DISCONTINUOUS SOLUTIONS OF VARIATIONAL PROBLEMS

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1. Introduction

The most elementary problem of the calculus of variations consists in finding a single-valued function $y(x)$, defined over an interval $[a, b]$ and taking given values at the end points, such that the integral

$$(1) \qquad I = \int_a^b f(x, y, y')dx$$

is stationary relative to all small weak variations of the function $y(x)$ consistent with the boundary conditions. Since y' occurs in the integrand, it is clear that I is only defined when $y(x)$ is differentiable and accordingly when $y(x)$ is continuous. Usually $y'(x)$ is also continuous. Occasionally, however, the boundary conditions can only be satisfied and a stationary value of I found, by permitting $y'(x)$ to be discontinuous at a finite number of points. The arc $y = y(x)$ will then possess 'corners' and the well-known *Weierstrass-Erdmann corner conditions* [1] must be satisfied at all such points by any function $y(x)$ for which I is stationary. Arcs $y = y(x)$ for which $y'(x)$ is continuous except at a finite number of points, are referred to as *admissible arcs*. In this paper, we shall extend the range of admissible arcs to include those for which $y(x)$ is discontinuous at a finite number of points.

Before this extension can be made, it is necessary to state the sense in which I exists when $y(x)$ is such a function. Replacing the discontinuous function $y(x)$ by a continuous function $y_\delta(x)$ dependent upon a continuous parameter δ, we define an integral $I(\delta)$. As $\delta \to 0$, we suppose $y_\delta(x) \to y(x)$. Then $\lim_{\delta \to 0} I(\delta)$ is accepted as our definition of I for the function $y(x)$. The conditions under which this procedure is valid are given in Section 2. In consequence of this definition, if we are able to show that a discontinuous function $y(x)$ minimises I, for all practical purposes this integral may be minimised by taking $y = y_\delta(x)$ with δ small. Such a solution to a physical problem is therefore of practical significance.

The author [2, 3] has shown that a mathematical statement of the problem of calculating rocket tracks of minimum propellant expenditure in a given gravitational field, requires the minimisation of the integral

$$P = \int_{t_0}^{t_1} \sqrt{\{\ddot{x} + f(x, y)\}^2 + \{\ddot{y} + g(x, y)\}^2}\, dt$$

with respect to variation of the functions $x(t)$, $y(t)$ when these are subjected to the usual boundary conditions. This type of integral can be made stationary only by choosing functions $x(t)$, $y(t)$ whose first derivatives are discontinuous. Such functions are not admissible according to the classical theory of the calculus of variations. This type of solution is, however, physically significant, since $x(t)$, $y(t)$ are the coordinates of the rocket vehicle at time t and discontinuities in their first derivatives correspond to impulsive thrusts from the motor, which can be approximated in practice. Our generalization of the classical theory is accordingly not purely academic.

It is essential to appreciate that a definition of I which proceeds by simply removing an infinitesimal neighbourhood of the discontinuity is unsatisfactory, since the discontinuity will then make no finite contribution to I. If, however, this integral represents a physical quantity, a discontinuity in y corresponds, effectively, to a large value of y' and, as in the case of an impulse applied to a massive body, it must be allowed to make a contribution to the value of I.

It may also be remarked at this stage ,that the introduction of discontinuous functions cannot be avoided by either reversing the roles of the variables x and y so that y becomes independent or by making these variables dependent upon a third variable t, for, if either step is taken, x is permitted to be non-monotonic, whereas, in the type of physical problem to which the theory has been applied, x is the time variable. Such a redrafting of the problem can accordingly result in a stationary value for I which is not physically attainable, e.g. x may be required to vary beyond the confines of the interval $[a, b]$. Alternatively, we observe that such an approach permits $y(x)$ to be multi-valued and we wish to avoid this possibility. This point will be further amplified in Section 4.

