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CLASSIFICATION OF REFLECTION SUBGROUPS MINIMALLY CONTAINING *p*-SYLOW SUBGROUPS

KANE DOUGLAS TOWNSEND

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Abstract

Let a prime p divide the order of a finite real reflection group. We classify the reflection subgroups up to conjugacy that are minimal with respect to inclusion, subject to containing a p-Sylow subgroup. For Weyl groups, this is achieved by an algorithm inspired by the Borel–de Siebenthal algorithm. The cases where there is not a unique conjugacy class of reflection subgroups minimally containing the p-Sylow subgroups are the groups of type F_4 when $p = 2$ and $I_2(m)$ when $m \geq 6$ is even but not a power of 2 for each odd prime divisor p of m. The classification significantly reduces the cases required to describe the p-Sylow subgroups of finite real reflection groups.

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1. Introduction

Throughout, let W be a finite real reflection group. For each prime p dividing the order of W, we classify the reflection subgroups up to W-conjugacy that are minimal with respect to inclusion, subject to containing a p-Sylow subgroup. By Sylow's theorems, the p-Sylow subgroups form a unique conjugacy class for each prime p dividing $|W|$. Since the class of reflection subgroups is closed under conjugation, we have conjugacy classes of reflection subgroups containing the p-Sylow subgroups if a reflection subgroup has the same p-adic valuation as $|W|$. By the classification of finite real reflection groups seen in [8, Ch. 2], we only need to classify for each irreducible type of W.

Definition 1.1. Call the conjugacy class of parabolic subgroups minimally containing the p-Sylow subgroups the *p-Sylow conjugacy class of parabolic subgroups*. We refer to a parabolic subgroup in such a conjugacy class as P_p. In the case that $W = P_p$ we say that p is *cuspidal* for W.

Definition 1.2. Call a conjugacy class of reflection subgroups minimally containing the p-Sylow subgroups a *p-Sylow conjugacy class of reflection subgroups*. We refer to a reflection subgroup in one of these conjugacy classes as R_p.

In [1, Table 4.1], a classification of P_p up to W-conjugacy for all Weyl groups is presented. We will first present a classification of P_p up to W-conjugacy for all finite real reflection groups, adding the classification for noncrystallographic reflection groups to [1, Table 4.1]. Then we will give a classification of R_p up to W-conjugacy. As suggested by Definition 1.1, the P_p conjugacy class is unique since the class of parabolic subgroups is closed under conjugation and intersection. However, as suggested by Definition 1.2, the R_p conjugacy class is not necessarily unique, since the class of reflection subgroups is not closed under intersection. An example of this is in the Weyl group of type G_2, which has two R_p conjugacy classes of type A_2 containing the 3-Sylow subgroup isomorphic to the cyclic group C_3.

In [5], a classification of reflection subgroups in finite real reflection groups is given up to W-conjugacy. A possible way to classify the p-Sylow conjugacy classes of reflection subgroups would be to use the given classification directly, while solving an inclusion minimisation problem where the p-adic valuation of $|W|$ is preserved. However, for Weyl groups we instead classify by adapting the Borel–de Siebenthal algorithm found in [9, page 136]. This algorithm is based on the work of Borel and de Siebenthal in [2].

We will see in Corollary 2.5 that any R_p has a P_p as its parabolic closure. Hence, R_p is a reflection subgroup minimally containing a p-Sylow subgroup in a P_p. We can then find R_p up to W-conjugacy for each cuspidal prime p, which will also give us R_p up to P_p-conjugacy in the noncuspidal cases of W. We then easily extend these noncuspidal classifications of R_p to W-conjugacy.

In Section 2 we show that any R_p has P_p as its parabolic closure and classify the p-Sylow conjugacy class of parabolic subgroups. In Section 3, we introduce the basic definitions required to state the Borel–de Siebenthal algorithm and turn it into Algorithm 3.9 which finds the p-Sylow conjugacy classes of reflection subgroups for the cuspidal cases. In Section 4 we classify the p-Sylow conjugacy classes of reflection subgroups for the cuspidal cases, allowing us to deduce R_p up to P_p-conjugacy for the noncuspidal cases and extend the classification to W-conjugacy. In Section 5 we make some general observations regarding our classifications and see how they reduce the cases required to describe the p-Sylow subgroups of finite real reflection groups.

