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FINITE COMPLEXES AND INTEGRAL REPRESENTATIONS II

BY

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ABSTRACT. In the paper "Finite complexes and integral representations" [Illinois Journal of Math, 26, (1982), p 442] an exact sequence relating homotopy types of (G, d)-complexes with objects of integral representation theory together with some known calculations seemed to imply that the group of homotopy types of (G, d)-complexes was always a subquotient of $(\mathbb{Z}/|g|)^*$. This paper gives a new characterization of one of the terms of the above sequence that allows one to conclude that this is not generally true.

0. **Introduction.** If one attempts to classify connected, finite, *m-dimensional CW* complexes with fundamental group G and *m-connected universal covering*, one is quickly led, via the work of MacLane–Whitehead [3] and Wall [11], to the study of chain homotopy types of truncated finitely generated $\mathbb{Z}G$-free resolutions of the trivial G-module $\mathbb{Z}$.

In his thesis, W. Browning [1] defined finite abelian groups $h^{d+1}(G, l)$ (respectively $\mathrm{cl}^{d+1}(G, l)$) which classify up to chain homotopy equivalence truncated finitely generated free (resp. projective) resolutions of $\mathbb{Z}$ of length d and Euler characteristic l provided there exists one such resolution $\mathbb{P}_*$ with $H_d(\mathbb{P}_*)$ an Eichler module [5]. In [6] these groups were related to objects of integral representation theory. More precisely, if

$$\mathbb{P}: 0 \to M \to P_d \to P_{d-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

is a truncated finitely generated projective resolution of length d and Euler characteristic l, there exists an exact sequence

$$(*) \qquad K_1(\mathrm{End}_{\mathbb{Z}G} M) \xrightarrow{\det} (\mathbb{Z}/|G|)^* \xrightarrow{\sigma} \mathrm{cl}^{d+1}(G, l) \to \tilde{\mathfrak{G}}(M) \to 0.$$

$\tilde{\mathfrak{G}}(M)$ is the reduced genus group of M consisting of all formal differences $\{[M]-[N],\ M_p \simeq N_p \text{ for all primes } p\}$. For $\alpha \in GL(n, \mathrm{End}_{\mathbb{Z}G} M) = \mathrm{Aut}_{\mathbb{Z}G} M^n$, $\det[\alpha]$ is the determinant of the automorphism α^* of $(\mathbb{Z}/|G|)^n$, where $\alpha^* = k_p^{-n} \mathrm{Ext}_{\mathbb{Z}G}^{d+1}(1, \alpha) k_p^n$ and $k_p : \mathbb{Z}/|G| \to \mathrm{Ext}_{\mathbb{Z}G}^{d+1}(\mathbb{Z}, M)$ is the canonical isomorphism. Moreover there exists a homorphism $t : \mathrm{cl}^{d+1}(G, l) \to \tilde{K}_0(\mathbb{Z}G)$ such that

$$\begin{array}{ccc} (\mathbb{Z}/|G|)^* & \xrightarrow{\sigma} & \mathrm{cl}^{d+1}(G, l) \\ & \searrow^{sw_G} \quad \swarrow_{t} & \\ & \tilde{K}(\mathbb{Z}G) & \end{array}$$

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commutes, where $SW_G : (\mathbb{Z}/|G|)^* \to \tilde{K}(\mathbb{Z}G)$ is the map which associates to each integer r relatively prime to $|G|$ the projective ideal (r, Σ) of $\mathbb{Z}G$ generated by r and the norm element Σ. [9].

