A TOPOLOGICAL FORMALISM FOR QUANTITATIVE ANALYSIS OF DESIGN SPACES

Ortiz, Joshua (1);
Summers, Joshua (2);
Coykendall, James (1);
Roberts, Travis (1);
Rai, Rahul (1)

1: Clemson University;
2: The University of Texas at Dallas

ABSTRACT

The objective of this paper is to present a mathematically grounded description of the two topological spaces for the design problem and the design solution. These spaces are derived in a generalized form such that they can be applied by researchers studying engineering design and developing new methods or engineers seeking to understand the influence that changes in the problem space have on the solution space. In addition to the formal definitions of the spaces, including assumptions and limitations, three types of supported reasoning are presented to demonstrate the potential uses. These include similarity analysis to compare spaces, an approach to sensitivity analysis of the solution space to changes in the problem space, and finally a distance measure to determine how far a current proposal is to the feasible solution space. This paper is presented to establish a common vocabulary for researchers when discussing, studying, and supporting the dyadic nature of engineering design (problem-solution co-evolution).

Keywords: Design theory, Requirements, Embodiment design

Contact:
Summers, Joshua
Clemson University
Mechanical Engineering
United States of America
joshua.summers@utdallas.edu

Cite this article: Ortiz, J., Summers, J., Coykendall, J., Roberts, T., Rai, R. (2021) 'A Topological Formalism for Quantitative Analysis of Design Spaces', in *Proceedings of the International Conference on Engineering Design (ICED21),* Gothenburg, Sweden, 16-20 August 2021. DOI:10.1017/pds.2021.30

1 INTRODUCTION

The design process focuses on two separate but interconnected areas: the problem space and the solution space, which represent everything known about the problem and all of its available solutions (Anandan et al., 2006; Maher et al., 1996). While prior research has yielded a variety of representations for design spaces, few lend themselves to a quantitative analysis of the spaces themselves or the effects they have on each other. Goel and Pirolli (1992) chose a graph network model, but focused only on the problem space. MacLean et al. (1991) also used a graphical structure but combined the two spaces into a single entity. Schätz et al. (2010) depicted the solution space in terms of set theory using Boolean rules to define constraints. Some researchers went beyond representation and studied the applications of design space analysis. Bowen and Dittmar (2017) used set theory to define a solution space that also incorporated the relationship between design artifacts and process stakeholders to trace design "states". Dinar et al. (2011) used p-maps to compare how designers think about the problem. It is noted that in each case, the definitions used were developed for narrow application and limited in scope, hindering subsequent use or extension.

Contrastingly, topological analysis has been used to make significant strides in an array of fields including science (Blonder, 2018; Hijmans et al., 2005), engineering (Ruiz-Pérez et al., 2016; Zhu and Gao, 2016), and data analysis (Wasserman, 2018). There has been previous work to bring the versatility of topology into engineering design, as well. For example, Siddique and Rosen (2001) developed a topological representation for discrete sets of designs for the purpose of exploring different solution configurations. Also, Taura and Yoshikawa (1994) presented a topological metric to relate the role of components in a system to its overall function. This paper proposes that a more versatile and extensible representation can be achieved by considering each design space as a subset of real-space. While it is not a panacea, the formalism put forth here is intended to promote broad application and utility exploration.

2 DESIGN SPACES REDEFINED

The problem and solution of the design process are intrinsically linked and interdependent (Maher et al., 1996; Gero and Kannengiesser, 2004). While not the focus here, this fact implies that any representations of the two design spaces should also be interconnected. In this section, the problem space $\mathcal{P}$ and the solution space $\mathcal{S}$ are shown to be linked via a set $\mathcal{M}$ of mapping functions. The problem space is represented as a subset of n-dimensional real-space, referred to here as the constraint space $\mathcal{C}$, where n is the number of parameters that are to be constrained by a set of requirements $\mathcal{R}$. Similarly, the solution space is embedded into a different subset of real-space having m dimensions, which is called the form space $\mathcal{F}$. Its dimensions are set by the number of inputs to $\mathcal{M}$, which serve as the design parameters. These spaces, and their mapping functions, are expected to evolve over the course of the design process, changing in both shape and dimensionality.

2.1 Requirements

To begin, the requirements must be constructed in such a way as to retain their conceptual meaning while also providing the mathematical basis necessary for topological spaces. First, let $\mathcal{A} = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a set of parameters which must be constrained in order to meet the needs of the stakeholders and create a viable product. These are the constraint parameters for the problem. As an example, if a project necessitated restrictions on volume V and mass m, then $\mathcal{A} = \{V, m\}$.

