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MARKUŠEVIČ BASES AND CORSON COMPACTA IN DUALITY

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ABSTRACT. We characterize Banach spaces that admit Markuševič bases with various properties connected with weak countable determining or weak Lindelöf determining of spaces or with various norming properties.

1. **Introduction.** The purpose of this note is to characterize Banach spaces that admit Markuševič bases with various properties connected with the notion of Corson compacta. The classes of spaces we obtain coincide with important classes of spaces like weakly countably or weakly Lindelöf determined spaces *etc.*

The results in this paper can be read also as characterizations of several important classes of spaces in terms of Markuševič bases.

We prove our results by using projectional resolutions of identity, which allow us to further develop certain orthogonalization methods for uncountable systems of elements.

A compact space is said to be *Corson compact* if it is homeomorphic to a subset K of a cube $[0,1]^\Gamma$ such that each element of K has at most countably many nonzero coordinates.

We follow [AM], and say a Banach space X is *weakly Lindelöf determined* (WLD) if there is a one-to-one bounded linear operator $T: X^* \longrightarrow l^c_\infty(\Gamma)$ which is w^* to pointwise continuous, where $l^c_\infty(\Gamma)$ denotes the subspace of countably supported elements in $l_\infty(\Gamma)$.

The following known result relates the notions of Corson compact spaces and WLD spaces.

THEOREM 1.1. *For a Banach space X, the following are equivalent.*
(a) X is WLD.
(b) B_{X^*} (the unit ball of X^*) is Corson compact in its w^*-topology.
(c) X admits an M-basis and for any M-basis $\{x_i, f_i\}_{i\in I}$ on X one has $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in X^*$.

We should point out that ideas of Argyros and Mercourakis [AM], Corson [C], Orihuela *et al.* [OSV], Pličko [P2], Pol [Po2], Valdivia ([V1], [V3]) and others contributed to this theorem and related results; see [OSV, Proposition 4.1] and [V3, Theorem 2 and Corollary 3.1] for an explicit proof of Theorem 1.1. Notice also that if X has an M-basis and B_{X^*} is w^*-*angelic* (*i.e.* the w^*-sequential closure and w^*-closure of subsets of B_{X^*} coincide), then an argument of Pličko shows that given any M-basis $\{x_i, f_i\}_{i\in I}$ on X, we

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have $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in X^*$ (see [P2, p. 386–387]; [V1, proof of Corollary 2]). Thus the second part of (c) follows from the fact that Corson compact spaces are angelic—this property can be checked directly from the definition of Corson compact sets.

We refer to [LT], [S] and [DGZ] for the reference on Markuševič bases and to [V4], [FaG], [FG], [DG], [AM], [GoT] and [Va] for some recent results on this topic.

With Theorem 1.1 as our motivation, we examine the connection between the existence of a one-to-one w^* to pointwise continuous operator $T: X^* \longrightarrow l_\infty(\Gamma)$ such that $TY \subset l_\infty^c(\Gamma)$ for some Y that is w^*-dense in X^* and the existence of an M-basis $\{x_i, f_i\}_{i\in I}$ on X such that each $f \in Y$ is countably supported on $\{x_i\}_{i\in I}$.

We begin in Section 2, by using a method of Pličko [P2] to decompose certain spaces into smaller pieces. These decompositions are used to construct biorthogonal systems.

The third section uses the results of Section 2 to characterize certain spaces in terms of Markuševič bases. In particular, an M-basis characterization of weakly countably determined Banach spaces based on Mercourakis' result from [M] is given in Theorem 3.2. Some further equivalent conditions for a space to be WLD are presented in Theorem 3.3. As applications of our characterization of WLD spaces, we show that certain "twisted sums" of weakly compactly generated spaces do not admit M-bases (Remark 3.6). Characterizations of spaces admitting M-bases and of spaces which linearly inject into $c_0(\Gamma)$ are given in Theorems 3.7 and 3.8.

Section 4 presents some new results concerning 1-norming M-bases. For instance, we show if X admits a countably 1-norming M-basis and X^* is WLD, then X admits a 1-norming M-basis. It is also shown if X has a 1-norming M-basis, then so does any WLD subspace of X.

