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MONOTONE SEMIFLOWS GENERATED BY NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATION TO COMPARTMENTAL SYSTEMS

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ABSTRACT. This paper is devoted to the machinery necessary to apply the general theory of monotone dynamical systems to neutral functional differential equations. We introduce an ordering structure for the phase space, investigate its compatibility with the usual uniform convergence topology, and develop several sufficient conditions of strong monotonicity of the solution semiflows to neutral equations. By applying some general results due to Hirsch and Matano for monotone dynamical systems to neutral equations, we establish several (generic) convergence results and an equivalence theorem of the order stability and convergence of precompact orbits. These results are applied to show that each orbit of a closed biological compartmental system is convergent to a single equilibrium.

1. **Introduction.** The purpose of this paper is to establish several strong monotonicity principles and (generic) convergence theorems for neutral functional differential equations and to apply these results to prove a convergence theorem for a mathematical model of biological compartmental systems.

Recently, monotone dynamical systems have received considerable attention in the literature. Many interesting results concerning the convergence of precompact orbits have been obtained, and applied to various evolution equations enjoying a strong comparison principle. For details, we refer to [1]–[4], [26]–[28], [36]–[38], [40]–[42], [45], [48], [49], [51] and [53].

In [49], Smith developed the necessary machinery to apply the general theory of monotone dynamical systems to retarded functional differential equations. It was shown that a cooperative and irreducible retarded equation generates a strongly monotone semiflow, and the qualitative behavior of solutions is generically the same as for the ordinary differential equation obtained by ignoring the delay.

It is a natural problem to extend Smith's results to the neutral functional differential equation

$$\frac{d}{dt} D(x_t) = f(x_t) \tag{1.1}$$

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where $D, f\colon C([-r,0],R^n) \longrightarrow R^n$ are continuous and D is linear, atomic and stable (see, e.g. [24]). However, this extension requires some new techniques since, roughly speaking, a neutral equation is a combination of a generalized difference equation and a differential equation, and therefore, the dynamical system generated by a neutral equation is essentially a combination of a discrete dynamical system generated by the generalized difference equation, and a continuous dynamical system (semiflow) generated by a differential equation. As a consequence, monotonicity is more delicate for neutral equations than for retarded equations. For example, the semiflow defined by a neutral equation does not preserve the usual functional ordering of the space $C = C([-r,0],R^n)$. To see this point, we consider the following initial value problem:

$$\begin{cases} \frac{d}{dt}[x(t) - \frac{1}{2}x(t-1)] = 0, & t \geq 0 \\ x(\theta) = 1 - 3\theta, & -1 \leq \theta \leq 0 \end{cases} \tag{1.2}$$

By using the method of steps, one can easily verify that the solution is $x(t) = -\frac{3}{2}t + 1$ for $t \in [0,1]$ and $x(t) = -\frac{3}{4}t + \frac{1}{4}$ for $t \in [1,2]$. Therefore a positive initial datum generates a negative solution on [1,2]. We will see that the source of the failure of the solution to preserve the nonnegative property is $x(0) - \frac{1}{2}x(-1) = -1 < 0$, and that this pathology can be removed by introducing a new ordering $\varphi \geq 0$ in C iff $\varphi(\theta) \geq 0$ for $\theta \in [-1,0]$ and $\varphi(0) - \frac{1}{2}\varphi(-1) \geq 0$. This is equivalent to considering the neutral equation on the product space $X = R \times C([-1,0],R)$ with the usual ordering, and relating the solution to a semiflow $\left(x(t) - \frac{1}{2}x(t-1), x_t\right)$ on X. This idea was used by many investigators for different purposes such as the state space description of retarded equations and neutral equations (see, [11]–[13], [46] and references therein).

Motivated by the above example and discussion, we will consider equation (1.1) on the space C endowed with the following ordering $\underset{D}{\geq}$

$$\varphi \underset{D}{\geq} 0 \text{ iff } \varphi \underset{C}{\geq} 0 \text{ and } D(\varphi) \underset{R^n}{\geq} 0$$

where $\underset{R^n}{\geq}$ and $\underset{C}{\geq}$ denote the componentwise ordering on R^n and the functional (pointwise) ordering on C.

This paper is organized as follows. In Section 2, we present a brief discussion about the compatibility of the ordering $\underset{D}{\geq}$ and the uniform convergence topology of C. It is shown that the state space C with the ordering $\underset{D}{\geq}$ and the uniform convergence topology is strongly ordered in the sense of Hirsch [28], and the zero element of C can be approximated by a sequence of positive elements (in $\underset{D}{\geq}$). In Section 3, we will prove that the solution semiflow is monotone if the following is satisfied

(H): whenever $\varphi \underset{D}{\leq} \psi$ and $D_i(\varphi) = D_i(\psi)$ it follows that $f_i(\varphi) \leq f_i(\psi)$.

This reduces to the quasimonotonicity condition for retarded equations (see, e.g. [31], [40]–[42], [45] and [49] for detailed references). We will borrow the technique from Hirsch [28] for ordinary differential equations and Smith [49] for retarded equations to develop a fairly general sufficient condition for equation (1.1) to generate an eventually strongly monotone semiflow. In Section 4, we state several direct applications of the generic convergence theory for monotone dynamical systems due to Hirsch [27, 28] to neutral equations and prove a result about the equivalence between order-stability and convergence of orbits.

The results in the first four sections are applied in Section 5 to a convergence problem of a mathematical model of biological compartmental systems which goes back to Bellman [9]. Assuming each compartment produces material itself, these systems can be modeled by neutral equations. A material conservation law is derived which implies the boundedness and order-stability of each orbit and the separation of ω-limits of ordered points. Under reasonable assumptions, it is shown that each orbit is convergent to a single equilibrium.

2. **A strong ordered space for NFDEs.** Let X be a metrizable topological space with a closed (partial) order relation $\Re \subset X \times X$. Such a space is called an order space. We define

$$x \leq y \ (y \geq x) \text{ if } (x,y) \in \Re,$$
$$x < y \ (y > x) \text{ if } x \leq y \text{ but } x \neq y,$$
$$x \ll y \ (y \gg x) \text{ if } x \neq y \text{ and } (x,y) \in \operatorname{Int} \Re$$

The ordered space X is said to be strongly ordered if, for every open set $U \subseteq X$ and for any $x \in U$, there exist $a, b \in U$ such that $a \ll x \ll b$. A set W is order-convex if $x, y \in W$ and $x < y$ imply that the closed order interval $[x,y] = \{a \in X, x \leq a \leq y\}$ is in W. Any pair (a,b) with $a \ll b$ defines an open order interval $[[a,b]] = \{x \in X;\ a \ll x \ll b\}$. The topology of X generated by order-convex open sets is called the order topology.

A typical ordered space is the so-called ordered topological vector space V together with a closed convex cone V_+ such that $V_+ \cap (-V_+) = \{0\}$. An ordering is defined by $y \geq x$ iff $y - x \in V_+$. Obviously, V is strongly ordered iff V_+ is solid, that is, $\operatorname{Int} V_+ \neq \emptyset$. Moreover, if V_+ is solid, then the order topology can be defined by the norm $\|x\|_e = \inf\{c \in R_+;\ x \in [[-ce, ce]]\}$, where $R_+ = [0, \infty)$, $e \gg 0$ is a fixed element.

Let $U \subset X$ be an open subset, $\Phi = \{\Phi_t\}_{t \geq 0}$ be a local semiflow on U. We say Φ is monotone if, for any $x,\ y \in U$ with $x \leq y$ we have $\Phi_t(x) \leq \Phi_t(y)$ for all $t \in [0, \tau(x,y))$, where $\tau(x)$ denotes the escape time of x, and $\tau(x,y) = \min\{\tau(x), \tau(y)\}$. Φ is eventually strongly monotone if it is monotone, $\tau(x,y) = \infty$ for any $x, y \in U$, and there exists a constant $T > 0$ such that $x, y \in U$ and $x < y$ imply $\Phi_t(x) \ll \Phi_t(y)$ for all $t \geq T$.

Let R^n_+ be the space of non-negative vectors in R^n. We use $\underset{R^n}{\leq}$ to denote the order relation in R^n defined by R^n_+. Given $r = (r_1, \ldots, r_n) \in R^n_+$, we define $|r| = \max_{1 \leq i \leq n} r_i$, $C_r = \prod_{i=1}^n C([-r_i, 0], R)$ and $C_r^+ = \prod_{i=1}^n C([-r_i, 0], R_+)$. Obviously, C_r is a strongly ordered space with the uniform convergence topology defined by the norm

$$\|\varphi\|_{C_r} = \max_{1 \leq i \leq n} \sup_{-r_i \leq \theta \leq 0} |\varphi_i(\theta)|, \varphi \in C_r$$

and the usual functional ordering $\underset{C_r}{\leq}$ defined by

$$\varphi \underset{C_r}{\leq} \psi \text{ iff } \varphi_i(\theta) \leq \psi_i(\theta) \text{ for } 1 \leq i \leq n \text{ and } \theta \in [-r_i, 0].$$

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Suppose $D: C_r \longrightarrow R^n$ is a given bounded linear operator. According to the Riesz representation theorem, there exists functions $\mu_{ij}: [-r_j, 0] \longrightarrow R$, $1 \le i,j \le n$, of bounded variation on $[-r_j, 0]$ such that

$$(2.1) \qquad D_i(\varphi) = \sum_{j=1}^{n} \int_{-r_j}^{0} d\mu_{ij}(\theta)\varphi_j(\theta), 1 \le i \le n,\ \varphi \in C_r.$$

We assume that D is atomic at zero, that is, the matrix

$$(2.2) \qquad A = \left(\mu_{ij}(0) - \mu_{ij}(0^-)\right)$$

is nonsingular and there exists a nonnegative scalar continuous function β on $[0, \min_{j\in J} r_j]$ with $\beta(0) = 0$ and

$$(2.3) \qquad \left|\sum_{j\in J} \int_{-s}^{0} d\overline{\mu_{ij}}(\theta)\varphi_j(\theta)\right| \le \beta(s)\|\varphi\|_{C_r},$$

where $N = \{1, \ldots, n\}$, $J = \{i \in N;\ r_i \neq 0\}$ and

$$(2.4) \qquad \overline{\mu_{ij}}(\theta) = \begin{cases} \mu_{ij}(\theta) & \text{for } \theta \in [-r_j, 0), \\ \mu_{ij}(0^-) & \text{for } \theta = 0. \end{cases}$$

We now define an ordering, denoted by $\underset{D}{\le}$, as follows

$$\varphi \underset{D}{\le} \psi \text{ iff } \varphi \underset{C_r}{\le} \psi \text{ and } D(\varphi) \underset{R^n}{\le} D(\psi),$$

where φ and ψ are generic points in C_r. The following Lemma provides a sufficient condition for the space C_r endowed with the norm $\|\cdot\|_{C_r}$ and the ordering $\underset{D}{\le}$ to be a strongly ordered space.

