THE SERIES $\Sigma_1^\infty f(n)/n$ FOR PERIODIC f

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1. <u>The Problem Posed.</u> We are here concerned with the problem of deciding when $\Sigma_{n=1}^{\infty} f(n)/n \neq 0$, given that f is periodic and the series convergent. In particular, we consider

CONJECTURE A. *Let* p *be a positive integer and* f *a (real- or complex-valued) number-theoretic function with period* p. *If* $f(n) \neq 0$ *for some positive integer* n, *then* $\Sigma_{n=1}^{\infty} f(n)/n \neq 0$ *whenever the series is convergent.*

The problem in this form was posed by S. Chowla in an address delivered before the Annual Meeting of the American Mathematical Society in 1949 and appeared subsequently as one of fourteen unsolved problems in number theory in the published version of that address [1, p. 300]. He (incorrectly) attributed Conjecture A to Paul Erdős. (See Section 5 below for Erdős' conjecture.)

A positive resolution of Conjecture A would include several results in number theory whose known proofs are decidedly non-trivial. In this connection, Chowla cites the Dirichlet formula

$$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right)/n \neq 0 ,$$

where $\left(\frac{n}{p}\right)$ is the Legendre "quadratic character" symbol (defined to be 0 when $p|n$), which is a special case of

THEOREM A [2, p. 93]. *If* p *is a positive integer and* χ *a non-principal character modulo* p, *then*

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$$\sum_{n=1}^{\infty} \chi(n)/n \neq 0 ,$$

the series being convergent.

This Theorem is the crucial step in the known proofs of Dirichlet's

THEOREM B [2, p. 96]. *If* a *and* b *are positive integers for which* $(a, b) = 1$, *then there are infinitely many primes in the sequence* $\{a+nb\}_{n=1}^{\infty}$.

Observe that the requirement of convergence for the series in Conjecture A is equivalent to the condition $\Sigma_{n=1}^{p} f(n) = 0$. It is then clear that when $p = 2$, the only functions under consideration are multiples of $(-1)^{n+1}$. In this case, Conjecture A is obviously true, the series in question being a multiple of $\ln 2$. We formalize this as

THEOREM 1.1. *Conjecture A is true when* $p = 2$.

One of the purposes of this note is to show that Theorem 1.1 is all that can be said. Indeed, we show that Conjecture A is false for every $p > 2$ (Corollary 2.1). Admittedly the counter-examples we exhibit do not have all the structure of characters modulo p, though in certain cases they are multiplicative. They do suggest, however, that Theorem A is sufficiently deep that an argument of the complexity of Dirichlet's cannot be avoided.

Since Conjecture A is generally false, is it possible that it will become true under sufficiently mild further restrictions on f? For example, can we get by with the additional assumption that $f(p-n) = -f(n)$, which is the case for $f(n) = (\frac{n}{p})$ when p is a prime of the form $4k+3$ [3, p. 38]? In particular, Chowla has (informally) posed [1, p. 300]

CONJECTURE B. *Let* p *be a positive integer and* f

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a (real- or complex-valued) number-theoretic function with period p. *If* $f(p-n) = -f(n)$ $(n = 1, 2, \ldots, p-1)$ *and* $f(n) \neq 0$ *for some positive integer* n, *then* $\sum_{n=1}^{\infty} f(n)/n \neq 0$ *whenever the series is convergent.*

In the case when $p = 2$, it is evident that the only number-theoretic function for which $f(n+2) = f(n)$ and $f(2-n) = -f(n)$ is the zero function; hence Conjecture B is (vacuously) true when $p = 2$. For $p = 3$ and $p = 4$, it is easily seen that the only functions under consideration in Conjecture B are multiples of $f(n) = \sin(2n\pi/3)$ and $f(n) = \sin(n\pi/2)$, respectively. The Conjecture is then obviously true for these cases; formally,

THEOREM 1.2. *Conjecture* B *is true for* $p \leq 4$.

Chowla claims to have proved the truth of Conjecture B under the additional assumption that p and $(p-1)/2$ are primes. "Professor Siegel, to whom I showed my proof, proved the result" in the form of Conjecture B[1, p. 300]. Apparently they implicitly imposed further restrictions on f, for we show that, as a matter of fact, Conjecture B is false for $p > 3$ (Corollary 2.2).

The number-theoretic functions of most interest are multiplicative. In view of the results already mentioned, we are naturally led to

CONJECTURE C. *Let* p *be a positive integer and* f *a (real- or complex-valued) multiplicative number-theoretic function of period* p. *If* $f(n) \neq 0$ *for some positive integer* n, *then* $\sum_{n=1}^{\infty} f(n)/n \neq 0$ *whenever the series is convergent.*

But even this conjecture, though true for prime periods p (Theorem 3.1), is almost always false (Corollary 3.2).