We work in real Banach spaces and use the following standard definitions and notation. The closed unit ball of X is denoted by B_X. The linear span of a set S is denoted by span(S) and aco(S) denotes the convex hull of $-S \cup S$. The density of X, dens(X), is the smallest cardinality of a dense subset of X. We will say a collection $S \subset X^*$ is *total* on X if for each $x \in X \setminus \{0\}$ there is an $f \in S$ such that $f(x) \neq 0$. A subspace $Y \subset X^*$ is called λ-*norming* if $\sup\{f(x) : f \in Y \cap B_{X^*}\} \geq \lambda\|x\|$ for all $x \in X$; if the above holds for some $\lambda > 0$, then we may refer to Y as *norming*. Note that Y is then 1-norming under the equivalent norm $|||x||| = \sup\{f(x) : x \in Y \cap B_{X^*}\}$. A system $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ is called a *Markuševič basis* (M-basis) if $f_i(x_j) = \delta_{ij}$ (the Kronecker delta), $\{f_i\}_{i\in I}$ is total on X and $\overline{\text{span}}(\{x_i : i \in I\}) = X$. An M-basis $\{x_i, f_i\}_{i\in I}$ of X is said to be *countably norming* (*countably λ-norming*) if there is a norming (λ-norming) subspace $Y \subset X^*$ such that $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in Y$. If $Y = \overline{\text{span}}(\{f_i\}_{i\in I})$ is norming (λ-norming), then the M-basis $\{x_i, f_i\}_{i\in I}$ is said to be *norming* (λ-*norming*). A *projective resolution of identity* (PRI) on X is a "long sequence" of projections $P_\beta: X \longrightarrow X$, for $\omega_0 \leq \beta \leq \mu$ where μ is the first ordinal of cardinality dens(X), satisfying $\|P_\beta\| \leq 1$, $P_\beta P_\gamma = P_\gamma P_\beta = P_{\min(\gamma,\beta)}$, dens$(P_\beta X) \leq |\beta|$, $P_\beta X = \overline{\bigcup\{P_{\gamma+1}X : \gamma < \beta\}}$, and $P_\mu = I$. Finally, a Banach space is said to be *weakly compactly generated* (WCG), if it contains a weakly compact fundamental subset. A compact space is called *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space. If X^{**} admits a countable collection of w^*-compact subsets K_n such that given any $x \in X$ and $y \in X^{**} \setminus X$ there is a K_n such that $x \in K_n$ and $y \notin K_n$, then X is said to be *weakly countably determined* (WCD).

2. **Orthogonalization techniques in nonseparable spaces.** In this section we will develop some basic techniques which will be needed throughout this paper.

The first result is a modification of [P2, Theorem 1].

THEOREM 2.1. *Let $\{x_i\}_{i\in I} \subset X$ and $Y \subset X^*$ be 1-norming on X. If $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in Y$ and $\overline{\text{span}}(\{x_i : i \in I\}) = X$, then there is a* PRI, $\{P_\beta\}_{\omega_0 \leq \beta \leq \mu}$, *such that for each β we have $P_\beta X = \overline{\text{span}}(\{x_i : i \in J_\beta \subset I\})$, $|J_\beta| \leq |\beta|$ and* $\ker P_\beta = \overline{\text{span}}(\{x_i : i \notin J_\beta\})$.

Moreover, if $Y = \overline{\text{span}}(\{f_i : i \in I\})$ is such that $|\{i : f_i(x) \neq 0\}| \leq \aleph_0$ for every $x \in X$, then P_β can be chosen so that it also satisfies $P^*_\beta Y = \overline{\text{span}}(\{f_i : i \in J_\beta\})$ and $P^*_\beta f = 0$ for $f \in \overline{\text{span}}(\{f_i : i \notin J_\beta\})$.

We will use the following elementary lemma in the proof of Theorem 2.1.

LEMMA 2.2. *Let $\{x_i\}_{i\in I}$ be such that $\overline{\text{span}}(\{x_i : i \in I\}) = X$ and let $Y \subset X^*$ be such that Y is 1-norming and $|i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in Y$. Then the following are satisfied.*

(a) $|I| \leq \text{dens}(X)$.

(b) If $Y = \overline{\text{span}}(\{f_j : j \in J\})$ and $|\{j : f_j(x) \neq 0\}| \leq \aleph_0$ for each $x \in X$, then $|J| = \text{dens}(X)$.

(c) Given $I_1 \subset I$, there exists I_2 such that $I_1 \subset I_2 \subset I$, $|I_1| = |I_2|$ and $d(S_{X_1}, X_2) = 1$ where $X_1 = \overline{\text{span}}(\{x_i : i \in I_1\})$, $X_2 = \overline{\text{span}}(\{x_i : i \in I \setminus I_2\})$, and $d(S_{X_1}, X_2)$ denotes the distance of S_{X_1} to X_2.

PROOF OF LEMMA 2.2. Let $\{z_\alpha\}_{\alpha\in A}$ be dense in X with $|A| = \text{dens}(X)$. To prove (a), fix $f_\alpha \in Y$ so that $\|f_\alpha\|^* = 1$ and $f_\alpha(z_\alpha) \geq \frac{1}{2}\|z_\alpha\|$. Then

$$I = \bigcup_{\alpha\in A} \{i : f_\alpha(x_i) \neq 0\}$$

and so $|I| \leq \aleph_0 |A| = |A|$.

For (b), notice that $J = \bigcup_{\alpha\in A}\{j : f_j(z_\alpha) \neq 0\}$ and $A = \bigcup_{j\in J}\{\alpha : f_i(z_\alpha) \neq 0\}$ and so $|J| = |A|$.