LEMMA 2.1. *Suppose that* $\operatorname{Int} R_+^n \cap A(\operatorname{Int} R_+^n) \neq \emptyset$. *Then* $\operatorname{Int} C_{r,D}^+ \neq \emptyset$, *where* $C_{r,D}^+ = \{\varphi \in C_r : \varphi \underset{D}{\ge} 0\}$.

PROOF. According to our assumption, there exists $v \in \operatorname{Int} R_+^n$ such that $Av \underset{R^n}{\gg} 0$. For any $i \in N \backslash J$, we define $\psi_i \in C([-r_i, 0], R)$ by $\psi_i(0) = v_i$, and for any given $i \in J$ and $s \in (0, \min\{\min_{j\in J} r_j, \min_{i\in N} v_i\})$ we define $\psi_i \in C([-r_i, 0], R)$ by

$$\psi_i(\theta) = \begin{cases} s & \text{if } -r_i \le \theta \le -s \\ v_i + \frac{v_i - s}{s}\theta & \text{if } -s \le \theta \le 0. \end{cases}$$

Then $\psi = (\psi_1, \ldots, \psi_n) \underset{C_r}{\gg} 0$, $\|\psi\|_{C_r} \le \max_{i\in N} v_i$ and

$$\begin{aligned} \left|\sum_{j=1}^{n} \int_{-rj}^{0} [d\bar\mu_{ij}(\theta)]\psi_j(\theta)\right| &= \left|\sum_{j\in J}\left[\int_{-s}^{0} d\bar\mu_{ij}(\theta)\left(v_j + \frac{v_j - s}{s}\theta\right) + s\int_{-r_j}^{-s} d\mu_{ij}(\theta)\right]\right| \\ &\le \beta(s) \max_{i\in N} v_i + \left(\sum_{j\in J} \operatorname{Var}_{[-r_j,0]} \mu_{ij}\right)s \longrightarrow 0 \text{ as } s \longrightarrow 0. \end{aligned}$$

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Therefore for sufficiently small $s > 0$ and any given $i \in N$, we have

$$D_i(\psi) = \sum_{j=1}^{n}[\mu_{ij}(0) - \mu_{ij}(0^-)]\psi_j(0) + \sum_{j=1}^{n}\int_{-r_j}^{0}[d\bar{\mu}_{ij}(\theta)]\psi_j(\theta)$$
$$= (Av)_i + \sum_{j=1}^{n}\int_{-rj}^{o}[d\bar{\mu}_{ij}(\theta)]\psi_j(\theta) > 0.$$

That is, $D(\psi) \underset{R^n}{\gg} 0$. Therefore, $\psi \in \operatorname{Int} C^+_{r,D}$. This completes the proof.

The following Lemma shows that the zero element in C_r can be approximated by a sequence of elements in $C^+_{r,D}$.

LEMMA 2.2. *Let* $\operatorname{Int} R^n_+ \cap A(\operatorname{Int} R^n_+) \neq \emptyset$. *Then for any* $\varepsilon > 0$ *there exists* $\varphi^\varepsilon \in C_r$ *such that* $0 \underset{C_r}{\ll} \varphi^\varepsilon \underset{C_r}{\ll} \varepsilon\hat{e}$ *and* $0 \underset{R^n}{\ll} D(\varphi^\varepsilon) \underset{R^n}{\ll} \varepsilon e$, *where* $\varepsilon e \in R^n$ *with* $(\varepsilon e)_i = \varepsilon$ *for* $i \in N$, *and* $\varepsilon\hat{e} \in C_r$ *with* $(\varepsilon\hat{e})_i(\theta_i) = \varepsilon$ *for* $i \in N$ *and* $\theta_i \in [-r_i, 0]$.

PROOF. Let $v \in \operatorname{Int} R^n_+$ be a fixed vector such that $Av \in \operatorname{Int} R^n_+$. For any given sufficiently small $\varepsilon > 0$, we define $\varphi^\varepsilon = (\varphi^\varepsilon_1, \ldots, \varphi^\varepsilon_n) \in C_r$ as follows

$$\varphi^\varepsilon_i(0) = v_i\delta \ \text{ if } i \in N\backslash J$$

and

$$\varphi^\varepsilon_i(\theta) = \begin{cases} s & \text{for } -r_i \leq \theta \leq -s \\ v_i\delta + \frac{v_i\delta - s}{s}\theta & \text{for } -s \leq \theta \leq 0 \text{ if } i \in J, \end{cases}$$

where $\delta > 0$ is given such that

$$\sum_{j=1}^{n}[\mu_{ij}(0) - \mu_{ij}(0^-)]v_j\delta < \frac{\varepsilon}{4}, \quad \max_{i\in N} v_i\delta < \varepsilon,$$

and $s > 0$ is given such that

$$s < \min\{\min_{i\in J} r_i, \ \min_{i\in N} v_i\delta\}$$

and

$$\beta(s)\max_{i\in N} v_i\delta + s\sum_{j=1}^{n}\operatorname{Var}_{[-r_j,0]}\mu_{ij} < \sum_{j=1}^{n}[\mu_{ij}(0) - \mu_{ij}(0^-)]v_j\delta.$$

Then $0 \underset{R^n}{\ll} \varphi^\varepsilon \underset{R^n}{\ll} v\delta \underset{R^n}{\ll} \varepsilon e$, and for any $i \in N$, one has

$$\left|\sum_{j=1}^{n}\int_{-rj}^{0} d\bar{\mu}_{ij}(\theta)\varphi^\varepsilon_j(\theta)\right| \leq \sum_{j\in J}\int_{-s}^{0} d\bar{\mu}_{ij}(\theta)\left[v_j\delta + \frac{v_j\delta - s}{s}\theta\right] + \left|\sum_{j\in J}\int_{-rj}^{0} d\mu_{ij}(\theta)\right|s$$
$$\leq \beta(s)\max_{i\in N} v_i\delta + \left(\sum_{j=1}^{n}\operatorname{Var}_{[-r_j,0]}\mu_{ij}\right)s$$
$$< \sum_{j=1}^{n}[\mu_{ij}(0) - \mu_{ij}(0^-)]v_j\delta$$

from which it follows that

$$\begin{aligned}|D_i(\varphi^\varepsilon)| &\leq \sum_{j=1}^n [\mu_{ij}(0)-\mu_{ij}(0^-)]\varphi_j^\varepsilon(0) + \left|\sum_{j=1}^n \int_{-r_j}^0 d\bar{\mu}_{ij}(\theta)\varphi_j^\varepsilon(\theta)\right| \\ &< \sum_{j=1}^n [\mu_{ij}(0)-\mu_{ij}(0^-)]v_j\delta + \sum_{j=1}^n [\mu_{ij}(0)-\mu_{ij}(0^-)]v_j\delta \\ &< \frac{\varepsilon}{2}\end{aligned}$$

and

$$\begin{aligned}D_i(\varphi^\varepsilon) &\geq \sum_{j=1}^n [\mu_{ij}(0)-\mu_{ij}(0^-)]v_j\delta - \left|\sum_{j=1}^n \int_{-r_j}^0 d\bar{\mu}_{ij}(\theta)\varphi_j^\varepsilon(\theta)\right| \\ &> \sum_{j=1}^n [\mu_{ij}(0)-\mu_{ij}(0^-)]v_j\delta - \sum_{j=1}^n [\mu_{ij}(0)-\mu_{ij}(0^-)]v_j\delta \\ &= 0.\end{aligned}$$

This completes the proof.

The following result provides a technical tool for establishing the strong monotonicity principle in the next section.

LEMMA 2.3. *Suppose that $A^{-1}(\mathrm{Int}\,R_+^n) \subseteq \mathrm{Int}\ R_+^n$ and $\mu_{ij}\colon [-r_j, 0) \longrightarrow R$ is nonincreasing and continuous from the left. Then for any $\tau > 0$ and any continuous function $x = (x_1, \ldots, x_n)$, $x_i\colon [-r_i, \tau] \longrightarrow R$, $i \in N$, with $x_0 \in C_r^+$, if $D(x_t) \underset{R^n}{\gg} 0$ for $t \in [0, \tau]$, then $x(t) \underset{R^n}{\gg} 0$ for $t \in [0, \tau]$.*

PROOF. Let $L\colon C_r \longrightarrow R^n$, $i \in N$, be defined by

$$\big(L(\varphi)\big)_i = \sum_{j=1}^n \int_{-r_j}^0 d\bar{\mu}_{ij}(\theta)\varphi_j(\theta), \ \varphi \in C_r, \ i \in N. \tag{2.5}$$

Then by the nonincreasing property of μ_{ij}, $L(\varphi) \underset{R^n}{\leq} 0$ for $\varphi \in C_r^+$. Evidently $D(x_t) = Ax(t) + L(x_t)$ from which it follows $x(t) = A^{-1}D(x_t) - A^{-1}L(x_t)$. Therefore by our assumption, $x(0) \underset{R^n}{\gg} 0$. If the conclusion in our lemma is not true, then there exists $s \in (0, \tau]$ such that $x_i(s) = 0$ for some $i \in N$, and $x(t) \underset{R^n}{\geq} 0$ for all $t \in [0, s]$. In this case, $L(x_s) \underset{R^n}{\leq} 0$, $-A^{-1}L(x_s) \underset{R^n}{\geq} 0$ and $A^{-1}D(x_s) \underset{R^n}{\gg} 0$ by our assumptions. Therefore $x(s) \underset{R^n}{\geq} A^{-1}D(x_s) \underset{R^n}{\gg} 0$. This is contrary to $x_i(s) = 0$. The proof is complete.

