

STRICTLY n -FINITE VARIETIES OF HEYTING ALGEBRASTAPANI HYTTINEN , MIGUEL MARTINS, TOMMASO MORASCHINI , AND
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Abstract. For any $n < \omega$ we construct an infinite $(n + 1)$ -generated Heyting algebra whose n -generated subalgebras are of cardinality $\leq m_n$ for some positive integer m_n . From this we conclude that for every $n < \omega$ there exists a variety of Heyting algebras which contains an infinite $(n + 1)$ -generated algebra, but which contains only finite n -generated algebras. For the case $n = 2$ this provides a negative answer to a question posed by G. Bezhanishvili and R. Grigolia in [4].

§1. Introduction. A Heyting algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a bounded distributive lattice with a binary operation \rightarrow such that

$$a \wedge b \leq c \iff a \leq b \rightarrow c,$$

for every $a, b, c \in H$ [1, 7, 9, 16]. Heyting algebras appear naturally in many areas of mathematics. For instance, the lattice of open sets of a topological space forms a Heyting algebra. The subobject classifier of a topos can also be endowed with the structure of a Heyting algebra. Lastly, every distributive algebraic lattice is a Heyting algebra.

In this paper we focus on finitely generated Heyting algebras. We recall that an algebra A is said to be n -generated when there is a subset $X \subseteq A$ of size $\leq n$ such that the least subalgebra of A containing X is A itself. Accordingly, we say that A is *finitely generated* when it is n -generated for some $n < \omega$. A class of similar algebras that can be axiomatised by (universally quantified) equations is called a *variety*. Examples of varieties include the class of all Heyting algebras, as well as that of all Boolean algebras. A variety is said to be n -finite when its n -generated members are finite, and *locally finite* when it is n -finite for every $n < \omega$. We call a variety *strictly n -finite* if it is n -finite, but not $(n + 1)$ -finite.

Dual characterisations of finitely generated Heyting algebras were obtained in [10] (see also [5, Section 3.1]), while locally finite varieties of Heyting algebras were studied by G. Bezhanishvili and R. Grigolia in [4]. In the same paper, they raise the following question [4, Problem 2.4]: is it true that a variety of Heyting algebras is locally finite iff it is 2-finite? While this holds in the restrictive context of varieties of Heyting algebras of width two [2], in this paper we establish that for any $n < \omega$

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there exists an infinite $(n + 1)$ -generated Heyting algebra H_n whose n -generated subalgebras are of size $\leq m_n$ for some $m_n < \omega$. From this we deduce the main result of this article: for every $n < \omega$ the variety generated by H_n fails to be locally finite, although it is n -finite (Theorem 5.1). For $n = 2$ this provides a negative answer to Bezhanishvili and Grigolia's question.

From the viewpoint of propositional logic, the key importance of varieties of Heyting algebras is that they provide an algebraic semantics to the axiomatic extensions of the intuitionistic propositional calculus, i.e., *superintuitionistic logics*. As a consequence of this fact, properties of varieties of Heyting algebras correspond to properties of superintuitionistic logics, and vice versa. It thus follows immediately from the main result of this paper that, for every $n < \omega$, there exists a superintuitionistic logic with only finitely many formulas up to logical equivalence in n variables, but which has infinitely many non-equivalent formulas in $(n + 1)$ variables. In Remark 5.5 we additionally make explicit that this logic can be chosen to be finitely axiomatisable.

The structure of this article is based around the proof of Theorem 5.1. In Section 2 we recall the Esakia duality between the category of Heyting algebras with homomorphisms and the category of Esakia spaces with Esakia morphisms. In Section 3 we introduce the notion of colouring of a poset, which is the essential combinatorial tool in the proof of our main result. We believe that everything from Section 3 is essentially folklore, or a slight variation on the standard ideas from bisimulations and back-and-forth systems, but we provide details of every proof as the finitary version of these claims are not common in the literature. In Section 4 we introduce the counterexamples to the question of Bezhanishvili and Grigolia. We describe for every $n < \omega$ an Esakia Space X_n and we use the Colouring Theorem to show that its dual H_n is an infinite $(n + 1)$ -generated Heyting algebra, but every n -generated subalgebras of H_n is bounded in size by some $m_n < \omega$. Finally, in Section 5 we use standard arguments from universal algebra to establish the main result of the article (Theorem 5.1) and we point out some immediate corollaries.

This result was first established in 2020, although it never appeared in print [14]. Independently, the first and fourth authors discovered an alternative simpler proof in 2023 [13]. To make the result available, we decided to publish the latter together.

§2. Esakia duality. We review in this section the Esakia duality [8, 9] between Heyting algebras and Esakia spaces. We start by fixing some notation: whenever (X, \leq) is a poset and $Y \subseteq X$ we let

$$\uparrow Y = \{x \in X \mid \exists y \in Y \text{ and } y \leq x\} \quad \text{and} \quad \downarrow Y = \{x \in X \mid \exists y \in Y \text{ and } y \geq x\}.$$

For $x \in X$, we write $\uparrow x$ and $\downarrow x$ for the sets $\uparrow\{x\}$ and $\downarrow\{x\}$, respectively. A subset $Y \subseteq X$ is said to be an *upset* when $U = \uparrow U$. We write $\text{Up}(X)$ for the set of upsets of a poset X . Also, given any subset $Y \subseteq X$ we write Y^c for its complement $X \setminus Y$. If $x \in \downarrow U$, we often say that x *sees* U . We will always use the convention that the “arrow operators” defined above bind stronger than other set theoretic operations. For example, the expressions $\uparrow U \setminus V$ and $\downarrow U \cap V$ are to be read as $(\uparrow U) \setminus V$ and $(\downarrow U) \cap V$, respectively.

We recall that an *Esakia space* is a triple $X = (X, \tau, \leq)$, where (X, τ) is a compact topological space and (X, \leq) a poset satisfying the following conditions:

- (i) *Priestley separation axiom*: For all $x, y \in X$ such that $x \not\leq y$, there is a clopen upset U such that $x \in U$ and $y \notin U$.
- (ii) If U is clopen, then also $\downarrow U$ is clopen.

Given Esakia spaces X and Y , an *Esakia morphism* $p: X \rightarrow Y$ is a continuous map satisfying the two following conditions:

- (i) For all $x, y \in X$ if $x \leq y$, then $p(x) \leq p(y)$.
- (ii) For all $x \in X$ and $y \in Y$ such that $p(x) \leq y$, there exists $z \in X$ such that $x \leq z$ and $p(z) = y$.