Tables 1–4 give the respective classifications of p-Sylow conjugacy classes of parabolic subgroups, the cuspidal cases for finite real reflection groups, the p-Sylow conjugacy classes of reflection subgroups for the cuspidal cases and the full classification of p-Sylow conjugacy classes of reflection subgroups. We note here the interpretations of types of groups found in the tables. We interpret B_1 as A_1, $I_2(2)$ as $A_1 \times A_1$, $I_2(3)$ as A_2 and $I_2(4)$ as B_2. The tilde above the type of group in the case of Weyl groups signifies that the roots in this subsystem are short, while for the noncrystallographic reflection groups it is simply used to signify a different conjugacy class. In the classical Weyl group cases, the base p expression for n is $(b_l b_{l-1} \ldots b_1 b_0)_p$. Finally, for the groups of type $I_2(m)$, the p-adic valuation of $2m$ is denoted by k.

2. Preliminaries and the p-Sylow conjugacy class of parabolic subgroups

For introductory definitions and results regarding finite real reflection groups and root systems we refer the reader to Humphreys [8, Ch. 1–2]. In this section, we use the definition of root systems that does not require the crystallographic property.

Definition 2.1. A *parabolic subgroup* of W is a reflection subgroup generated by any subset of any set of simple reflections.

Let V be a real euclidean space on which W acts essentially. The following lemma gives an equivalent definition of parabolic subgroups as stabilisers of subspaces of V.

Lemma 2.2 [9, pages 60–61]. *A subgroup W' of W is a parabolic subgroup if and only if $W' = W_U$ for some subspace U of V, where $W_U = \{w \in W \mid w(v) = v$ for all $v \in U\}$.*

Lemma 2.3. *Let W be a finite real reflection group. The intersection of a parabolic subgroup P and a reflection subgroup R is a reflection subgroup of W.*

Proof. By Lemma 2.2, $P = W_U$ for some subspace U of V, so $P \cap R$ is the stabiliser in R of the intersection of U and the space on which R acts essentially. Hence, $P \cap R$ is a parabolic subgroup of R generated by a subset of reflections found in R and we conclude that $P \cap R$ is a reflection subgroup of W. □

Definition 2.4. The *parabolic closure* of some subset X of W is the minimal parabolic subgroup of W with respect to inclusion that contains X.

Corollary 2.5. *For any R_p, its parabolic closure is a P_p.*

Proof. Take P_p containing the same p-Sylow subgroup as R_p. By Lemma 2.3 and minimality arguments, P_p contains R_p since $P_p \cap R_p$ is a reflection subgroup containing a p-Sylow subgroup. Then by the minimality of P_p it is the parabolic closure of R_p. □

We conclude this section by classifying P_p up to W-conjugacy for all finite real reflection groups by using the classification of parabolic subgroups from [5]. Let

$$v_p(n) := \max\{k \in \mathbb{N} \mid p^k \text{ divides } n\}.$$

We use the same notation and definitions for partitions as in [5].

Lemma 2.6. *Let the base p expression of n be $(b_l b_{l-1} \dots b_1 b_0)_p$. Then the partition $\lambda \vdash n$ with length r that provides the minimum of the set*

$$\left\{ \prod_{i=1}^{r} \lambda_i! \;\middle|\; v_p(n!) = \sum_{i=1}^{r} v_p(\lambda_i!) \right\}$$

is given by the join $\lambda = \lambda^l \cup \lambda^{l-1} \cup \dots \cup \lambda^1 \cup \lambda^0$, where $\lambda^j = (p^j, \dots, p^j)$ with length b_j for each j with $0 \le j \le l$.

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TABLE 1. Type of P_p in W.

