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A remark on semi-stable fibrations over $\mathbb{P}^1$ in positive characteristic*

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Abstract. We prove a lower bound for the number of singular fibres of a non-isotrivial semi-stable family of curves over the projective line, in positive characteristic. The result is the same obtained previously by Beauville in characteristic zero, namely that the number is at least 4. Furthermore some consequences are discussed in the case that the number is exactly 4.

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This note is a supplement to my previous papers [N1–2] and deals with the following problem originally posed by Szpiro ([Sz1]): what is the minimum s_0 for the number s of singular fibres of a non-isotrivial semi-stable fibration of curves of genus $g \geqslant 1$ over $\mathbb{P}^1$? (The isotrivial case is easy, since a such fibration must be smooth in view of the uniqueness of the semi-stable model). All examples have $s \geqslant 4$ and there are examples $(g = 1)$ having 4 singular fibres. This suggests the estimate $s_0 = 4$. In characteristic zero it is known as Beauville's theorem ([B] – for finer results, see [N1–2]). The point in proving this estimate is that any abelian scheme over $\mathbb{P}^1$ in characteristic zero is constant (another proof is due to M.-H. Saito and also involves characteristic zero properties). In contrast in positive characteristic the constancy of families of abelian varieties over $\mathbb{P}^1$ is no longer true: such nonconstant families of abelian surfaces were constructed by Moret-Bailly (the so-called Moret-Bailly families; [M-B1-2]). For our use later one can see that these are the only special cases due to a remarkable result of Ekedahl ([E]). As noted in [B] the numerical part in the first step of proving $s_0 = 4$ remains valid in positive characteristic without difficulty. Hereafter we follow the notation and definition of [Sz2], [N2-4], e.g., g is the genus of the generic fibre, S is the set of critical values, s is the cardinality of S, $g(\widetilde{X}_t)$ is the genus of the normalization of the fibre X_t over t, etc.

LEMMA 1 (cf. [N2, A.4.1], [B]). *Let $f\colon X \to \mathbb{P}^1$ be a semi-stable fibration over an algebraically closed field. Then we have*

$$\rho_2 = (g - g_0)(s - 4) - \sum_{t \in S} (g(\widetilde{X}_t) - g_0) - r, \qquad (*)$$

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where ρ_2 *is the Lefschetz number,* $g_0 := \dim \mathrm{Pic}(X) = \dim \mathrm{Alb}(X)$ *and* r *is the so-called 'virtual' Mordell–Weil rank.*

The proof is similar to that given in [N2]. Below we just indicate the essential point in deducing (*). In fact the proof of A.3.1 in [N2] needs to be slightly modified to yield the same result formulated above as (*). First recall the definition of the Lefschetz number: $\rho_2 = b_2 - \rho$, where b_2 and ρ are the 2nd Betti number and the Picard number of X respectively. Further we remark that the equality $b_1(X) = 2g_0$ holds true for any characteristic (cf. [B-M]). So that one can write

$$\chi(X) = 2 - 4g_0 + b_2.$$

On the other hand it is easy to see that (A3), (A5) and (A10) of [N2] hold true without using characteristic zero properties. Combining all these together one obtains (*). It should be noted that formula (*) is a special case of the general formula of Ogg–Shafarevich–Grothendieck for families of curves which will be discussed in a future paper.

The aim of this note is to give a proof of the estimate $s_0 = 4$ in the positive characteristic case. More precisely, all the statements of [N2, A.4.2] remain valid in this case.

THEOREM 2. *Let* $f: X \to \mathbb{P}^1$ *be a nonisotrivial semi-stable fibration with* $g \geqslant 1$. *Then* $s \geqslant 4$. *Moreover,* $s = 4$ *is equivalent to the following three conditions*

(1) $\rho_2 = 0$, *i.e. the surface* X *is supersingular,*
(2) $g(\widetilde{X}_t) = g_0,\ \forall t \in S$,
(3) $r = 0$.