Esakia duality is a dual categorical equivalence between the category of Heyting algebras with homomorphisms and the category of Esakia spaces with Esakia morphisms, which generalizes Stone duality. We shall review the definition of the two contravariant functors $(-)^*$ and $(-)_*$ witnessing Esakia duality. On the one hand, with every Heyting algebra H we associate an Esakia space H_* as follows. A *prime filter* F of H is a proper filter for which $x \vee y \in F$ entails $x \in F$ or $y \in F$. Then the Esakia space H_* is obtained by endowing the poset of prime filters of H ordered under the inclusion relation with the topology generated by the subbasis

$$\{\phi(a) \mid a \in H\} \cup \{\phi(a)^c \mid a \in H\},$$

where $\phi(a)$ is the set of prime filters of H containing a . Furthermore, every homomorphism $h : H \rightarrow H'$ between Heyting algebras is associated with the Esakia morphism $h_* : H'_* \rightarrow H_*$ defined as $h_*(F) = h^{-1}[F]$. On the other hand, with every Esakia space X we associate a Heyting algebra X^* as follows. Let $\text{ClUp}(X)$ be the set of clopen upsets of X . Then

$$X^* = (\text{ClUp}(X), \cap, \cup, \rightarrow, \emptyset, X),$$

where \rightarrow is defined by letting $U \rightarrow V = (\downarrow(U \setminus V))^c$. Furthermore, every Esakia morphism $p : X \rightarrow Y$ is associated with the homomorphism $p^* : Y^* \rightarrow X^*$ defined as $p^*(U) = p^{-1}[U]$.

§3. Poset colourability. We introduce in this section the colouring technique, which is the key combinatorial technique used in our subsequent proofs. The first item in the next definition exemplifies the idea of a *back-and-forth system*, and can also be seen as a version of the notion of layered bisimulation from [18]. The connection between back and forth systems and types goes back to Fraïssé [11].

REMARK 3.1. In this paper, we always identify natural numbers with finite ordinals, i.e., we identify each natural number $n < \omega$ with the set $\{m \in \omega \mid m < n\}$. In particular, we stress that when we write $n < \omega$, it may be the case that $n = 0$.

DEFINITION 3.2. Let X be a poset and $G \subseteq \text{Up}(X)$.

- (i) For every $n < \omega$ we define recursively an equivalence relation \sim_n^G on X as follows: for every $x, y \in X$,

$$x \sim_0^G y \iff \forall g \in G (x \in g \iff y \in g);$$

$$x \sim_{n+1}^G y \iff \forall z \geq x \exists v \geq y (z \sim_n^G v) \wedge \forall v \geq y \exists z \geq x (z \sim_n^G v).$$

Moreover, we consider the following equivalence relation on X :

$$\sim_\omega^G = \bigcap_{n \in \omega} \sim_n^G.$$

(ii) The n -type and the ω -type over G of an element $x \in X$ are, respectively, the sets

$$\{y \in X \mid x \sim_n^G y\} \text{ and } \{y \in X \mid x \sim_\omega^G y\}.$$

(iii) We say an element $x \in X$ is G -isolated if $x \sim_\omega^G y$ entails $x = y$.

(iv) We say that X is G -coloured (or coloured by G) if every element of X is G -isolated.

LEMMA 3.3. *Let X be a poset. For every $G \subseteq \text{Up}(X)$ and $n < \omega$, the equivalence relation \sim_{n+1}^G refines \sim_n^G .*

PROOF. We proceed by induction. For $n = 0$ we assume $x \sim_0^G y$. Without loss of generality there is some $g \in G$ such that $x \in g$ and $y \notin g$. Since g is an upset, it follows that $z \in g$ for every $z \geq x$ but $y \notin g$, showing $x \sim_1^G y$.

For $n = m + 1$ assume $x \sim_n^G y$. Without loss of generality there is some $z \geq x$ such that for all $v \geq y$ it holds $z \sim_{n-1}^G v$. Hence, by the induction hypothesis, we obtain $z \sim_n^G v$ and therefore $x \sim_{n+1}^G y$. □

By a *term* we understand a first-order term in the language of Heyting algebras.

DEFINITION 3.4. The *implication rank* $\text{rank}(\phi)$ of a term ϕ is defined as follows:

- (i) If ϕ is a constant or a variable, then $\text{rank}(\phi) = 0$;
- (ii) $\text{rank}(\psi \wedge \chi) = \max\{\text{rank}(\psi), \text{rank}(\chi)\}$;
- (iii) $\text{rank}(\psi \vee \chi) = \max\{\text{rank}(\psi), \text{rank}(\chi)\}$;
- (iv) $\text{rank}(\psi \rightarrow \chi) = \max\{\text{rank}(\psi), \text{rank}(\chi)\} + 1$.

Let H be a Heyting algebra. Given a subset $G \subseteq H$, we denote by $\langle G \rangle$ the subalgebra of H generated by G . We recall that the universe of $\langle G \rangle$ is

$$\{\phi^H(\vec{g}) \mid \vec{g} \in G \text{ and } \phi \text{ is a term}\},$$

where $\phi^H(\vec{g})$ is the interpretation of ϕ in H under the assignment \vec{g} .

DEFINITION 3.5. Let X be an Esakia space and $G = \{g_i \mid i < k\} \subseteq X^*$. The *implication rank* $\text{rank}(U)$ of an element $U \in \langle G \rangle$ is

$$\min\{\text{rank}(\phi) \mid \phi \text{ is a term such that } U = \phi^{X^*}(g_0, \dots, g_{k-1})\}.$$

In addition, with every U as above we associate a term $\phi_U(x_0, \dots, x_{k-1})$ such that $U = \phi_U^{X^*}(g_0, \dots, g_{k-1})$ and $\text{rank}(U) = \text{rank}(\phi)$.

The following lemma and the subsequent Colouring Theorem generalize [12, Proposition 34] and are essentially a reformulation of [5, Theorem 3.1.5]. The relation between the implication rank of a term and the existence of a back-and-forth system of corresponding length was established in [18, Theorems 4.7 and 4.8].

LEMMA 3.6. *Let X be an Esakia space and $x, y \in X$. The following condition holds for every finite $G \subseteq X^*$:*

$$x \sim_n^G y \iff \forall U \in \langle G \rangle \text{ with } \text{rank}(U) \leq n : (x \in U \iff y \in U).$$

PROOF. We fix an enumeration $G = \{g_i \mid i < k\}$ and let $\vec{g} = (g_0, \dots, g_{n-1})$. Moreover, given a term $\phi(x_0, \dots, x_{n-1})$, we will write $\phi(\vec{g})$ as a shorthand for $\phi^{X^*}(\vec{g})$. Both implications in the statement will be proven by induction on n .

(\Rightarrow) For the case where $n = 0$, suppose that $x \sim_0^G y$ and consider $U \in \langle G \rangle$ such that $\text{rank}(U) = 0$. Then we may assume that ϕ_U is a meet of joins of constants and variables. If $\phi_U = 0$ or $\phi_U = 1$, then $U = \emptyset$ or $U = X$ and the claim follows immediately. On the other hand, if $\phi_U = g_i$ for some $i < k$, we have that $x \in g_i$ if and only if $y \in g_i$ by the definition of \sim_0^G . As ϕ_U is a meet of joins of constants and variables, this implies that the claim holds.