Type of W	$\lvert W\rvert$	p	Type of P_p	$\lvert P_p\rvert$
$A_{n-1}, n \geq 2$	$n!$	Any	$\prod_{i=1}^{l} A_{p^i-1}^{b_i}$	$\prod_{i=1}^{l} [(p^i)!]^{b_i}$
$B_n, n \geq 2$	$2^n n!$	2	B_n	$2^n n!$
		>2	$\prod_{i=1}^{l} A_{p^i-1}^{b_i}$	$\prod_{i=1}^{l} [(p^i)!]^{b_i}$
$D_n, n \geq 4$	$2^{n-1} n!$	2	D_n	$2^{n-1} n!$
		>2	$\prod_{i=1}^{l} A_{p^i-1}^{b_i}$	$\prod_{i=1}^{l} [(p^i)!]^{b_i}$
E_6	$2^7 \cdot 3^4 \cdot 5$	2	D_5	$2^7 \cdot 3 \cdot 5$
		3	E_6	$2^7 \cdot 3^4 \cdot 5$
		5	A_4	$2^3 \cdot 3 \cdot 5$
E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	2	E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
		3	E_6	$2^7 \cdot 3^4 \cdot 5$
		5	A_4	$2^3 \cdot 3 \cdot 5$
		7	A_6	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	≤ 5	E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
		7	A_6	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
F_4	$2^7 \cdot 3^2$	2, 3	F_4	$2^7 \cdot 3^2$
G_2	$2^2 \cdot 3$	2, 3	G_2	$2^2 \cdot 3$
H_3	$2^3 \cdot 3 \cdot 5$	2	H_3	$2^3 \cdot 3 \cdot 5$
		3	A_2	$2 \cdot 3$
		5	$I_2(5)$	$2^2 \cdot 5^2$
H_4	$2^6 \cdot 3^2 \cdot 5^2$	2, 3, 5	H_4	$2^6 \cdot 3^2 \cdot 5^2$
$I_2(m), m = 5$ or $m \geq 7$	$2m$	2	$\begin{cases} I_2(m) & \text{for } m \text{ even,} \\ A_1 & \text{for } m \text{ odd} \end{cases}$	$\begin{cases} 2m & \text{for } m \text{ even,} \\ 2 & \text{for } m \text{ odd} \end{cases}$
		>2	$I_2(m)$	$2m$

PROOF. By Kummer's theorem for binomial coefficients, $v_p((\sum_{i=1}^{r} n_i)!) = \sum_{i=1}^{r} v_p(n_i!)$ for any $n_i \in \mathbb{N}$ if and only if the sum of n_i in base p has no carries. Hence, the result follows. □

We achieve the classification of P_p up to W-conjugacy by observing the type of parabolic subgroup that has smallest order as well as the same p-adic valuation as $|W|$. The subgroup minimal with respect to inclusion will be the same as the subgroup minimal with respect to order, since the p-Sylow conjugacy class of parabolic subgroups is unique.

THEOREM 2.7. *The classification of the p-Sylow conjugacy classes of parabolic subgroups in finite real reflection groups is as given in Table 1.*

PROOF. This classification can be found in [1, Table 4.1] for Weyl groups. For completeness, we give an independent proof for all finite real reflection groups.

TABLE 2. Cases when p is cuspidal for W.

Type of W	p cuspidal for W
A_n	$n = p^s - 1$, where $s \in \mathbb{N}$
B_n	$p = 2$
D_n	$p = 2$
E_6	$p = 3$
E_7	$p = 2$
E_8	$p = 2, 3, 5$
F_4	$p = 2, 3$
G_2	$p = 2, 3$
H_3	$p = 2$
H_4	$p = 2, 3, 5$
$I_2(m), m = 5$ or $m \geq 7$	$p = 2$ when m is even and $p > 2$ for all m

Let W be of type A_{n-1} with $n \geq 2$. We claim that the P_p conjugacy class consists of subgroups of type $\prod_{i=1}^{l} A_{p^i-1}^{b_i}$, where the base p expression of n is $(b_l b_{l-1} \ldots b_1 b_0)_p$. The order of the reflection group of type A_{n-1} is $n!$. From [5, Theorem 3.1] the parabolic subgroups are unique conjugacy classes of type $\prod_i A_{\lambda_i - 1}$ for each partition $\lambda \vdash n$. Hence, the result follows by Lemma 2.6.

Let W be of type B_n with $n \geq 2$. We claim that the P_p conjugacy class has a subgroup of type B_n when $p = 2$ and subgroups of type $\prod_{i=1}^{l} A_{p^i-1}^{b_i}$ when $p > 2$, where the base p expression of n is $(b_l b_{l-1} \ldots b_1 b_0)_p$. The order of the reflection group of type B_n is $2^n n!$. From [5, Section 3] the parabolic subgroups are unique conjugacy classes of type $B_{n-m} \times \prod_i A_{\lambda_i - 1}$ for each partition $\lambda \vdash m$ with $0 \leq m \leq n$. Hence, the result follows by Lemma 2.6.