Furthermore if there exists a truncated finitely generated free resolution of length d and Euler characteristic l, then $\ker t \simeq h^{d+1}(G, l)$. Hence if TG (resp. $SW(G)$) denotes the image (resp. kernel) of the map SW_G, it follows one has a commutative diagram

$$(**)\qquad \begin{array}{ccccccccc}
 & & SW(G) & \longrightarrow & h^{d+1}(G, l) & \longrightarrow & \ker \bar{t} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
K_1(\mathrm{End}_{\mathbb{Z}G}(M)) & \xrightarrow{\det} & (\mathbb{Z}/|G|)^* & \longrightarrow & \mathrm{cl}^{d+1}(G, l) & \longrightarrow & \tilde{\mathfrak{S}}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle SW_G} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \bar{t}} & & \\
 & 0 \longrightarrow & TG & \longrightarrow & \tilde{K}_0(\mathbb{Z}G) & \longrightarrow & \tilde{K}_0(\mathbb{Z}G)/TG & \longrightarrow & 0.
\end{array}$$

Since in all the known calculations $h^{d+1}(G, l)$ was a quotient of a sub-group of $(\mathbb{Z}/|G|)^*$, this seemed to indicate $\ker \bar{t}$ was always equal to zero. In this paper we give the following interpretation of $\ker \bar{t}$. Suppose $0 \to M \to F_d \to \cdots \to F_0 \to \mathbb{Z} \to 0$ is an truncated finitely generated free resolution of $\mathbb{Z}$ of length d and Euler characteristic l. Let $F_M = \{[M]-[N] \in \tilde{\mathfrak{S}}(M) \mid M \oplus \mathbb{Z}G \simeq N \oplus \mathbb{Z}G\}$ and let $\Gamma_M = \{[M]-[N] \in \tilde{\mathfrak{S}}(M) |$ there exists a truncated finitely generated free resolution $\mathbb{F}_*$ of $\mathbb{Z}$ of length d and Euler characteristic l with $H_d(\mathbb{F}_*) \simeq N.\}$.

THEOREM. $\ker \bar{t} = F_M = \Gamma_M$.

This allows us to use a result of Sieradski–Dyer [8] to show $\ker \bar{t}$ must in general be different from zero. A result of Ullam [12] can be used to show the same for $SW(G)/\det$.

1. **Notation and terminology.** Let G be a finite group and $\Lambda = \mathbb{Z}G$ its integral group ring. We will refer to a truncated finitely generated projective (free) resolution $\mathbb{P}_*(\mathbb{F}_*)$ of the trivial G module $\mathbb{Z}$ as a syzygy (free syzygy). When no confusion is possible we will write $H_d(\mathbb{P}_*)$ as M. We will work with syzygies of a fixed length $d \geq 1$. The Euler characteristic $\chi(\mathbb{P}_*)$ is

$$\frac{1}{|G|} \sum_{j=0}^{d} (-1)^j rk_{\mathbb{Z}} P_{d-j} \in \mathbb{Z}$$

and the Euler class

$$e(\mathbb{P}_*) = \sum_{j=0}^{d} (-1)^j [P_{d-j}] \in \tilde{K}_0(\Lambda).$$

Each syzygy $0 \to M \to \mathbb{P}_* \to \mathbb{Z} \to 0$ determines a canonical isomorphism $k_{\mathbb{P}} : \mathbb{Z}/|G| \to \mathrm{Ext}_\Lambda^{d+1}(\mathbb{Z}, M)$ where $\mathrm{Ext}_\Lambda^{d+1}(\mathbb{Z}, M)$ is computed with respect to the resolution $P_* \to \mathbb{Z}$, and the image of $[1] \in \mathbb{Z}/|G|$ is also denoted $k_{\mathbb{P}_*}$ and is the k-invariant of $\mathbb{P}_*$. It is shown in [6] that the map $\bar{t} : \mathfrak{G}(M) \to \tilde{K}(\mathbb{Z}G)/T(G)$ in (**) is given as follows: Let $[M]-[N] \in \mathfrak{G}(M)$. By Roiter's lemma we may embed $N \hookrightarrow M$ with cokernel X finite and of $\mathbb{Z}$-annihilator relatively prime to $|G|$. If we projectively resolve X, i.e., if $0 \to Q \to P \to X \to 0$ is exact with P (and hence Q) finitely generated Λ-projective, then $\bar{t}([M]-[N]) = ([P]-[Q]) + TG \in \tilde{K}_0(\mathbb{Z}G)/TG$. Note also that since $\operatorname{ann}_{\mathbb{Z}} X$ is of order relatively prime to $|G|$ and N, M are Λ-lattices, we have by the generalized Schanuel's lemma [13, V2.6; VII3.5] that $M \oplus Q \simeq N \oplus P$. Two syzygies P_*, $P'_{\mathscr{B}}$ are equivalent, i.e., equal in $\mathrm{cl}^{d+1}(G, l)$ if and only if they are chain homotopy equivalent (as complexes with two non zero homology groups) by a chain map inducing the identity on $\mathbb{Z} = H_0(\mathbb{P}_*) = H_*(\mathbb{P}'_*)$. It is obvious by making elementary changes that $\mathbb{P}_*$ is equivalent to a free syzygy if and only if $e(\mathbb{Z}_*) = 0$.