2. Definition of I for Discontinuous y(x)

Suppose that $y(x)$ is continuous throughout the interval $[a, b]$ with the exception of the point $x = c$. We define a function $y_\delta(x)$ $(\delta > 0)$ thus:

$$(2) \qquad \begin{cases} y_\delta(x) = y(x), \quad a \leqq x \leqq c - \delta, \quad c + \delta \leqq x \leqq b, \\ \qquad\quad = \dfrac{1}{2\delta}\left[(x - c + \delta)y(c + \delta) - (x - c - \delta)y(c - \delta)\right], \\ \qquad\qquad\qquad\qquad\qquad c - \delta \leqq x \leqq c + \delta, \end{cases}$$

i.e. $y_\delta(x)$ is identical with $y(x)$, except over the interval $(c - \delta, c + \delta)$

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where it varies linearly between the values $y(c-\delta)$, $y(c+\delta)$. $y_\delta(x)$ is continuous. If $y(x)$ is replaced by $y_\delta(x)$ in the integral (1), we shall denote its value by $I(\delta)$.

Let R be any closed region of the xy-plane containing the arc $y = y_\delta(x)$ for all values of δ satisfying $0 < \delta < \Delta$. If $y(c+0) > y(c-0)$, $y'_\delta(c) \to +\infty$ as $\delta \to +0$. In these circumstances, we shall denote by E the assemblage of points (x, y, y') such that (x, y) belongs to R and $y' \geqq p$, where p is a lower bound of $y'_\delta(x)$ for values of x in $[a, b]$ and δ in $(0, \Delta)$. If $y(c+0) < y(c-0)$, the inequality $y' \geqq p$ is replaced by $y' \leqq P$, where P is an upper bound of $y'_\delta(x)$ for these values of x and δ.

To avoid many alternative statements in the theorem which follows, we shall suppose that $y(c-0) < y(c+0)$.

THEOREM 1. *If $f(x, y, y')$ is continuous over E and, as $y' \to +\infty$, $f(x, y, y') \sim y'k(x, y)$ uniformly with respect to x and y satisfying*

$$c-\xi \leqq x \leqq c+\xi,\ y(c-0)-\eta \leqq y \leqq y(c+0)+\eta,$$
$$I(\delta) \to \int_a^{c-0} + \int_{c+0}^{b} f(x, y, y')dx + \int_{y(c-0)}^{y(c+0)} k(c, y)dy$$

as $\delta \to +0$.

Since $f/y' \to k(x, y)$ uniformly as $y' \to +\infty$, $k(x, y)$ is continuous for values of (x, y) lying in the intervals given in the statement of the theorem. Also, given $\varepsilon > 0$, $Y(\varepsilon)$ (> 0) can be found such that

$$|f(x, y, y') - y'k(x, y)| < \varepsilon y', \tag{3}$$

provided $y' > Y$ and (x, y) lie in their intervals.

If x lies in the interval $[c-\delta, c+\delta]$, by choosing δ sufficiently small, x and $y_\delta(x)$ will lie in the intervals $[c-\xi, c+\xi]$, $[y(c-0)-\eta, y(c+0)+\eta]$ respectively and $y'_\delta(x)$ will be large. Hence we can choose $\delta_1 = \delta_1(Y) = \delta_1(\varepsilon) > 0$ such that

$$|f(x, y_\delta, y'_\delta) - y'_\delta k(x, y_\delta)| < \varepsilon y'_\delta, \tag{4}$$

provided $0 < \delta < \delta_1$ and x lies in the interval $[c-\delta, c+\delta]$.

Thus

$$\begin{aligned}\left|\int_{c-\delta}^{c+\delta} [f(x, y_\delta, y'_\delta) - y'_\delta k(x, y_\delta)]dx\right| &\leqq 2\,\epsilon\,\delta y'_\delta \\ &= \varepsilon\{y(c+\delta) - y(c-\delta)\} \\ &\leqq A\varepsilon\end{aligned} \tag{5}$$

since $y(c+\delta) - y(c-\delta)$ is bounded.