Let W be of type D_n with $n \geq 4$. We claim that the P_p conjugacy class has a subgroup of type D_n when $p = 2$ and subgroups of type $\prod_{i=1}^{l} A_{p^i-1}^{b_i}$ when $p > 2$, where the base p expression of n is $(b_l b_{l-1} \ldots b_1 b_0)_p$. The order of the reflection group of type D_n is $2^{n-1} n!$. From [5, Section 3] the parabolic subgroups are unique conjugacy classes of type $D_{n-m} \times \prod_i A_{\lambda_i - 1}$ for each partition $\lambda \vdash m$ with $0 \leq m \leq n-2$, two classes of type $\prod_i W(A_{\lambda_i - 1})$ for each even partition $\lambda \vdash n$ and a unique class of type $\prod_i W(A_{\lambda_i - 1})$ for each noneven partition $\lambda \vdash n$. Hence, the result follows by Lemma 2.6.

When W is of type $E_6, E_7, E_8, F_4, G_2, H_3$, or H_4, the classification is found by observing the type of parabolic subgroup with the smallest order while preserving the p-adic valuation of $|W|$. The classifications of the parabolic subgroups of W are found in [5, Tables 3–9].

Let W be of type $I_2(m)$ with $m = 5$ or $m \geq 7$. We claim that the P_p conjugacy class consists of subgroups of type $I_2(m)$ when m is even and A_1 when m is odd. This is clear since the parabolic subgroups of W are simply $\emptyset$, A_1, or $I_2(m)$. □

We now have the complete classification of the p-Sylow conjugacy class of parabolic subgroups in finite real reflection groups presented in Table 1. This matches the classification in [1, Table 4.1] for all Weyl groups. Table 2 summarises when p is cuspidal for W by extracting the cases with $W = P_p$.

3. Adapting the Borel–de Siebenthal algorithm

We now use the convention that root systems require the crystallographic property as seen in the work by Bourbaki [3]. We introduce some definitions required to present the Borel–de Siebenthal algorithm as in [9, page 136].

Definition 3.1. A *subsystem* Ψ of root system Φ is a subset that is also a root system.

Definition 3.2. A subsystem $\Psi \subset \Phi$ is called *closed* if for all $\alpha, \beta \in \Psi$ we have $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Psi$.

Definition 3.3. A *maximal subsystem* of a root system Φ is a proper subsystem of Φ that is not properly contained in any other proper subsystem of Φ.

The following theorem is the core of the Borel–de Siebenthal algorithm which takes an irreducible root system Φ and returns all maximal closed subsystems up to $W(\Phi)$-conjugacy.

Theorem 3.4 (Borel–de Siebenthal). *Let Φ be an irreducible crystallographic root system with simple system $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and highest root $\tilde{\alpha} = \sum_{i=1}^n h_i\alpha_i$. The maximal closed subsystems of Φ up to $W(\Phi)$-conjugacy are those with simple systems*

(i) $\{\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n\}$ *where $h_i = 1$;*
(ii) $\{-\tilde{\alpha}, \alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_n\}$, *for each $1 \le i \le n$, such that h_i is prime.*

The positive integers h_i in the expression of the highest root in terms of simple roots $\tilde{\alpha} = \sum_{i=1}^n h_i\alpha_i$ are called the *weights*. In Figure 1 we present the extended Dynkin diagrams with the weights added to their corresponding vertices. These can all be found in [3, pages 200–222].

We will now adapt the Borel–de Siebenthal algorithm to classify the p-Sylow conjugacy classes of reflection subgroups in the cuspidal cases. The *rank* of a root system Φ is the number of roots found in its simple system, or equivalently, the dimension of real euclidean space V on which $W(\Phi)$ acts essentially.

Lemma 3.5 [5, Lemma 2.1]. *Let R be a reflection subgroup of W. Then R and its parabolic closure have the same rank.*

Definition 3.6. A *maximal-rank subsystem* of a root system Φ is a subsystem with rank equal to the rank of Φ.

Since we are concerned here with the classification in the cuspidal cases, by Corollary 2.5 and Lemma 3.5, we need only consider the maximal-rank reflection subgroups of W since they will have W as their parabolic closure. Hence, (i) of the Borel–de Siebenthal algorithm will not be needed. Let W be an irreducible Weyl group with root system Φ and dual system $\Phi^\vee$, so $W = W(\Phi) = W(\Phi^\vee)$. We use the following property of maximal subsystems of Weyl groups.