2. **The results.** The equality of $\ker \bar{t}$ and Γ_M is a consequence of the following proposition.

PROPOSITION 1. *Suppose* $0 \to M \to \mathbb{P}_* \to \mathbb{Z} \to 0$ *is a syzygy. For all* s *with* $(s, |G|) = 1$, *there exists a syzygy* $0 \to M \to \mathbb{P}'_* \to \mathbb{Z} \to 0$ *such that*

(i) $k_{\mathbb{P}_*} = s k_{\mathbb{P}_*} \in \operatorname{Ext}_\Lambda^{d+1}(\mathbb{Z}, M)$.

(ii) $e(\mathbb{P}'_*) = e(\mathbb{P}_*) + (-1)^{d+1}[(s, \Sigma)]$

(iii) $\chi(\mathbb{P}'_*) = \chi(\mathbb{P}_*)$.

Proof. Let $\mathbb{P}''_*$ be the pushout of $\mathbb{P}_*$ by the map $s : M \to M$. In [6] it is shown that one obtains $(-1)^{d+1}[(s, \Sigma)] \in \tilde{K}(\mathbb{Z}G)$ by projectively resolving M/sM. On the other hand from the above pushout and Schanuel's lemma this is $[P''_d]-[P_d]$ and hence $P''_d \oplus \Lambda \simeq P_d \oplus (s^{(-1)^{d+1}}, \Sigma)$ by the Bass cancellation theorem. In $\mathbb{P}''_*$ add Λ to P''_d and P''_{d-1} and then replace $P''_d \oplus \Lambda$ by $P_d \oplus (s^{(-1)^{d+1}}, \Sigma)$ to obtain a syzygy P'_* and a map

$$\begin{array}{llllllllllllllll}
\mathbb{P}_* & 0 & \longrightarrow & M & \longrightarrow & P_d & \longrightarrow & P_{d-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle i_1} & & & & \| & & & & \\
\mathbb{P}'_* & 0 & \longrightarrow & M & \longrightarrow & P_d \oplus (s^{(-1)^{d+1}}, \Sigma) & \longrightarrow & P_{d-1} \oplus \Lambda & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Since $s : M \to M$ induces multiplication by s on $\operatorname{Ext}^{d+1}(\mathbb{Z}, M)$, $k_{\mathbb{P}'_*} = s k_{\mathbb{P}_*}$. Clearly $e(\mathbb{P}'_*) = e(\mathbb{P}_*) + (-1)^{d+1}(s, \Sigma)$ and $\chi(\mathbb{P}'_*) = \chi(\mathbb{P}'_*)$. ||.

THEOREM. *Suppose* $0 \to M \to \mathbb{F}_* \to \mathbb{Z} \to 0$ *is a free syzygy, then* $\ker \bar{t} = \Gamma_M$.

Proof. Let $[M]-[N] \in \ker \bar{t}$. Embed $h : M \hookrightarrow N$ by Roiter's lemma with cokernel h finite and annihilator relatively prime to $|G|$. Then $h_* \mathbb{P}_* : 0 \to N \to P'_d \to F_{d-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ has Euler class $[P'_d] = [(r, \Sigma)]$ for some r with $(r, |G|) = 1$ by the generalized Schanuel lemma [13] since $\bar{t}([M]-[N]) = 0$. By

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Proposition 1, there exists a syzygy $0 \to N \to \mathbb{P}''_* \to \mathbb{Z} \to 0$ with Euler class $e[\mathbb{P}'_*]-[(r, \Sigma)]=0$. Hence there exists a free syzygy $\mathbb{F}'_*$ with $H_*(\mathbb{F}'_*)=N$.