Having come this far, we might just as well combine the assumptions of Conjectures B and C and set forth

CONJECTURE D. *Let* p *be a positive integer and* f *a (real- or complex-valued) multiplicative number-theoretic function of period* p, *for which* $f(p-n) = -f(n)$ $(n = 1, 2, \ldots, p-1)$.

If $f(n) \neq 0$ *for some* n , *then* $\Sigma_{n=1}^{\infty} f(n)/n \neq 0$ *whenever the series is convergent.*

Again, this Conjecture suffers the same fate as the earlier ones: It is sometimes true, sometimes false (Theorems 4.1 and 4.2).

REMARK. It is perfectly clear that Theorems 1.1 and 1.2 are equally valid for series $\Sigma_{n=1}^{\infty} a_n f(n)$ if the sequence $\{a_n\}_1^{\infty}$ is sufficiently manageable; for example, if $a_{n+1} < a_n$ $(n = 1, 2, 3, \ldots)$ and $\lim_{n\to\infty} a_n = 0$. We see no point in formalizing these results.

2. *Conjectures A and B.* A number-theoretic function is nothing more nor less than a (real- or complex-valued) sequence. Consequently, with the exception of Corollaries 2.1 and 2.2, we formulate our statements in terms of sequences.

We need the more or less obvious

LEMMA 2.1. *Let* $\{a_n\}_1^{\infty}$ *and* $\{b_n\}_1^{\infty}$ *be (real- or complex-valued) sequences and* p *a positive integer greater than* 1. *If* $\lim_{n\to\infty} a_n = 0$,

$$\alpha_r = \sum_{n=0}^{\infty} \{a_{np+r} - a_{np+r+1}\} \qquad (r = 1, 2, \ldots, p-1)$$

is convergent, $b_{n+p} = b_n$ $(n = 1, 2, 3, \ldots)$, *and* $\Sigma_{k=1}^{p} b_k = 0$, *then*

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{p-1} \alpha_k B_k$$

where $B_k = \Sigma_{j=1}^{k} b_j$.

Proof. It is clear that the periodicity of $\{b_n\}_1^{\infty}$ and the condition $\Sigma_{k=1}^{p} b_k = 0$ imply that $B_{mp+k} = B_k$ for $k = 1, 2, \ldots, p$ and $m = 0, 1, 2, \ldots$. Writing $\Delta a_j = a_j - a_{j+1}$, we therefore obtain

$$\sum_{n=1}^{mp+r} B_n \Delta a_n = \sum_{k=1}^{p-1} B_k \sum_{j=0}^{m-1} \Delta a_{jp+r} + \sum_{k=1}^{r} B_k \Delta a_{jp+k}$$

by expressing the sum on the left as a sum over intervals of "length" p plus a residual, making the obvious change of summing index, and using the periodicity of $\{B_n\}_1^\infty$ and the fact that $B_p = 0$. Summing by parts, we then find that

$$\sum_{n=1}^{mp+r} a_n b_n = a_{mp+r+1} B_{mp+r} + \sum_{n=1}^{mp+r} B_n \Delta a_n$$

$$= a_{mp+r+1} B_r + \sum_{k=1}^{p} B_k \sum_{j=0}^{m-1} \Delta a_{jp+k} + \sum_{k=1}^{r} B_k \Delta a_{jp+k}$$

$$\to \sum_{k=1}^{p} \alpha_k B_k = \sum_{k=1}^{p-1} \alpha_k B_k$$

as $m \to \infty$ $(1 \leq r \leq p)$, since a_n, and hence Δa_n, tend to zero as $n \to \infty$.

THEOREM 2.1. *Let $\{a_n\}_1^\infty$ be a (real- or complex-valued) sequence and p a positive integer greater than* 2. *If $\lim_{n\to\infty} a_n = 0$ and*

$$\alpha_r = \sum_{n=0}^{\infty} (a_{np+r} - a_{np+r+1}) \qquad (r = 1, 2, \ldots, p-1)$$

is convergent, then there is a non-zero (real- or complex-valued) sequence $\{b_n\}_1^\infty$ for which $b_{n+p} = b_n$ $(n = 1, 2, 3, \ldots)$,

$\Sigma_{n=1}^{p} b_n = 0$ and $\Sigma_{n=1}^{\infty} a_n b_n = 0$.