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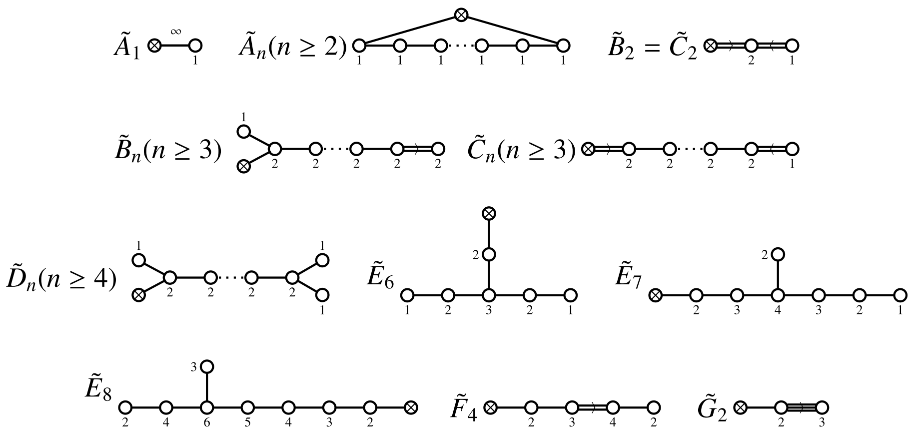


FIGURE 1. Extended Dynkin diagrams with weights.

LEMMA 3.7 [6, Corollary 2]. *If Ψ is a maximal subsystem of Φ, then either Ψ is closed in Φ or $\Psi^\vee$ is closed in $\Phi^\vee$.*

Carter [4] attributes the following property of maximal-rank closed subsystems to Dynkin [7, page 146] who states it in terms of subalgebras of simple Lie algebras.

LEMMA 3.8 [4, Proposition 32]. *Any two maximal-rank closed subsystems are of the same type if and only if they are W-conjugate.*

ALGORITHM 3.9. Let p be a cuspidal prime for the irreducible Weyl group W.

(1) Take the extended Dynkin diagram of W and remove a vertex with prime weight.
(2) Apply (1) to the dual of the Dynkin diagram of W and take the dual of the resulting diagram.
(3) Check if the p-adic valuation of the resulting diagrams are the same as $|W|$. If not, discard. Repeat steps (1) and (2) on each component of the remaining diagrams until no new diagram is found.
(4) Take the diagrams on which step (3) halts as the output.

THEOREM 3.10. *Let p be a cuspidal prime for the Weyl group W. Then the output of Algorithm 3.9 is the types of R_p.*

PROOF. By elementary maximality arguments it is clear that for any maximal-rank subsystem Ψ_1 of Φ, we have a finite chain $\Psi_1 \subsetneq \cdots \subsetneq \Psi_k \subsetneq \Phi$ such that Ψ_i is a maximal subsystem of Ψ_{i+1} for all $1 \leq i \leq k$, taking $\Psi_{k+1} = \Phi$. By Lemma 3.7, the chain is such that either Ψ_i is closed in Ψ_{i+1} or $\Psi_i^\vee$ is closed in $\Psi_{i+1}^\vee$ for $1 \leq i \leq k$. Hence, repeating steps (1) and (2) on each new subsystem will find all the subsystems of Φ. However, we do not need all subsystems, so in step (3) we filter the process so that we only follow chains that have group orders with the same p-adic valuation as $|W|$. Naturally, in step (4) we take the minimal subsystems which are the diagrams that step (3) is last applied to. □

TABLE 3. Type of R_p for cuspidal cases.

Type of W	p	Type of R_p
$A_{p^s-1}, s \in \mathbb{N}$	Any	A_{p^s-1}
$B_n, n \geq 2$	2	$\prod_{i=0}^{l} B_{2^i}^{b_i}$
$D_n, n \geq 4$	2	D_n
E_6	3	E_6
E_7	2	$A_1 \times D_6$
E_8	2	D_8
	3	$A_2 \times E_6$
	5	A_4^2
	7	A_6
F_4	2	B_4 and C_4
	3	$A_2 \times \tilde{A}_2$
G_2	2	$A_1 \times \tilde{A}_1$
	3	A_2 and $\tilde{A}_2$
H_3	2	A_1^3
H_4	2	D_4
	3	A_2^2
	5	$I_2(5)^2$
$I_2(m), m \geq 8$ with m even	2	$I_2(2^{k-1})$
$I_2(m), m = 5$ or $m \geq 7$	>2	$\begin{cases} I_2(p^k) \text{ and } \tilde{I}_2(p^k) & \text{for } m \text{ even,} \\ I_2(p^k) & \text{for } m \text{ odd} \end{cases}$

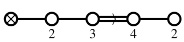
4. Classification of p-Sylow conjugacy classes of reflection subgroups

We begin this section by classifying R_p up to W-conjugacy for the cuspidal cases. We use Algorithm 3.9 for the Weyl group cases and the reflection subgroup classifications given in [8] for the noncrystallographic cases.