Let $[M]-[N] \in \Gamma_M$. i.e., suppose there exists a free syzygy $0 \to N \to \mathbb{F}'_* \to \mathbb{Z} \to 0$. Let $k'=k_{\mathbb{F}'_*}$ be the k-invariant. Embed $h : M \hookrightarrow N$ by Roiter's lemma and let $\mathbb{P}''_* = h_* \mathbb{F}_*$. As we have seen $\bar{t}([N]-[M])=[P''_d]+TG$. Let $k''=k_{\mathbb{P}_*}$ be the k-invariant. Then $k''=sk'$ for some s with $(s, |G|)=1$. By Proposition 1 there exists a syzygy $0 \to N \to \mathbb{P}'_* \to \mathbb{Z} \to 0$ with $k_{\mathbb{P}'_*}=sk'$ and $e(\mathbb{P}'_*)=(-1)^{d+1}[(s, \Sigma)]$. Since $\mathbb{P}'_*$ and $\mathbb{P}''_*$ have the same k-invariant, there exists a chain map from $\mathbb{P}'_*$ to $\mathbb{P}''_*$ inducing the identity on H_0 and H_d. Hence $[P''_d]=e(\mathbb{P}''_*)=e(\mathbb{P}'_*)=(-1)^{d+1}[(s, \Sigma)]$, i.e. $\bar{t}([M]-[N])=0$. ||

The equality of Γ_M and F_M depends on the following result.

Proposition 2. *Suppose* $0 \to M \xrightarrow{\rho} D \xrightarrow{\partial} E \to 0$ *is an exact sequence of* Λ-*lattices and* $M \oplus X \underset{\beta}{\cong} N \oplus Y$ *for some* Λ-*lattice* N *and some finitely generated* Λ-*projective* Y. *Then there exists an exact sequence* $0 \to N \to D \oplus X \to E \oplus Y \to 0$ *and a map*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\rho} & D & \xrightarrow{\partial} & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_1\beta\iota_1} & & \downarrow & & \downarrow{\scriptstyle \iota_1} & & \\
0 & \longrightarrow & N & \longrightarrow & D \oplus X & \xrightarrow{(\partial, g)} & E \oplus Y & \longrightarrow & 0
\end{array}$$

for some map $g : D \oplus X \to Y$.

Proof. Consider $0 \to M \oplus X \xrightarrow{\rho \oplus 1} D \oplus X \xrightarrow{(\partial, 0)} E \to 0$. Replacing $M \oplus X$ by $N \oplus Y$ via β gives the following.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & D & \xrightarrow{(\partial,0)} & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta\iota_1} & & \downarrow{\scriptstyle \iota_1} & & \|{\scriptstyle \iota} & & \\
0 & \longrightarrow & N \oplus Y & \xrightarrow{u} & D \oplus X & \xrightarrow{(\partial,0)} & E & \longrightarrow & 0.
\end{array}$$

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & W & \xrightarrow{k} & E & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle p_1} & & {\scriptstyle h}\,\downarrow\uparrow{\scriptstyle \nu} & & \| & & \\
0 & \longrightarrow & N \oplus Y & \xrightarrow{u} & D \oplus X & \xrightarrow{(\partial,0)} & E & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i_2} & & {\scriptstyle g}\,\downarrow\uparrow{\scriptstyle u \circ i_2} & & & & \\
 & & Y & = & Y & & & &
\end{array}$$

Since Y is Λ-projective, and all are Λ-lattices, the middle vertical sequence splits [9] with splitting maps $h : W \to D \oplus X$ and $g : D \oplus X \to Y$.