With these parameters identified, let $\mathcal{R} = \{\mathcal{R}_\alpha : \alpha \in \mathcal{A}\}$ be a set of numerical requirements, indexed by $\mathcal{A}$, so that there is a one-to-one correspondence between the set of constraint parameters and the set of requirements. Furthermore, let each

$$\mathcal{R}_\alpha = \{y_\alpha \in \mathbb{R} : \text{stakeholder needs are met when } \alpha = y_\alpha, \alpha \in \mathcal{A}\} \quad (1)$$

This definition characterizes each constraint $\mathcal{R}_\alpha$ as the set of all numerical values which the parameter α could take without violating any stakeholder needs. To continue the example above, $\mathcal{A} = \{V, m\}$ would have requirement $\mathcal{R}_V$ for volume and another $\mathcal{R}_m$ for mass. If the problem necessitated an upper limit of 100 kg for mass, then $\mathcal{R}_m = \{x : 0 \text{ kg} < x \leq 100 \text{ kg}\}$. In this manner, each requirement may be composed of one or more continuous intervals or discrete values. When necessary, complex-valued constraints can be obtained by separating them into two requirements and representing the imaginary term as a real multiplier on i. Besides being numerical, a requirement must also be testable, meaning it should

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

always be possible to determine whether a value is allowable or not. Excepting notation, this is not significantly different from much of the existing guidance for writing engineering requirements (INCOSE, 2015; Hirshorn et al., 2017). Additionally, certain topological constructs may make it possible to extend this definition to accept categorical, or non-numerical, requirements in the future.

2.2 Problem Space

The constraint parameters in $\mathcal{A}$ are also used to define the constraint space. A generic topological space is represented by an ordered pair $(X,\mathcal{T})$, where X is a set and $\mathcal{T}$ is a collection of subsets of X meeting three criteria:

1. *$\mathcal{T}$ contains the empty set $\varnothing$ and the set X*
2. *The union of any family of members of $\mathcal{T}$ is a member of $\mathcal{T}$*
3. *The intersection of any finite family of members of $\mathcal{T}$ is a member of $\mathcal{T}$*

To ensure the constraint space meets these three criteria, let it be defined as the pair $(\mathcal{C},\mathcal{T}_{\mathcal{C}})$ so that

$$\mathcal{C} = \prod_{\alpha\in\mathcal{A}} \mathbb{R} = \{y = (y_1,y_2,...,y_n) \mid y_i \in \mathbb{R}, 1 \leq i \leq n\} \tag{2}$$

$$\mathcal{T}_{\mathcal{C}} = \{\bigcup_{y\in\mathcal{C}} B(y,\varepsilon_y) \mid y \in \mathcal{C},\ \varepsilon_y \in \mathbb{R}^+\} \tag{3}$$

where $n = |\mathcal{A}|$ and

$$B(y,\varepsilon_y) = \{y' \in \mathcal{C} \mid d(y,y') < \varepsilon_y\} \tag{4}$$

The function d referenced above is known as a metric, or distance function, for $\mathcal{C}$. This function has the properties that

1. $d(x,y) = 0$ iff $x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$

In real-space, the standard metric is given by the Euclidean distance function

$$d(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}},\ x = (x_1,\ldots,x_n), y = (x_1,\ldots,x_n) \in \mathbb{R}^n \tag{5}$$

This is known as the "usual topology" on the real-space, and when applied to $\mathcal{C}$, makes it equivalent to n-dimensional real-space. The reader is referred to Croom (2016) for in-depth proof of how metrics generate a topology. As is common in topology, this paper will refer to spaces simply by the symbol for their set, with the presence of its topology implied.

It is worth noting that, while the topology presented above results in one of the more familiar types of topological space, there are many other options available to the engineer as well. Any distance function meeting the criteria may be used, and topologies can also be defined without a metric. These provide more general types of space that have properties different from Euclidean space. Furthermore, if multiple topologies or metrics are desired, disparate spaces can be combined to form what are known as product spaces. Such constructs could be useful in the future to incorporate categorical requirements alongside numerical ones.

Since the axes of $\mathcal{C}$ are designated according to the constraint parameters in $\mathcal{A}$, each coordinate of a point $y \in \mathcal{C}$ corresponds to a value for the parameter associated with that axis. Certain points in $\mathcal{C}$ exist for which every coordinate value is a member of its respective requirement. In other words, they meet all of the requirements, and the problem space $\mathcal{P}$ is the collection of all of those points. This collection forms a subspace of $\mathcal{C}$ and can be formally defined as

$$\mathcal{P} = \prod_{\alpha\in\mathcal{A}} \mathcal{R}_\alpha = \{y = (y_1,y_2,...,y_n) \mid y_i \in \mathcal{R}_{\alpha_i}, 1 \leq i \leq n\} \tag{6}$$

2.3 Map Set

To construct the solution space, a way of testing candidate designs against requirements is needed. Topologically, this means mapping between the spaces. Essentially, engineers do this when defining