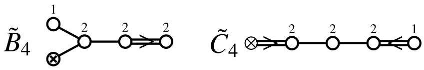
THEOREM 4.1. *The classification of R_p up to W-conjugacy in the cuspidal cases is given in Table 3.*

PROOF. Let W be a Weyl group. Then the classification of R_p up to type is just a corollary of Theorem 3.10. For illustrative purposes we include worked examples of applying Algorithm 3.9 when $p = 2$ to types F_4 and B_n with $n \geq 2$. We then explain how this algorithm leads to a classification up to W-conjugacy.

Let W have type F_4. Consider the extended Dynkin diagram of F_4 whose dual is also F_4:

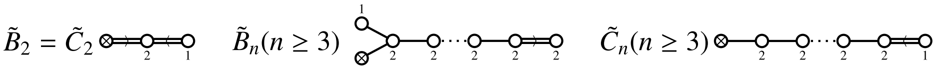
$\tilde{F}_4$ 

Applying steps (1) and (2), gives types $A_1 \times C_3$, B_4, $\tilde{A}_1 \times B_3$, C_4 and $A_2 \times \tilde{A}_2$. In step (3) we discard $A_1 \times C_3$, $\tilde{A}_1 \times B_3$ and $A_2 \times \tilde{A}_2$, leaving us with B_4 and C_4.



Applying steps (1) and (2) to B_4 and C_4 gives types $A_1^2 \times B_2$, $A_3 \times \tilde{A}_1$, D_4, $\tilde{A}_1 \times B_3$ and B_2^2 from B_4 and the duals from C_4. By step (3) of the algorithm we discard all of these. Finally, in step (4) we conclude there are two types of R_p in $W(F_4)$, namely B_4 and C_4.

Let W have type B_n with $n \geq 2$. We claim that R_2 has type $\prod_{i=0}^{l} B_{2^i}^{b_i}$, where the base 2 expression of n is $(b_l b_{l-1} \ldots b_1 b_0)_2$ and B_1 is interpreted as A_1. Consider the following extended Dynkin diagrams:



Applying step (1) of Algorithm 3.9 gives $B_{n-m} \times D_m$ for some $2 \leq m \leq n$, which reduces the 2-adic valuation of the group order, so we discard all of these in step (3). Note we interpret D_2 as $A_1 \times A_1$ and D_3 as A_3. Applying step (2) gives $B_{n-m} \times B_m$ for some $1 \leq m \leq n$, which preserves the 2-adic valuation if and only if the sum of $n - m$ and m in base 2 has no carries (by Kummer's theorem). Repeating steps (1)–(3) and using Lemma 2.6, we find that in step (4) there are 2-Sylow conjugacy classes of reflection subgroups of the form $\prod_{i=0}^{l} B_{2^i}^{b_i}$, where the base 2 expression of n is $(b_l b_{l-1} \ldots b_1 b_0)_2$.

Now we will settle the W-conjugacy of the types of R_p found for the cuspidal cases of the Weyl groups. In all cases other than B_n with $p = 2$, the type found from Algorithm 3.9 comes from one application of either step (1) or step (2). By the Borel–de Siebenthal Algorithm 3.4, each type of R_p, other than B_n with $p = 2$, has a unique W-conjugacy class. For the case of type B_n with $p = 2$ we claim there is a unique W-conjugacy class of R_p with type $\prod_{i=0}^{l} B_{2^i}^{b_i}$. When applying Algorithm 3.9, each new subsystem achieved from steps (1)–(3) stems from step (2). Hence, the chains are subsystems $\Psi_1 \subset \cdots \subset \Psi_k \subset \Phi$ such that $\Psi_i^\vee$ is closed in $\Psi_{i+1}^\vee$ for $1 \leq i \leq k$. By the definition of closed, $\Psi_1^\vee = (\prod_{i=0}^{l} B_{2^i}^{b_i})^\vee$ is closed in $B_n^\vee$ and, by Lemma 3.8, the groups of type $\prod_{i=0}^{l} B_{2^i}^{b_i}$ have a unique W-conjugacy class in $W(B_n^\wedge) = W(B_n)$.