Changing the variable of integration from x to y_δ, we obtain

$$\int_{c-\delta}^{c+\delta} y'_\delta k(x, y_\delta)dx = \int_{y(c-\delta)}^{y(c+\delta)} k(x, y_\delta)dy_\delta = \int_{y(c-\delta)}^{y(c+\delta)} k(x, y)dy, \tag{6}$$

where, by equations (2),

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(7) $$x = \frac{2\delta y + (c-\delta)y(c+\delta) - (c+\delta)y(c-\delta)}{y(c+\delta) - y(c-\delta)}$$

Thus

(8) $$\left| \int_{c-\delta}^{c+\delta} f(x,\ y_\delta,\ y'_\delta)\mathrm{d}x - \int_{y(c-\delta)}^{y(c+\delta)} k(x,\ y)\mathrm{d}y \right| \leqq A\varepsilon,$$

provided δ is small.

Now

(9) $$\begin{aligned} &\left| \int_{y(c-\delta)}^{y(c+\delta)} k(x,\ y)\mathrm{d}y - \int_{y(c-0)}^{y(c+0)} k(c,\ y)\mathrm{d}y \right| \\ &\leqq \left| \int_{y(c-\delta)}^{y(c-0)} k(x,\ y)\mathrm{d}y \right| + \left| \int_{y(c+0)}^{y(c+\delta)} k(x,\ y)\mathrm{d}y \right| \\ &+ \left| \int_{y(c-0)}^{y(c+0)} [k(x,\ y) - k(c,\ y)]\mathrm{d}y \right|. \end{aligned}$$

For sufficiently small δ

(10) $$|y(c-\delta) - y(c-0)| < \varepsilon, \quad |y(c+\delta) - y(c+0)| < \varepsilon.$$

Thus, $k(x,\ y)$ being continuous,

(11) $$\left| \int_{y(c-\delta)}^{y(c-0)} k(x,\ y)\mathrm{d}y \right| < B\varepsilon, \quad \left| \int_{y(c+0)}^{y(c+\delta)} k(x,\ y)\mathrm{d}y \right| < B\varepsilon,$$

provided $0 < \delta < \delta_2$.

Again, by making δ sufficiently small, it is clear from equation (7) that we can make x lie in the interval $(c-\xi,\ c+\xi)$ for all y satisfying $y(c-0) \leqq y \leqq y(c+0)$. Then, since $k(x,\ y)$ is continuous for such values of $(x,\ y)$, it follows that

(12) $$|k(x,\ y) - k(c,\ y)| < \varepsilon$$

provided that δ and therefore $|x - c|$ is sufficiently small. Thus

(13) $$\left| \int_{y(c-0)}^{y(c+0)} [k(x,\ y) - k(c,\ y)]\mathrm{d}y \right| \leqq \{y(c+0) - y(c-0)\}\varepsilon.$$

Combining inequalities (9), (11) and (13), we prove that

(14) $$\left| \int_{y(c-\delta)}^{y(c+\delta)} k(x,\ y)\mathrm{d}y - \int_{y(c-0)}^{y(c+0)} k(c,\ y)\mathrm{d}y \right| < C\varepsilon.$$

It now follows from (8) and (14) that

(15) $$\left| \int_{c-\delta}^{c+\delta} f(x,\ y_\delta,\ y'_\delta)\mathrm{d}x - \int_{y(c-0)}^{y(c+0)} k(c,\ y)\mathrm{d}y \right| < D\varepsilon$$

provided δ is small. Hence

(16) $$\int_{c-\delta}^{c+\delta} f(x,\ y_\delta,\ y'_\delta)\mathrm{d}x \to \int_{y(c-0)}^{y(c+0)} k(c,\ y)\mathrm{d}y$$

as $\delta \to 0$.

But

$$\int_a^b f(x, y_\delta, y'_\delta)\mathrm{d}x = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c-\delta}^{b} f(x, y_\delta, y'_\delta)\mathrm{d}x$$
$$= \int_a^{c-\delta} + \int_{c+\delta}^{b} f(x, y, y')\mathrm{d}x + \int_{c-\delta}^{c+\delta} f(x, y_\delta, y'_\delta)\mathrm{d}x$$
$$\to \int_a^{c-0} + \int_{c+0}^{b} f(x, y, y')\mathrm{d}x + \int_{y(c-0)}^{y(c+0)} k(c, y)\mathrm{d}y.$$

Thus the theorem is proved.