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From the sequence $0 \to N \xrightarrow{(1,0)} W \oplus Y \xrightarrow{k \oplus 1} E \oplus Y \to 0$ and the isomorphism $(h, ul_2): W \oplus Y \xrightarrow{\approx} D \oplus X$, one obtains the desired sequence $0 \to N \xrightarrow{h1} D \oplus X \xrightarrow{(\partial, g)} E \oplus Y \to 0$. Moreover it is clear there are maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \oplus Y & \xrightarrow{u} & D \oplus X & \xrightarrow{(\partial,0)} & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \nu} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{1} & W & \xrightarrow{k} & E & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_1} & & \\
0 & \longrightarrow & N & \xrightarrow{(1,0)} & W \oplus Y & \xrightarrow{k \oplus 1} & E \oplus Y & \longrightarrow & 0 \\
 & & \| & & \approx \downarrow{\scriptstyle (h, u\iota_2)} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{h1} & D \oplus X & \xrightarrow{(\partial, g)} & E \oplus Y & \longrightarrow & 0.
\end{array}$$

COROLLARY 1. *If* $0 \to M \to \mathbb{P}_* \to \mathbb{Z} \to 0$ *is a syzygy and* $\beta : M \oplus P \simeq N \oplus Q$ *is an isomorphism with* P, Q Λ*-projective; then there exists a syzygy* $\mathbb{Q} : 0 \to N \to P_d \oplus P \to P_{d-1} \oplus Q \to \cdots \to P_0 \to \mathbb{Z} \to 0$ *and a chain map* $f : \mathbb{P}_* \to \mathbb{Q}_*$ *inducing Id on* H_0 *and* $p_1 \beta \iota_1$ *on* H_d.

Remark. It is obvious that $e(\mathbb{Q}_*) = e(\mathbb{P}_*) + [P] - [Q]$ and that if M and N have the same $\mathbb{Z}$-rank, for example are in the same genus, then $\chi(\mathbb{P}_*) = \chi(\mathbb{Q}_*)$.

COROLLARY 2. *If* $0 \to M \to \mathbb{F}_* \to \mathbb{Z} \to 0$ *is a free syzygy, then* $\Gamma_M = F_m$.

Proof. From Schanuel's lemma and the Bass cancellation theorem it is obvious that $\Gamma_M \subseteq F'_M$. The converse is an immediate consequence of Corollary 1 and the remarks following. ||

In [8] Sieradaki and Dyer prove the following theorem. Let G be a finite abelian group with torsion coefficients $t_1 | \cdots | t_n$.

THEOREM. *If* $0 \to M \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$ *is a free syzygy of length* 2. *and minimal Euler characteristic* $= 1 + \binom{n}{2}$, *then there exists an epimorphism* $B : \Gamma_M \to (\mathbb{Z}/t_1)^* / \pm(\mathbb{Z}/t_1)^*)^{\binom{n}{2}} = \sum_{t_1}^{\binom{n}{2}}$.

It is easily seen that this group is in general non-zero. For example if $t_1 = p$ prime then it is easy to see $\sum_p^s$ is cyclic of order s if $p \equiv 1 \bmod 2s$. W. Browning [1], based on work of Metzler [4] and Sieradski [7], has shown that for G finite abelian with torsion coefficients $t_1 | \cdots | t_n$, $h^3(G, 1 + \binom{n}{2}) \approx (\mathbb{Z}/t_1)^* / (\pm 1)$. Hence if we consider the elementary abelian group $\mathbb{Z}/7 \times \mathbb{Z}/7 \times \mathbb{Z}/7$ it follows from the above considerations of $\sum_p^s$ that the map $\det : K_1(\text{End}\, M) \to SW(G)$ in $(*\,*)$ must be onto.

Ullam [12], see also Taylor [10], has shown that for an elementary abelian

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group of rank k the map $SW_G:(\mathbb{Z}/p^k)^*\to TG$ restricted to the subgroup generated by $(1+p)$ is an isomorphism, hence $SW(G)\cap(1+p)=1$. However the following result shows that for only two torsion coefficients the image of the determinant map is equal to $(-1)\subset SW(G)$.