Let W be of type H_3 or H_4. From [5, Table 8] or [5, Table 9], respectively, we find R_p up to W-conjugacy.

Let W be of type $I_2(m)$ with $m = 5$ or $m \geq 7$, where the p-adic valuation of $2m$ is k. We claim that $I_2(m)$ with $m = 5$ or $m \geq 7$ has the following classifications of R_p up to W-conjugacy for the cuspidal cases.

(i) When m is even and $p = 2$, there is a unique conjugacy class of R_p with type $I_2(2^{k-1})$.

(ii) When $p > 2$, there are two conjugacy classes of R_p with types $I_2(p^k)$ and $\tilde{I}_2(p^k)$.

By [5, Theorem 5.1], the reflection subgroups of $I_2(m)$ up to conjugacy are of types $\emptyset$, A_1 and $I_2(d)$ where $d > 1$ is a divisor of m. Note, we interpret $I_2(2)$ as $A_1 \times A_1$, $I_2(3)$

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as A_2, $I_2(4)$ as B_2 and $I_2(6)$ as G_2. Also note the order of the reflection group of type $I_2(m)$ is $2m$. Hence, the claim follows. □

From the classification of the cuspidal cases in Table 3, we can easily deduce R_p up to P_p-conjugacy for the noncuspidal cases since the R_p of W have the same type as the R_p of P_p. We wish to extend this to a classification of R_p up to W-conjugacy. If a type of R_p is unique up to P_p-conjugacy, then it is clear by the uniqueness of P_p up to W-conjugacy that this type of R_p is unique up to W-conjugacy.

Theorem 4.2. *The classification of the p-Sylow conjugacy classes of reflection subgroups in finite real reflection groups is as given in Table 4.*

Proof. We deduce R_p up to P_p-conjugacy from Tables 1 and 3, since the R_p of W in the noncuspidal cases have the same type as the R_p of P_p. From this, all the noncuspidal cases have each type of R_p unique up to P_p-conjugacy, and so unique up to W-conjugacy. □

Remark 4.3. When applying steps (1) and (2) in Algorithm 3.9 to the cuspidal cases of the Weyl groups, removing a vertex of prime weight not equal to the prime p will always result in a reflection subgroup with lower p-adic valuation. If we do not restrict ourselves to the cuspidal cases and apply step (ii) of the Borel–de Siebenthal Algorithm 3.4, we find that if the p-adic valuation is preserved when removing a vertex of prime weight not equal to the prime p, then the parabolic subsystem found by removing this same vertex and not adding on the negative of the highest root will also preserve the p-adic valuation.

5. Observations and *p*-Sylow subgroups in finite real reflection groups

In this section, we give some comments on the classifications found in Tables 1–4 and show how they relate to the p-Sylow subgroups of finite real reflection groups.

Corollary 5.1. *The cases when a p-Sylow conjugacy class of reflection subgroups is not unique occur in the groups of type F_4 when $p = 2$ and $I_2(m)$ when $m \geq 6$ is even but not a power of 2 for each odd prime divisor of m, including G_2 when $p = 3$.*

As with parabolic subgroups, the reflection subgroups minimally containing p-Sylow subgroups are the same when minimality is with respect to order instead of inclusion. This is because the R_p in the nonunique conjugacy classes have the same order.

A special case of our classification is when the p-Sylow subgroups S_p are reflection subgroups and so $S_p = R_p$. This can only happen when $p = 2$, since the order of any group containing reflections is even. We check the orders of the R_2 found in Table 4 to find the cases when the order is a power of 2. We do the same for P_2 found in Table 1.

Corollary 5.2. *The cases of W when the 2-Sylow conjugacy class of reflection subgroups is itself the 2-Sylow conjugacy class are in types A_1, A_2, B_2, G_2, H_3 and $I_2(m)$ for $m = 5$ and all $m \geq 7$.*

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TABLE 4. Type of R_p in W.