Proof. Take any vector $(B_1, B_2, \ldots, B_{p-1}) \neq (0, 0, \ldots, 0)$ which is perpendicular to $(\alpha_1, \alpha_2, \ldots, \alpha_{p-1})$ in $(p-1)$-space. (There are at least $p-2$ linearly independent such vectors.) Next, define $\{b_n\}_1^{\infty}$ in the obvious way: $b_1 = B_1$, $b_p = -B_{p-1}$,

$$b_k = B_k - B_{k-1} \quad (k = 2, 3, \ldots, p-1),$$

and

$$b_{n+p} = b_n \qquad (n = 1, 2, 3, \ldots).$$

(In the complex case, $\bar{B}_n$ is used in place of B_n in prescribing b_n.) It is immediately evident that $b_n \neq 0$ for at least one n. Now appeal to Lemma 2.1.

COROLLARY 2.1. *Conjecture A is false for* $p > 2$: *Given a positive integer* p *greater than* 2 , *there is a non-trivial number-theoretic function* f *with period* p *for which* $\Sigma_{n=1}^{\infty} f(n)/n = 0$.

Proof. Take $a_n = 1/n$ in Theorem 2.1, and set $f(n) = b_n$.

LEMMA 2.2. *If* p *is a positive integer greater than* 1 *and* $b_1, b_2, \ldots, b_p$ *are (real or complex) numbers for which* $\Sigma_{k=1}^{p} b_k = 0$, *then*

$$b_{p-j} = -b_j \qquad (j = 1, 2, \ldots, p-1)$$

if and only if $b_p = 0$ *and*

$$B_k = B_{p-k-1} \qquad (k = 1, 2, \ldots, p-2) ,$$

where $B_k = \Sigma_{n=1}^{k} b_n$, *in which case* $b_{p/2} = 0$ *if* p *is even.*

Proof. For the "only if", we have

$$0 = B_p = B_k + \sum_{n=k+1}^{p} b_k = B_k + b_p - \sum_{n=k+1}^{p-1} b_{p-n}$$

$$= B_k + b_p - \sum_{m=1}^{p-k-1} b_m = B_k + b_p - B_{p-k-1}$$

and, hence,

$$B_k + b_p = B_{p-k-1} \qquad (k = 1, 2, \ldots, p-2) .$$

For $p = 2q+1$, it follows that $B_q + b_p = B_q$ and, hence, that $b_p = 0$. If $p = 2q$, we find that

$$B_{q-1} + b_p = B_q \qquad \text{and} \qquad B_q + b_p = B_{q-1} ,$$

whence $B_q - B_{q-1} = 0$ and, then, $b_p = 0$.

For the converse,

$$b_k = B_k - B_{k-1} = B_{p-k-1} - B_{p-k} = - b_{p-k}$$

for $k = 2, 3, \ldots, p-2$, while

$$b_1 = B_1 = B_{p-2} = B_{p-1} - b_{p-1} = - b_{p-1}$$

($B_{p-1} = 0$ since $b_p = B_p = 0$!).

THEOREM 2.2. *Let* p *be a positive integer greater than* 4 *and* $\{a_n\}_1^{\infty}$ *a real-valued sequence for which* $\lim_{n\to\infty} a_n = 0$ *and*

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$$\alpha_k = \sum_{n=0}^{\infty} (a_{np+k} - a_{np+k+1})$$

is convergent $(k = 1, 2, \ldots, p-1)$. Then there is a non-zero (real- or complex-valued) sequence $\{b_n\}_1^\infty$ for which $b_{n+p} = b_n$ $(n = 1, 2, 3, \ldots)$, $b_{p-k} = -b_k$ $(k = 1, 2, \ldots, p-1)$, $\Sigma_{j=1}^{p} b_j = 0$, and $\Sigma_{n=1}^{\infty} a_n b_n = 0$.

Proof. In view of the assumptions on $\{b_n\}_1^\infty$, Lemmas 2.1 and 2.2 tell us that $B_k = B_{p-k-1}$ $(k = 1, 2, \ldots, p-2)$, $b_p = B_{p-1} = 0$, and $\Sigma_{n=1}^{\infty} a_n b_n = \Sigma_{k=1}^{p-2} \alpha_k B_k$, where $B_k = \Sigma_{j=1}^{k} b_j$. If p is even, then

$$\sum_{k=p/2}^{p-2} \alpha_k B_k = \sum_{n=1}^{(p-2)/2} \alpha_{p-n-1} B_{p-n-1} = \sum_{n=1}^{(p-2)/2} \alpha_{p-n-1} B_n ,$$

while, for p odd,

$$\sum_{k=(p+1)/2}^{p-2} \alpha_k \beta_k = \sum_{n=1}^{(p-3)/2} \alpha_{p-n-1} B_n .$$

Consequently,

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{[\frac{p-1}{2}]} \beta_k B_k$$

with $\beta_k = \alpha_k + \alpha_{p-k-1}$ for $k = 1, 2, \ldots, [(p-3)/2]$ and

$$\beta_{[(p-1)/2]} = \begin{cases} \alpha_{[(p-1)/2]} + \alpha_{[(p+1)/2]}, & p \text{ even} , \\ \alpha_{[(p-1)/2]} , & p \text{ odd} . \end{cases}$$