If $y(c-0) > y(c+0)$, the result just obtained remains true provided $f(x, y, y') \sim y' k(x, y)$ as $y' \to -\infty$.

3. First Variation of I

To simplify the statement of our argument, we shall assume that $y(x)$ possesses one discontinuity only in the interval $[a, b]$, viz. that at $x = c$. It will be convenient to suppose x and y dependent upon a variable t, x being a monotonic function of this variable with a continuous derivative so that x increases from a to b as t increases from α to β. Thus

(17) $$x = x(t),\ y = y(t),\ \alpha \leqq t \leqq \beta.$$

Let $x(\gamma) = c$. The values of y at the end points of the interval $[a, b]$, supposed given, will be denoted by A and B. Thus $y(\alpha) = A$, $y(\beta) = B$. Then I can be expressed in the form

(18) $$I = \int_\alpha^{\gamma-0} + \int_{\gamma+0}^{\beta} f(x, y, \dot{y}/\dot{x})\dot{x}\,\mathrm{d}t + \int_{y_-}^{y_+} k(c, y)\mathrm{d}y,$$

where dots denote differentiations with respect to t and $y_- = y(c-0)$, $y_+ = y(c+0)$. This notation will be employed quite generally throughout the following argument, i.e. if $X(t)$ is any quantity whose value can be determined at all points of the arc whose parametric equations are (17), then $X_- = X(\gamma - 0)$, $X_+ = X(\gamma + 0)$.

Suppose that the equations (17) specify an admissible arc relative to which I is stationary and let the equations

(19) $$x = x(t) + \varepsilon p(t),\ y = y(t) + \varepsilon q(t),$$

represent a neighbouring arc. In equations (19), ε is small and $p(t)$, $q(t)$ are continuous and differentiable, except that $q(t)$ may be discontinuous at $t = \gamma$. We also assume that

(20) $$p(\alpha) = p(\beta) = q(\alpha) = q(\beta) = 0,$$

i.e. the neighbouring arc passes through the points (a, A), (b, B) and hence satisfies the boundary conditions. Substituting from equations (19) into equation (18) and expanding in a series of powers of ε, we obtain for the new value of I,

$$I + \delta I = I + \varepsilon \int_{\alpha}^{\gamma-0} + \int_{\gamma+0}^{\beta} \left\{ \frac{\partial f}{\partial x} \dot{x} p + \frac{\partial f}{\partial y} \dot{x} q + \frac{\partial f}{\partial y'} (\dot{q} - y'\dot{p}) + f\dot{p} \right\} \mathrm{d}t \tag{21}$$
$$+ \varepsilon(q_+ k_+ - q_- k_-) + \varepsilon p(\gamma) \int_{y_-}^{y_+} \frac{\partial k}{\partial c} \mathrm{d}y + O(\varepsilon^2),$$

where $k_+ = k(c, y_+)$, $k_- = k(c, y_-)$.

Integrating by parts those terms containing $\dot{p}$ and $\dot{q}$ as factors which appear in the integrands in equation (21) and employing equations (20), we obtain in the usual manner the result

$$\frac{\delta I}{\varepsilon} = \left(\frac{\partial f}{\partial y'}\right)_- q_- - \left(\frac{\partial f}{\partial y'}\right)_+ q_+ + \left(f - y' \frac{\partial f}{\partial y'}\right)_- p_- - \left(f - y' \frac{\partial f}{\partial y'}\right)_+ p_+$$
$$+ \int_{\alpha}^{\gamma-0} + \int_{\gamma+0}^{\beta} \left[\left\{ \frac{\partial f}{\partial x} \dot{x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(f - y' \frac{\partial f}{\partial y'}\right) \right\} p + \left\{ \frac{\partial f}{\partial y} \dot{x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial y'}\right) \right\} q \right] \mathrm{d}t \tag{22}$$
$$+ q_+ k_+ - q_- k_- + p(\gamma) \int_{y_-}^{y_+} \frac{\partial k}{\partial c} \mathrm{d}y + O(\varepsilon).$$