Type of W	$\lvert W\rvert$	p	Type of R_p	$\lvert R_p\rvert$
$A_{n-1}, n \geq 2$	$n!$	Any	$\prod_{i=1}^{l} A_{p^i-1}^{b_i}$	$\prod_{i=1}^{l}[(p^i)!]^{b_i}$
$B_n, n \geq 2$	$2^n n!$	2	$\prod_{i=0}^{l} B_{2^i}^{b_i}$	$2^n \prod_{i=0}^{l}[(2^i)!]^{b_i}$
		>2	$\prod_{i=1}^{l} A_{p^i-1}^{b_i}$	$\prod_{i=1}^{l}[(p^i)!]^{b_i}$
$D_n, n \geq 4$	$2^{n-1} n!$	2	D_n	$2^{n-1} n!$
		>2	$\prod_{i=1}^{l} A_{p^i-1}^{b_i}$	$\prod_{i=1}^{l}[(p^i)!]^{b_i}$
E_6	$2^7 \cdot 3^4 \cdot 5$	2	D_5	$2^7 \cdot 3 \cdot 5$
		3	E_6	$2^7 \cdot 3^4 \cdot 5$
		5	A_4	$2^3 \cdot 3 \cdot 5$
E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	2	$A_1 \times D_6$	$2^{10} \cdot 3^2 \cdot 5$
		3	E_6	$2^7 \cdot 3^4 \cdot 5$
		5	A_4	$2^3 \cdot 3 \cdot 5$
		7	A_6	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	2	D_8	$2^{14} \cdot 3^2 \cdot 5 \cdot 7$
		3	$A_2 \times E_6$	$2^8 \cdot 3^5 \cdot 5$
		5	A_4^2	$2^6 \cdot 3^2 \cdot 5^2$
		7	A_6	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
F_4	$2^7 \cdot 3^2$	2	B_4 and C_4	$2^7 \cdot 3$
		3	$A_2 \times \tilde{A}_2$	$2^2 \cdot 3^2$
G_2	$2^2 \cdot 3$	2	$A_1 \times \tilde{A}_1$	2^2
		3	A_2 and $\tilde{A}_2$	$2 \cdot 3$
H_3	$2^3 \cdot 3 \cdot 5$	2	A_1^3	2^3
		3	A_2	$2 \cdot 3$
		5	$I_2(5)$	$2 \cdot 5$
H_4	$2^6 \cdot 3^2 \cdot 5^2$	2	D_4	$2^6 \cdot 3$
		3	A_2^2	$2^2 \cdot 3^2$
		5	$I_2(5)^2$	$2^2 \cdot 5^2$
$I_2(m), m = 5$ or $m \geq 7$	$2m$	2	$\begin{cases} I_2(2^{k-1}) & \text{for } m \text{ even,} \\ A_1 & \text{for } m \text{ odd} \end{cases}$	2^k
		>2	$\begin{cases} I_2(p^k) \text{ and } \tilde{I}_2(p^k) & \text{for } m \text{ even,} \\ I_2(p^k) & \text{form odd} \end{cases}$	$2 \cdot p^k$

COROLLARY 5.3. *The cases of W when the 2-Sylow conjugacy class of parabolic subgroups is itself the 2-Sylow conjugacy class are in types A_1, A_2, B_2 and $I_2(m)$ where $m \geq 7$ is odd or a power of 2.*

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When the prime p is not cuspidal for W and so R_p does not have maximal-rank, the p-Sylow conjugacy classes of reflection subgroups is in fact the unique p-Sylow conjugacy class of parabolic subgroups.

COROLLARY 5.4. *If $W \neq P_p$, then $P_p = R_p$.*

Our classification of p-Sylow conjugacy classes of reflection subgroups leads to a significant reduction in the cases required to describe the p-Sylow subgroups of finite real reflection subgroups. For example, from our classification for the reflection group of type E_8, the p-Sylow subgroups of $W(E_8)$ are the same as the p-Sylow subgroups of its reflection subgroups $W(D_8)$ for $p = 2$, $W(A_2) \times W(E_6)$ for $p = 3$, $W(A_4)^2$ for $p = 5$ and $W(A_6)$ for $p = 7$. Checking all the cases with $W = R_p$ gives the next result.

COROLLARY 5.5. *The p-Sylow subgroups of finite real reflection groups are products of the p-Sylow subgroups of A_{p^i-1} for all primes p where $i \in \mathbb{N}$, B_{2^j} for $p = 2$ where $j \in \mathbb{N}$, D_n for $p = 2$, E_6 for $p = 3$ and $I_2(p^k)$ for all primes p and $k \in \mathbb{N}$.*

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KANE DOUGLAS TOWNSEND, Sydney, Australia
e-mail: kane.d.townsend@gmail.com