Let $A^{-1} = (b_{ij})$. Then $A^{-1}(\mathrm{Int}\,R_+^n) \subseteq \mathrm{Int}(R_+^n)$ is equivalent to the assumption that $b_{ij} \geq 0$ and $\sum_{j=1}^n b_{ij} > 0$ for $1 \leq i, j \leq n$. This assumption is satisfied, if the conditions $b_{ii} > 0$ and $b_{ij} \geq 0$ hold for $1 \leq i, j \leq n$. In fact, with these conditions we have the following:

LEMMA 2.4. *Let $b_{ii} > 0$ and $b_{ij} \geq 0$, and $\mu_{ij}\colon [-r_j, 0) \longrightarrow R$ be nonincreasing and continuous from the left for $1 \leq i, j \leq n$. Then for any $\tau > 0$ and any continuous* *function* $x = (x_1, \ldots, x_n)$, $x_i\colon [-r_i, \tau] \longrightarrow R$, $i \in N$, *with* $x_0 \in C_r^+$, *if* $D(x_t) \underset{R^n}{\geq} 0$ *and for some* j, $D_j(x_t) > 0$ *for* $t \in [0, \tau]$, *then* $x_j(t) > 0$ *for* $t \in [0, \tau]$.

The proof is similar to that of Lemma 2.3 and therefore is omitted.

For ease of reference, we introduce

DEFINITION 2.1. A bounded linear operator $D\colon C_r \longrightarrow R^n$ defined by (2.1) is said to be *quasimonotone,* if

(i): it is atomic at zero;

(ii): $b_{ii} > 0$ and $b_{ij} \geq 0$ for $i, j \in N$, where $(b_{ij}) = \big(\mu_{ij}(0) - \mu_{ij}(0^-)\big)^{-1}$;

(iii): $\mu_{ij}\colon [-r_j, 0) \longrightarrow R$, $i, j \in N$, is nonincreasing and continuous from the left.

Evidently, for the quasimonotone operator D, $A(\operatorname{Int} R_+^n) \cap \operatorname{Int} R_+^n \neq \emptyset$. Therefore the conclusion of Lemma 2.1–2.4 are valid for D as well. As a final remark of this section, we note that the D-operator associated with usual equations has the form $D(\varphi) = \varphi(0) + L(\varphi)$, and therefore it is quasimonotone if (iii) is satisfied, where $L(\varphi)$ is defined by (2.4) and (2.5).

3. **Strong monotonicity principles.** We consider the following neutral functional differential equation

$$\frac{d}{dt} D(x_t) = f(t, x_t) \tag{3.1}$$

where $D\colon C_r \longrightarrow R^n$ is a quasimonotone linear operator, $f\colon \Omega \longrightarrow R^n$, where Ω is an open subset in $R_+ \times C_r$, is continuous and Lipschitz in the second variable on any compact subsets of Ω. Under these assumptions, the initial value problem of (3.1) is well posed. That is, for any $(\sigma, \varphi) \in \Omega$, there exists $\tau(\sigma, \varphi) > \sigma$ and a continuous function, the solution of (3.1) through (σ, φ), $x = (x_1, \ldots, x_n)$ with $x_i\colon [\sigma - r_i, \tau(\sigma, \varphi)) \longrightarrow R$, $i \in N$, such that $(t, x_t) \in \Omega$, the mapping $t \in [\sigma, \tau(\sigma, \varphi)) \longrightarrow D(x_t) \in R^n$ is differentiable and (3.1) holds for all $t \in [\sigma, \tau(\sigma, \varphi))$. Here and in what follows, $x_t = (x_t^1, \ldots, x_t^n)$ with $x_t^i(\theta) = x_i(t + \theta)$, $-r_i \leq \theta \leq 0$, $x(\cdot) = x(\cdot; \sigma, \varphi)$ denotes the unique solution of (3.1) through (σ, φ). For details, we refer to [24].

We start with the following assumption:

(M): *If* (t, φ), $(t, \psi) \in \Omega$ *with* $\varphi \underset{D}{\leq} \psi$ *and* $D_i(\varphi) = D_i(\psi)$ *for some* $i \in N$, *then* $f_i(t, \varphi) \leq f_i(t, \psi)$.

Under this assumption, we have the following monotonicity principle:

LEMMA 3.1. *Let* (M) *hold. Then for any* $(\sigma, \varphi), (\sigma, \psi) \in \Omega$ *with* $\varphi \underset{D}{\leq} \psi$, *we have* $x_t(\sigma, \phi) \underset{D}{\leq} x_t(\sigma, \psi)$ *for all* $t \in [\sigma, \tau^*)$, *where* $\tau^* = \min\{\tau(\sigma, \phi), \tau(\sigma, \psi)\}$.

PROOF. By Lemma 2.2, for any given positive integer m we can choose $\psi_m \in C_r$ such that $\psi_m = \psi + \delta_m$, $\delta_m \in \operatorname{Int} C_{r,D}^+$ and $D(\psi_m) = D(\psi) + \varepsilon_m$, $\varepsilon_m \in \operatorname{Int} R_+^n$ and $\delta_m \longrightarrow 0$ as $m \longrightarrow \infty$. Let $x^m(\cdot, \sigma, \psi_m)$ be a solution of the following initial value problem

$$\frac{d}{dt} D(x_t) = f(t, x_t) + \frac{1}{m} \tag{3.2}$$

$$x_\sigma = \psi_m \tag{3.3}$$

By the continuous dependence of solutions, it suffices to prove that $x_t(\sigma,\varphi) \underset{D}{\ll} x_t^m(\sigma,\psi_m)$ for any positive integer m, any given $t_1 \in [\sigma,\tau^*)$ and all $t \in [\sigma,t_1]$. By way of contradiction, if the above statement is not true, then by Lemma 2.3, there exists an integer $m > 0$ and a constant $t^* \in (\sigma,t_1]$ such that $x_t(\sigma,\varphi) \underset{C_r}{\ll} x_t^m(\sigma,\psi_m)$, $D\big(x_t(\sigma,\varphi)\big) \underset{R^n}{\ll} D\big(x_t^m(\sigma,\psi_m)\big)$ for all $t \in [\sigma,t^*)$ and $D_i\big(x_{t^*}(\sigma,\varphi)\big) = D_i\big(x_{t^*}^m(\sigma,\psi_m)\big)$ for some integer $i \in N$. Therefore at $t = t^*$, we have

$$\frac{d}{dt}D_i\big(x_t^m(\sigma,\psi_m)\big) \leq \frac{d}{dt}\,D_i\big(x_t(\sigma,\varphi)\big).$$

On the other hand, $D\big(x_{t^*}(\sigma,\phi)\big) \underset{R^n}{\leq} D\big(x_{t^*}^m(\sigma,\psi_m)\big)$ and $x_{t^*}(\sigma,\phi) \underset{C_r}{\leq} x_{t^*}^m(\sigma,\psi_m)$ by continuity. Therefore at $t = t^*$, by the assumption (M) we have

$$\begin{aligned}\frac{d}{dt}D_i\big(x_t(\sigma,\varphi)\big) &= f_i\big(t^*,x_{t^*}(\sigma,\varphi)\big)\\ &\leq f_i\big(t^*,x_{t^*}^m(\sigma,\psi_m)\big)\\ &< f_i\big(t^*,x_{t^*}^m(\sigma,\psi_m)\big)+\frac{1}{m}\\ &= \frac{d}{dt}D_i\big(x_{t^*}^m(\sigma,\psi_m)\big)\end{aligned}$$

which yields a contradiction, completing the proof.

REMARK 3.1. For retarded equations, $D(\varphi) = \varphi(0)$, assumption (M) reduces to the usual pseudo-monotonicity condition that $\varphi \leq \psi$ and $\varphi_i(0) = \psi_i(0)$ imply that $f_i(t,\varphi) \leq f_i(t,\psi)$. A monotonicity principle for retarded equations (usually called a comparison principle) has been proved by many authors. We refer to [31],[33],[40]–[42],[45],[47] and [49] for detailed references. However, to the best of our knowledge, Lemma 3.1 is new for neutral equations.

REMARK 3.2. The idea employed in the argument of Lemma 3.1 can be used to prove the following invariance result: suppose $\Omega = R_+ \times U$ and U is an open subset of C_r containing $C_{r,D}^+$. If for any $\varphi \in C_r$ with $\varphi \underset{D}{\geq} 0$ and $D_i(\varphi) = 0$ for some $i \in N$, one has $f_i(t,\varphi) \geq 0$ for $t \in R$, then the set $C_{r,D}^+$ is invariant. That is, for any $(\sigma,\varphi) \in R_+ \times C_r$ with $\varphi \underset{D}{\geq} 0$, we have $x_t(\sigma,\varphi) \underset{D}{\geq} 0$ for all $t \in [\sigma,\tau(\sigma,\varphi))$.