Then we consider the case where $n = m + 1$. Suppose that $x \sim_{m+1}^G y$ and consider $U \in \langle G \rangle$ such that $\text{rank}(U) \leq m + 1$. First suppose that $\text{rank}(U) \leq m$. By Lemma 3.3 we have $x \sim_m^G y$ and, therefore, the claim holds by the induction hypothesis. Then we consider the case where $\text{rank}(U) = m + 1$. We may assume that ϕ_U is a conjunction of disjunctions of terms of the form $\alpha \rightarrow \beta$ of implication rank $\leq m + 1$ and with variables among x_0, \dots, x_{n-1} .

For each of these implications $\alpha \rightarrow \beta$, let

$$U_{\alpha\beta} := \alpha(\vec{g}) \rightarrow \beta(\vec{g}) = (\downarrow(\alpha(\vec{g}) \setminus \beta(\vec{g})))^c.$$

We will show that $x \in U_{\alpha\beta}$ if and only if $y \in U_{\alpha\beta}$. By symmetry, it suffices to prove the implication from right to left. Accordingly, suppose $x \notin (\downarrow(\alpha(\vec{g}) \setminus \beta(\vec{g})))^c$. Then there is some $z \geq x$ such that $z \in \alpha(\vec{g}) \setminus \beta(\vec{g})$ and, since $x \sim_{m+1}^G y$, there is some $v \geq y$ such that $z \sim_m^G v$. As $\alpha(\vec{g}), \beta(\vec{g}) \in \langle G \rangle$ and $\text{rank}(\alpha(\vec{g})), \text{rank}(\beta(\vec{g})) \leq m$, we can apply the induction hypothesis obtaining $v \in \alpha(\vec{g}) \setminus \beta(\vec{g})$. Hence, we conclude that $y \notin (\downarrow(\alpha(\vec{g}) \setminus \beta(\vec{g})))^c = U_{\alpha\beta}$.

Since U is a meet of joins of sets of the form $U_{\alpha\beta}$, the claim follows from the fact no $U_{\alpha\beta}$ separates x and y .

(\Leftarrow) For the case where $n = 0$, suppose that $x \approx_0^G y$. Without loss of generality, we may assume that there is some $g_i \in G$ such that $x \in g_i$ and $y \notin g_i$. Since $\text{rank}(g_i) = 0$, the claim follows immediately.

For the case where $n = m + 1$, suppose that $x \approx_{m+1}^G y$. We may assume that there is $z \geq x$ such that for all $v \geq y$ we have $z \approx_m^G v$. By the induction hypothesis, for every $v \geq y$, there is either $\psi_v(\vec{g}) \in \langle G \rangle$ such that $z \in \psi_v(\vec{g})$ and $v \notin \psi_v(\vec{g})$, or $\chi_v \in \langle G \rangle$ such that $z \notin \chi_v(\vec{g})$ and $v \in \chi_v(\vec{g})$, with $\text{rank}(\psi_v), \text{rank}(\chi_v) \leq m$. We let

$$I_0 := \{v \in \uparrow y \mid z \in \psi_v(\vec{g}) \text{ and } v \notin \psi_v(\vec{g})\};$$

$$I_1 := \{v \in \uparrow y \mid z \notin \chi_v(\vec{g}) \text{ and } v \in \chi_v(\vec{g})\}.$$

By construction we have $\uparrow y = I_0 \cup I_1$. Then we define

$$Z := \bigcap_{v \in I_0} \psi_v(\vec{g}) \rightarrow \bigcup_{v \in I_1} \chi_v(\vec{g}) = \left(\downarrow \left(\bigcap_{v \in I_0} \psi_v(\vec{g}) \setminus \bigcup_{v \in I_1} \chi_v(\vec{g}) \right) \right)^c.$$

Notice that by the previous direction the number of terms of rank $\leq m$ is finite, whence the intersections and unions above are finitary and thus Z is a well-defined element of $\langle G \rangle$. Furthermore, $\text{rank}(Z) \leq m + 1$ because each ψ_v and χ_v has implication rank $\leq m$. Therefore, to conclude the proof, it suffices to show that $x \notin Z$ and $y \in Z$.

Since for every $v \in I_0$ we have $z \in \psi_v(\vec{g})$ and for every $v \in I_1$ we have $z \notin \chi_v(\vec{g})$, it follows that $z \in \bigcap_{v \in I_0} \psi_v(\vec{g}) \setminus \bigcup_{v \in I_1} \chi_v(\vec{g})$. As $x \leq z$, we obtain

$$x \notin \left(\downarrow \left(\bigcap_{v \in I_0} \psi_v(\vec{g}) \setminus \bigcup_{v \in I_1} \chi_v(\vec{g}) \right) \right)^c = Z.$$

To prove that $y \in Z$, suppose the contrary. Then there is some $w \geq y$ such that $w \in \bigcap_{v \in I_0} \psi_v(\vec{g}) \setminus \bigcup_{v \in I_1} \chi_v(\vec{g})$. As $w \geq y$ and $\uparrow y = I_0 \cup I_1$, either $w \in I_0$ or $w \in I_1$. If $w \in I_0$, then $w \notin \psi_w(\vec{g})$. While if $w \in I_1$, then $w \in \chi_w(\vec{g})$. In both cases, we obtain $w \notin \bigcap_{v \in I_0} \psi_v(\vec{g}) \setminus \bigcup_{v \in I_1} \chi_v(\vec{g})$, a contradiction. \dashv

Let X be an Esakia space and $G \subseteq X^*$. In view of Esakia duality, the subalgebra $\langle G \rangle$ of X^* is proper if and only if the relation

$$R = \{ \langle x, y \rangle \in X \times X \mid x \in U \text{ iff } y \in U, \text{ for every } U \in \langle G \rangle \}$$

differs from the identity relation on X (see, e.g., [5, Corollary 2.3.10]). As a consequence, we deduce the following:

LEMMA 3.7. *Let X be an Esakia space and $G \subseteq X^*$. Then $X^* = \langle G \rangle$ if and only if for every $x, y \in X$,*

$$\{ U \in \langle G \rangle \mid x \in U \} = \{ U \in \langle G \rangle \mid y \in U \} \text{ implies } x = y.$$

In view of the next result, the concept of subalgebra generation can be studied through that of colouring.

COLOURING THEOREM 3.8. *Let X be an Esakia space and $G \subseteq X^*$ finite. Then $X^* = \langle G \rangle$ if and only if X is G -coloured.*

PROOF. (\Rightarrow) To prove that every element of X is G -isolated, it suffices to show that for every pair of distinct $x, y \in X$ we have $x \sim_\omega^G y$. Accordingly, consider two distinct $x, y \in X$. By symmetry we may assume that $x \not\leq y$. The Priestley separation axiom implies that there is $U \in X^*$ such that $x \in U$ and $y \notin U$. From the assumption that $X^* = \langle G \rangle$ it follows that $U \in \langle G \rangle$. By Lemma 3.6 we obtain that $x \sim_n^G y$ for $n = \text{rank}(U)$. Therefore, the definition of \sim_ω^G guarantees that $x \sim_\omega^G y$.