The construction of a sequence $\{b_n\}_1^\infty$ satisfying the requirements of the Theorem proceeds somewhat as in Theorem 2.1. Letting $q = [(p-1)/2]$, choose any vector $(B_1, B_2, \ldots, B_q) \neq (0, 0, \ldots, 0)$ which is perpendicular to $(\beta_1, \beta_2, \ldots, \beta_q)$ in (real or complex) q-space. (There are at least $q-1$ linearly independent such vectors.) Then define $B_{q+1}, B_{q+2}, \ldots, B_{p-2}$ by means of the formula $B_k = B_{p-k-1}$, set $B_{p-1} = B_p = 0$, and prescribe $b_1, b_2, b_3, \ldots$ in the obvious way (using $\bar{B}_n$ instead of B_n for the complex case). It is apparent that not all the b_n's are zero. Finally, Lemma 2.2 guarantees that $b_{p-k} = -b_k$ $(k = 1, 2, \ldots, p-1)$.

COROLLARY 2.2. <u>Conjecture B is false for</u> $p > 4$: <u>Given a positive integer</u> $p > 4$, <u>there is a non-trivial number-theoretic function</u> f <u>with period</u> p <u>for which</u> $f(p-n) = -f(n)$ $(n = 1, 2, \ldots, p-1)$ <u>and</u> $\Sigma_{n=1}^\infty f(n)/n = 0$.

<u>Proof.</u> Take $a_n = 1/n$ in Theorem 2.2 and set $f(n) = b_n$.

3. <u>The Multiplicative Case.</u> This section is devoted to the discussion of Conjecture C. The number-theoretic functions of interest are accordingly multiplicative: If a and b are positive integers for which $(a, b) = 1$, then $f(ab) = f(a) f(b)$.

LEMMA 3.1. <u>Let</u> p <u>be a positive integer and</u> f <u>a (real- or complex-valued) multiplicative number-theoretic function with period</u> p. <u>Then</u>

(i) $f(ab) = f(a) f(b)$ <u>whenever</u> $(a, p) = 1$ or $(b, p) = 1$.

(ii) <u>If</u> $f(p) \neq 0$, <u>then</u> $f(a) = 1$ <u>whenever</u> $(a, p) = 1$.

<u>Proof.</u> Suppose that a and b are positive integers with $(a, p) = 1$. For a quick proof of the first assertion, Theorem B tells us that $\{a + np\}_{n=1}^\infty$ contains arbitrarily large primes and, hence, that $(a + np, b) = 1$ for some positive

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integer n. Since f has period p,

$$f(ab) = f\{(a+np)b\} = f(a+np)\, f(b) = f(a)\, f(b),$$

which is what we wished to prove.

Leo Moser points out that the very deep Theorem B can be avoided in the verification of this first assertion, and we reproduce (essentially) his very simple argument here.

Let β denote a prime divisor of b. If $\beta \mid p$ and $\beta \mid (a+np)$, then $\beta \mid (a,p)$. Since $(a,p) = 1$, it follows that $(a+np, \beta) = 1$ for each common prime divisor of b and p and each positive integer n. For the remaining prime divisors of b (if there are any), define π by $p\pi \equiv 1 \pmod{\prod_{\beta \nmid p} \beta}$. By the Chinese Remainder Theorem, there is a positive integer n such that $n \equiv \pi(1-a) \pmod{\beta}$, $\beta \nmid p$. It follows that $a + pn \equiv 1 \pmod{\beta}$ or $(a+pn, \beta) = 1$ for each such β. But then $(a+np, b) = 1$ for this n.

The second assertion of Lemma 3.1 follows easily from the first and the periodicity of f.

THEOREM 3.1. <u>Conjecture C is true when</u> p <u>is a prime.</u>

<u>Proof.</u> Our assumptions are that f is multiplicative with period p, p is a prime, $f(n) \neq 0$ for some positive integer n, and $\sum_{k=1}^{p} f(k) = 0$.

If $f(p) \neq 0$, Lemma 3.1 (ii) tells us that $f(k) = 1$ for $k = 1, 2, \ldots, p-1$. Since $\sum_{k=1}^{p} f(k) = 0$, it follows that $f(p) = 1-p$. It is then apparent that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{k=0}^{\infty} \left(\frac{1}{pk+1} + \frac{1}{pk+2} + \ldots + \frac{1}{pk+p-1} - \frac{p-1}{pk+p}\right) > 0 .$$

If $f(p) = 0$, then $f(a) = 0$ whenever $(a,p) = 1$ because of the periodicity of f and the assumption that p is a prime, while $f(1) \neq 0$ because of the multiplicativity. This and

Lemma 3.1 (i) show that f is a character modulo p [3, p. 83]. Finally, the condition $\Sigma_{k=1}^{p} f(k) = 0$ guarantees that f is not the principal character [3, p. 84]. Now, appeal to Theorem A.