Now $p_- = p_+ = p(\gamma)$ and $p(\gamma)$, q_-, q_+, $p(t)$, $q(t)$ are arbitrary. Hence $\delta I = O(\varepsilon^2)$ and I is stationary only if

$$\begin{cases} \left(\dfrac{\partial f}{\partial y'}\right)_- = k_-, \quad \left(\dfrac{\partial f}{\partial y'}\right)_+ = k_+, \\ \left(f - y' \dfrac{\partial f}{\partial y'} - \displaystyle\int_h^y \frac{\partial k}{\partial c} \mathrm{d}y\right)_- = \left(f - y' \dfrac{\partial f}{\partial y'} - \displaystyle\int_h^y \frac{\partial k}{\partial c} \mathrm{d}y\right)_+ \\ \dfrac{\partial f}{\partial x} \dot{x} - \dfrac{\mathrm{d}}{\mathrm{d}t}\left(f - y' \dfrac{\partial f}{\partial y'}\right) = 0, \\ \dfrac{\partial f}{\partial y} \dot{x} - \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial f}{\partial y'}\right) = 0, \end{cases} \tag{23}$$

h being arbitrary. The final pair of conditions for a stationary value must be satisfied over the continuous portions of the arc $x = x(t)$, $y = y(t)$ and are together equivalent to the single equation

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{24}$$

We have therefore proved the following theorem:

THEOREM 2. *Subject to the provisions of Theorem* 1, *the integral*

$$I = \int_a^b f(x, y, y')\mathrm{d}x$$

is stationary relative to weak variations of the function $y = y(x)$ satisfying the boundary conditions

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$$y(a) = A, \; y(b) = B,$$

provided that,

(i) *where $y(x)$ and its derivative are continuous, this function satisfies the equation*

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

(ii) *where $y'(x)$ is discontinuous, the Weierstrass-Erdmann corner conditions are satisfied,*

and (iii) *at a point $x = c(a < c < b)$ where $y(x)$ is discontinuous, the quantity*

$$f - y' \frac{\partial f}{\partial y'} - \int \frac{\partial k}{\partial c}\, dy$$

is continuous and

$$\frac{\partial f}{\partial y'} = k$$

on either side of the discontinuity.

If the discontinuity is supposed to occur at an end point of the interval $[a, b]$, it is restricted to move in one direction only during any small variation. This implies that it is not necessary that I should be stationary (i.e. $\delta I = O(\varepsilon^2)$) in order that it shall be a maximum or a minimum and the conditions for I to be stationary are then of no interest. On the other hand, the conditions for I to be a maximum or a minimum appear to be somewhat complicated and we shall not consider them further here.

4. An Example

Consider the integral

$$J = \int_0^1 (y'^3 - 1)^{1/3}\, \mathrm{d}x, \tag{25}$$

the boundary conditions on $y(x)$ being $y(0) = 0$, $y(1) = 2$.

In this case $k = 1$ identically, and the conditions of Theorem 1 are satisfied over the whole of the xy-plane.

Referring to Theorem 2, it may be verified that the conditions which have to be satisfied at a discontinuity are equivalent to the requirement

$$y'_- = y'_+ = 2^{-1/3}. \tag{26}$$

Equation (24) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'^2}{(y'^3 - 1)^{2/3}}\right] = 0. \tag{27}$$

Hence

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(28) $$y' = \text{constant}$$

and all the extremals are straight lines.

It is clear that J is stationary relative to all small variations from the discontinuous arc $OPQA$ shown in the figure. This arc γ satisfies the boundary conditions and, the gradients of its straight portions being $2^{-1/3}$,

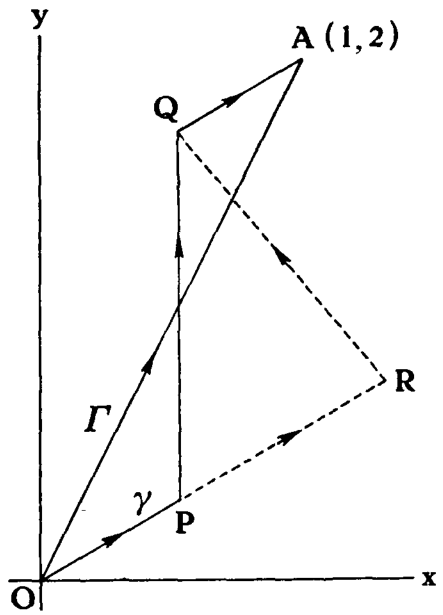

Fig. 1.