Note that (c) is proved in [P2, Lemma 1]. ■

PROOF OF THEOREM 2.1. We follow Plička's argument (*cf.* [P2, Theorem 1]) and only prove the "moreover" part. By Lemma 2.2(a), we have $|I| = \text{dens}(X)$ and so we arrange the elements $\{x_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ in transfinite sequences $\{x_\beta\}$ and $\{f_\beta\}$ so that $\omega_0 < \beta < \mu$ where $\text{dens}(X) = |\mu|$. If all of the projections P_γ have been constructed for $\gamma < \beta$ and β is a limit ordinal, then let $J_\beta = \bigcup_{\gamma<\beta} J_\gamma$. We denote $X_{J_\beta} = \overline{\text{span}}\{x_\gamma : \gamma \in J_\beta\}$ and $X^{J_\beta} = \overline{\text{span}}\{x_\gamma : \gamma \notin J_\beta\}$. It follows easily that the projection P_β onto X_{J_β} with null space X^{J_β} satisfies the required properties.

Now suppose that β is not a limit ordinal. We construct a sequence of subsets $J_\beta^n \subset J$ with $|J_\beta^n| \leq |\beta|$ for all $n \in \mathbb{N}$ such that the following four conditions are satisfied.

(1) $x_\beta \in \{x_\gamma : \gamma \in J_\beta^1\}$, $J_{\beta-1} \subset J_\beta^1$, $J_\beta^n \subset J_\beta^{n+1}$.

(2) If $\gamma \notin J_\beta^{n+1}$, then $f_\gamma(x_\alpha) = 0$ for all $\alpha \in J_\beta^n$.

(3) If $\gamma \notin J_\beta^{n+1}$, then $f_\alpha(x_\gamma) = 0$ for all $\alpha \in J_\beta^n$.

(4) $d(S_{X_{J_\beta^n}}, X^{J_\beta^{n+1}}) = 1$.

To do this, let $J_\beta^1 = J_{\beta-1} \cup \beta$. Suppose J_β^n has been constructed. Now let $I_1 = J_\beta^n \bigcup\{\gamma : f_\gamma(x_\alpha) \neq 0$ for some $\alpha \in J_\beta^n\} \cup \{\gamma : f_\alpha(x_\gamma) \neq 0$ for some $\alpha \in J_\beta^n\}$. By the countable support properties in the hypotheses, we have $|I_1| \leq |J_\beta^n| \leq |\beta|$. Using Lemma 2.2(c), we choose $J_\beta^{n+1} = I_2$ such that $I_1 \subset I_2$, $|I_1| = |I_2|$ and $d(S_{X_{I_1}}, X^{I_2}) = 1$. The sets J_β^n selected thus satisfy conditions (1) through (4).

Now let $J_\beta = \bigcup_{n=1}^\infty J_\beta^n$ and P_β be the projection onto X_{J_β} with null space X^{J_β}. By condition (4), we have $d(S_{X_{J_\beta}}, X^{J_\beta}) = 1$ and so $\|P_\beta\| = 1$. From condition (3), we have $f_\gamma(x) = 0$ if $\gamma \in J_\beta$ and $x \in X^{J_\beta}$; condition (2) guarantees $f_\gamma(x) = 0$ if $\gamma \notin J_\beta$ and $x \in X_{J_\beta}$. From this one can easily verify the conclusion of the theorem. ∎

The first part of the following theorem was shown by Valdivia ([V2, Theorem 2]) using different methods. The "moreover" part has been obtained recently and independently by Valdivia [V4]; we refer the reader to [V4] for further results concerning 1-norming M-bases. We will present a proof based on Theorem 2.1.

THEOREM 2.3. *Let X be a Banach space and Y be a* 1*-norming subspace of X^*. If $\{z_j\}_{j\in J}$ is such that* $\overline{\operatorname{span}}(\{z_j : j \in J\}) = X$ *and* $|\{j : f(z_j) \neq 0\}| \leq \aleph_0$ *for each $f \in Y$, then X admits an M-basis $\{x_i,f_i\}_{i\in I}$ such that* $\operatorname{span}(\{x_i\}_{i\in I}) = \operatorname{span}(\{z_j\}_{j\in J})$ *and* $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ *for each $f \in Y$.*

Moreover, if $Y = \overline{\operatorname{span}}(\{g_j : j \in J\})$ *and* $|\{j : g_j(x) \neq 0\}| \leq \aleph_0$ *for each $x \in X$, then the M-basis $\{x_i,f_i\}_{i\in I}$ can be chosen to also satisfy,* $\operatorname{span}(\{f_i : i \in I\}) = \operatorname{span}(\{g_j : j \in J\})$. *In particular, this M-basis is* 1*-norming.*

PROOF. We prove the "moreover" part first. Notice that the theorem is trivial if X is finite dimensional; for X separable the result follows from the proof of [LT, Proposition 1.f.3], because Lemma 2.2(a) ensures that $|J| = \aleph_0$.