PROPOSITION 3. *Let* $G=\mathbb{Z}/t_1\times\mathbb{Z}/t_2$, $t_1 \mid t_2$ *and suppose* $0\to M\to\mathbb{F}_*\to\mathbb{Z}\to 0$ *is the free syzygy of length 2 based on the standard presentation of G. If* $h: M^k\to M^k$ *is an automorphism, then* $\det h\equiv\pm 1 \bmod t_1$.

Proof. It is easy to see there exists an exact sequence $0\to\mathbb{Z}\to M\to IG^2\to 0$ where $\mathbb{Z}\approx M^G$ and IG is the augmentation ideal of G. Since $H^2(G, IG)=0$, $H^3(G, IG)\approx H^2(G,\mathbb{Z})\approx G$, $H^3(G,\mathbb{Z})\approx\mathbb{Z}/t_1$, $H^3(G, M)\approx\mathbb{Z}/|G|$, and $H^4(G, M)=0$ the exact cohomology sequence reduces to

$$0\to\mathbb{Z}/t_1\xrightarrow{j}\mathbb{Z}/|G|\to H^2(G,\mathbb{Z})^2\to H^4(G,\mathbb{Z})\to 0$$

We may choose generators u of $\mathbb{Z}/t_1$ and v of $\mathbb{Z}/|G|$ so that $j(u)=t_2v$ since there is only one subgroup of $\mathbb{Z}|G|$ of order t_1. If $g: M^k\to M^k$ is an automorphism then h induces an automorphism $h^G:(M^G)^k\to(M^G)^k$ and hence a commutative diagram

$$\begin{array}{ccccccccc}
0\to & (\mathbb{Z}/t_1)^k & \xrightarrow{j} & (\mathbb{Z}/|G|)^k & \to & H^2(G,\mathbb{Z})^{2k} & \to & H^4(G,\mathbb{Z})^k & \to 0\\
 & \downarrow h_*^G & & \downarrow h_* & & \downarrow & & \downarrow & \\
0\to & (\mathbb{Z}/t_1)^k & \xrightarrow{j} & (\mathbb{Z}/|G|)^k & \to & H^2(G,\mathbb{Z})^{2k} & \to & H^4(G,\mathbb{Z})^k & \to 0
\end{array}$$

Since $M^G\approx\mathbb{Z}$, $\det h^G=\pm 1$ and hence $\det h_*^G\equiv\pm 1 \pmod{t_1}$. If (b_{ij}) is the matrix representing h_*^G with respect to the basis $u_1,\ldots,u_k$ of $(\mathbb{Z}/t_1)^k$ in $GL_k(\mathbb{Z}/t_1)$ and (a_{ij}) that representing h_* with respect to the basis $v_1,\ldots,v_k$ of $(\mathbb{Z}/|G|)^k$ in $GL_k(\mathbb{Z}/|G|)$ then since $j(u_i)=t_2v_i$ it is easy to see from the above diagram that $t_2a_{ij}\equiv t_2b_{ij} \bmod (t_1t_2)$, i.e. $a_{ij}\equiv b_{ij} \bmod t_1$. But this surely 'implies $\det h_*\equiv\pm 1 \pmod{t_1}$. ||

Let $G=\mathbb{Z}/P\times\mathbb{Z}/P$. From the above result we have $\operatorname{im}(\det)=\pm 1$. The work of Ullam and Taylor shows $(1+p)\cap SW(G)=1$ and hence reduction mod p defines an isomorphism of $SW(G)$ with $(\mathbb{Z}/p)^*$ since $(1+p)$ is the kernel of the reduction map. Therefore $SW(G)/\det(K_1)\simeq(\mathbb{Z}/p)^*/\pm 1$. Since Browning's result says $h^3(G,2)\simeq(\mathbb{Z}/p)^*/\pm 1$ we must have $\Gamma_M=0$. Hence for an elementary p group of rank 2, all of $h^3(G, l)$ arises from the Swan subgroup of $(\mathbb{Z}/|G|)^*$, while for rank 3 (at least for $p=7$) it arises entirely from Γ_M. This discussion shows that even in the finite abelian case where $h^3(G, 1+\binom{n}{2})$ is known, the image of the determinant map is somewhat of a mystery.

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