the conditions (26) are satisfied at the discontinuity. The 'jump' at the discontinuity is of magnitude $2 - 2^{-1/3}$ and this is the contribution of the discontinuity to the value of J computed over γ. It follows that

(29) $$J(\gamma) = 2 - 2^{2/3}.$$

Instead of a single discontinuity, any number may be introduced without altering the integral's value from that stated in the last equation.

The classical theory yields the straight line OA as the only arc over which J is stationary, for it will be found that the Weierstrass-Erdmann corner conditions cannot be satisfied unless y' is continuous. Over this arc Γ,

(30) $$J(\Gamma) = 7^{1/3}$$

and, applying Legendre's test, it will be found that this is a maximum value of J. Amongst the admissible arcs of the classical theory, however, there is none for which J is minimised. But it is easy to show that $J(\gamma)$ is an absolute minimum of J, for we have the inequality

(31) $$(y'^3 - 1)^{1/3} \geqq y' - 2^{2/3}$$

and this implies that

(32) $$J \geqq \int_0^1 (y' - 2^{2/3})\mathrm{d}x = 2 - 2^{2/3} = J(\gamma).$$

When discussing the minimisation of J, it might be expected that the introduction of a discontinuity could be avoided by exchanging the roles of the x- and y-axes. If y is the independent variable, the arc γ is admissible according to the classical theory, the points P and Q being simple corners. Changing the variable of integration to y, we obtain

$$-J = \int_0^2 (x'^3 - 1)^{1/3} \mathrm{d}y, \tag{33}$$

where $x' = \mathrm{d}x/\mathrm{d}y$. We have to maximise the left hand member of this equation, a problem which is of exactly the same type as our original problem. As stated above, this integral cannot be stationary over an arc possessing corners and hence it is not stationary with respect to γ. This appears to contradict our previous result. However, arcs such as $ORQA$, which were not admissible previously since it was required that $y(x)$ should be single-valued, can now be admitted and over such arcs J can assume values less than $J(\gamma)$.

5. Generalizations

The results of the preceding Sections can be extended in two ways, (i) by permitting second and higher order derivatives of the unknown function $y(x)$ to occur in the integrand of I and (ii) by introducing further unknown functions into the integrand. The former generalization can be made in the obvious manner, but the latter requires more careful consideration.

Thus, if a second derivative y'' is present in the integrand, $y = y(x)$ is an admissible arc according to the theory of this paper, provided $y'(x)$ is continuous except at a finite number of points. If $x = c$ is a point of discontinuity, it is required that, in an appropriate neighbourhood, we shall have

$$f(x, y, y', y'') \sim y''k(x, y, y') \tag{34}$$

uniformly with respect to (x, y, y'). Then the contribution of the discontinuity to the integral is taken to be

$$\int_{y'(c-0)}^{y'(c+0)} k(c, C, y')\mathrm{d}y', \tag{35}$$

where C is the value taken by y at the discontinuity in y'.

Then the integral

$$\int_a^b f(x, y, y', y'')\mathrm{d}x \tag{36}$$

is stationary relative to an admissible arc if, (i) over the segments of the arc for which y' is continuous $y(x)$ satisfies the usual characteristic equation, (ii) at points where y' is continuous but y'' is discontinuous the Weierstrass-