Suppose that $\operatorname{dens}(X) = |\mu|$ and that the result holds for all Banach spaces E with $\operatorname{dens}(E) < |\mu|$. By Theorem 2.1 there is a PRI, $\{P_\beta\}_{\omega_0 \leq \beta \leq \mu}$ such that:

$$P_\beta X = \overline{\operatorname{span}}(\{z_j : j \in J_\beta\}, \quad \ker P_\beta X = \overline{\operatorname{span}}(\{z_j : j \notin J_\beta\},$$
$$P_\beta^* Y = \overline{\operatorname{span}}(\{g_j : j \in J_\beta\} \quad \text{and} \quad \ker P_\beta^* g_j = 0 \quad \text{if } j \notin J_\beta.$$

We may assume $P_{\omega_0} = 0$ and we denote $X_\beta = (P_{\beta+1} - P_\beta)(X)$ and $Y_\beta = (P_{\beta+1}^* - P_\beta^*)(Y)$ for $\omega_0 \leq \beta < \mu$. Thus we have $X_\beta = \overline{\operatorname{span}}(\{z_j : j \in J_{\beta+1} \setminus J_\beta\})$, and $Y_\beta = \overline{\operatorname{span}}(\{g_j : j \in J_{\beta+1} \setminus J_\beta\})$. According to Lemma 2.2(a) we have $\operatorname{dens}(X_\beta) = |J_{\beta+1} \setminus J_\beta|$. Hence by the induction hypothesis we can choose an M-basis $\{x_{i,\beta}, f_{i,\beta}\}$ of X_β such that

$$\operatorname{span}(\{x_{i,\beta}\}) = \operatorname{span}(\{z_j : j \in J_{\beta+1} \setminus J_\beta\}), \text{ and} \tag{1}$$

$$\operatorname{span}(\{f_{i,\beta}\}) = \operatorname{span}(\{g_j : j \in J_{\beta+1} \setminus J_\beta\}). \tag{2}$$

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Since $(P^*_{\beta+1} - P^*_\beta)f_{i,\beta} = f_{i,\beta}$ it follows that

$$\{x_i, f_i\}_{i\in I} = \bigcup_{\omega_0 \leq \beta < \mu} \{x_{i,\beta}, f_{i,\beta}\}$$

is the desired M-basis on X.

In order to prove the first part of the theorem, one could set $Y_\beta = \{f|_{X_\beta} : f \in Y\}$ and obtain an M-basis $\{x_{i,\beta}, f_{i,\beta}\}$ on X_β which satisfies (1) and every element of Y_β would be countably supported on $\{x_{i,\beta}\}$. Letting

$$\{x_i, f_i\}_{i\in I} = \bigcup_{\omega_0 \leq \beta < \mu} \{x_{i,\beta}, (P^*_{\beta+1} - P^*_\beta)f_{i,\beta}\}$$

we see that each $f \in Y$ is countably supported on $\{x_i\}_{i\in I}$ since $f|_{X_\beta} = 0$ except for countably many β. ∎

3. **Spaces with Markušević bases.** We now use the results in Section 2 to characterize Banach spaces that admit Markušević bases with various properties related to the notion of Corson compacta.

The following is a known result which follows from [AL]. Note that the conditions in Theorem 3.1 are equivalent to X being WCG; see [AL].

THEOREM 3.1 ([AL]). *For a Banach space X the following are equivalent.*

(a) There is a bounded linear injection $T: X^* \longrightarrow c_0(\Gamma)$ such that T is w^* to pointwise continuous.

(b) X admits an M-basis $\{x_i, f_i\}_{i\in I}$ such that $\{x_i\}_{i\in I} \cup \{0\}$ is weakly compact.

PROOF. (a) ⇒ (b): Let $z_\gamma = \pi_\gamma \circ T$ where π_γ is the projection onto the γ-th coordinate in $c_0(\Gamma)$. Since T is w^* to pointwise continuous, it follows that $z_\gamma \in X$. Moreover $\overline{\text{span}}(\{z_\gamma\}_{\gamma\in\Gamma}) = X$ by the Hahn-Banach theorem since T is one-to-one. In addition, $\{z_\gamma\}_{\gamma\in\Gamma} \cup \{0\}$ is weakly compact since $TX^* \subset c_0(\Gamma)$. Now we can certainly apply Proposition 2.3 since $\{f(z_\gamma)\}_{\gamma\in\Gamma} \in c_0(\Gamma)$ for each $f \in X^*$. Therefore, there is an M-basis $\{y_i, g_i\}_{i\in I}$ such that $\text{span}(\{y_i\}_{i\in I}) = \text{span}(\{z_\gamma\}_{\gamma\in\Gamma})$. For each i, choose $c_i > 0$ so that $c_i y_i \in \text{aco}(\{z_\gamma\}_{\gamma\in\Gamma})$. By the Krein-Šmulyan theorem $\overline{\text{aco}}(\{z_\gamma\}_{\gamma\in\Gamma})$ is weakly compact. Let $x_i = c_i y_i$ and $f_i = c_i^{-1} g_i$. It follows that $\{x_i\}_{i\in I} \cup \{0\}$ is weakly compact since 0 is its only weak limit point.