(It is unfortunate that our proof of Theorem 3.1 depends upon Theorem A. If it did not, we would of course have a new proof of Theorem A for the case where p is a prime. What has actually been proved is that Conjecture C and Theorem A are equivalent when p is a prime.)

LEMMA 3.2. *Suppose that* $p = \Pi_{k=1}^{r} p_i^{\pi_i}$ *with* $p_1, p_2, \ldots, p_r$ *distinct (when* $r > 1$*) primes and* $\pi_1, \pi_2, \ldots, \pi_r$ *positive integers. For* $0 < a \leq p$, *let*

$$f(a) = \begin{cases} 0 & \text{if } (a,p) = \prod_{i=1}^{r} p_i^{\alpha_i} \text{ and } 0 < \alpha_j < \pi_j \text{ for some } j, \\ \prod_{p_i \mid a} x_i & \text{if } (a,p) = \prod_{i=1}^{r} p_i^{\alpha_i} \text{ and } \alpha_j = 0 \text{ or } \pi_j \text{ for each } j, \end{cases}$$

where $x_1, x_2, \ldots, x_r$ *are any (real or complex) numbers (and the empty product has its usual meaning); define* $f(n+p) = f(n)$ *for* $n = 1, 2, 3, \ldots$. *Then* f *is a non-zero multiplicative number-theoretic function with period* p.

Proof. That f is a non-zero number-theoretic function with period p is clear, so we concentrate on the multiplicativity. To this end, suppose that a and b are positive integers and that A, B, and C are the least positive residues modulo p of a, b, and ab, respectively. Then

$$(A,p) = \prod_{i=1}^{r} p_i^{\alpha_i}, \quad (B,p) = \prod_{i=1}^{r} p_i^{\beta_i}, \quad (C,p) = \prod_{i=1}^{r} p_i^{\alpha_i+\beta_i}$$

with $\alpha_i \beta_i = 0$.

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Case 1: $0 < \alpha_j < \pi_j$ for some j. Since $\alpha_i\beta_i = 0$ for each i, $0 < \alpha_j + \beta_j = \alpha_j < \pi_j$, so that $f(A) = f(C) = 0$ and, hence,

$$f(ab) = f(C) = 0 = f(A) = f(A)f(B) = f(a)f(b) .$$

A similar argument subsists if $0 < \beta_k < \pi_k$ for some k.

Case 2: $\alpha_i = 0$ or π_i, and $\beta_i = 0$ or π_i for each i. Then $\alpha_i + \beta_i = 0$ or π_i because $\alpha_i\beta_i = 0$, and

$$f(C) = \prod_{p_i | C} x_i = \Big(\prod_{p_i | A} x_i\Big) \prod_{p_i | B} x_j = f(A)\, f(B) .$$

This completes the proof that f is multiplicative.

THEOREM 3.2. Let p be a composite positive integer and $\{a_n\}_1^\infty$ a real-valued sequence for which $\lim_{n\to\infty} a_n = 0$ and

$$\alpha_k = \sum_{n=0}^{\infty} (a_{np+k} - a_{np+k+1})$$

is convergent $(k = 1, 2, \ldots, p-1)$. If α_k does not change sign for $k = 1, 2, \ldots, p-1$, then there is a non-zero multiplicative number-theoretic function f with period p for which $\Sigma_{n=1}^{\infty} a_n f(n) = 0$.

Proof. If $\alpha_1 = \alpha_2 = \ldots = \alpha_{p-1} = 0$, we may take f to be any multiplicative function of period p (in particular, any character modulo p) since $\Sigma_{n=1}^{\infty} a_n f(n) = \Sigma_{k=1}^{p-1} \alpha_k F(k) = 0$ by Lemma 2.1, where $F(k) = \Sigma_{j=1}^{k} f(j)$. We therefore assume henceforth that $\alpha_i \neq 0$ for at least one i.