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

a plan for the verification and validation of their designs to ensure requirements are met. Here, these checks are gathered in their various forms into a set $\mathcal{M}$ defined as

$$\mathcal{M} = \{f_\alpha : \mathcal{F} \to \mathcal{C}_\alpha\} \tag{7}$$

where $\mathcal{C}_\alpha$ represents the 1-dimensional axis of $\mathcal{C}$ corresponding to the constraint parameter α and $\mathcal{F}$ is the form space mentioned earlier. Each element in $\mathcal{M}$ is a function corresponding to a specified constraint parameter α in $\mathcal{A}$, which outputs a value of α for each point in $\mathcal{F}$. In general, a function $f : X \to Y$ is any rule which assigns a member of X to a unique member of Y. Often, functions are thought of as mathematical equations, but in this context the term also includes other tools used by engineers to assess the adherence of a design to a constraint, such as optimization algorithms, machine learning models, or lookup tables. Since the functions used will largely depend on the conceptual form chosen for the design, $\mathcal{M}$ is an aggregation of the design decisions that have been made thus far. Once constructed, $\mathcal{M}$ may be treated as a single function $\mathcal{M} : \mathcal{F} \to \mathcal{C}$. In the context of a design space, this implies that every design $x \in \mathcal{F}$ maps to one and only one point $y \in \mathcal{C}$. However, in general, $\mathcal{M}$ is not assumed to be either surjective or injective; meaning some points in $\mathcal{C}$ may have two or more corresponding designs, while others may have none.

2.4 Solution Space

Just as the constraint parameters defined the axes of the constraint space, the input parameters for $\mathcal{M}$ likewise define the axes of the form space. Let the collection of the input parameters for $\mathcal{M}$ be referred to as the set $\mathcal{B} = \{\beta_1, \beta_2, ..., \beta_m\}$, where each $\beta \in \mathcal{B}$ represents a design parameter to be chosen by the designer and m is the number of unique design parameters needed to calculate an output from each $f_\alpha \in \mathcal{M}$. Every member function of $\mathcal{M}$ will take some subset $\mathcal{B}' \subseteq \mathcal{B}$ as its inputs, and any individual β may serve as an input for multiple functions. The set $\mathcal{B}$ only contains those parameters which are essential to validate the current set of requirements and will likely not incorporate all of the parameters required for production until the end of the design process has been reached. The solution space can therefore be determined at any time during the design process provided $\mathcal{M} \neq \varnothing$, which also entails $\mathcal{R} \neq \varnothing$. These parameters shall be elements of the design that can be independently and directly controlled, such as the physical dimensions of a component or number of windings in a transformer. Parameters such as mass or volume would not, in general, be considered design parameters in this context since their value is usually dependent on other independent parameters.
Let the embedding space $\mathcal{F}$, from the previous section, be defined as $(\mathcal{F}, \mathcal{T}_\mathcal{F})$ such that

$$\mathcal{F} = \prod_{\beta \in \mathcal{B}} \mathbb{R} = \{x = (x_1, x_2, ..., x_m) \mid x_i \in \mathbb{R}, 1 \leq i \leq m\} \tag{8}$$

$$\mathcal{T}_\mathcal{F} = \{\bigcup_{x \in \mathcal{F}} B(x, \varepsilon_x) \mid x \in \mathcal{F},\ \varepsilon_x \in \mathbb{R}^+\} \tag{9}$$

where $m = |\mathcal{B}|$. As with $\mathcal{C}$ before, $\mathcal{F}$ is also equivalent to the real-space but with a different number of dimensions. Every point $x \in \mathcal{F}$ maps to a corresponding point in $\mathcal{C}$ according to $y = \mathcal{M}(x)$. Further, let the solution space be

$$\mathcal{S} = \mathcal{M}^{-1}(\mathcal{P}) = \prod_{\alpha \in \mathcal{A}} f_\alpha^{-1}(\mathcal{P}) = \{x \in \mathcal{F} : \mathcal{M}(x) \in \mathcal{P}\} \tag{10}$$

That is to say, $\mathcal{S}$ is the inverse image of $\mathcal{P}$ under $\mathcal{M}$. Equivalently, $\mathcal{S}$ can be described as all points in $\mathcal{F}$ that map to $\mathcal{P}$ via $\mathcal{M}$. The space $\mathcal{S}$ must be defined according to this inverse relationship owing to the assumption that $\mathcal{M}$ may not be surjective. Conceptually, $\mathcal{S}$ is a collection of all of the possible designs that are considered viable solutions to the given engineering problem, according to the design decisions that have been made.

2.5 Constructing the Space

A simple example may help to illustrate the process of obtaining these spaces, from requirements definition through solution space visualization. Assuming an early stage in the design process where only two constraints have been placed on the design problem:

1. The volume of the product must be between 10 and 20 m^3, inclusive.
2. The mass of the product must be between 30 and 75 kg, inclusive.