(b) ⇒ (a): For $f \in X^*$, let $Tf = \{f(x_i)\}_{i\in I}$, then $Tf \in c_0(I)$. ∎

We now look at spaces X such that there is a one-to-one bounded linear map $T: X^* \longrightarrow c_1(\Sigma' \times \Gamma)$ which is w^* to pointwise continuous; these spaces coincide with the class of WCD Banach spaces (see [M, Theorem 4.1]). Recall that $\Sigma' \subset \mathbb{N}^{\mathbb{N}}$, $c_1(\Sigma' \times \Gamma) \subset l^c_\infty(\Sigma' \times \Gamma)$ and for $K \in \mathcal{K}(\Sigma')$, $c_1(K \times \Gamma) \subset c_0(K \times \Gamma)$ where $\mathcal{K}(\Sigma')$ denotes the compact subsets of Σ'. See [DGZ, Chapter VI] for a detailed discussion of the spaces $c_1(\Sigma' \times \Gamma)$.

The aim of the following result is to give an M-basis version of condition (b) in Theorem 4.1 of [M].

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THEOREM 3.2. *For a Banach space X the following are equivalent.*

(a) There is a bounded linear injection $T: X^* \longrightarrow c_1(\Sigma' \times \Gamma)$ such that T is w^* to pointwise continuous.

(b) There is an M-basis $\{x_\alpha, f_\alpha\}_{\alpha \in A}$ of X such that

$$\{x_\alpha\}_{\alpha\in A} \bigcup \{0\} = \bigcup\{L_K : K \in \mathcal{K}(\Sigma')\}$$

where L_K is weakly compact and $K_1 \subset K_2$ implies $L_{K_1} \subset L_{K_2}$.

PROOF. We only prove (a) $\Rightarrow$ (b) since the reverse implication follows from [M, Theorem 4.1]. Set $z_i = \pi_i \circ T$ where π_i is the projection onto the i-th coordinate in $c_1(\Sigma' \times \Gamma)$. The Hahn-Banach theorem and the fact that T is one-to-one imply $\overline{\text{span}}(\{z_i : i \in \Sigma' \times \Gamma\}) = X$. Because $c_1(\Sigma' \times \Gamma) \subset l^c_\infty(\Sigma' \times \Gamma)$, as in the proof of Theorem 3.1, we use Theorem 2.3 to conclude there is an M-basis $\{x_\alpha, f_\alpha\}_{\alpha\in A}$ such that $\{x_\alpha\}_{\alpha \in A} \subset \text{aco}(\{z_i : i \in \Sigma' \times \Gamma\})$. For each $K \in \mathcal{K}(\Sigma')$, let $S_K = \{z_i : i \in K \times \Gamma\}$. Now let

$$L_K = \{x_\alpha : x_\alpha \in \text{aco}(S_K)\} \cup \{0\}.$$

Notice that $S_K \cup \{0\}$ is weakly compact since $c_1(K \times \Gamma) \subset c_0(K \times \Gamma)$. Because $L_K \subset \text{aco}(S_K)$ it follows as in the proof of Theorem 3.1 that L_K is weakly compact. Finally, observe for a fixed α that $x_\alpha \in \text{aco}(\{z_{i_1}, \ldots, z_{i_n}\})$ for some $\{i_j\}_{j=1}^n$. Now, $z_{i_j} \in S_{K_j}$ for some $K_j \in \mathcal{K}(\Sigma')$ so it follows that $i_j \in (K_{j_1} \cup \cdots \cup Kj_n) \times \Gamma$ for $1 \le j \le n$, that is

$$x_\alpha \in \text{aco}(S_{(K_{i_1} \cup \cdots \cup K_{i_n})}) \quad \text{and} \quad x_\alpha \in L_{(K_{i_1} \cup \cdots \cup K_{i_n})}.$$

This shows that $\{x_\alpha\}_{\alpha\in A} \bigcup\{0\} = \bigcup\{L_K : K \in \mathcal{K}(\Sigma')\}$. ∎

DEFINITION. Following [Po1], we shall say a Banach space X has *property* (C) if every collection of closed convex bounded sets with empty intersection contains a countable subcollection with empty intersection.

The following result extends [AM, Theorem 1.6].

THEOREM 3.3. *For a Banach space X the following are equivalent.*

(a) X is WLD.

(b) X admits an M-basis $\{x_i, f_i\}_{i \in I}$ such that $|\{i : f(x_i) \neq 0\}| \le \aleph_0$ for each $f \in X^*$.

(c) X admits an M-basis $\{x_i, f_i\}_{i\in I}$ such that $\{x_i\}_{i\in I} \cup \{0\}$ is weakly Lindelöf in its restricted topology.

(d) X is weakly Lindelöf and admits an M-basis.

(e) X has property (C) and admits an M-basis.

PROOF. (a) $\Rightarrow$ (b): Let $T: X^* \longrightarrow l^c_\infty(\Gamma)$ be as in the definition of WLD spaces. As in the proof of Theorem 3.1, $\overline{\text{span}}(\{z_\gamma : \gamma \in \Gamma\}) = X$ for $z_\gamma = \pi_\gamma \circ T$. Applying Theorem 2.3 with $Y = X^*$ we obtain an M-basis $\{x_i, f_i\}_{i\in I}$ of X such that $|\{i : f(x_i) \neq 0\}| \le \aleph_0$ for each $f \in X^*$.