(\Leftarrow) By Lemma 3.7 it suffices to prove that if $x, y \in X$ are such that $\{ U \in \langle G \rangle \mid x \in U \} = \{ U \in \langle G \rangle \mid y \in U \}$, then $x = y$. Together with Lemma 3.6, the assumption that $\{ U \in \langle G \rangle \mid x \in U \} = \{ U \in \langle G \rangle \mid y \in U \}$ implies $x \sim_\omega^G y$. Since X is G -coloured, we conclude that $x = y$. \dashv

For the following definition to make sense, recall that we always identify natural numbers with finite ordinals, i.e., we identify $n < \omega$ with the set $\{ m \in \omega \mid m < n \}$.

DEFINITION 3.9. Let X be an Esakia space.

- (i) A *colouring* of X is a function $c : n \rightarrow X^*$ where $n < \omega$ and X is coloured by $c[n]$;
- (ii) X is said to be *n -colourable* if there is a colouring $c : n \rightarrow X^*$.

The following is an immediate consequence of the Colouring Theorem 3.8:

COROLLARY 3.10. *An Esakia space X is n -colourable if and only if X^* is n -generated.*

§4. The counterexamples. Our aim is to construct for each $n < \omega$ an infinite $(n + 1)$ -generated Heyting algebra whose n -generated subalgebras are of size $\leq m_n$ for some $m_n < \omega$. We will do this by exhibiting their dual Esakia spaces X_n .

DEFINITION 4.1. For every $n < \omega$, let $X_n = (X_n, \tau, \leq)$ be the ordered topological space where

$$X_n = \{x_i^l \mid l \leq 2^n \text{ and } i < \omega\} \cup \{x_\infty\},$$

$$\tau = \{U \in \wp(X_n) \mid \text{if } x_\infty \in U, \text{ then } U \text{ is cofinite}\},$$

and \leq is the unique partial order with minimum x_∞ such that for every $x_i^l, x_{i'}^{l'} \in X_n$,

$$x_i^l \leq x_{i'}^{l'} \iff \text{either } i \geq i' + 2 \text{ or } (i = i' + 1 \text{ and } l' \neq l + 1).$$

Lastly, for each $i < \omega$ we let $L_n^i = \{x_i^l \mid l \leq 2^n\}$ and we refer to this as the i th level/layer of X_n .

Notice that X_0 is the dual of the Rieger–Nishimura lattice, i.e., the free Heyting algebra on one generator [15, 17]. On the other hand, X_2 is depicted in Figure 1 as an exemplification.

LEMMA 4.2. For every $n < \omega$, X_n is an Esakia space.

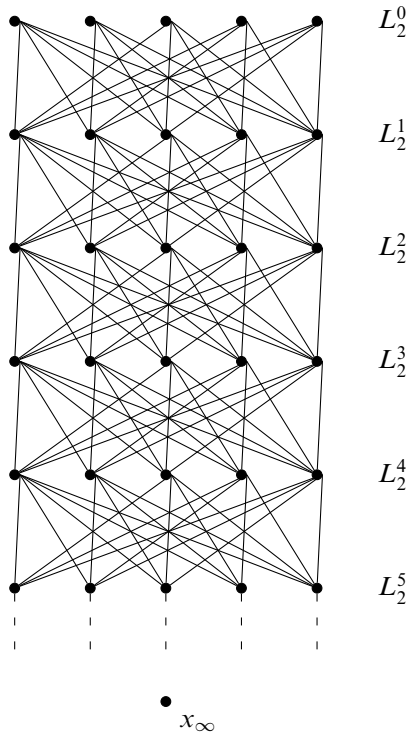


FIGURE 1. The Esakia space X_2 .

PROOF. First, X_n is compact because it is the Alexandroff extension of the countable discrete space $X_n \setminus \{x_\infty\}$. To prove that the Priestley separation axiom holds, consider $x, y \in X_n$ such that $x \not\leq y$. Then x differs from the minimum x_∞ . Consequently, $\uparrow x$ is finite and omits x_∞ . It follows that $\uparrow x$ is a clopen upset which, obviously, omits y . It only remains to prove that the downset of a nonempty clopen set U is clopen. Since U is open and nonempty, the definition of the topology guarantees that U contains an element of the form x_i^l . Therefore, $\downarrow U$ contains $X_n \setminus (L_n^0 \cup \dots \cup L_n^{i+1})$. It follows that $\downarrow U$ is cofinite and contains x_∞ , whence it is clopen. \dashv

By the previous lemma, the dual Heyting algebra X_n^* of X_n is well-defined. The goal of this section is to show that X_n^* is an infinite $(n + 1)$ -generated Heyting algebra whose n -generated subalgebras are of size $\leq m_n$, for some $m_n < \omega$ which depends only on n . Firstly, notice that to prove that X_n^* is $(n + 1)$ -generated, it suffices by Corollary 3.10 to show that X_n is $(n + 1)$ -colourable. Intuitively, the latter holds because of two reasons. On the one hand, since X_n has strictly less than 2^{n+1} maximal elements, we can assign to each maximal element a different subset of colours. On the other hand, the construction of X_n makes sure any two points at level $i + 1$ see different elements at level i , and thus have different ω -types. We make this intuition precise in the proof of the next proposition.

PROPOSITION 4.3. *The Esakia space X_n is $(n + 1)$ -colourable.*

PROOF. Since $n < n + 1$ there is an injection $e : 2^n + 1 \rightarrow \wp(n + 1)$. Then let $c : n + 1 \rightarrow X_n^*$ be the map defined by letting

$$c(k) = \{x_0^l \in X_n \mid k \in e(l)\}.$$

By the definition of X_n we have

$$X_n = \{x_\infty\} \cup \bigcup_{i < \omega} L_n^i.$$

Therefore, to prove that X_n is $(n + 1)$ -colourable, it suffices to show that for every $i < \omega$ the points in L_n^i are all $c[n + 1]$ -isolated. We proceed by induction on i , noting that, by the definition of c and of $\sim_0^{c[n+1]}$, it is clear that every point in L_n^0 is $c[n + 1]$ -isolated. Now, let $i > 0$ and assume that, for all $j < i$, every point in L_n^j is $c[n + 1]$ -isolated.

Let us first show that $x_i^l \sim_\omega^{c[n+1]} x_i^{l'}$, for every $l \neq l' \leq 2^n$. By the construction of X_n we can suppose, without loss of generality, that there exists $z \in L_n^{i-1}$ lying above x_i^l but not above $x_i^{l'}$. As z is $c[n + 1]$ -isolated by our induction hypothesis, it follows that for all $v \geq x_i^l$, there exists m_v satisfying $z \sim_{m_v}^{c[n+1]} v$. Take $m := \max\{m_v \in \omega : v \geq x_i^l\}$, which exists, as $\uparrow x_i^l$ is finite. By Lemma 3.3 we have that $z \sim_m^{c[n+1]} v$ for every $v \geq x_i^l$. It is now clear that $x_i^l \sim_{m+1}^{c[n+1]} x_i^{l'}$, thus $x_i^l \sim_\omega^{c[n+1]} x_i^{l'}$.