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

First, determine the parameters being constrained to establish the set $\mathcal{A}$ define the appropriate requirements. In this case, volume V and mass m are the parameters. So

$$\mathcal{A} = \{V, m\} \tag{11}$$

$$\mathcal{R}_V = \{x \mid 10\text{m}^3 \leq x \leq 20\text{m}^3\} \tag{12}$$

$$\mathcal{R}_m = \{x \mid 30\text{kg} \leq x \leq 75\text{kg}\} \tag{13}$$

$$\mathcal{R} = \{\mathcal{R}_V, \mathcal{R}_m\} \tag{14}$$

The constraint space $\mathcal{C}$ in this case is the 2-dimensional real space with m along one axis and V on the other. Within $\mathcal{C}$, the Cartesian product $\mathcal{R}_m \times \mathcal{R}_V$ provides the problem space $\mathcal{P}$. Figure 1a depicts $\mathcal{P}$ as the shaded region residing within the plane of $\mathcal{C}$.

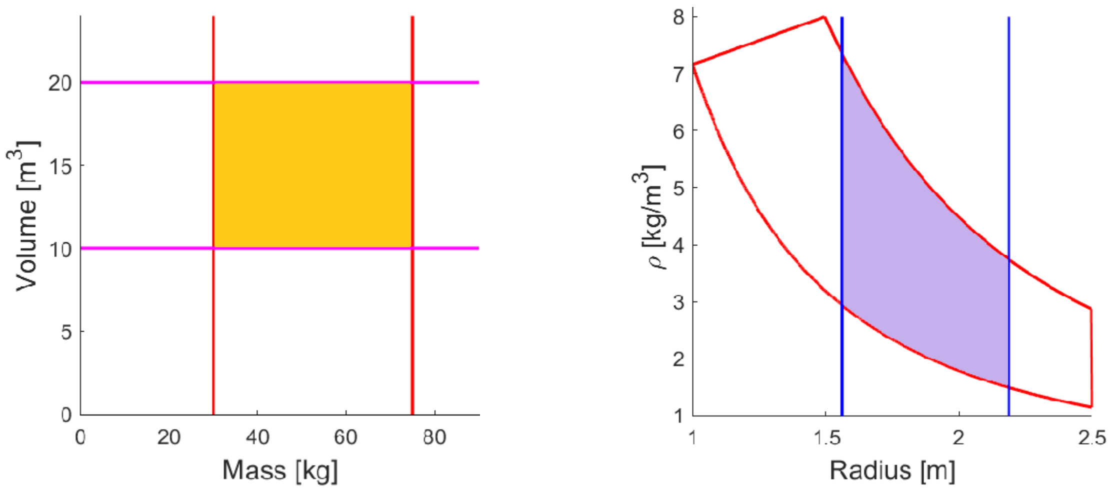
To develop the map and solution space, design decisions need to be made. These can be changed later without sacrificing analyzability, if desired. Selecting a solid sphere as the initial design concept gives a map set $\mathcal{M}$ as follows.

$$f_V = \frac{4}{3}\pi r^3 \tag{15}$$

$$f_m = \frac{4}{3}\rho\pi r^3 = \rho V = \rho f_V(r) \tag{16}$$

$$\mathcal{M} = \left\{ \begin{array}{c} f_m(r) \\ f_V(\rho, r) \end{array} \right\} \tag{17}$$

From $\mathcal{M}$, a set of design parameters is obtained that can be directly manipulated when designing the product, the density ρ and radius r of the sphere. This provides the design parameter set $\mathcal{B} = \{\rho, r\}$ and gives $\mathbb{R}^2$ as the form space, with ρ and r as the axes. In a simple case, such as this, it is relatively simple to analytically determine the solution space, leading to the plot in Fig 1b.



(a) *(b)*

Figure 1. (a) Lines show the bounds on constraint parameters, shaded region depicts problem space. (b) Lines bound the region meeting the volume constraint. Irregular outline bounds the region of allowable mass. Shaded region is the solution space.

Fig 2 demonstrates how the solution space gets its shape by plotting each constraint parameter against its associated design variables. The projection of the surface in 2a and that of the curve in 2b each bound the shaded region in 1b to form the solution space for this problem under the given design decisions.