(b) $\Rightarrow$ (c): Let $\{x_i, f_i\}_{i\in I}$ be an M-basis as in (b). If U is a weak neighborhood of 0, then $|\{i : x_i \notin U\}| \le \aleph_0$. Thus $\{x_i\}_{i\in I} \cup \{0\}$ is weakly Lindelöf.

(c) $\Rightarrow$ (a): Let $\{x_i,f_i\}_{i\in I}$ be an M-basis as in (c). For $\epsilon > 0$ and $f \in X^*$, let $U(f,\epsilon) = \{x : |f(x)| < \epsilon\}$ and let $U_i = \{x : f_i(x) > 0\}$. By the hypothesis, this collection has a countable subcover. Since $x_j \notin U_i$ for $i \neq j$, it follows that all but countably many x_i are in $U(f,\epsilon)$. Thus $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$. Now normalize $\{x_i\}_{i\in I}$ so that $\{x_i\}_{i\in I} \subset B_X$. It follows that $T\colon X^* \longrightarrow l^c_\infty(I)$ where T is defined by $Tf = \{f(x_i)\}_{i\in I}$ for $f \in X^*$ is the desired operator.

(a) $\Rightarrow$ (d): From [AP] it follows that a WLD space is weakly Lindelöf (see *e.g.* [AM, Theorem 1.3]). The implication (a) $\Rightarrow$ (b) shows X admits an M-basis.

(d) $\Rightarrow$ (e): This follows immediately from the definitions.

(e) $\Rightarrow$ (a): Let $\{x_i,f_i\}_{i\in I}$ be an M-basis of X with $\{x_i\}_{i\in I} \subset B_X$. Define $T\colon X^* \longrightarrow l_\infty(I)$ by $Tf = \{f(x_i)\}_{i\in I}$. If $Tf \notin l^c_\infty(I)$ then replacing f by $-f$ if necessary, $|\{i : f(x_i) \geq \epsilon\}| > \aleph_0$ for some $\epsilon > 0$. By reindexing we can write $f(x_\beta) \geq \epsilon$ for $0 \leq \beta < \omega_1$ for some subcollection $\{x_\beta\}_{0\leq\beta<\omega_1} \subset \{x_i\}_{i\in I}$. Define $C_\beta = \overline{\operatorname{span}}(\{x_\nu : \beta \leq \nu < \omega_1\} \bigcup \{x : f(x) \geq \epsilon\} \cap B_X)$. Thus C_β is bounded, closed and convex for each β. Moreover $\bigcap_{\beta<\omega_1} C_\beta = \emptyset$ while no countable subcollection of $\{C_\beta\}_\beta$ has empty intersection. This cannot happen since X has property (C), therefore, $Tf \in l^c_\infty(I)$. ∎

COROLLARY 3.4. *(a) Suppose Z and X/Z have property (C). If X admits an M-basis, then X is* WLD.

(b) Suppose Z and X/Z are both WLD *and Asplund. Then X is* WCG *if X admits an M-basis.*

PROOF. (a) By [Po1, Proposition 1], X has property (C). Invoking Theorem 2.6 completes the proof.

(b) By part (a), X is WLD. Since Z and X/Z are Asplund, it follows that X is Asplund [NP]). Hence X is WCG, because [V1, Proposition 2] shows that a WLD Asplund space is WCG. ∎

Notice that (e) in Theorem 3.3 cannot be replaced with the weaker condition: "X has property (C) and injects into some $c_0(\Gamma)$". Moreover, it is necessary to assume X admits an M-basis in Corollary 3.4 because there are spaces such that both Z and X/Z are WLD and Asplund (in particular X has property (C) by [Po1, Proposition 1]), yet X fails to admit an M-basis. Indeed let $X =$ JL be the Johnson-Lindenstrauss space constructed in [JL]. Then is a subspace Z of X such that Z is isomorphic to $c_0(\mathbb{N})$ and X/Z is isomorphic to a nonseparable Hilbert space. However X does not admit an M-basis (see Remark 3.6(a)) but X linearly injects into $c_0(\Gamma)$ since X^* is WCG.

The following test is useful in showing certain spaces do not admit M-bases.

COROLLARY 3.5. *Suppose X has property (C). If there is a subset $S \subset X^*$ such that S is total on X and $|S| <$* dens(X), *then X does not admit an M-basis.*

PROOF. Suppose such an X admits an M-basis $\{x_i,f_i\}_{i\in I}$, then X is WLD by Theorem 3.3. Now let S be any total subset of X^* and observe that

$$I = \bigcup_{f\in S} \{i : f(x_i) \neq 0\}.$$

By Theorem 1.1 it follows that $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in B_{X^*}$. Therefore $|S| \geq |I|$ and the corollary is proved. ∎

REMARK 3.6. (a) [P1] The space JL does not admit an M-basis. This easily follows from Corollary 3.5 because JL has property (C) and is nonseparable while JL* has a countable total set.