To guarantee the *ignition* of some component of $D\big(x_t(\sigma,\psi)\big) - D\big(x_t(\sigma,\varphi)\big)$ in the case where $\varphi \underset{D}{<} \psi$, we assume

(I): *If* (t,φ), $(t,\psi) \in \Omega$ *with* $\varphi \underset{D}{\leq} \psi$ *and* $\varphi_i(-r_i) < \psi_i(-r_i)$ *for some* $i \in N$, *then there exists* $j \in N$ *such that either* $D_j(\varphi) < D_j(\psi)$ *or* $D_j(\varphi) = D_j(\psi)$ *and* $f_j(t,\varphi) < f_j(t,\psi)$ *hold.*

LEMMA 3.2. *Let* (M) *and* (I) *hold. Then for any* (σ,φ), $(\sigma,\psi) \in \Omega$ *with* $\varphi \underset{D}{<} \psi$ *and* $\tau^* = \min\{\tau(\sigma,\varphi),\tau(\sigma,\psi)\} > \sigma + |r|$, *there exist* $j \in N$ *and* $t \in [\sigma,\sigma+|r|]$ *such that* $D_j\big(x_t(\sigma,\varphi)\big) < D_j\big(x_t(\sigma,\psi)\big)$.

PROOF. By the assumption $\varphi \underset{D}{<} \psi$, there exist $i \in N$ and $\theta \in [-r_i, 0]$ so that $\varphi_i(\theta) < \psi_i(\theta)$. Let $t_1 = \sigma + \theta + r_i$. Without loss of generality, we may assume that $t_1 > \sigma$. By Lemma 3.1, we have $x_{t_1}(\sigma, \varphi) \underset{D}{\leq} x_{t_1}(\sigma, \psi)$. Since $x_i(t_1 - r_i; \sigma, \varphi) = \varphi_i(\theta) < \psi_i(\theta) = x_i(t_1 - r_i, \sigma, \psi)$, by the assumption (I) there exists $j \in N$ so that either $D_j\big(x_{t_1}(\sigma, \varphi)\big) < D_j\big(x_{t_1}(\sigma, \psi)\big)$, or $D_j\big(x_{t_1}(\sigma, \varphi)\big) = D_j\big(x_{t_1}(\sigma, \psi)\big)$ and $f_j\big(t_1, x_{t_1}(\sigma, \varphi)\big) < f_j\big(t_1, x_{t_1}(\sigma, \psi)\big)$. However, the latter is impossible, for otherwise, at $t = t_1$,

$$\begin{aligned}\frac{d}{dt} D_j\big(x_t(\sigma, \varphi)\big) &= f_j\big(t_1, x_{t_1}(\sigma, \varphi)\big) < f_j\big(t_1, x_{t_1}(\sigma, \psi)\big) \\ &= \frac{d}{dt} D_j\big(x_t(\sigma, \psi)\big)\end{aligned}$$

which implies that $D_j\big(x_t(\sigma, \varphi)\big) > D_j\big(x_t(\sigma, \psi)\big)$ for $t < t_1$ and near t_1. This is contrary to Lemma 3.1, completing the proof.

To guarantee the component $D_j\big(x_t(\sigma, \psi)\big) - D_j\big(x_t(\sigma, \varphi)\big)$ remains positive, we require

(P): *There exists a continuous functional $F: R \times C_r^2 \longrightarrow R$ such that for any $(t, \varphi), (t, \psi) \in \Omega$ with $\varphi \underset{D}{\leq} \psi$, we have*

$$f_i(t, \psi) - f_i(t, \varphi) \geq F(t, \varphi, \psi)\big(D_i(\psi) - D_i(\varphi)\big), \quad i \in N$$

LEMMA 3.3. *Let (P) hold. Then for any (σ, φ), $(\sigma, \psi) \in \Omega$ with $\varphi \underset{D}{\leq} \psi$ and $D_j\big(x_{t_1}(\sigma, \varphi)\big) < D_j\big(x_{t_1}(\sigma, \psi)\big)$ for some $j \in N$ and $t_1 \in [\sigma, \tau^*)$, then $D_j\big(x_t(\sigma, \varphi)\big) < D_j\big(x_t(\sigma, \psi)\big)$ and $x_j(t; \sigma, \varphi) < x_j(t; \sigma, \psi)$ for all $t \in [t_1, \tau^*)$, where $\tau^* = \min\{\tau(\sigma, \varphi), \tau(\sigma, \psi)\}$.*

PROOF. Clearly, the assumption (P) implies (M). Therefore by Lemma 3.1, $x_t(\sigma, \varphi) \underset{D}{\leq} x_t(\sigma, \psi)$ for $t \in [\sigma, \tau^*)$. According to the assumption (P), we have

$$\frac{d}{dt}[D_j\big(x_t(\sigma, \psi)\big) - D_j\big(x_t(\sigma, \varphi)\big)] \geq F\big(t, x_t(\sigma, \varphi), x_t(\sigma, \psi)\big)\big[D_j\big(x_t(\sigma, \psi)\big) - D_j\big(x_t(\sigma, \varphi)\big)\big].$$

Hence by the well known comparison principle of ordinary differential equations (see, e.g. [33]), we have

$$\begin{aligned}& D_j\big(x_t(\sigma, \psi)\big) - D_j\big(x_t(\sigma, \psi)\big) \\ &= [D_j\big(x_{t_1}(\sigma, \psi)\big) - D_j\big(x_{t_1}(\sigma, \varphi)\big)] e^{\int_{t_1}^{t} F\left(s, x_s(\sigma, \varphi), x_s(\sigma, \psi)\right) ds}\end{aligned}$$

for all $t \in [t_1, \tau^*)$. Therefore $D_j\big(x_t(\sigma, \psi)\big) - D_j\big(x_t(\sigma, \varphi)\big) > 0$ for all $t \in [t_1, \tau^*)$. This implies $x_j(t; \sigma, \psi) - x_j(t; \sigma, \varphi) > 0$ by Lemma 2.4. The proof is completed.

Putting together (I) and (P), we see that if $\varphi, \psi \in C_r$ with $\varphi \underset{D}{<} \psi$, then some component of $D\big(x_t(\sigma, \psi)\big) - D\big(x_t(\sigma, \varphi)\big)$ becomes and remains positive. To turn on the other components (in the case where $n > 1$), we assume

(T): *For any proper subset $K \subsetneq N$ and any $\varphi, \psi \in C_r$ with $\varphi \underset{D}{\leq} \psi, \varphi_j(\theta) < \psi_j(\theta)$ and $D_j(\varphi) < D_j(\psi)$ for $j \in K$ and $\theta \in [-r_j, 0]$, there exists $i \in N \backslash K$ such that either (i) $D_i(\psi) > D_i(\varphi)$ or (ii) $D_i(\psi) = D_i(\varphi)$ and $f_i(t, \psi) > f_i(t, \varphi)$.*

Now, we are in the position to state our main result of this section.

THEOREM 3.1. *Let (I), (P) and (T) hold. If $\varphi, \psi \in C_r$ with $\varphi \underset{D}{<} \psi$ and $\tau^* = \min\{\tau(\sigma, \varphi), \tau(\sigma, \psi)\} > \sigma + n|r|$. Then $x(t; \sigma, \varphi) \underset{R^n}{\ll} x(t, \sigma, \psi)$ and $D\big(x_t(\sigma, \varphi)\big) \underset{R^n}{\ll} D\big(x_t(\sigma, \psi)\big)$ for all $t \in (\sigma + n|r|, \tau^*)$.*

PROOF. By Lemma 3.1, $x_t(\sigma, \varphi) \underset{D}{\leq} x_t(\sigma, \psi)$ for all $t \in [\sigma, \tau^*)$. By Lemmas 3.2 and 3.3, there exists $i \in N$ such that $D_i\big(x_t(\sigma, \varphi)\big) < D_i\big(x_t(\sigma, \psi)\big)$ and $x_i(t; \sigma, \varphi) < x_i(t; \sigma, \psi)$ for all $t \in (\sigma + |r|, \tau^*)$. Let $K = \{i\}$. If $N = K$, then the proof is complete. If $N \neq K$, then $D_i\big(x_{\sigma+2|r|}(\sigma, \varphi)\big) < D_i\big(x_{\sigma+2|r|}(\sigma, \psi)\big)$ and $x_i(\sigma + 2|r| + \theta, \varphi) < x_i(\sigma + 2|r| + \theta, \psi)$ for all $\theta \in [-r_i, 0]$. Therefore by the assumption (T), there exists $j \in N \backslash K$ such that either $D_j\big(x_{\sigma+2|r|}(\sigma, \varphi)\big) < D_j\big(x_{\sigma+2|r|}(\sigma, \psi)\big)$ or $D_j\big(x_{\sigma+2|r|}(\sigma, \varphi)\big) = D_j\big(x_{\sigma+2|r|}(\sigma, \psi)\big)$ and $f_j\big(\sigma + 2|r|, x_{\sigma+2|r|}(\sigma, \varphi)\big) < f_j\big(\sigma + 2|r|, x_{\sigma+2|r|}(\sigma, \psi)\big)$. The latter is impossible, for otherwise, we have

$$\frac{d}{dt} D_j\big(x_t(\sigma, \varphi)\big) < \frac{d}{dt} D_j\big(x_t(\sigma, \psi)\big) \text{ at } t = \sigma + 2|r|$$

which implies $D_j\big(x_t(\sigma, \varphi)\big) > D_j\big(x_t(\sigma, \psi)\big)$ for $t < \sigma + 2|r|$ and near $\sigma + 2|r|$ which is contrary to $x_t(\sigma, \varphi) \underset{D}{\leq} x_t(s, \psi)$ for $t \in [\sigma, \tau^*)$. Therefore $D_j\big(x_{\sigma+2|r|}(\sigma, \varphi)\big) < D_j\big(x_{\sigma+2|r|}(\sigma, \psi)\big)$.

By Lemma 3.3, we get $D_j\big(x_t(\sigma, \varphi)\big) < D_j\big(x_t(\sigma, \psi)\big)$ and $x_j(t; \sigma, \varphi) < x_j(t; \sigma, \psi)$ for $t \in (\sigma + 2|r|, \tau^*)$. If $N = \{i, j\}$, then the proof is complete. Otherwise, we continue our argument with $K = \{i, j\}$ to yield $k \in N|K$ so that $x_k(t; \sigma, \varphi) < x_k(t; \sigma, \psi)$ and $D_k\big(x_t(\sigma, \varphi)\big) < D_k\big(x_t(\sigma, \psi)\big)$ for all $t \in (\sigma + 3|r|, \tau^*)$. Continuing in this manner, we obtain that $x(t; \sigma, \varphi) \underset{R^n}{\ll} x(t; \sigma, \psi)$ and $D\big(x_t(\sigma, \varphi)\big) \underset{R^n}{\ll} D\big(x_t(\sigma, \psi)\big)$ for $t \in (\sigma + n|r|, \tau^*)$. This completes the proof.