In many cases, it will not be feasible or even possible to analytically determine the solution space, as was done in this example. In those instances, the solution space can be approximated via sampling. To do this, the engineer must use their judgment to determine an appropriate sampling method, search space, and sample size. One approach would be to use available manufacturing capabilities to limit the search, a technique that can lead to new insights regarding the problem as well. To illustrate, consider that if manufacturing limitations in this case had necessitated a density below 7 kg/m^3 then the solution

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

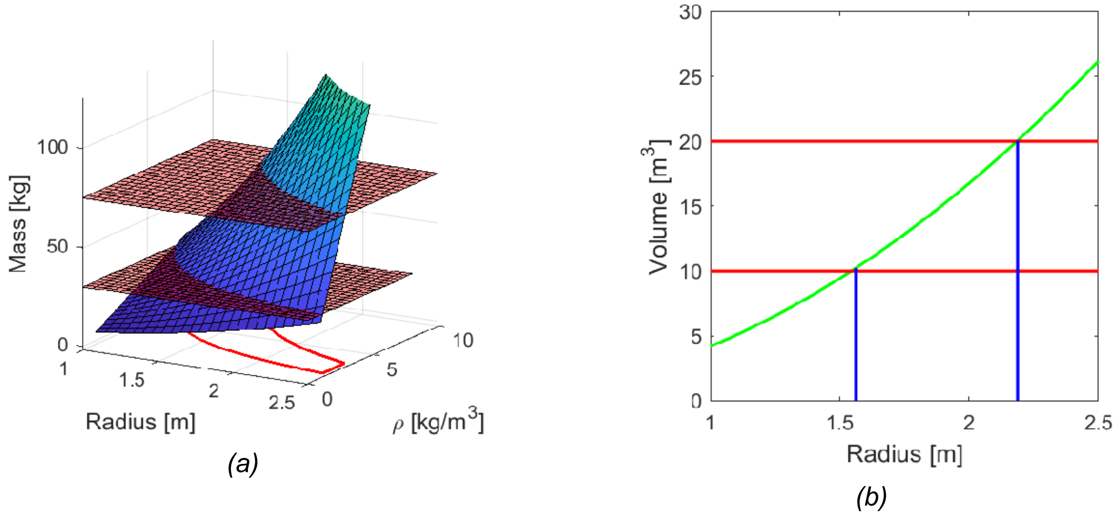


Figure 2. (a) Outline at the base of the plot is the projection of the portion of the surface which lies between the min and max allowable mass. This is the irregular shape shown in 1b. (b) The vertical lines show the projection of the curve between the min and max Volume onto the horizontal. These are the same as the vertical lines shown in 1b.

space would have been truncated. This also highlights how the choice of search area can be critical to obtaining the full space when sampling. Figure 3 shows another important aspect, which is the difference in the resolution of the solution space obtained from various sample sizes.

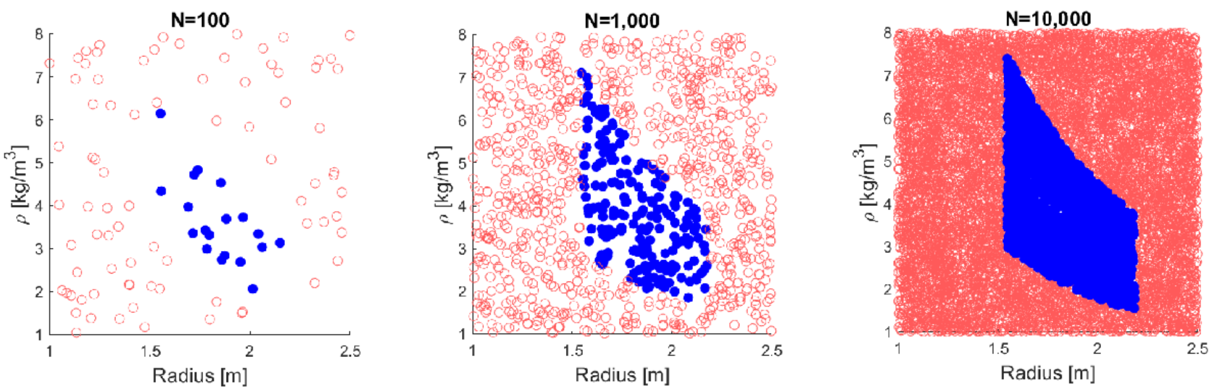


Figure 3. The same solution space as previously shown, but constructed through sampling techniques. By N = 10,000, ta fairly high-resolution image of the solution space boundaries can be seen.

3 METHODS

Having laid the topological foundations for the design spaces, various applications can be explored. This section will focus on three potential applications of these techniques: similarity, sensitivity, and conformity.

3.1 Similarity

There are a variety of similarity measures available for comparing discrete sets. Two of the most common, the Jaccard index (Jaccard, 1912) and the Overlap Coefficient (Simpson, 1943), are respectively defined as

$$\mathrm{oc}(A,B) = \frac{|A \cap B|}{\min(|A|,|B|)} \tag{18}$$

$$J(A,B) = \frac{|A \cap B|}{|A \cup B|} \tag{19}$$

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

Since design spaces may contain an infinite number of points, these indices can be extended to accommodate any topological measures μ for size, to include cardinality (Counting Measure) – which returns the adapted index to the original definition – and n-dimensional volume (Lebesgue Measure), among others. The measure may be selected as appropriate by the engineer. In cases where analytical measurement is not possible or practical, the measure can be approximated by sampling increasingly large subsets until convergence. Letting the representative samples of a set be denoted by