(b) Let X be the space of right continuous functions on $[0, 1]$ which have left limits. Recall that $X/C[0, 1]$ is isomorphic to $c_0(\Gamma)$ with $|\Gamma| = |\mathbb{R}|$. Because $c_0(\Gamma)$ and $C[0, 1]$ both have property (C), it follows from [Po1, Proposition 1] that X has property (C). However X^* contains a countable set total on X and thus Corollary 3.5 ensures that X has no M-basis.

We should point out that one can use Theorem 2.3 (as in Theorem 3.3) to show that X admits a countably norming M-basis if and only if there is a bounded linear w^* to pointwise continuous injection $T: X^* \longrightarrow l_\infty(\Gamma)$ such that $TY \subset l^c_\infty(\Gamma)$ where Y is norming on X.

The following theorem characterizes Banach spaces which admit M-bases.

THEOREM 3.7. *For a Banach space X, the following are equivalent.*

(a) X admits an M-basis.

(b) There is a subspace $Y \subset X^*$ total on X and a bounded linear one-to-one operator $T: X^* \longrightarrow l_\infty(\Gamma)$ which is w^* to pointwise continuous such that $TY \subset l^c_\infty(\Gamma)$.

(c) There is a subset $\{x_\alpha\}_{\alpha \in A}$ of X and a subspace $Y \subset X^*$ total on X such that $\overline{\mathrm{span}}(\{x_\alpha : \alpha \in A\}) = X$ and $|\{\alpha : f(x_\alpha) \neq 0\}| \leq \aleph_0$ for each $f \in Y$.

PROOF. (a) $\Rightarrow$ (b): Let $T: X^* \longrightarrow l_\infty(\Gamma)$ be defined by $Tf = \{f(x_\gamma)\}_{\gamma \in \Gamma}$ where $\{x_\gamma, f_\gamma\}_{\gamma \in \Gamma}$ is an M-basis of X and $\{x_\gamma\}_{\gamma \in \Gamma} \subset B_X$. We see that this T with $Y = \overline{\mathrm{span}}(\{f_\gamma\}_{\gamma \in \Gamma})$ satisfy the conditions in (b).

(b) $\Rightarrow$ (c): Let T and Y be as in (b) and let $e_\gamma = \pi_\gamma \circ T$ where π_γ is the projection of $l_\infty(\Gamma)$ onto its γ-th coordinate. Clearly $e_\gamma \in X$, $\overline{\mathrm{span}}(\{e_\gamma : \gamma \in \Gamma\}) = X$ (by the Hahn-Banach theorem since T is one-to-one) and $|\{\gamma : f(e_\gamma) \neq 0\}| \leq \aleph_0$ for each $f \in Y$.

(c) $\Rightarrow$ (a): Define a new norm $|\cdot|$ on X by

$$|x| = \sup\{f(x) : f \in Y \cap B_{X^*}\}.$$

Let $(\tilde{X}, |\cdot|)$ be the completion of $(X, |\cdot|)$ and $\tilde{Y}$ be all Hahn-Banach extensions of Y on $\tilde{X}$. Observe that $\overline{\mathrm{span}}^{|\cdot|}(\{x_\alpha : \alpha \in A\}) = \tilde{X}$, $|\{\alpha : f(x_\alpha) \neq 0\}| \leq \aleph_0$ for each $f \in \tilde{Y}$ and $\tilde{Y}$ is 1-norming on $(\tilde{X}, |\cdot|)$. According to Theorem 2.3 there is a countably 1-norming M-basis $\{z_i, f_i\}_{i \in I}$ of $(\tilde{X}, |\cdot|)$ such that $\mathrm{span}(\{z_i : i \in I\}) = \mathrm{span}(\{x_\alpha : \alpha \in A\})$. Since $|\cdot| \leq \|\cdot\|$ where $\|\cdot\|$ is the original norm on X it follows that $\hat{f}_i = f_i|X \in (X, \|\cdot\|)^*$. Therefore $\{z_i, \hat{f}_i\}$ is an M-basis of $(X, \|\cdot\|)$. ∎

We outline how the methods used in the proof of Theorem 3.7 can be used to obtain

THEOREM 3.8. *For a Banach space $(X, \|\cdot\|)$, the following are equivalent.*

(a) $(X, \|\cdot\|)$ continuously linearly injects into $c_0(\Gamma)$ for some Γ.

(b) There is a total linear subspace $Y \subset X^*$ and a convex symmetric $K \subset Y$ which is Corson compact in its w^*-topology such that $\operatorname{span}(K) = Y$.