As an immediate consequence of this theorem, for the autonomous neutral equation

$$\frac{d}{dt} D(x_t) = f(x_t) \tag{3.4}$$

where $f: C_r \longrightarrow R^n$ is continuous, we get the following strong monotonicity principle:

COROLLARY 3.1. *Suppose $\tau(0, \varphi) = \infty$ for all $\varphi \in C_r$. If (M) holds, then the solution semiflow $\{x_t(0, \varphi)\}_{t \geq 0}$ defined by the equation (3.4) is monotone, moreover, if (I), (P) and (T) holds, then the semiflow is eventually strongly monotone.*

EXAMPLE 3.1. As an illustrative example, we consider the following scalar neutral equation

$$\frac{d}{dt}[x(t) - cx(t - r)] = f\big(x(t), x(t - r)\big) \tag{3.5}$$

where c is a constant, $f \in C^1(R^2, R)$. The associated D-operator is defined by $D(\varphi) = \varphi(0) - c\varphi(-r)$. Let

$$d = \inf_{(x,y)\in R^2} \frac{\partial}{\partial y} f(x,y) + c \inf_{(x,y)\in R^2} \frac{\partial}{\partial x} f(x,y).$$

We claim that

(i) if $c \geq 0$, then D is quasimonotone;

(ii) if $c \geq 0$ and $d \geq 0$, then (M) and (P) are satisfied;

(iii) if $c \geq 0$ and $d > 0$, then (M), (I) and (P) are satisfied.

The conclusion (i) is trivial. To prove (ii) and (iii), we use the intermediate value theorem to obtain

$$\begin{aligned} &f\big(\psi(0), \psi(-r)\big) - f\big(\varphi(0), \varphi(-r)\big) \\ &\quad = \int_0^1 \frac{\partial}{\partial x} f\big(s\varphi(0) + (1-s)\psi(0), \psi(-r)\big)\, ds[D(\psi) - D(\varphi)] \\ &\quad + \Big[\int_0^1 \frac{\partial}{\partial y} f\big(\varphi(0), s\varphi(-r) + (1-s)\psi(-r)\big)\, ds \\ &\quad + c\int_0^1 \frac{\partial}{\partial x} f\big(s\varphi(0) + (1-s)\psi(0), \psi(-r)\big)\, ds\Big][\psi(-r) - \varphi(-r)] \end{aligned}$$

for all $\varphi, \psi \in C([-r,0], R)$. Therefore, if $\varphi \underset{D}{\leq} \psi$, $D(\varphi) = D(\psi)$ and $c \geq 0$, then

$$f\big(\psi(0), \psi(-r)\big) - f\big(\varphi(0), \varphi(-r)\big) \geq d[\psi(-r) - \varphi(-r)].$$

Hence if $d \geq 0$, then (M) holds; if $d > 0$, then (I) holds. Moreover, if $\varphi \underset{D}{\leq} \psi$, $c \geq 0$ and $d \geq 0$, then

$$f\big(\psi(0), \psi(-r)\big) - f\big(\varphi(0), \varphi(-r)\big) \geq F(\varphi, \psi)[D(\psi) - D(\varphi)]$$

with $F(\varphi, \psi) = \int_0^1 \frac{\partial}{\partial x} f\big(s\varphi(0) + (1-s)\psi(0), \psi(-r)\big)\, ds$. This proves (P).

EXAMPLE 3.2. To illustrate that our results include those results contained in [49], we consider the autonomous system (3.4) with a quasimonotone D-operator and $f \in C^1(C_r, R^n)$. According to the Riesz representation theorem, for any given $\zeta \in C_r$ there exists $\eta_{ij}(\zeta)\colon [-r_j, 0] \longrightarrow R$, $i, j \in N$, of bounded variation on $[-r_j, 0]$ such that

$$[df(\zeta)\varphi]_i = \sum_{j=1}^{n} \int_{-r_j}^{0} \varphi_j(\theta)\, d_\theta \eta_{ij}(\zeta, \theta), \quad i \in N, \varphi \in C_r.$$

Let

$$B(\zeta) = \big(\eta_{ij}(\zeta, 0) - \eta_{ij}(\zeta, 0^-)\big) \tag{3.6}$$

and define $K(\zeta)\colon C_r \longrightarrow R^n$ by

$$[K(\zeta)\varphi]_i = \sum_{j=1}^{n} \int_{-r_j}^{0} \varphi_j(\theta) d_\theta \bar{\eta}_{ij}(\xi, \theta), \quad i \in N, \quad \varphi \in C_r \tag{3.7}$$

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where

$$\bar{\eta}_{ij}(\zeta,\theta) = \begin{cases} \eta_{ij}(\zeta,\theta) & \text{if } \theta \in [-r_j, 0), \\ \eta_{ij}(\zeta, 0^-) & \text{if } \theta = 0. \end{cases} \tag{3.8}$$

Then

$$df(\zeta)\varphi = B(\zeta)\varphi(0) + K(\zeta)\varphi. \tag{3.9}$$

Let A and L be defined by (2.2) and (2.5). For reference purposes, the following is introduced

DEFINITION 3.1. The neutral equation (3.4) is *cooperative* if

(i): for any $\zeta \in C_r$, all off-diagonal elements of $B(\zeta)A^{-1}$ are nonnegative and $\big(K(\zeta) - B(\zeta)A^{-1}L\big)C_r^+ \subset R_+^n$. The neutral equation (3.4) is *irreducible,* if the following hold

(ii): for any $j \in J$, there exists $i \in N$ such that for sufficiently small $\varepsilon > 0$, $\eta_{ij}(\zeta, -r_j + \varepsilon) - \sum_{k=1}^n c_{ik}(\zeta)\mu_{k_j}(-r_j + \varepsilon) > 0$, where $B(\zeta)A^{-1} = \big(c_{ij}(\zeta)\big)$;

(iii): the matrix $\Big(\big(K(\zeta) - B(\zeta)A^{-1}L\big)(\hat{e}_1), \ldots, \big(K(\zeta) - B(\zeta)(A^{-1}L)(\hat{e}_n)\Big)$ is irreducible, where $(e_1, \ldots e_n)$ is a standard basis of R^n and ^denotes the inclusion $R^n \longrightarrow C_r$.

The following is a consequence of Theorem 3.1.

COROLLARY 3.2. *Suppose that $\tau(0,\varphi) = \infty$ for $\varphi \in C_r$. If equation (3.4) is cooperative, then the semiflow $\{x_t(0,\varphi)\}_{t\geq 0}$ is monotone, moreover, if equation (3.4) is irreducible, then the semiflow is eventually strongly monotone.*

PROOF. By definition, we have

$$\begin{aligned} df(\zeta)\varphi &= B(\zeta)\varphi(0) + K(\zeta)\varphi \\ &= B(\zeta)A^{-1}D\varphi + [K(\zeta) - B(\zeta)A^{-1}L]\varphi. \end{aligned}$$

Therefore,

$$\begin{aligned} f(\psi) - f(\varphi) &= \int_0^1 df\big(s\varphi + (1-s)\psi\big)(\psi - \varphi)\, ds \\ &= \int_0^1 B\big(s\varphi + (1-s)\psi\big)A^{-1}\, ds[D(\psi) - D(\varphi)] \\ &\quad + \int_0^1 [K\big(s\varphi + (1-s)\psi\big) - B\big(s\varphi + (1-s)\psi\big)A^{-1}L][\psi - \varphi]\, ds, \end{aligned}$$

from which we can verify that if equation (3.4) is cooperative, then (M) and (P) holds; and that (ii) implies (I) and (iii) implies (T). Therefore our conclusion follows from Corollary 3.1.

To conclude this section, we remark that for delay equations, $D(\varphi) = \varphi(0)$, assumptions (i)–(iii) in Definition 3.1 reduces to the assumption (K), (R) and (I) in [49], respectively. Therefore our conclusions include the results of Section 2 of [49] as a special case.

4. **Order-stability and convergence.** In this section, with the help of the established strong motonicity principle, we obtain several (generic) convergence and order-stability theorems for neutral equations as direct applications of the powerful results due to Hirsch [27, 28] for general monotone semiflows.

We should mention that the results of Hirsch which we are going to use below for strongly monotone semiflows are true for what we term eventually strongly monotone semiflows (see, e.g. [49]).

Let us recall that a bounded linear operator D: $C_r \longrightarrow R^n$ is stable if the zero solution of the *generalized difference equation*

$$\begin{cases} D(y_t) = 0 \\ y_0 = \varphi \end{cases}$$

is uniformly asymptotically stable. A simple sufficient condition to guarantee the stability of the D-operator defined by $D_i(\varphi) = \varphi_i(0) - \sum_{j=1}^n \int_{-r_j}^0 \varphi_j(\theta)\, dv_{ij}(\theta)$, where v_{ij}: $[-r_j, 0] \longrightarrow R$ is of bounded variation and $\mathrm{Var}_{[-s,0]}\, v_{ij} \longrightarrow 0$ as $s \longrightarrow 0$, is $\sum_{j=1}^n \mathrm{Var}_{[-r_j,0]}\, v_{ij} < 1$ for $i \in N$. See [16] and [44] for details.