$$\breve{X} = \{x_i \mid x_i \in \Omega_X, i = 1, ..., N\} \tag{20}$$

where Ω_X is sample space of X and N is the number of points in the sample. Then the indices can be modified as

$$\tilde{J}(A,B) = \frac{\mu(A \cap B)}{\mu(A \cup B)} \approx \frac{|\breve{A} \cap \breve{B}|}{|\breve{A} \cup \breve{B}|} \tag{21}$$

$$\tilde{oc}(A,B) = \frac{\mu(A \cap B)}{\min(\mu(A), \mu(B))} \approx \frac{|\breve{A} \cap \breve{B}|}{\min(|\breve{A}|, |\breve{B}|)} \tag{22}$$

both of which have a range of [0, 1]. The respective values given by these formulas provide different information about the sets or spaces in question. While a value of zero indictates disjoint sets for both measures – assuming finite cardinality, $J(A,B) = 1$ indicates $A = B$ whereas $\mathrm{oc}(A,B) = 1$ signifies either $A \subseteq B$ or $B \subseteq A$.

For the purposes of design, both of these measures convey information that can help the designer understand the commonality of two spaces. However, they also have weaknesses. The Jaccard index fails to differentiate between situations where the sizes of the individual sets vary but the sizes of the union and intersection remain the same. Overlap, on the other hand, cannot distinguish any changes in relative set size when one set is a subset of the other. Fig 4 demonstrates these circumstances graphically.

Due to their limitations, a combination of these equations is proposed that will more appropriately quantify similarity for use in design spaces. Let

$$\mathrm{ss}(A,B) = \frac{\sqrt{\tilde{J}(A,B)^2 + \tilde{oc}(A,B)^2}}{\sqrt{2}} \tag{23}$$

This new equation, referred to as spatial similarity ss, still offers a range of [0,1] with $\mathrm{ss} = 0$ indicating no similarity and $\mathrm{ss} = 1$ if and only if $A = B$, as with the Jaccard index. However, it also captures differences in relative size when neither of the other two are able to. Table 1 illustrates how the three formulas handle ambiguous cases.

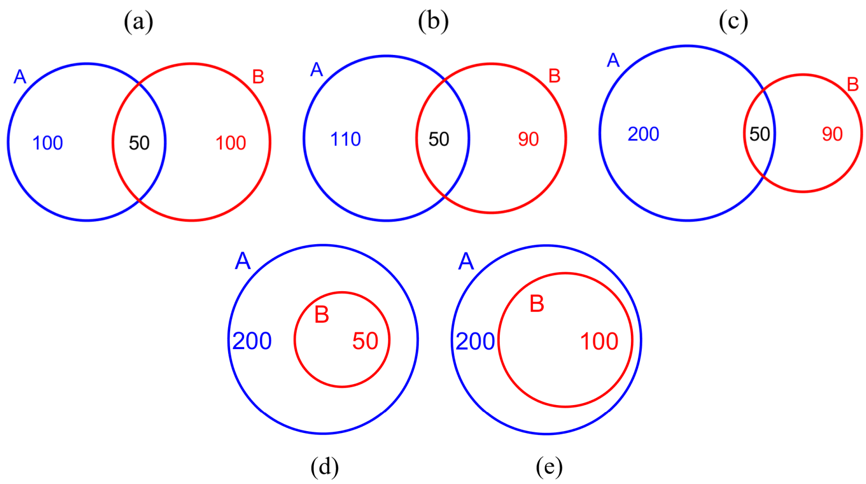


Figure 4. Examples of set relations which Jaccard and Overlap have difficulty differentiating. Numbers indicate the size of the space given by $\mu(X)$. The size of the intersections are included in both set sizes.

It should be noted that certain choices of size measure μ may introduce scenarios wherein all of the similarity equations discussed result in a value of 0 when the intersection is in fact non-empty. For

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

Table 1. Similarity values for sets in Figure 4. Bolded values denote similarity scores shared with another case for the same index.

	$\tilde{J}$	$\tilde{oc}$	ss
a	**0.333**	0.500	0.425
b	**0.333**	**0.556**	0.458
c	0.208	**0.556**	0.420
d	0.250	**1.000**	0.729
e	0.500	**1.000**	0.790

example, using volume gives a similarity of 0 when the intersection is a lower dimension than the sets themselves, as in circles intersecting at a point or cubes intersecting on a face. However, it is generally expected that these occasions are rare and will be known to the designer when the intersection is calculated, as they would be indicated by all points in the intersection sharing the same value for at least one coordinate.

