $p_1|A_k$ is null-homotopic, $p_n|A_k$ is null-homotopic. This is a contradiction. Hence for some t_0 ($0 < t_0 < \omega(X)$), each element of $\omega^{-1}(t_0)$ is a tree-like continuum. By (3.1), the shape morphism $f_{0t}:X \to \omega^{-1}(t)$ is a shape equivalence for $0 < t \leqq t_0$.

It is well known that a continuum X is an FANR (e.g., see [**1**] or [**17**]) if and only if X is movable and calm (see [**4**]). Hence we have the following

(3.3) Corollary. *Let X be a curve and let ω be any Whitney map for $C(X)$. If X is an* FANR, *then there is t_0* ($0 < t_0 < \omega(X)$) *such that $f_{0t}:X \to \omega^{-1}(t)$ is a shape equivalence for* $0 < t \leqq t_0$.

(3.4) Corollary ([**7**], [**8**] and [**14**]). *Suppose that X is one of a circle-like continuum, a tree-like continuum, or the Case-Chamberlin curve* [**2**, p. 79]. *If ω is any Whitney map for $C(X)$, then* $\text{Sh } \omega^{-1}(t) = \text{Sh } X$ *for* $0 < t < \omega(X)$.

Proof. Note that each proper subcontinuum of X is tree-like. Hence, (3.4) follows from (3.1).

(3.5) *Remark*. In (3.2) (resp. (3.3)), we can not omit the condition that X is calm (resp. X is an FANR) (see [**8**, (3.12)]). In [**9**], we proved that Whitney continua of 1-dimensional connected polyhedra admit all homotopy types of compact connected ANRs. Also, we proved that if P is a compact connected polyhedron with dim $P \geqq 2$ and $n \geqq 2$, then there is a Whitney map ω for $C(P)$ such that for some t $(0 < t < \omega(P))$ the n-sphere S^n is homotopically dominated by $\omega^{-1}(t)$, in particular, Fd $\omega^{-1}(t) \geqq n$ (see [**10**]). In [7], we showed that the property of being pointed 1-movable is a Whitney Property. But the property of being 2-movable is not a Whitney property for curves (see [**11**]).

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https://doi.org/10.4153/CJM-1988-009-5 Published online by Cambridge University Press