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Erdmann corner conditions are satisfied, and (iii) at a discontinuity $x = c$ in y' the expressions

$$(37) \qquad \begin{cases} \text{(a)} \quad \dfrac{\partial f}{\partial y'} - \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\partial f}{\partial y''}\right) \\ \text{(b)} \quad f - y'\left\{\dfrac{\partial f}{\partial y'} - \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\partial f}{\partial y''}\right)\right\} - y''\dfrac{\partial f}{\partial y''} - \displaystyle\int \dfrac{\partial}{\partial c} k(c, C, y')\mathrm{d}y' \end{cases}$$

are continuous and, on either side of the discontinuity

$$(38) \qquad \frac{\partial f}{\partial y''} = k.$$

Consider now an integral of the form

$$(39) \qquad \mathscr{I} = \int_a^b f(x, y, z, y', z')\mathrm{d}x$$

where $y(x)$, $z(x)$ are to be chosen to satisfy the usual boundary conditions and so that the integral is stationary. The definition of the integral over any interval within which y and z are not discontinuous simultaneously presents no difficulty. If, however, both these functions are discontinuous at $x = c$, we shall first require that

$$(40) \qquad f(x, y, z, y', z') \sim y'k(x, y, z, z'/y')$$

as $y' \to \pm\infty$, $z' \to \pm\infty$ independently and uniformly with respect to the remaining variables. Since

$$y'k = z'\left(\frac{y'}{z'}k\right),$$

this condition clearly implies that

$$(41) \qquad f \sim z'K(x, y, z, y'/z')$$

under the same circumstances, i.e. this condition is symmetrical with respect to y and z. The contribution of the discontinuity to the value of the integral may then be taken to be

$$(42) \qquad \int_{y(c-0)}^{y(c+0)} k(c, y, z, \mathrm{d}z/\mathrm{d}y)\mathrm{d}y.$$

This integral is not fully determined until the relationship $z = z(y)$ has been specified. Since $\mathscr{I}$ is to be stationary, it is necessary to choose the form of this relationship in such a way that the integral (42) is also stationary relative to small variations. This is a problem of the type considered in the earlier sections.

If $\mathscr{I}$ represents a physical quantity, we can approximate a discontinuity in the functions $y(x)$, $z(x)$ by arranging for them to be very rapidly variable relative to x. If, however, $\mathscr{I}$ is to be stationary, it follows from the above

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analysis that such variations must be related in the precise fashion necessary to ensure that the integral (42) is stationary.

Consider, for example, the integral P referred to in Section (1). This involves second order derivatives of the unknown functions $x(t)$, $y(t)$, but the principles involved are unaltered. We have

$$\sqrt{\{(\ddot{x}+f)^2+(\ddot{y}+g)^2\}} \sim \ddot{x}\sqrt{1+(\ddot{y}/\ddot{x})^2}, \tag{43}$$

as $\ddot{x} \to \infty$, $\ddot{y} \to \infty$. Hence, simultaneous discontinuities in $\dot{x}$ and $\dot{y}$ make a contribution to P of

$$\int_{X_-}^{X_+} \sqrt{1+(\mathrm{d}Y/\mathrm{d}X)^2}\,\mathrm{d}X, \tag{44}$$

where $X = \dot{x}$, $Y = \dot{y}$. This integral represents the length of an arc joining two points in the XY-plane and it is minimised by the straight line path between these points. Thus, if P is to be minimised, across any discontinuity $\dot{x}$ and $\dot{y}$ must be linearly related. In the context of the problem leading to the integral P, this is interpreted as requiring that any impulsive thrust applied by the motors to the rocket must be maintained constant in direction during its small, but necessarily finite, duration.

It is now easy to prove, as in Section 3, that the conditions to be satisfied by the functions $y(x)$, $z(x)$ across a discontinuity, if $\mathscr{I}$ is to be stationary are, (i) the expression

$$f - y'\frac{\partial f}{\partial y'} - z'\frac{\partial f}{\partial z'} - \int \frac{\partial k}{\partial c}\,\mathrm{d}y \tag{45}$$

must be continuous and (ii) on either side of the discontinuity the following equations must be true:

$$\begin{cases} \dfrac{\partial f}{\partial y'} = k - \dot{z}\dfrac{\partial k}{\partial \dot{z}}, \\[2ex] \dfrac{\partial f}{\partial z'} = \dfrac{\partial k}{\partial \dot{z}}, \end{cases} \tag{46}$$

where $\dot{z} = \mathrm{d}z/\mathrm{d}y$. The quantities $\dot{z}_+$, $\dot{z}_-$ involved in this second condition are to be computed at the ends of the arc $z = z(y)$ making the integral (42) stationary.

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