Next we show that, given $l \leq 2^n$, then for every $t > i$ and $y \in L_n^t$, we have $x_i^l \sim_\omega^{c[n+1]} y$. By the construction of X_n we know that $x_i^{l'} > y$, for some $l' \neq l \leq 2^n$. It follows from our induction hypothesis and from what we just proved above that for every $z \geq x_i^l$, there exists $m_z \in \omega$ such that $z \sim_{m_z}^{c[n+1]} x_i^{l'}$. As $\uparrow x_i^l$ is finite, taking

$m' := \max\{m_z \in \omega : z \geq x'_i\}$ and applying Lemma 3.3 yields $z \sim_{m'}^{c[n+1]} x'_i$ for every $z \geq x'_i$. Since $x'_i \geq y$, this implies $x'_i \sim_{m'+1}^{c[n+1]} y$, hence also $x'_i \sim_{\omega}^{c[n+1]} y$.

Again using our induction hypothesis, we can now conclude that every point in L_n^i is $c[n + 1]$ -isolated. \dashv

COROLLARY 4.4. *The Heyting algebra X_n^* is infinite and $(n + 1)$ -generated.*

PROOF. Since X_n is infinite, the Heyting algebra X_n^* is also infinite. Furthermore, it is $(n + 1)$ -generated by Corollary 3.10 and Proposition 4.3. \dashv

Therefore, it only remains to prove that the n -generated subalgebras of X_n^* are of size $\leq m_n$ for some $m_n < \omega$. This result follows from two facts. Firstly, in Lemmas 4.5 and 4.7, we show that for every n -colouring $c : n \rightarrow X_n^*$ we can find a level $j < \omega$ such that at least two elements of L_n^j have the same type under $c[n]$ and every element in $\downarrow L_n^{j+1}$ has the same 0-type. Then, in Lemma 4.9, we use the specific configuration of X_n to show that there is a level $j' > j$ such that every element in $\downarrow L_n^{j'}$ has the same ω -type under $c[n]$. We then conclude in Proposition 4.10 that every subalgebra of X_n^* is bounded in size by some $m_n < \omega$ depending only on n . We will now proceed to prove three technical lemmas.

LEMMA 4.5. *Let $c : m \rightarrow X_n^*$ be a function and $i, k < \omega$ such that the following conditions hold:*

- (i) $|L_n^i / \sim_{\omega}^{c[m]}| \leq k \leq 2^n$.
- (ii) For all $x, y \in L_n^{i+1}$ we have $x \sim_0^{c[m]} y$.

Then $|L_n^{i+1} / \sim_{\omega}^{c[m]}| \leq k \leq 2^n$.

PROOF. By condition (i) we can enumerate $L_n^i / \sim_{\omega}^{c[m]}$ as $\{A_j \mid j < k\}$. Recall that we say that an element $x \in L_n^{i+1}$ sees some A_j when $x \in \downarrow A_j$.

CLAIM 4.6. *If two elements of L_n^{i+1} see the same A_j 's, then they have the same ω -type.*

PROOF OF THE CLAIM. Consider $x, y \in L_n^{i+1}$ and suppose that they see the same A_j 's. We need to show that $x \sim_p^{c[m]} y$ for every $p < \omega$. The proof proceeds by induction on p . The case where $p = 0$ holds by condition (ii). For the case where $p = q + 1$, the induction hypothesis guarantees that $x \sim_q^{c[m]} y$. Then consider some $z \geq x$. We need to find some $v \geq y$ such that $z \sim_q^{c[m]} v$. If $z = x$, then we are done taking $v := y$. Then we consider the case where $x < z$. If $z \in \uparrow L_n^{i-1}$, the definition of X_n and the assumption that $x, y \in L_n^{i+1}$ guarantee that $y \leq z$, in which case we take $v := z$. It only remains to consider the case where $z \in L_n^i$. Clearly, there exists $j < k$ such that $z \in A_j$. Therefore, x sees A_j and so does y by assumption. Let $v \in A_j$ be such that $y \leq v$. As $z, v \in A_j$, the elements z and v have the same ω -type, whence $z \sim_q^{c[m]} v$. \dashv

Now, observe that if A_j contains at least two elements of L_n^i , then every element of L_n^{i+1} sees A_j because of the structure of X_n . Furthermore, as L_n^i has $2^n + 1$ elements and the A_j 's are exactly $k \leq 2^n$, we may assume without loss of generality that A_{k-1} contains at least two elements of L_n^i . Together with the claim, this implies that if two elements of L_n^{i+1} see the same elements of $\{A_j \mid j < k - 1\}$, then they have the same

ω -type. As the structure of X_n guarantees that every element of L_n^{i+1} sees every A_j except possibly one, we conclude that $L_n^{i+1} / \sim_\omega^{c[l]}$ has $\leq k$ elements. \dashv

LEMMA 4.7. *For every function $c : n \rightarrow X_n^*$, there is $j < \omega$ such that at least two elements in L_n^j have the same ω -type and every element in $\downarrow L_n^{j+1}$ has the same 0-type.*

PROOF. For each $l < n$, let $U_l := c(l)$. We may assume without loss of generality that every U_l is finite and nonempty, for otherwise the topology of X_n would yield $U_l = \emptyset$ or $U_l = X_n$, in which case U_l does not contribute to distinguish between the ω -type or 0-type of the elements of X_n . Furthermore, for each $l < n$ we denote by i_l the least i such that $U_l \cap L_n^{i+1} = \emptyset$, which exists because U_l is finite. Lastly, we may assume without loss of generality that $i_l \leq i_{l'}$ for each $l < l'$.

CLAIM 4.8. *For each $l < n$, the number of distinct ω -types over $c[l + 1]$ of the elements of L_n^i is bounded above by 2^{l+1} . Furthermore, every member of $\downarrow L_n^{i+1}$ has the same 0-type over $c[l + 1]$.*

PROOF OF THE CLAIM. From the definition of i_0, \dots, i_l and the assumption that $i_0 \leq \dots \leq i_l$ it follows that $\downarrow L_n^{i+1} \cap (U_0 \cup \dots \cup U_l) = \emptyset$. Therefore, the last part of the claim holds. We prove the first part of the claim by induction on l .