3.2 Sensitivity

Since the similarity indices presented all have ranges on the interval [0,1], it is possible to quantify change in a space X from state 1 to state 2 as

$$\Delta X_{1\to 2} = 1 - \text{Similarity}(X_1, X_2) \tag{24}$$

It should be noted that any similarity index may be substituted in this equation, provided that it produces a measure of sameness between the spaces X_1 and X_2 on the interval $[0,1]$. This concept of change can then be used to define a new measure of sensitivity between related spaces, such as the sensitivity of the Solution space to changes in the Problem space. Since $\mathcal{S}$ is dependent on $\mathcal{P}$, their states are intrinsically linked. And the sensitivity of that link can be quantified by

$$\frac{\Delta\mathcal{S}}{\Delta\mathcal{P}} = \frac{1 - \text{Similarity}(\mathcal{S}_1, \mathcal{S}_2)}{1 - \text{Similarity}(\mathcal{P}_1, \mathcal{P}_2)} \tag{25}$$

This idea can be extended to determine the impact of each individual requirement on the solution space as well. Since each $\mathcal{R}_\alpha$ is a set and the similarity indices are set-based equations, the sensitivity measure can be use to compare $\mathcal{S}$ to $\mathcal{R}_\alpha$ for any requirement in $\mathcal{R}$. By extension, a notion of change gradient can also be determined such that

$$\nabla\mathcal{S} = \begin{bmatrix} \frac{\Delta\mathcal{S}}{\Delta\mathcal{R}_{\alpha_1}} \\ \vdots \\ \frac{\Delta\mathcal{S}}{\Delta\mathcal{R}_{\alpha_n}} \end{bmatrix}, \; n = \mid \mathcal{A} \mid \tag{26}$$

which can provide an indication of those requirements, or combinations of requirements, which would result in the greatest change in the solution space.

3.3 Conformity

The third technique is a measure on the conformity of a design, which indicates the smallest adjustment necessary to bring an infeasible design into the solution space. That is, the distance from a point in $\mathcal{F}$ which is not a member of $\mathcal{S}$ to the nearest boundary of $\mathcal{S}$.

The specific distance metric used can be tailored to the topology of the Form Space as well as to the needs of the engineer. In the example presented in Section 2.5, we have $\mathbb{R}^2$ as our Form Space and the Euclidean distance as our metric. Determining the nearest point in the solution space analytically may not be a trivial task in most cases. However, when sampling, there are a number of algorithms for finding nearest neighbors which may be used. Fig 5b demonstrates how this could be done for an infeasible design in the previously defined Form Space.

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

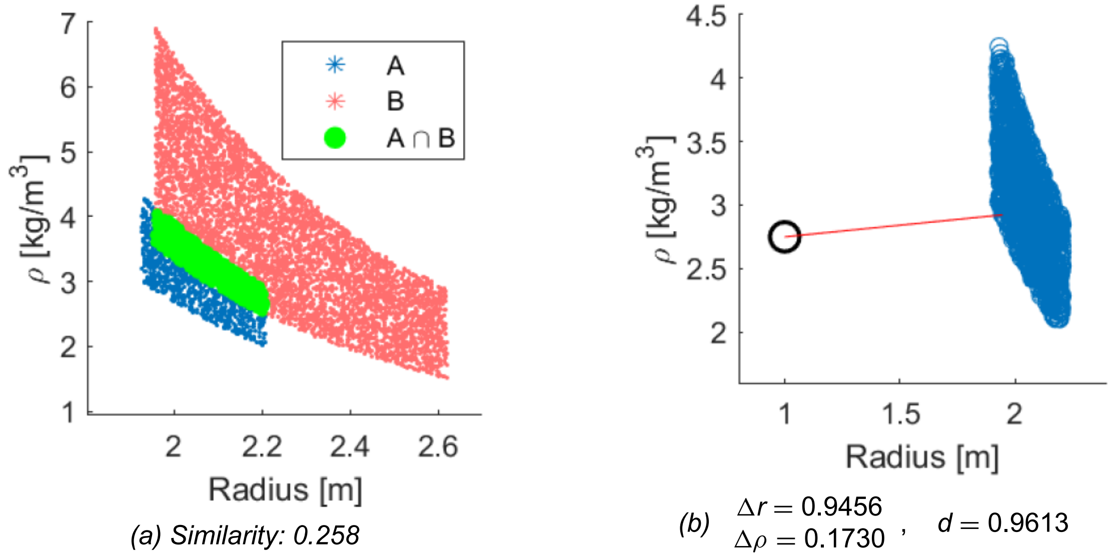


Figure 5. (a) Shows the similarity of the Solution Space from Figure 1b to one that would result from adjusting $\mathfrak{R}_V$ to [7.5 m^3, 18m^3] and $\mathfrak{R}_m$ to [27 kg/m^3, 52 kg/m^3]; (b) Minimum adjustment to bring a design having $r = 1$ and $\rho = 2.75$ into conformity with the requirements.