Case 1: $p = \Pi_{i=1}^{r} p_i^{\pi_i}$ with $r \geq 2$, $p_1, p_2, \ldots, p_r$ distinct primes, and $\pi_1, \pi_2, \ldots, \pi_r$ positive integers. Let f be the function defined in Lemma 3.2. According to Lemma 2.1., $\Sigma_{n=1}^{\infty} a_n f(n) = \Sigma_{k=1}^{p-1} \alpha_k F(k)$ if the series is convergent, where $F(k) = \Sigma_{j=1}^{k} f(j)$. The conditions $F(p) = 0$ and $\Sigma_{k=1}^{p-1} \alpha_k F(k) = 0$ become

$$\sum_{S_p} \Bigl(\sum_{\substack{k \leq p \\ (k,p) = \prod_{i=1}^{r} p_i^{\alpha_i}}} 1 \Bigr) \prod_{j=1}^{r} x_j^{\alpha_j/\pi_j} = 0 ,$$

$$\sum_{T_p} \Bigl\{ \sum_{n=1}^{p-1} \alpha_n \Bigl(\sum_{\substack{k \leq n \\ (k,p) = \prod_{i=1}^{r} p_i^{\alpha_i}}} 1 \Bigr) \Bigr\} \prod_{j=1}^{r} x_j^{\alpha_j/\pi_j} = 0 ,$$

where each α_i is 0 or π_i, $S_p = \{(\epsilon_1, \ldots, \epsilon_r) \mid \epsilon_i = 0 \text{ or } 1\}$, and $T_p = S_p - \{(1, \ldots, 1)\}$. The left members here are polynomials in $x_1, x_2, \ldots, x_r$, the first of degree r and the second of degree $r-1$. It follows that the equations have a solution. For example, taking $x_3 = x_4 = \ldots = x_r = 1$ gives a quadratic and a linear polynomial equation in x_1 and x_2.

Case 2: $p = q^{\alpha}$, q a prime and α a positive integer greater than 1. We define

$$f(a) = \begin{cases} 1 , & q \nmid a \\ x , & q \mid a \\ y , & a = p . \end{cases} \qquad (0 < a < p) ,$$

$$f(n+p) = f(n) \qquad (n = 1, 2, 3, \ldots) .$$

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That f is a non-zero number-theoretic function with period p is clear, and an argument somewhat redolent of that in the proof of Lemma 3.2 shows it to be multiplicative.

Set $F(n) = \Sigma_{k=1}^{n} f(k)$. The conditions $F(p) = 0$ and $\Sigma_{k=1}^{p-1} \alpha_n F(n) = 0$ become

$$\phi(p) + \{p-1-\phi(p)\} x+y = 0 ,$$

$$\sum_{n=1}^{p-1} \alpha_n \sum_{\substack{k\le n \\ (k,p)=1}} 1 + \Big(\sum_{n=1}^{p-1} \alpha_n \sum_{\substack{k\le n \\ (k,p)>1}} 1\Big)x = 0 ,$$

for which it is obvious that there is a solution (x, y). Now appeal to Lemma 2.1.

COROLLARY 3.2. <u>Conjecture C is false when p is composite: If p is composite, there is a non-trivial multiplicative number-theoretic function f with period p for which $\Sigma_{n=1}^{\infty} f(n)/n = 0$.</u>

<u>Proof.</u> Take $a_n = 1/n$ in Theorem 3.2.

4. <u>Odd Multiplicative Functions.</u> Our concern now is Conjecture D, so the functions f to be considered are multiplicative, have a positive integer period p, and satisfy $f(p-n) = -f(n)$ $(n = 1, 2, \ldots, p-1)$ and $\Sigma_{n=1}^{p} f(n) = 0$.

It is easy to see that $f(1) = f(3) = 1$, $f(2)$ and $f(4)$ arbitrary, and g a character modulo 4 give the only multiplicative number-theoretic functions with period 4. This observation, the condition $f(p-n) = -f(n)$ $(n = 1, 2, \ldots, p-1)$, and Theorem 3.1 prove

THEOREM 4.1. <u>Conjecture D is true when $p = 4$ or p is a prime.</u>

It is easy to construct examples for $p = 6, 9, 10, 14$, and 15 to demonstrate that Conjecture D is false for these composite periods. The author feels that this conjecture is indeed false for any composite period other than $p = 4$, but he is able to offer only one general result, namely,

THEOREM 4.2. Conjecture D is false when $p = 4q$, q an integer greater than 1: There is a non-trivial multiplicative number-theoretic function f with period $p = 4q$, $q > 1$, for which $f(p-n) = -f(n)$ $(n = 1, 2, \ldots, p-1)$ and

$$\Sigma_{n=1}^{\infty} f(n)/n = 0 .$$

Proof. For $0 < a \leq p = 4q$, define

$$f(q) = -f(3q) = x ,$$

$$f(a) = \begin{cases} (-1)^{\frac{a-1}{2}} & \text{if } (a, p) = 1 , \\ 0 & \text{if } (a, p) > 1 \text{ but } a \neq q \text{ and } a \neq 3q , \end{cases}$$

$$f(n+p) = f(n) \quad (n = 1, 2, 3, \ldots) .$$

The condition $f(p-n) = -f(n)$ is clear for $n = q$ and $n = 3q$. For $(n, p) = 1$, we have $(p-n, p) = 1$ and

$$f(p-n) = (-1)^{\frac{p-n-1}{2}} = (-1)^{\frac{4q-n-1}{2}} = (-1)^{\frac{n+1}{2}} = -(-1)^{\frac{n-1}{2}} = -f(n) ;$$

while if $(n, p) > 1$ but $n \neq q$ and $n \neq 3q$, then the same is true of $(p-n, p)$ and $n-p$, and so $f(p-n) = 0 = -f(n)$ even for this situation.