First note that one has $\rho_2 \geqslant 0$ (this fact is due to Igusa [I]). Further since $g(\widetilde{X}_t) \geqslant g_0$ and $r \geqslant 0$ (cf. [B], [N2]) the second part of the theorem is obvious in view of its first part and formula (*). It remains to establish $s \geqslant 4$. We first recall that the Kodaira–Spencer class K–S(f) can be defined as the tangent map to the induced map $\mathbb{P}^1 - S \to M_g$, or as in the classical manner due to Kodaira–Spencer. In general the Kodaira–Spencer class can be defined for a proper morphism of algebraic varieties $h: Y \to Z$ as the canonical section of the sheaf

$$\mathit{Hom}\,(h_*\Omega^1_{Y/Z}, R^1h_*(h^*\Omega^1_Z))$$

obtained by taking higher direct images of the following standard exact sequence:

$$0 \to h^*\Omega^1_Z \to \Omega^1_Y \to \Omega^1_{Y/Z} \to 0.$$

The important property in positive characteristic we shall use is that the Kodaira–Spencer class of a smooth morphism is zero iff it is either isotrivial, or the pull-back by the absolute Frobenius F of another (smooth) morphism (cf. [Sz1–2]).

It is easy to see that the statement in case $g = 1$ is trivial. We now consider the case $g \geqslant 2$ assuming $s \leqslant 3$. From Lemma 1 it follows that $g = g_0$, or the

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associated Jacobian fibration $j\colon \mathrm{Pic}^0(X/\mathbb{P}^1) \to \mathbb{P}^1$ is an abelian scheme. If we denote by $A_t = \mathrm{Jac}(X_t)$ for generic t and by Θ the tangent sheaf, then from the canonical homomorphism

$$H^1(X_t, \Theta_{X_t}) \cong H^0(X_t, \omega_t^{\otimes 2})^\vee \to T_{A_t,0} \otimes T_{A_t,0} \cong H^1(A_t, \Theta_{A_t})$$

one can see easily that K–S$(f) = 0$ implies K–S$(j) = 0$. Since $\mathrm{Pic}^0(X/\mathbb{P}^1)$ is nonconstant and smooth over $\mathbb{P}^1$, it is the pull-back by a power F^n of another (abelian) scheme over $\mathbb{P}^1$. By theorems of Raynaud and Deligne-Mumford ([D-M, Thms 2.4–2.5]) we conclude that $f\colon X \to \mathbb{P}^1$ is the pull-back by F^n of another semi-stable family over $\mathbb{P}^1$. Thus one reduces to the case K–S$(f) \neq 0$.

Next from [E, Sect. 3, Thm. 1.1, Prop. 2.1] it follows that either $f\colon X \to \mathbb{P}^1$ is the pull-back of a Moret-Bailly family (this is the case if $g = 2$), or X is of general type. Since a Moret-Bailly family has at least 5 singular fibres ([M-B1–2]) it remains to consider the second possibility which was essentially treated in [Sz2, Th. 4]. For the completeness we recall briefly Szpiro's argument in this case. The extension of K–S(f) over $\mathbb{P}^1$ gives us a nonzero homomorphism

$$T_{\mathbb{P}^1}(-S) \to R^1 f_* \omega_{X/\mathbb{P}^1}^{\otimes -1},$$

or by Leray's spectral sequence

$$H^1(X, \mathcal{L}^{\otimes -1}) \neq 0,$$

where $\mathcal{L} = \omega_{X/\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(2-s)$ and so in view of vanishing Theorem 2.1′ of [Sz1], $\mathcal{L}$ is not numerically positive. We get a contradiction: for X of general type $\mathcal{L}$ is numerically positive ([Sz2, Lemma 6]).

Remarks. (1) In the formulation of Theorem 2.5.1 of [N2] (the improved canonical class inequality announced in [N1]) the condition $s > 0$ was omitted (and was used implicitly in the proof). Using the opportunity here I would like to correct this inaccuracy. For further discussions of the smooth case see [N3].

(2) One can ask a question similar to Beauville's conjecture ([B], [N1–3]), or at least describe all non-isotrivial semi-stable fibrations over $\mathbb{P}^1$ with $s = 4$. This seems a very difficult problem because of various pathologies in positive characteristic. However the elliptic case with $s = 4$ is more or less easier to be handled (at least for characteristic > 3). It will appear somewhere else. Thus in this case one gets a series of unirational elliptic surfaces via pull-backs by powers of the absolute Frobenius *à la* Shioda (cf. [Sh]).

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