Induction Base. We need to prove that the number of distinct ω -types over $c[1] = \{U_0\}$ of the elements of $L_n^{i_0}$ is ≤ 2 . It suffices to show that

$$(x \in U_0 \iff y \in U_0) \text{ implies } x \sim_\omega^{c[1]} y,$$

for every pair of distinct $x, y \in L_n^{i_0}$. If $i_0 = 0$, this is clear, as the sole possible ω -types over $\{U_0\}$ of the elements of L_n^0 are U_0 and $\{x \in X_n \mid \uparrow x \cap U_0 = \emptyset\}$. Then we consider the case where $i_0 > 0$. Clearly, if $x, y \in U_0$, the ω -type over $\{U_0\}$ of x and y is U_0 . Then we consider the case where $x, y \notin U_0$. If $L_n^{i_0-1} \subseteq U_0$, then $x, y \notin U_0$, but $\uparrow x \setminus \{x\}, \uparrow y \setminus \{y\} \subseteq U_0$, so that $x \sim_\omega^{c[1]} y$. Then we consider the case where $L_n^{i_0-1} \not\subseteq U_0$. By assumption there is an element $z \in U_0 \cap L_n^{i_0}$. The definition of X_n guarantees that the set $\uparrow z \cap L_n^{i_0-1}$ is either $L_n^{i_0-1}$ or $L_n^{i_0-1} \setminus \{v\}$ for some $v \in X_n$. Since $\uparrow z \subseteq U_0$ and $L_n^{i_0-1} \not\subseteq U_0$, we obtain $U_0 \cap L_n^{i_0-1} = L_n^{i_0-1} \setminus \{v\}$ for some $v \in X_n$. As x and y are distinct from z (because $x, y \notin U_0$ and $z \in U_0$) and $z \not\subseteq v \in L_n^{i_0-1}$, the definition of X_n guarantees that $x, y < v$. As $x, y, v \notin U_0$ and $\uparrow x \setminus \{x, v\}, \uparrow y \setminus \{y, v\} \subseteq \uparrow z \subseteq U_0$, we conclude that $x \sim_\omega^{c[1]} y$ as desired.

Induction Step. Suppose that the statement holds for l , i.e., that the number of distinct ω -types over $c[l + 1]$ of the elements of L_n^i is bounded above by 2^{l+1} . We will prove that this also holds when l is replaced by $l + 1$. If $i_{l+1} = 0$, this is clear, as the sole possible ω -types over $\{U_0, \dots, U_{l+1}\}$ of the elements of $L_n^{i+1} = L_n^0$ are the sets of the form $\bigcap_{j \in J} U_j \cap \bigcap_{j \notin J} U_j^c$ for some $J \subseteq l + 1$. Thus we may assume that $i_{l+1} > 0$.

We will prove that the number of distinct ω -types over $c[l + 1]$ of the elements of L_n^{i+1-1} is at most 2^{l+1} , that is,

$$|L_n^{i+1-1} / \sim_\omega^{c[l+1]}| \leq 2^{l+1}. \tag{1}$$

We have two cases: either $i_l < i_{l+1}$ or $i_l = i_{l+1}$. Suppose first that $i_l < i_{l+1}$. If $i_l = i_{l+1} - 1$ we are done by the inductive assumption. Then we may assume that

$i_l < i_{l+1} - 1$. Recall that $i_0 \leq \dots \leq i_l < i_l + 1 \leq i_{l+1}$. Moreover, from the definition of i_0, \dots, i_l it follows

$$(U_0 \cup \dots \cup U_l) \cap \downarrow L_n^{i_l+1} = \emptyset.$$

Therefore, for each $x, y \in \downarrow L_n^{i_l+1}$ it holds $x \sim_0^{c[l+1]} y$. Consequently, the result follows from the induction hypothesis and $i_{l+1} - i_l$ applications of Lemma 4.5. It only remains to consider the case where $i_l = i_{l+1}$.

Let m be the greatest integer such that $i_m < i_l = i_{l+1}$, if it exists, or -1 otherwise. If $m \neq -1$, then $m < l$ and by the induction hypothesis, the number of distinct ω -types over $c[m + 1]$ of the elements of L_n^m is at most 2^{m+1} . Since

$$(U_0 \cup \dots \cup U_m) \cap \downarrow L_n^{i_m+1} = \emptyset,$$

by the definition of the i_i 's and by the structure of X_n , it follows from $i_l - 1 - i_m$ applications of Lemma 4.5 that $|\downarrow L_n^{i_l-1} / \sim_\omega^{c[m+1]}| \leq 2^{m+1}$.

Now, for an arbitrary m satisfying the above definition, notice that if $m < k < l + 1$, then $i_k = i_l = i_{l+1}$. Again by the definition of the i_i 's, it follows that, for any such k , we have that $L_n^j \subseteq U_k = c(k)$, for every $j < i_l - 1$. As $m < m + 1 < l + 1$ because $m < l$, the ω -type of an element x of $L_n^{i_l-1}$ over $c[l + 1]$ is totally determined by its ω -type over $c[m + 1]$ (of which there are none if $m = -1$, and at most 2^{m+1} otherwise, by above) together with whether or not x belongs to U_k , for each $m < k < l + 1$. Thus, there are at most $2^{m+1} \cdot 2^{l-m} = 2^{l+1}$ possible ω -types over $c[l + 1]$ that $x \in L_n^{i_l-1}$ can have, as desired. This concludes the proof of condition (1).

To conclude the proof of the claim, it is convenient to separate the following cases.

Case A. Suppose that $L_n^{i_{l+1}-1} \subseteq U_{l+1}$. This entails that the ω -types over $c[l + 2]$ of the elements in $L_n^{i_{l+1}-1}$ are the same as those over $c[l + 1]$. Hence, the elements from $L_n^{i_{l+1}}$ which see the same ω -types over $c[l + 1]$ from $L_n^{i_{l+1}-1}$ also see the same ω -types over $c[l + 2]$ from $L_n^{i_{l+1}-1}$. Consequently, the ω -types over $c[l + 2]$ of the elements in $L_n^{i_{l+1}}$ are determined by their ω -types over $c[l + 1]$ (of which there are at most 2^{l+1} , by induction hypothesis and by possibly repeatedly applying Lemma 4.5 if $i_l < i_{l+1}$) together with whether or not they belong to the set $U_{l+1} \cap L_n^{i_{l+1}}$, in the sense that for $x, y \in L_n^{i_{l+1}}$,

$$x \sim_\omega^{c[l+2]} y \iff x \sim_\omega^{c[l+1]} y \text{ and } x \sim_0^{\{U_{l+1}\}} y.$$

This gives us at most 2^{l+2} possible ω -types over $c[l + 2]$ for elements of $L_n^{i_{l+1}}$.

Case B. Suppose that $L_n^{i_{l+1}-1} \not\subseteq U_{l+1}$. Recall that $U_{l+1} \cap L_n^{i_{l+1}} \neq \emptyset$ by the definition of i_{l+1} . Moreover, by the definition of X_n the upset generated by any pair of distinct elements of $L_n^{i_{l+1}}$ contains the whole $L_n^{i_{l+1}-1}$. Therefore, the assumption that $L_n^{i_{l+1}-1} \not\subseteq U_{l+1}$ allows us to assume, without loss of generality, that $L_n^{i_{l+1}} \cap U_{l+1} = \{x_{i_{l+1}}^0\}$ and $L_n^{i_{l+1}-1} \cap U_{l+1} = L_n^{i_{l+1}-1} \setminus \{x_{i_{l+1}-1}^1\}$. By condition (1), the elements in $L_n^{i_{l+1}-1} \cap U_{l+1} = \{x_{i_{l+1}-1}^t \mid t \neq 1\}$ have at most 2^{l+1} different ω -types over $c[l + 1]$ and, since they all belong to U_{l+1} , they have at most 2^{l+1} different ω -types also over $c[l + 2]$.