We show next that f is multiplicative. Suppose that $(a, b) = 1$, and let A, B, and C be the least positive residues of a, b, and ab, respectively. Writing $q = 2^{\pi} \Pi_{i=1}^{r} p_i^{\pi_i}$ where π is a non-negative integer, $p_1, p_2, \ldots, p_r$ are distinct (if $r > 1$) odd primes, and $\pi_1, \pi_2, \ldots, \pi_r$ are positive integers (the empty product is 1, as usual), we have

$$(A,p) = 2^{\alpha} \prod_{i=1}^{r} p_i^{\alpha_i}, \ (B,p) = 2^{\beta} \prod_{i=1}^{r} p_i^{\beta_i}, \ (C,p) = 2^{\alpha+\beta} \prod_{i=1}^{r} p_i^{\alpha_i+\beta_i}$$

with $\alpha\beta = \alpha_i\beta_i = 0$, $0 \leq \alpha, \beta \leq 2 + \pi$, and $0 \leq \alpha_i, \beta_i \leq \pi_i$.

Case 1: $0 < \alpha + \beta < \pi$ or $\alpha + \beta = \pi + 1$ or $\alpha + \beta = \pi + 2$ or $0 < \alpha_j + \beta_j < \pi_j$ for some j, or $\alpha + \beta = 0$ and $\alpha_i + \beta_i = \pi_i$ for each i , or $\alpha + \beta = \pi$ and $\alpha_i + \beta_i = 0$ for each i. Here $(C,p) > 1$ but $C \neq q$ and $C \neq 3q$, and the same is true of one of A and B. Referring to the definition of f, we see that f(C) and one of f(A) and f(B) is zero, so that $f(ab) = f(C) = 0 = f(A)f(B) = f(a)f(b)$.

Case 2: $\alpha + \beta = \alpha_i + \beta_i = 0$ $(i = 1, 2, \ldots, r)$. In this case, $(A,p) = (B,p) = (C,p) = 1$. Since A and B are odd, $(A-1)(B-1) \equiv 0 \pmod 4$. Thus, modulo 4, $0 \equiv AB - A - B + 1 \equiv C - A - B + 1 = C - 1 - (A + B - 2)$, and so

$$f(C) = (-1)^{\frac{C-1}{2}} = (-1)^{\frac{A+B-2}{2}} = (-1)^{\frac{A-1}{2}} (-1)^{\frac{B-1}{2}} = f(A)f(B) .$$

Case 3: $\alpha + \beta = \pi$ and $\alpha_i + \beta_i = \pi_i$ for $i = 1, 2, \ldots, r$. We have $C = q$ or $C = 3q$ and $(A,p) = q$ or $(B,p) = q$, but not both (since $(A,B) = 1$). We may assume that $(A,p) = q$ and, hence, that $A = q$ or $A = 3q$.

Take first the case where $C = q$. If $A = q$, then $AB = qB \equiv C = q \pmod 4$, so that $B \equiv 1 \pmod 4$.

Thus, $f(C) = x = x(-1)^{\frac{B-1}{2}} = f(A)f(B)$. On the other hand, if $A = 3q$, then $B \equiv 3 \pmod 4$, and $f(C) = x = (-x)(-1)^{\frac{B-1}{2}} = f(A)f(B)$.

A similar argument takes care of the situation when $C = 3q$, and the proof of the multiplicativity is now complete.

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Writing $B_n = \Sigma_{k=1}^{n} f(k)$, the requirement that $B_p = 0$ is automatic:

$$B_p = x + (-x) + \sum_{\substack{k=1 \\ (k,p)=1}}^{p} (-1)^{\frac{k-1}{2}} = \sum_{j=1}^{2q} (-1)^{k-1} = 0 .$$

We find that

$$B_n = \begin{cases} \displaystyle\sum_{\substack{k \le n \\ (k,p)=1}} (-1)^{\frac{k-1}{2}} & (1 \le n < q) \\ \displaystyle\sum_{\substack{k \le n \\ (k,p)=1}} (-1)^{\frac{k-1}{2}} + x & (q \le n < 2q) . \end{cases}$$

Referring to the proof of Theorem 2.2, we therefore have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n=1}^{2q-1} \beta_n B_n = \sum_{n=1}^{2q-1} \beta_n \sum_{\substack{k \le n \\ (k,p)=1}} (-1)^{\frac{k-1}{2}} + x \sum_{n=q}^{2q-1} \beta_n$$

with $\beta_n > 0$ $(n = 1, 2, \ldots, 2q-1)$. It is immediate that x may be chosen so that $\Sigma_{n=1}^{\infty} f(n)/n = 0$.