THEOREM 4.1. *Suppose that the neutral equation (3.4) is cooperative, the operator D is stable and f maps bounded sets of C_r into bounded sets of R^n. Let $\varphi \in C_r$ be a given element such that the orbit $\{x_t(\varphi)\}_{t \geq 0}$ is bounded and $x_T(\varphi) \underset{D}{\ll} \varphi$ (or $x_T(\varphi) \underset{D}{\gg} \varphi$) for some real $T > 0$, then $x_t(\varphi)$ converges to an equilibrium as $t \longrightarrow \infty$, here and in what follows, $x_t(\varphi)$ denotes $x_t(0, \varphi)$.*

PROOF. Since the equation (3.4) is cooperative, the semiflow $\{x_t(\phi)\}_{t \geq 0}$ is monotone by Corollary 3.2. Moreover, the stability of D and the assumption that f maps bounded sets of C_r to bounded sets of R^n imply that each bounded orbit has compact closure (see, e.g. [24, Theorem 6.1]). Therefore, the conclusion follows directly from [27, Theorem 2.3].

As an immediate consequence to the above theorem, we obtain the following:

COROLLARY 4.1. *Suppose that the neutral equation (3.4) is cooperative, the operator D is stable and f maps bounded sets of C_r into bounded sets of R^n, then equation (3.4) has no attracting periodic orbit, i.e., a nonconstant closed orbit which attracts one of its open neighborhoods.*

In the case where an open set contains a unique equilibrium point which is asymptotically stable, we have the following global convergence.

THEOREM 4.2. *Suppose that*

(i) the neutral equation (3.4) is cooperative and irreducible, the operator D is stable and f maps bounded sets of C_r into bounded sets of R^n;

(ii) there exists an open subset U of C_r such that for each $\varphi \in U$, $\{x_t(\varphi);\ t \geq 0\}$ is bounded, and there exists a unique equilibrium ψ in $c\ell\ U$ with $\psi \in$

$\omega(\zeta)$ for some $\zeta \in U$, where $\omega(\zeta)$ denotes the ω-limit set of ζ, i.e., $\omega(\zeta) = \bigcap_{\tau \geq 0} \mathrm{Cl}\Big(\bigcup_{t \geq \tau} x_t(\zeta)\Big)$.

Then for any $\varphi \in U$, $x_t(\varphi) \longrightarrow \psi$ as $t \longrightarrow \infty$.

PROOF. Since each orbit in U is bounded, D is stable and f maps bounded sets of C_r into bounded sets of R^n, each orbit in U has compact closure. Moreover, (ii) implies that there exists a unique equilibrium point in $\bigcup_{\varphi \in U} \omega(\varphi)$. Therefore our conclusion follows from [28, Theorem 10.3].

REMARK 4.1. A simple criterion for conditions (ii) is that each orbit in U is bounded and Cl U contains a unique equilibrum which is locally asymptotically stable.

The above theorem indicates that for a cooperative and irreducible neutral equation, a complicated dynamics is generated by the existence of multi-eqiulibria and the interactions between stability properties of equilibria.

The following theorem provides a generic convergence result for a cooperative and irreducible neutral equation.

THEOREM 4.3. *Suppose that*

(i) the neutral equation (3.4) *is cooperative and irreducible, the operator D is stable and f maps bounded sets of C_r into bounded sets of R^n;*

(ii) for each $\varphi \in C_r$, the set $\{x_t(\varphi); t \geq 0\}$ is bounded.

Then the subset $Q \subseteq C_r$ of points convergent to the set of equilibria is dense in C_r.

PROOF. This is an immediate consequence of Corollary 7.6 in [28], Corollary 3.2 in this paper and Theorem 6.1 in [24].

Finally, we focus on an interesting equivalence theorem due to Hirsch [28] relating order-stability and the convergence of precompact orbits. This equivalence is important, since in applications, the order stability can be proved by using a Liapunov function or functional with semidefinite derivative, but the convergence to a single equilibrium requires further restrictions on the Liapunov function and needs a sophisticated analysis.

We notice that if the neutral equation (3.4) defines a monotone semiflow, then an orbit $\{x_t(\varphi); t \geq 0\}$ is order-stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\psi, \eta \in C_r$ with $\varphi_i(\theta) - \delta \leq \eta_i(\theta) \leq \varphi_i(\theta) \leq \psi_i(\theta) \leq \varphi_i(\theta) + \delta$ and $D_i(\varphi) - \delta \leq D_i(\eta) \leq D_i(\varphi) \leq D_i(\psi) \leq D_i(\varphi) + \delta$ for $i \in N$ and $\theta \in [-r_i, 0]$, one has $x_i(t, \varphi) - \varepsilon \leq x_i(t, \eta) \leq x_i(t, \varphi) \leq x_i(t, \psi) \leq x_i(t, \varphi) + \varepsilon$ and $D_i\big(x_t(\varphi)\big) - \varepsilon \leq D_i\big(x_t(\eta)\big) \leq D_i\big(x_t(\varphi)\big) \leq D_i\big(x_t(\psi)\big) \leq D_i\big(x_t(\varphi)\big) + \varepsilon$ for $i \in N$ and $t \geq 0$.

THEOREM 4.4. *Let D be stable, f map bounded sets of C_r to bounded sets of R^n, and let equation (3.4) define an eventually strongly monotone semiflow. Suppose that for any $\varphi \in C_r$, the orbit $\{x_t(\varphi); t \geq 0\}$ is bounded and order-stable. Then each orbit is convergent to the set $\mathcal{E}$ of equilibria as $t \longrightarrow \infty$. More precisely, for any $\varphi \in C_r$ one of the following holds: either*

(i) *there exists* $\delta > 0$ *such that if* $\psi \in C_r|\{\varphi\}$ *is given with* $0 \leq \psi_i(\theta) - \varphi_i(\theta) < \delta$ *and* $D_i(\varphi) \leq D_i(\psi) < D_i(\varphi) + \delta$ *for* $\theta \in [-r_i, 0]$ *and* $i = 1, \dots n$, *then* $x_t(\psi)$ *is convergent to* $\mathcal{E}$ *as* $t \longrightarrow \infty$ *and* $\omega(\varphi) = \omega(\psi) \subseteq \mathcal{E}$; *or else*

(ii) $x_t(\varphi)$ *is convergent to a single equilibrium as* $t \longrightarrow \infty$.

PROOF. According to Lemma 2.2, for any given $\varphi \in C_r$, we can find a sequence $\varphi_m \in C_r$ with $\varphi_m \underset{D}{\gg} \varphi$ and $\varphi_m \longrightarrow \varphi$ as $m \longrightarrow \infty$. Moreover for any $\varphi_m \in C_r$, the orbit $\{x_t(\varphi_m); t \geqq 0\}$ has compact closure, since boundedness of orbits implies relative compactness for a neutral equation with stable D-operator (see, Theorem 6.1 in [24]). Therefore, the conclusion follows from Theorem 8.3 of [28].

As a final remark, we point out that Theorems 4.1–4.4 are stated in the whole state space C_r for simplicity. It is easy to verify that these results still hold if we replace C_r by any positively invariant closed subset of C_r.

5. **An application to compartmental systems.** As an application of the results in previous sections, we consider a mathematical model of biological compartmental systems which have been extensively studied in the literature [5], [6], [9], [10], [18]–[21], [29], [32], [34], [39], [43], and [45].

Denote by $C_1, \dots, C_n$ the components of a compartmental system, by $x_i(t)$ the amount of the material in compartment C_i at time t, and by C_0 the environment of the compartmental system. We assume the following:

(H1): at time $t \geq 0$, the rate of material outflow from C_i in the direction of C_j is given by the so-called transport function $g_{ji}\big(x_i(t)\big)$, $j = 0, 1, \dots, n$ and $i = 1, \dots, n$; which is nondecreasing, continuously differentiable and $g_{ji}(0) = 0$,

(H2): at time $t \geq 0$, the compartment C_i produces material itself at a rate $\sum_{j=1}^n \int_{-r_j}^0 \dot{x}_j(t + \theta)\, dv_{ij}(\theta)$, where $r_j > 0$ is a constant, $v_{ij}: [-r_j, 0] \longrightarrow R$ is nondecreasing, continuous from the left and $\mathrm{Var}_{[-s,0]}\, v_{ij} \longrightarrow 0$ as $s \longrightarrow 0$, $i, j = 1, \dots, n$;

(H3): material flows from compartment C_j into compartment C_i through a pipe P_{ij} having a transit time distribution function $\eta_{ij}: [-r_j, 0] \longrightarrow R$ which is monotone nondecreasing, continuous from the left on $[-r_j, 0)$ with $\mathrm{Var}_{[-r_j,0]}\, \eta_{ij} = 1$ for $i, j = 1, \dots, n$.

Under this set of assumptions, since the change of the amount of material of any compartment C_i, $1 \leq i \leq n$, in any interval of time equals to the difference between the amount of total influx into and total outflux from C_i in the same time interval, we obtain the model equation

$$
\begin{aligned}
\frac{d}{dt}\Big[x_i(t) - \sum_{j=1}^n \int_{-r_j}^0 x_j(t+\theta)\, dv_{ij}(\theta)\Big]& \\
= -\sum_{j=0}^n g_{ji}\big(x_i(t)\big) + \sum_{j=1}^n \int_{-r_j}^0 g_{ij}\big(x_j(t+\theta)\big)\, d\eta_{ij}(\theta)& \qquad (5.1)
\end{aligned}
$$

For details, we refer to [19] and [21]. The convergence problem of solutions to a single equilibrium point was first raised by Bellman [9] and it has been referred to as Bellman's conjecture [18]. It has been proved that this conjecture is true for linear systems by Bellman [9] and for nonlinear systems (donor controlled systems) by Jacquez [29] in the special case where $v_{ij} \equiv 0$ and $\eta_{ij}(0) = 1, \eta_{ij}(\theta) = 0$ for $\theta \in [-r_j, 0)$ (i.e. for ordinary differential equations). In the case where $v_{ij} = 0$ (retarded equation), Gÿori [19] proved that all nonnegative solutions tend to an equilibrium as $t \longrightarrow \infty$ if there is only one equilibrium point. For related results, we refer to [6], [18], [21], [29], [32], [34], [39] and [43].