Now, the definition of X_n guarantees that for every $m > 0$ we have that $x_{i_{l+1}}^m \leq x_{i_{l+1}-1}^1$. Furthermore, $x_{i_{l+1}}^m$ does not belong to U_{l+1} because $L_n^{i_{l+1}} \cap U_{l+1} = \{x_{i_{l+1}}^0\}$.

Therefore, the ω -type over $c[l + 2]$ of an element of the form x_{i+1}^m with $m > 0$ is determined by the fact that x_{i+1}^m does not belong to U_{l+1} and by the elements of L_n^{i+1-1} with distinct ω -types over $c[l + 2]$ it sees. Since x_{i+1}^m sees all but possibly one of the element of L_n^{i+1-1} and the elements of L_n^{i+1-1} have at most 2^{l+1} different ω -types, this implies that

$$|L_n^{i+1} \setminus \{x_{i+1}^0\} / \sim_\omega^{c[l+2]}| \leq 2^{l+1} + 1.$$

Consequently,

$$|L_n^{i+1} / \sim_\omega^{c[l+2]}| \leq 2^{l+1} + 2 \leq 2^{l+2},$$

thus finishing the proof of the claim. ←

From the claim it follows that the number of ω -types of the elements in $L_n^{i_{n-1}}$ over $c[n]$ is bounded above by 2^n . Since $L_n^{i_{n-1}}$ has $2^n + 1$ elements, it follows that at least two elements of $L_n^{i_{n-1}}$ have the same ω -type. Moreover, the second part of the claim guarantees that every element of $\downarrow L_n^{i_{n-1}+1}$ has the same 0-type over $c[n]$. Thus, the statement holds for $j := i_{n-1}$. ←

The proof of the following lemma shows concretely why, in the Esakia space X_n , we defined each element $x_{i+1}^{2^n}$ so that it sees every element at level L_n^i . The key point is that, if two elements at level i have the same ω -type, then $x_{i+1}^{2^n+1}$ will have exactly the same type of some other element at level $i + 1$. Proceeding inductively, we find a level from which every element has the same type under the given colouring. We make this reasoning explicit in the next proof.

LEMMA 4.9. *Let $c : m \rightarrow X_n^*$ be a function and $i < \omega$. Suppose two distinct elements of L_n^i have the same ω -type over $c[m]$ and that for every $q > i$ the elements of L_n^q have the same 0-type over $c[m]$. Then every element of $\downarrow L_n^{i+2^n+1}$ has the same ω -type over $c[m]$.*

PROOF. Since two distinct elements of L_n^i have the same ω -type over $c[m]$ and all the elements of L_n^{i+1} have the same 0-type over $c[m]$, then it follows from the construction of X_n that $x_{i+1}^{2^n} \sim_\omega^{c[m]} x_{i+1}^j$ for some $j < 2^n$. Therefore, the construction of X_n guarantees that the elements $x_{i+2}^{2^n-1}$ and $x_{i+2}^{2^n}$ see the same equivalence classes of $L_n^{i+1} / \sim_\omega^{c[m]}$. Since by assumption $x_{i+2}^{2^n-1}$ and $x_{i+2}^{2^n}$ have the same 0-type over $c[m]$, this implies that $x_{i+2}^{2^n-1} \sim_\omega^{c[m]} x_{i+2}^{2^n}$. For the same reason, we have that $x_{i+3}^{2^n-2} \sim_\omega^{c[m]} x_{i+3}^{2^n-1} \sim_\omega^{c[m]} x_{i+3}^{2^n}$. By proceeding in this way, we obtain that every element of $L_n^{i+2^n+1}$ has the same ω -type over $c[m]$. Since for $t \geq i + 2^n + 1$ every element of L_n^t has the same 0-type over $c[m]$, this is enough to conclude that every element of $\downarrow L_n^{i+2^n+1}$ has the same ω -type over $c[m]$. ←

We can now conclude by showing that the Heyting algebra X_n^* satisfies the desired property.

PROPOSITION 4.10. *There exists $m_n < \omega$ such that the n -generated subalgebras of X_n^* are of size $\leq m_n$.*

PROOF. We begin by the following observation.

CLAIM 4.11. *There exists $k < \omega$ such that for every function $c : n \rightarrow X_n^*$ the number of ω -types over $c[n]$ of elements of X_n is $\leq k$.*

PROOF OF THE CLAIM. Recall the definition of the integers i_0, \dots, i_{n-1} associated with c in the proof of Lemma 4.7. In view of Claim 4.8 and the fact that each $L_n^{i_l}$ has $2^n + 1$ elements, we obtain that for each i_l at least two elements of $L_n^{i_l}$ have the same ω -type over $c[l + 1]$ and every point in $\downarrow L_n^{i_l+1}$ has the same 0-type over $c[l + 1]$. Hence, it follows from Lemma 4.9 that all the elements of $\downarrow L_n^{i_l+2^n+1}$ have the same ω -type over $c[l + 1]$. Furthermore, from the construction of X_n and the definition of i_l it follows that the upset $\uparrow L_n^{i_l-2}$ (which is the emptyset if $i_l \leq 1$) is contained in $U_l := c(l)$. Let us partition X_n as the union

$$\uparrow L_n^{i_l-2} \cup L_n^{i_l-1} \cup L_n^{i_l} \cup \dots \cup L_n^{i_l+2^n} \cup \downarrow L_n^{i_l+2^n+1},$$

and note that the above discussion entails that the effect of the clopen U_l in the determination of the ω -type over $c[n]$ of a point x is trivial if $x \in \uparrow L_n^{i_l-2} \cup \downarrow L_n^{i_l+2^n+1}$, and noticeable only if $x \in L_n^{i_l-1} \cup L_n^{i_l} \cup \dots \cup L_n^{i_l+2^n}$. Since each of these $(i_l + 2^n) - (i_l - 2) = 2^n + 2$ layers has $2^n + 1$ many elements, we conclude that the clopen U_l can only contribute to distinguish at most $(2^n + 1)(2^n + 2) + 2$ ω -types over $c[n]$ in X_n .