4 CONCLUSION

In this paper, a formalism has been proposed for the Problem Space and the Solution Space as true topological spaces. It was then shown that this representation can be paired with a variety of topologies to fit the needs of the project. This characterization offers new avenues of comparison and decision-making through topological analysis methods. Although the methods included are simple in their complexity, it is intended that the definitions put forth will be used to explore the breadth of topological analysis techniques available. The primary goal of this endeavor has been to provide the field with a formalization of the design spaces that is applicable beyond this work and that can be used by others in ways of their own imagining.

This preliminary work has demonstrated the foundational concepts of this formalism and illustrated several possible applications. In subsequent work, steps will be taken to show how this can be extended to more complex scenarios involving hierarchical systems, discrete and non-numeric design variables, as well as interdependent requirement sets. Integration with existing design techniques and tools will also be addressed. Such capabilities are necessary to allow engineers to incorporate this methodology into their existing workflow with little disruption, thereby augmenting current design strategies.

Potential areas for future work include the adaptation of persistent homology to identify features in high-dimensional spaces, the use of design criteria as either gradients on the Solution Space or a separate "performace space" which may allow for direct optimization of the design, and the development of software to ease the use of these techniques in complex engineering projects.

REFERENCES

Anandan, S., Teegavarapu, S. and Summers, J.D. (2006), "Issues of similarity in engineering design", in: *International design engineering technical conferences and computers and information in engineering conference*, Vol. 4255, pp. 73–82.

Blonder, B. (2018), "Hypervolume concepts in niche-and trait-based ecology", *Ecography*, Vol. 41 No. 9, pp. 1441–1455.

Bowen, J. and Dittmar, A. (2017), "Formal definitions for design spaces and traces", in: *2017 24th Asia-Pacific Software Engineering Conference (APSEC)*, IEEE, pp. 600–605.

Croom, F.H. (2016), *Principles of topology*, Courier Dover Publications.

Dinar, M., Shah, J., Hunt, G., Campana, E. and Langley, P. (2011), "Towards a formal representation model of problem formulation in design", in: *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, Vol. 54860, pp. 263–272.

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press

Gero, J.S. and Kannengiesser, U. (2004), “The situated function–behaviour–structure framework”, *Design studies*, Vol. 25 No. 4, pp. 373–391.
Goel, V. and Pirolli, P. (1992), “The structure of design problem spaces”, *Cognitive science*, Vol. 16 No. 3, pp. 395–429.
Hijmans, R.J., Cameron, S.E., Parra, J.L., Jones, P.G. and Jarvis, A. (2005), “Very high resolution interpolated climate surfaces for global land areas”, *International Journal of Climatology: A Journal of the Royal Meteorological Society*, Vol. 25 No. 15, pp. 1965–1978.
Hirshorn, S.R., Voss, L.D. and Bromley, L.K. (2017), *Nasa systems engineering handbook*, Technical Report NASA/SP-2016-6105 REV 2, NASA.
INCOSE (2015), “Guide for writing requirements”, *Version 2. Prepared by: Requirements Working Group*.
Jaccard, P. (1912), “The distribution of the flora in the alpine zone. 1”, *New phytologist*, Vol. 11 No. 2, pp. 37–50.
MacLean, A., Young, R.M., Bellotti, V.M. and Moran, T.P. (1991), “Questions, options, and criteria: Elements of design space analysis”, *Human–computer interaction*, Vol. 6 No. 3-4, pp. 201–250.
Maher, M.L., Poon, J. and Boulanger, S. (1996), “Formalising design exploration as co-evolution”, in: *Advances in formal design methods for CAD*, Springer, pp. 3–30.
Ruiz-Pérez, L., Messager, L., Gaitzsch, J., Joseph, A., Sutto, L., Gervasio, F.L. and Battaglia, G. (2016), “Molecular engineering of polymersome surface topology”, Science advances, Vol. 2 No. 4, p. e1500948.
Schätz, B., Hölzl, F. and Lundkvist, T. (2010), “Design-space exploration through constraint-based model-transformation”, in: *2010 17th IEEE International Conference and Workshops on Engineering of Computer Based Systems*, IEEE, pp. 173–182.
Siddique, Z. and Rosen, D.W. (2001), “On combinatorial design spaces for the configuration design of product families”, *Artificial Intelligence for Engineering Design, Analysis and Manufacturing: AI EDAM*, Vol. 15 No. 2, p. 91.
Simpson, G.G. (1943), “Mammals and the nature of continents”, *American Journal of Science*, Vol. 241 No. 1, pp. 1–31.
Taura, T. and Yoshikawa, H. (1994), “Managing function concepts in the design process”, in: *Management of Design*, Springer, pp. 179–203.
Wasserman, L. (2018), “Topological data analysis”, *Annual Review of Statistics and Its Application*, Vol. 5, pp. 501–532.
Zhu, J. and Gao, T. (2016), *Topology optimization in engineering structure design*, Elsevier.

https://doi.org/10.1017/pds.2021.30 Published online by Cambridge University Press