REMARK. It is of course clear that the same function (with x chosen suitably) will also make $\Sigma_{n=1}^{\infty} a_n f(n) = 0$ if $\lim_{n\to\infty} a_n = 0$ and $\alpha_k = \Sigma_{n=0}^{\infty} (a_{np+k} - a_{np+k+1})$ is convergent $(k = 1, 2, \ldots, p-1)$.

5. Erdős' Conjecture. Written communication with

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Erdős brought forth the statement: "If I remember correctly, when I made the conjecture, I assumed that $f(n) = \pm 1$ and $f(m) \equiv 0$ if $m \equiv 0 \pmod p$." Formally, then, we have the

CONJECTURE (Erdős). *If p is a positive integer and f is a number-theoretic function with period p for which $f(n) \in \{-1, 1\}$ when $n = 1, 2, \ldots, p-1$ and $f(p) = 0$, then $\sum_{n=1}^{\infty} f(n)/n \neq 0$ whenever the series is convergent.*

The author is unable to settle the truth status of this conjecture. About all that he can say is that Erdős' conjecture is true if

$$\pi,\ \ln(2 \sin \frac{\pi}{p}),\ \ln(2 \sin \frac{2\pi}{p}),\ \ldots,\ \ln(2 \sin \frac{(p-1)\pi}{2p})$$

when p is odd, and

$$\pi,\ \ln(2 \sin \frac{\pi}{p}),\ \ln(2 \sin \frac{2\pi}{p}),\ \ldots,\ \ln(2 \sin \frac{(p-2)\pi}{2p}),\ \ln 2$$

when p is even, are linearly independent over the algebraic numbers. (As a matter of fact, this linear independence would prove Erdős' Conjecture under the weaker assumption that $f(p) = 0$ and $f(n)$ is algebraic $(n = 1, 2, \ldots, p-1)$.)

To see this, recall that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right) = \gamma + \frac{1}{z} + \psi(z)$$

for $z \neq 0, -1, -2, \ldots,$ where γ is the Euler-Mascheroni constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ [4, p. 247]. It follows that

$$p \sum_{n=0}^{\infty} \left(\frac{1}{np+k} - \frac{1}{np+k+1}\right) = \psi\left(\frac{k+1}{p}\right) - \psi\left(\frac{k}{p}\right) \qquad (k = 1, 2, \ldots, p-1)$$

and, hence, that

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$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = F(p-1)\,\psi(1) - \sum_{k=1}^{p-1} f(k)\,\psi(\frac{k}{p}) .$$

But now

$$\psi(\frac{k}{p}) = \psi(1) - \frac{\pi}{2} \cot \frac{k\pi}{p} - \ln p + S_p(k) \qquad (k = 1, 2, \ldots, p-1) ,$$

where

$$S_p(k) = \sum_{j=1}^{[(p-1)/2]} \cos \frac{2kj\pi}{p} \ln(4 \sin^2 \frac{j\pi}{p}) + \frac{1+(-1)^p}{2} (-1)^k \ln 2$$

[2, pp. 34-35], so that

$$\sum_{n=1}^{\infty} f(n)/n = \frac{\pi}{2} \sum_{k=1}^{p-1} f(k) \cot \frac{k\pi}{p} + F(p-1) \ln p$$

$$- \sum_{k=1}^{p-1} f(k) \sum_{j=1}^{[(p-1)/2]} \cos \frac{2kj\pi}{p} \ln(4 \sin^2 \frac{j\pi}{p}) - T_p$$

with $T_p = 0$ if p is odd, and

$$T_p = (\ln 2) \sum_{k=1}^{p-1} (-1)^k f(k) \qquad (p \text{ even}) .$$

If now $f(p) = 0$ (and, of course, $F(p) = 0$), then the term involving $\ln p$ vanishes. Since the factors $\cot k\pi/p$ and $\cos 2kj\pi/p$ are algebraic, we see that then $\Sigma_{n=1}^{\infty} f(n)/n$ is an algebraic linear combination of π, $\ln(2 \sin \pi/p)$, $\ln(2 \sin 2\pi/p)$, ..., $\ln(2 \sin [(p-1)/2]\, \pi/p)$ (and $\ln 2$ if p is even).

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Alberta, and subsequently making the author aware of Chowla's paper [1].

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