Since the i_l in the above argument was arbitrary, it now follows that each clopen in $c[n] = \{U_0, \dots, U_{n-1}\}$ can only contribute to distinguish at most $(2^n + 1)(2^n + 2) + 2$ ω -types over $c[n]$ in X_n . Therefore, there are at most $k := n[(2^n + 1)(2^n + 2) + 2]$ distinct ω -types over $c[n]$ in X_n . As this bound is independent of the choice of the function c , we have found the desired uniform upper bound. \dashv

In order to conclude the proof, it suffices to show that the n -generated subalgebras of X_n^* are of size $\leq 2^k$, for in this case the statement holds for $m_n := 2^k$. Suppose, on the contrary, that there is an n -generated subalgebra of X_n^* containing distinct elements U_0, \dots, U_{2^k} . Moreover, let $m < \omega$ be such that $\text{rank}(U_i) \leq m$ for every $i \leq 2^k$. Since every family of $2^k + 1$ distinct subsets of a set Y separates at least $k + 1$ elements of Y , there are distinct $x_0, \dots, x_k \in X_n$ that are separated by U_0, \dots, U_{2^k} . By Lemma 3.6 the elements x_0, \dots, x_k are unrelated by $\sim_m^{c[n]}$. By the definition of $\sim_\omega^{c[n]}$, this implies that x_0, \dots, x_k are also unrelated by $\sim_\omega^{c[n]}$. Therefore, there are $k + 1$ distinct ω -types over $c[n]$ (that is, $x_0/\sim_\omega^{c[n]}, \dots, x_k/\sim_\omega^{c[n]}$), but this contradicts Claim 4.11. \dashv

§5. The main result. Given a class K of similar algebras, we let

- $\mathbb{V}(K) :=$ the variety generated by K ;
- $\mathbb{S}(K) :=$ the class of subalgebras of the members of K .

The aim of this section is to establish the main result of the paper, namely:

THEOREM 5.1. *For each $n < \omega$, the variety $\mathbb{V}(X_n^*)$ is strictly n -finite, i.e., it is n -finite but contains an infinite $(n + 1)$ -generated algebra.*

In particular, the variety $\mathbb{V}(X_n^*)$ is n -finite, but not locally finite. For the case where $n = 2$, this provides a negative answer to [4, Problem 2.4]. The proof of Theorem 5.1 relies on the next observation [3, Lemma 1.3].

PROPOSITION 5.2. *Let \mathbb{K} be a variety and Var a set of variables. Moreover, let $\{v_i \mid i \in I\}$ be a family of functions $v_i: Var \rightarrow A_i$ with $A_i \in \mathbb{K}$ such that for every pair of terms φ and ψ with variables in Var it holds that*

$$\mathbb{K} \not\models \varphi \approx \psi \text{ implies that there exists } i \in I \text{ such that } \varphi^{A_i}(v_i(\vec{x})) \neq \psi^{A_i}(v_i(\vec{x})).$$

Then the n -generated free algebra of \mathbb{K} embeds into the direct product $\prod_{i \in I} A_i$.

As a consequence, we deduce the following:

COROLLARY 5.3. *Let \mathbb{K} be a class of similar algebras of finite type and $n < \omega$. If the cardinality of the n -generated members of $\mathbb{S}(\mathbb{K})$ is bounded above by some $m_n < \omega$, then $\mathbb{V}(\mathbb{K})$ is n -finite.*

PROOF. Since the type of \mathbb{K} is finite and the cardinality of the n -generated members of $\mathbb{S}(\mathbb{K})$ is bounded above by some $m_n < \omega$, up to isomorphism there are only finitely many n -generated algebras in $\mathbb{S}(\mathbb{K})$, all of whom are finite. We enumerate them as H_0, \dots, H_k .

Now, consider the set of variables $Var := \{x_0, \dots, x_{n-1}\}$. Clearly, if two terms φ and ψ with variables in Var differ when interpreted in the variety $\mathbb{V}(\mathbb{K})$, then there exist some $i \leq k$ and a function $v: Var \rightarrow H_i$ such that $\varphi^{H_i}(v(\vec{x})) \neq \psi^{H_i}(v(\vec{x}))$. Therefore, we can apply Proposition 5.2 obtaining that the free n -generated algebra F_n of $\mathbb{V}(\mathbb{K})$ embeds into $H := H_0^{Var} \times \dots \times H_k^{Var}$. Since both Var and each H_i are finite, so is H and, therefore, F_n . As every n -generated member of $\mathbb{V}(\mathbb{K})$ is a homomorphic image of F_n , we conclude that $\mathbb{V}(\mathbb{K})$ is n -finite. \dashv

We are now ready to prove Theorem 5.1.

PROOF. By Corollary 4.4 the Heyting algebra X_n^* is infinite and $(n + 1)$ -generated. As $X_n^* \in \mathbb{V}(X_n^*)$, it only remains to prove that the variety $\mathbb{V}(X_n^*)$ is n -finite. But this is an immediate consequence of Proposition 4.10 and Corollary 5.3. \dashv

REMARK 5.4. In view of Theorem 5.1, for each $n < \omega$ there is an n -finite variety of Heyting algebras that contains an infinite $(n + 1)$ -generated algebra. We will prove that such a variety can be chosen finitely axiomatisable.

First, recall from Theorem 5.1 that $\mathbb{V}(X_n^*)$ is n -finite. Therefore, there is some $m < \omega$ such that every n -generated member of $\mathbb{V}(X_n^*)$ has size $\leq m$ (for instance, m can be taken to be the size of the free n -generated algebra of $\mathbb{V}(X_n^*)$). This property can be expressed by a first-order sentence, namely,

$$\theta_n = \forall (x_i)_{i < n} \exists (y_j)_{j < m} \left(\left(\bigwedge_{i < n} x_i = y_i \right) \wedge \left(\bigvee_{j < m} y_j = 0 \right) \wedge \left(\bigvee_{j < m} y_j = 1 \right) \wedge \left(\bigwedge_{\odot \in \{\wedge, \vee, \rightarrow\}} \bigwedge_{i, i' < m} \bigvee_{j < m} y_i \odot y_{i'} = y_j \right) \right).$$

Then, let Σ_n be the set of universally quantified equations valid in $\mathbb{V}(X_n^*)$. Since $\mathbb{V}(X_n^*)$ is a variety, it is axiomatized by Σ_n . Together with the fact that every

n -generated member of $\mathbb{V}(X_n^*)$ has size $\leq m$, this implies $\Sigma_n \models \theta_n$ (where \models stands for the consequence relation of first-order logic). By the compactness theorem there is a finite $T \subseteq \Sigma_n$ such that $T \models \theta_n$. Let \mathcal{V} be the class of all the Heyting algebras satisfying T . Clearly, \mathcal{V} is a variety of Heyting algebras. Furthermore, from $T \subseteq \Sigma_n$ it follows immediately that $X_n^* \in \mathcal{V}$. Therefore, \mathcal{V} contains an infinite $(n + 1)$ -generated algebra, namely, X_n^* (Corollary 4.4). Lastly, \mathcal{V} is n -finite because $T \models \theta_n$ and, therefore, every n -generated algebra in \mathcal{V} is of size $\leq m$.

REMARK 5.5. From a logical standpoint, the importance of Heyting algebras is that they algebraize the *intuitionistic propositional calculus* IPC in the sense of [6]. As a consequence, the axiomatic extensions of IPC (known as *superintuitionistic logics*, or si-logics for short) form a lattice that is dually isomorphic to that of varieties of Heyting algebras (see, e.g., [7]). Because of this, Remark 5.4 can be rephrased as follows: for every $n < \omega$ there is a finitely axiomatisable si-logic that has only finitely many formulas in variables x_0, \dots, x_{n-1} up to logical equivalence, but that has infinitely many nonequivalent formulas in variables x_0, \dots, x_n .

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