Unfortunately, the uniqueness assumption of equilibrium is usually not satisfied. In particular, for closed compartmental systems (systems for which $g_{0i} = 0$ for $i \in N$), with $n = 1$, each constant function is an equilibrium point and it has been shown that each solution approaches to a single equilibrium (see, e.g. [6]–[8], [14], [15], [17], [22], [23], [30], [52] and [53]). Our purpose is to extend this result to the higher dimensional case. Thus throughout the remainder of this paper, we assume that $g_{oi} \equiv 0$ for $i = 1, \ldots, n$.

Our first task is to insure that solutions with nonnegative initial conditions are defined for all future time $t \geq 0$ and remain nonnegative. The following assumptions will be useful:

(H4): $h_{ij}\eta_{ij}(\theta) - \sum_{k=1}^{n} d_{ki}v_{ij}(\theta)$ is nondecreasing on $[-r_j, 0]$, $1 \leq i,\ j \leq n$, where $d_{ij} = \sup_{x \in R^+} g'_{ij}(x)$ and $h_{ij} = \inf_{x \in R^+} g'_{ij}(x)$ for $1 \leq i,\ j \leq n$;

(H5): $\sum_{j=1}^{n} \mathrm{Var}_{[-r_j,0]}\, v_{ij} < 1$ for $1 \leq i \leq n$.

PROPOSITION 5.1. *Let (H1)–(H5) hold. Then for any $\varphi \in C_r$ with $\varphi \underset{D}{\geq} 0$, $\tau(\varphi) := \tau(0, \varphi) = +\infty$, $x_t(\varphi) \underset{D}{\geq} 0$ for $t \geq 0$ and further $\{x_t(\varphi); t \geq 0\}$ has compact closure, where $D_i(\varphi) = \varphi_i(0) - \sum_{j=1}^{n} \int_{-r_j}^{0} \varphi_j(\theta)\, dv_{ij}(\theta)$ for $1 \leq i \leq n$.*

PROOF. We first prove that $x_t(\varphi) \underset{D}{\geq} 0$ for all $t \in [0, \tau(\varphi))$. According to Remark 3.2, it suffices to prove that $f_i(\varphi) \geq 0$ whenever $\varphi \underset{D}{\geq} 0$ and $\varphi_i(0) = \sum_{j=1}^{n} \int_{-r_j}^{0} \varphi_j(\theta)\, dv_{ij}(\theta)$, where

$$f_i(\varphi) = -\sum_{j=1}^{n} g_{ji}\big(\varphi_i(0)\big) + \sum_{j=1}^{n} \int_{-r_j}^{0} g_{ij}\big(\varphi_j(\theta)\big)\, d\eta_{ij}(\theta).$$

This is true, because of the assumption (H4) and

$$\begin{aligned} f_i(\varphi) &\geq -\sum_{j=1}^{n} d_{ji}\varphi_i(0) + \sum_{j=1}^{n} \int_{-r_j}^{0} h_{ij}\varphi_j(\theta) d\eta_{ij}(\theta) \\ &= -\sum_{j=1}^{n} \int_{-r_j}^{0} \sum_{k=1}^{n} d_{ki}\varphi_j(\theta)\, dv_{ij}(\theta) + \sum_{j=1}^{n} \int_{-r_j}^{0} h_{ij}\varphi_j(\theta)\, d\eta_{ij}(\theta) \\ &= \sum_{j=1}^{n} \int_{-r_j}^{0} \varphi_j(\theta)\, d\Big[h_{ij}\eta_{ij}(\theta) - \sum_{k=1}^{n} d_{ki}v_{ij}(\theta)\Big]. \end{aligned}$$

To prove $\tau(\varphi) = +\infty$ it suffices to prove that $\sup_{t \in [0, \tau(\varphi))} \|x_t(\varphi)\|_{C_r} < \infty$. We note that equation (5.1) is equivalent to the following integral equation

$$\begin{aligned} x_i(t) - \sum_{j=1}^{n} \int_{-r_j}^{0} x_j(t+\theta)\, d\nu_{ij}(\theta) &= x_i(0) - \sum_{j=1}^{n} \int_{-r_j}^{0} x_j(\theta)\, d\nu_{ij}(\theta) - \sum_{j=1}^{n} \int_0^t g_{ji}\big(x_i(s)\big)\, dx \\ &\quad + \sum_{j=1}^{n} \int_0^t \int_{-r_j}^{0} g_{ij}\big(x_j(s+\theta)\big)\, d\eta_{ij}(\theta)\, ds, \quad i \in N, \end{aligned}$$

hence,

$$\begin{aligned} \sum_{i=1}^{n} x_i(t) - \sum_{i,j=1}^{n} \int_{-r_j}^{0} x_j(t+\theta)\, d\nu_{ij}(\theta) + \sum_{i,j=1}^{n} \int_0^t g_{ji}\big(x_i(s)\big)\, ds& \\ - \sum_{i,j=1}^{n} \int_0^t \int_{-r_j}^{0} g_{ij}\big(x_j(s+\theta)\big)\, d\eta_{ij}(\theta)\, ds& \\ = \sum_{i=1}^{n} x_i(0) - \sum_{i,j=1}^{n} \int_{-r_j}^{0} x_j(\theta)\, d\nu_{ij}(\theta).& \end{aligned} \tag{5.2}$$

By interchanging integration order, we obtain

$$\begin{aligned} \int_0^t \int_{-r_j}^{0} g_{ij}\big(x_j(s+\theta)\big)\, d\eta_{ij}(\theta)\, ds &= \int_{-r_j}^{0} \int_0^t g_{ij}\big(x_j(s+\theta)\big)\, ds\, d\eta_{ij}(\theta) \\ &= \int_{-r_j}^{0} \int_{\theta}^{t+\theta} g_{ij}\big(x_j(u)\big)\, du\, d\eta_{ij}(\theta) \\ &= \int_{-r_j}^{0} \int_{\theta}^{0} g_{ij}\big(x_j(u)\big)\, du\, d\eta_{ij}(\theta) \\ &\quad + \int_{-r_j}^{0} \int_{0}^{t+\theta} g_{ij}\big(x_j(u)\big)\, du\, d\eta_{ij}(\theta). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^{n} x_i(t) - \sum_{i,j=1}^{n} \int_{-r_j}^{0} x_j(t+\theta)\, d\nu_{ij}(\theta) + \sum_{i,j=1}^{n} \int_0^t g_{ji}\big(x_i(s)\big)\, ds& \\ - \sum_{i,j=1}^{n} \int_0^t \int_{-r_i}^{0} g_{ji}\big(x_i(s+\theta)\big)\, d\eta_{ji}(\theta)\, ds& \\ = \sum_{i=1}^{n} x_i(t) - \sum_{i,j=1}^{n} \int_{-r_j}^{0} x_j(t+\theta)\, d\nu_{ij}(\theta)& \\ + \sum_{i,j=1}^{n} \int_0^t g_{ji}\big(x_i(s)\big)\, ds& \\ - \sum_{i,j=1}^{n} \int_{-r_i}^{0} \int_{\theta}^{0} g_{ji}\big(x_i(u)\big)\, du\, d\eta_{ji}(\theta)& \\ - \sum_{i,j=1}^{n} \int_{-r_i}^{0} \int_{0}^{t+\theta} g_{ji}\big(x_i(u)\big)\, du\, d\eta_{ji}(\theta).& \end{aligned}$$

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Noting that $\int_{-r_i}^0 d\eta_{ji}(\theta) = 1$, we obtain

$$\int_0^t g_{ji}\big(x_i(s)\big)\,ds = \int_{-r_i}^0 \int_0^t g_{ji}\big(x_i(u)\big)\,du\,d\eta_{ji}(\theta),$$

and so

$$\begin{aligned}
&\sum_{i=1}^n x_i(t) - \sum_{i,j=1}^n \int_{-r_j}^0 x_j(t+\theta)\,dv_{ij}(\theta) \\
&\qquad + \sum_{i,j=1}^n \int_0^t g_{ji}\big(x_i(s)\big)\,ds - \sum_{i,j=1}^n \int_0^t \int_{-r_i}^0 g_{ji}\big(x_i(s+\theta)\big)d\eta_{ji}(\theta)\,ds \\
&= \sum_{i=1}^n x_i(t) - \sum_{i,j=1}^n \int_{-r_j}^0 x_j(t+\theta)\,dv_{ij}(\theta) \\
&\qquad + \sum_{i,j=1}^n \int_{-r_i}^0 \int_{t+\theta}^t g_{ji}\big(x_i(u)\big)\,du\,d\eta_{ji}(\theta) \\
&\qquad - \sum_{i,j=1}^n \int_{-r_i}^0 \int_{\theta}^0 g_{ji}\big(x_i(u)\big)\,du\,d\eta_{ji}(\theta),
\end{aligned}$$

from which and the equality (5.2) we get the following law of material conservation

$$E\big(x_t(\varphi)\big) = E(\varphi) \tag{5.3}$$

where

$$\begin{aligned}
E(\varphi) = \sum_{i=1}^n &\Big[\varphi_i(0) - \sum_{j=1}^n \int_{-r_j}^0 \varphi_j(\theta)\,dv_{ij}(\theta)\Big] \\
&+ \sum_{i,j=1}^n \int_{-r_i}^0 \int_{\theta}^0 g_{ji}\big(\varphi_i(u)\big)\,du\,d\eta_{ji}(\theta).
\end{aligned}$$

Since $x_i(t) \geq 0$, and $x_i(t) - \sum_{j=1}^n \int_{-r_j}^0 x_j(t+\theta)\,dv_{ij}(\theta) \geq 0$ for $t \geq 0$ and $i \in N$, according to (5.3) we have

$$0 \leq x_i(t) - \sum_{j=1}^n \int_{-r_j}^0 x_j(t+\theta)\,dv_{ij}(\theta) \leq E(\varphi).$$

Hence by (H5) we get

$$0 \leq x_i(t) \leq \max\Big\{\frac{E(\varphi)}{1 - \sum_{j=1}^n \mathrm{Var}_{[-r_j,0]}\, v_{ij}},\ |\varphi|_{C_r}\Big\} \text{ for } t \geq 0.$$

Therefore $\{x_t(\varphi); t \geq 0\}$ is bounded. (H5) guarantees that the D-operator is stable (see, e.g. [16]), and hence $\{x_t(\varphi); t \geq 0\}$ has compact closure (see, e.g. [24, Theorem 6.1]). This completes the proof.

