On a Certain Matrix Product with Specified Latent Roots

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1. Vajda's paper[1] in this volume has suggested to the author the following problem:

Let A be a $n \times m$ matrix, $m \leqq n$, let B be a $m \times n$ matrix, and let $M = I - AB$, where I is the unit matrix of order n. Given A, to find B such that of the n latent roots of $M'M$, k are unity, and the remaining $n - k$ are zero.

The case considered by Vajda is that in which both A and $I - M$ are of maximum rank m; the present note takes the solution a little further without, however, answering the problem completely.

2. Consider first the more stringent requirement

$$M'M = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

where I_k is the unit matrix of order k, defining by its position a mode of partitioning of the $n \times n$ matrix $M'M$. In the same mode of partitioning let

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}.$$

We then have

$$m_1' m_1 + m_3' m_3 = I_k, \tag{2}$$

$$m_2' m_2 + m_4' m_4 = 0. \tag{3}$$

If, as we shall assume, M is real, (3) gives $m_2 = 0$, $m_4 = 0$, since both $m_2' m_2$ and $m_4' m_4$ are non-negative definite matrices (ref. (1), p. 97, ex. 2). Hence

$$M = \begin{bmatrix} m_1 & 0 \\ m_3 & 0 \end{bmatrix}$$

and $I - M$ is at least of rank $n - k$; but since it is at most of the rank of A, which we shall denote by r, where $r \leqq m$, we have $n - k \leqq r$, or $k \geqq n - r$.

[1] Ref. (2).

3. Let $k - (n - r) = s$, where s is one of the integers 0; 1, ..., r, and put

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

where m_{11} is a $(n - r) \times (n - r)$ matrix, m_{22} is a $s \times s$ matrix, while the partitioning of A, to accord with that of M, is such as to let a_1, a_2 and a_3 have $n - r$, s and $r - s$ rows respectively. Without loss of generality we may assume that r linearly independent rows of A are formed by those of a_2 and a_3. The following theorem holds:

For the existence of a solution B it is necessary and sufficient for the rows of a_1 to be independent of those of a_3.

The proof is an application of a general theorem on linear equations by which there exists a solution B if and only if the linear relations between the rows of A hold also for the rows of $I - M$.

Assume first that the rows of a_1 depend linearly on those of a_2 and a_3, so that there is a relation of the form $a_1 = Ca_2 + Da_3$, where C and D are matrices. Considering the last $r - s$ columns of $I - M$ we must then have $0 = C \times 0 + D \times I$, or $D = 0$. Hence $a_1 = Ca_2$, which establishes the necessity of the condition. It is also sufficient, for when it is satisfied we can determine m_{11} and m_{12} from the condition

$$[I_{n-r} - m_{11}, \qquad - m_{12}] = C\,[- m_{21}, \qquad I_s - m_{22}],$$

where I_{n-r}, I_s are unit matrices of the orders of their suffixes.

In order that condition (2) should also be satisfied we must further have

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}' \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + [m_{31} \quad m_{32}]'\,[m_{31} \quad m_{32}] = \begin{bmatrix} I_{n-r} & 0 \\ 0 & I_s \end{bmatrix}.$$

The preceding requirements can all be seen to be satisfied if we take

$$M = \begin{bmatrix} I_{n-r} & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

B is then obtained by solving the system of equations

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{r-s} \end{bmatrix}.$$

This completes the demonstration of the sufficiency of the condition enunciated in the theorem.

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4. In this section we shall discuss the particular case where $s = 0$ or $n - r = k$. Then a_2 has no rows and $a_1 = 0$. Hence we have to solve

$$\begin{bmatrix} 0 \\ a_3 \end{bmatrix} B = \begin{bmatrix} I_{n-r} - m_1 & 0 \\ - m_3 & I_r \end{bmatrix};$$

and, by considering the rank of the matrix formed by adjoining to A columns of $I - M$, we see that $I_{n-r} - m_1 = 0$. By (2) it then follows that $m_3 = 0$, and we solve for B from the system of equations $a_3 B = [0 \; I_r]$. In this case the determination (4) of M is the only possible one.

We now return to the original problem, proposed in § 1, where M is a matrix such that $M'M$ has k latent roots which are unity, and $n - k$ which are zero. In this case there exists an orthogonal matrix H such that, in analogy with (1),

$$H'M'MH = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}. \tag{5}$$

Since $H'H = I$ the equations for solution, $I - AB = M'M$, are equivalent to $I - H'ABH = H'M'MH$, which are of the form already solved except that where previously we had A and B we now have $H'A$ and BH. But by a previous result a solution BH exists if and only if there is a relation of the form

$$H'A = \begin{bmatrix} 0 \\ a_3 \end{bmatrix}. \tag{6}$$

If we partition H by writing $H = [h_1 \; h_2]$, when h_2 has r columns, we obtain from (6) the relations

$$h_2'A = a_3 \tag{7}$$

and
$$A = H \begin{bmatrix} 0 \\ a_3 \end{bmatrix} = h_2 a_3. \tag{8}$$

Now as a_3 is of rank r the matrix product $a_3 a_3'$, which is of order $r \times r$, is non-singular and so possesses an inverse (ref. (1), p. 97, ex. 3).

Hence we may write $Aa_3' = h_2 a_3 a_3'$, or $h_2 = Aa_3' (a_3 a_3')^{-1}$. On the other hand, A being of rank r, let A_1 be the matrix formed by some r linearly independent columns contained in A. Then there exists a matrix K such that $A = A_1 K$. Hence

$$h_2 = A_1 K a_3' (a_3 a_3')^{-1} = A_1 D, \tag{9}$$

say. Since h_2 and A_1 are both of rank r, the $r \times r$ matrix $D = K a_3' (a_3 a_3')^{-1}$ is also of rank r and hence possesses an inverse. Combining (7) and (9) we obtain

$$a_3 = D' A_1' A. \tag{10}$$

Now B is found from the relation $H'ABH = I - H'M'MH$, which, in virtue of (5) and (6), reduces to

$$\begin{bmatrix} 0 \\ a_3 \end{bmatrix} BH = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$

or $a_3 BH = [0 \ I_r]$, or $a_3 B = [0 \ I_r] H' = h_2'$. Substituting in this from (9) and (10) we have

$$D' A_1' AB = D' A_1'.$$

Since D, and so D', possesses an inverse this reduces finally to

$$A_1' AB = A_1'. \tag{11}$$

This is an assemblage of n systems of r equations each and m ($\geqq r$) unknowns each; and can therefore be solved. Although A_1', consisting of any r linearly independent rows of A', is not determinate, equations (11) are not therefore indeterminate; for any two different determinations of A_1' can be transformed into each other by premultiplying by a non-singular matrix of order r; and similarly the corresponding versions of equation (11) can be transformed into each other by premultiplying its two sides by the same matrix. This premultiplication leaves the value of B unchanged.

If $m = r$, or A has maximum rank, A_1 is identical with A, and the solution for B is then unique, being

$$B = (A'A)^{-1} A'.$$

REFERENCES

(1) Turnbull, H. W. and Aitken, A. C., *An introduction to the theory of canonica matrices* (Glasgow, 1932).

(2) Vajda, S., *Proc. Edinburgh Math. Soc.* (2), **10**, 1-15.

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