We now turn to the monotonicity of the solution semiflow. A similar argument to that for the first part of Proposition 5.1 leads to the following inequality

$$\begin{aligned}
f_i(\psi) - f_i(\varphi) \geq &-\sum_{j=1}^n d_{ji}[D_i(\psi) - D_i(\varphi)] \\
&+ \sum_{j=1}^n \int_{-r_j}^0 d[h_{ij}\eta_{ij}(\theta) - \sum_{k=1}^n d_{ki}v_{ij}(\theta)][\psi_j(\theta) - \varphi_j(\theta)]
\end{aligned} \tag{5.4}$$

for all $\varphi,\ \psi \in C_r$ with $\varphi \underset{D}{\leq} \psi$. Therefore (H4) implies (M) and (P), and we have the following:

THEOREM 5.1. *Let (H1)–(H5) hold. Then the solution semiflow defined by equation (5.1) is monotone and equation (5.1) has no attracting periodic orbit.*

PROOF. This is a direct consequence of Corollary 4.1 since (H4) implies (M) and (H5) implies that the operator D is stable.

In order to guarantee (I) and (T), the following assumptions are essential:

(H6): For any $j \in J$, there exists $i \in N$ such that for sufficiently small $\varepsilon > 0$,

$$h_{ij}\eta_{ij}(-r_j+\varepsilon) - \sum_{k=1}^{n} d_{ki}v_{ij}(-r_j+\varepsilon) > 0.$$

(H7): The matrix $\left(\mathrm{Var}_{[-r_j,0]}\left[h_{ij}\eta_{ij} - \sum_{k=1}^{n} d_{ki}v_{ij}\right]\right)$ is irreducible.

REMARK 5.1. In the case where $v_{ij} \equiv 0$ for $i,j = 1,\ldots,n$, assumptions (H4), (H6) and (H7) imply that the components of the compartmental system are connected directly or indirectly by the flow of materials. In the case where v_{ij} may not vanish, assumptions (H4), (H6) and (H7) are also satisfied if the components of the compartmental system are connected directly or indirectly by the flow of materials and each component produces material itself at a rate smaller than the rate at which material flows from one component to another.

PROPOSITION 5.2. *Let (H1)–(H7) hold. Then assumptions (M),(P),(I) and (T) are satisfied. Further the solution semiflow defined by equation (5.1) is eventually strongly monotone.*

PROOF. By (5.4) it is easy to verify that (H4) implies (M) as well as (P), and (H6) implies (I). To prove (T), we suppose that $K \subseteq N$ is a proper subset. By the irreducibility of the matrix $\left(\mathrm{Var}_{[-r_j,0]}\left[h_{ij}\eta_{ij} - \sum_{k=1}^{n} d_{ki}v_{ij}\right]\right)$, for any positive constants $\delta_j, j \in K$, there exists an integer $i \in N|K$ such that $\sum_{j\in K} \mathrm{Var}_{[-r_j,0]}\left[h_{ij}\eta_{ij} - \sum_{k=1}^{n} d_{ki}v_{ij}\right]\delta_j > 0$.

Therefore for any $\varphi, \psi \in C_r$ with $\varphi \underset{D}{\leq} \psi$, $\varphi_j(\theta) < \varphi_j(\theta)$ and $D_j(\varphi) < D_j(\psi)$ for $j \in K$ and $\theta \in [-r_j, 0]$, there exists $i \in N|K$ such that if $D_i(\varphi) = D_i(\psi)$, then

$$\begin{aligned} f_i(\psi) - f_i(\varphi) &\geq \sum_{j=1}^{n} \int_{-r_j}^{0} d\left[h_{ij}\eta_{ij}(\theta) - \sum_{k=1}^{n} d_{ki}v_{ij}(\theta)\right][\psi_j(\theta) - \varphi_j(\theta)] \\ &\geq \sum_{j\in K} \mathrm{Var}_{[-r_j,0]}\left[h_{ij}\eta_{ij} - \sum_{k=1}^{n} d_{ki}v_{ij}\right]\delta_j \\ &> 0, \end{aligned}$$

where $\delta_j = \min\{\psi_j(\theta) - \varphi_j(\theta); \theta \in [-r_j, 0]\}$ for $j \in K$. This proves (T). Therefore our conclusion follows from Corollary 3.1.

To apply Theorem 4.4 to equation (5.1), we need the following result about the order-stability of orbits.

PROPOSITION 5.3. *Let (H1)–(H5) hold. Then for any $\varphi, \psi \in C_r$ with $\psi \underset{D}{\geq} \varphi$, we have*

$$0 \leq x_i(t, \psi) - x_i(t, \varphi) \leq \max\left\{\frac{F(\varphi, \psi)}{\operatorname{Var}_{[-r_j, 0]} v_{ij}}, \ |\psi - \varphi|_{C_r}\right\}, \quad i \in N$$

and

$$0 \leq D_i\big(x_t(\psi)\big) - D_i\big(x_t(\varphi)\big) \leq F(\varphi, \psi), \quad i \in N$$

for $t \geq 0$, where

$$F(\varphi, \psi) = \sum_{i=1}^{n}\Big[\psi_i(0) - \sum_{j=1}^{n}\int_{-r_j}^{0}\psi_j(\theta)\, dv_{ij}(\theta) - \Big(\varphi_i(0) - \sum_{j=1}^{n}\int_{-r_j}^{0}\varphi_j(\theta)\, dv_{ij}(\theta)\Big)\Big]$$
$$+ \sum_{i,j=1}^{n}\int_{-r_i}^{0}\int_{\theta}^{0}\big[g_{ji}\big(\psi_i(u)\big) - g_{ji}\big(\varphi_i(u)\big)\big]\, du\, d\eta_{ji}(\theta),$$

implying that each orbit is order-stable.

PROOF. Let $z(t) = x(t, \psi) - x(t, \varphi)$. By Proposition 5.2, $z(t) \underset{R^n}{\geq} 0$ and $D(z_t) \underset{R^n}{\geq} 0$ for $t \geq 0$. Let $G_{ij}\colon R \times R \to R$ be defined by $G_{ij}(t, z_j) = g_{ij}\big(x_j(t, \varphi) + z_j\big) - g_{ij}\big(x_j(t, \varphi)\big)$ for $i, j = 1, \ldots, n$. Then $z(t)$ satisfies the following system of equations

$$\frac{d}{dt}[z_i(t) - \sum_{j=1}^{n}\int_{-r_j}^{0} z_j(t + \theta)\, dv_{ij}(\theta)]$$
$$= -\sum_{j=1}^{n} G_{ji}\big(t, z_i(t)\big) + \sum_{j=1}^{n}\int_{-r_j}^{0} G_{ij}\big(t + \theta, z_j(t + \theta)\big)\, d\eta_{ij}(\theta).$$

Employing a similar argument to the law of material conservation (5.3) in Proposition 5.1, we obtain

$$\sum_{i=1}^{n}\Big[z_i(t) - \sum_{j=1}^{n}\int_{-r_j}^{0} z_j(t + \theta)\, dv_{ij}(\theta)\Big] + \sum_{i,j=1}^{n}\int_{-r_i}^{0}\int_{t+\theta}^{t} G_{ji}\big(u, z_i(u)\big)\, du\, d\eta_{ji}(\theta) \tag{5.5}$$
$$= F(\varphi, \psi).$$

According to the monotonicity of g_{ij}, $G_{ji}\big(u, z_i(u)\big) \geq 0$. Hence $\sum_{i=1}^{n}\Big[z_i(t) - \sum_{j=1}^{n}\int_{-r_j}^{0} z_j(t + \theta)\, dv_{ij}(\theta)\Big] \leq F(\varphi, \psi)$, from which our conclusion follows.

The following result indicates that ordered points have disjoint ω-limit sets.

PROPOSITION 5.4. *Let (H1)–(H7) hold. Then for any $\varphi, \psi \in C_r^+$ with $\varphi \underset{D}{<} \psi$, we have $\omega(\varphi) \cap \omega(\psi) = \emptyset$.*

PROOF. By Proposition 5.1, we have $E\big(x_t(\varphi)\big) = E(\varphi)$ and $E\big(x_t(\psi)\big) = E(\psi)$ for $t \geq 0$. Therefore, $\omega(\varphi) \subseteq Q(\varphi)$ and $\omega(\psi) \subseteq Q(\psi)$, where for each $\xi \in C_r^+, Q(\xi)$ is defined by $Q(\xi) = \{\zeta \in C_r^+;\ E(\zeta) = E(\xi)\}$.

On the other hand, because of the eventual strong monotonicity of the solution semiflow of the system (5.1), for $z(t) = x(t, \psi) - x(t, \varphi)$ we have

$$z_i(t) - \sum_{j=1}^{n}\int_{-r_j}^{0} z_j(t + \theta)\, dv_{ij}(\theta) > 0 \text{ and } G_{ji}\big(t, z_i(t)\big) \geq 0$$

for $t \geq n|r|$, where G_{ji} is defined in the argument of Proposition 5.3, $i,j = 1,\ldots,n$. Therefore, from the equality (5.5) at $t = (n+1)|r|$ it follows that $F(\varphi,\psi) > 0$, i.e., $E(\psi) > E(\varphi)$. This implies that $Q(\varphi) \cap Q(\psi) = \emptyset$, and thus $\omega(\varphi) \cap \omega(\psi) = \emptyset$.

Now, we are in the position to state our major result in this section.

THEOREM 5.2. *Let (H1)–(H7) hold. Then each solution of the system (5.1) is convergent to a single equilibrium point as $t \longrightarrow \infty$.*

PROOF. This is an immediate consequence of Propositions 5.1–5.4 and Theorem 4.4.

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