PROOF. (b) $\Rightarrow$ (a): Consider $(X, |\cdot|)$ where $|x| = \sup\{f(x) : f \in K\}$, and let $(\tilde{X}, |\cdot|)$ be the completion of $(X, |\cdot|)$. Then $\tilde{K} = \{\tilde{f} \in (\tilde{X}, |\cdot|)^* : \tilde{f}|_X \in K\}$ is the unit ball of $(\tilde{X}, |\cdot|)^*$ and is Corson compact in its w^*-topology since K is. By Theorem 1.1, $(\tilde{X}, |\cdot|)$ has an M-basis, say, $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ with $\{f_\gamma\}_{\gamma\in\Gamma}$ bounded. Then the mapping $Tx = \{f_\gamma(x)\}_{\gamma\in\Gamma}$ is a continuous injection of $(\tilde{X}, |\cdot|)$ into $c_0(\Gamma)$. Certainly T is continuous on $(X, \|\cdot\|)$.

(a) $\Rightarrow$ (b): Let T be a bounded linear injection of X into $c_0(\Gamma)$. Set $K = T^* B_{c_0^*(\Gamma)}$ and $Y = \operatorname{span}(K)$; K is Corson (even Eberlein) compact in its w^*-topology and Y is total on X since T is one-to-one. ∎

4. **Norming Markuševič bases.** We begin with a characterization of spaces that admit 1-norming Markuševič bases.

THEOREM 4.1. *For a Banach space X, the following are equivalent.*

(a) X admits a 1-norming M-basis.

(b) There is a 1-norming subspace $Y = \overline{\operatorname{span}}(\{f_i : i \in I\})$ such that $|\{i \in I : f_i(x) \neq 0\}| \leq \aleph_0$ for every $x \in X$ and a bounded linear one-to-one w^* to pointwise continuous operator $T: X^* \longrightarrow l_\infty(I)$ such that $TY \subset l^c_\infty(I)$.

(c) There is a collection $\{x_i\}_{i\in I}$ norm dense in X and a 1-norming subspace $Y = \overline{\operatorname{span}}(\{f_j : j \in I\})$ such that $|\{j \in J : f_j(x) \neq 0\}| \leq \aleph_0$ for every $x \in X$ and $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in Y$.

PROOF. One can prove (a) implies (b) and (b) implies (c) exactly as in Theorem 3.7. Using Lemma 2.2, we have $|I| = |J|$ and thus (c) implies (b) follows from Theorem 2.3. ∎

COROLLARY 4.2. *Let X be a* WLD *space. Then the following are equivalent:*

(a) there is a bounded operator $T: X \longrightarrow l^c_\infty(\Gamma)$ for some Γ such that $T^*\big(l^c_\infty(\Gamma)\big)^*$ is a 1-norming subspace of X^*;

(b) X admits a 1-norming M-basis.

PROOF. (a) $\Rightarrow$ (b): Let $f_\gamma = T^* e_\gamma$ and $Y = \overline{\operatorname{span}}(\{f_\gamma\}_{\gamma\in\Gamma})$. Then $|\{\gamma : f_\gamma(x) \neq 0\}| \leq \aleph_0$ for all $x \in X$. Because X is WLD, X has an M-basis $\{x_i, f_i\}_{i\in I}$ such that each $f \in X^*$ is countably supported on it (see *e.g.* Theorem 3.3). Invoking Theorem 4.1(c) shows that (a) implies (b).

(b) $\Rightarrow$ (a): Fix a 1-norming M-basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ with $\|f_\gamma\| \leq 1$. The operator $T: X \longrightarrow l^c_\infty(\Gamma)$ defined by $Tx = \{f_\gamma(x)\}_{\gamma\in\Gamma}$ does the job. ∎

The following corollary gives a sufficient condition for a space to admit a 1-norming M-basis.

COROLLARY 4.3. *Suppose $Y \subset X^*$ is 1-norming and each $f \in Y$ is countably supported on $\{z_j\}_{j\in J}$ where $\overline{\operatorname{span}}(\{z_j\}_{j\in J}) = X$. If Y is* WLD, *then X admits a 1-norming M-basis.*

PROOF. Since Y is WLD, we choose (using Theorem 3.3) and M-basis $\{g_i, h_i\}_{i\in I}$ of Y so that $|\{i : g_i(x) \neq 0\}| \leq \aleph_0$ for each $x \in X \subset X^{**}$. By Theorem 4.1, X admits a 1-norming M-basis. ∎

In particular, according to Corollary 4.3, if X admits a countably 1-norming M-basis and X^* is WLD, then X admits a 1-norming M-basis. This follows because subspaces of WLD spaces are WLD which is a consequence of the fact that continuous images of Corson compact spaces are Corson compact (see *e.g.* [N, Theorem 6.26]).

The following result was obtained some time ago by R. Deville, G. Godefroy, S. Troyanski and the third named author. It has also been obtained independently by M. Valdivia [V4]. For the reader's convenience, we outline a proof of this result here. Recall that a norm $\|\cdot\|$ is said to be *locally uniformly rotund* (LUR) if $\|x - x_n\| \longrightarrow 0$ whenever $2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \longrightarrow 0$.

THEOREM 4.4. *Let* JL *denote the Johnson-Lindenstrauss space of [JL] such that if* $X = \mathrm{JL}^*$, *then* X *is isomorphic to* $l_2(\aleph_1) \oplus l_1(\mathbb{N})$. *Then* $(X, \|\cdot\|^*)$ *does not admit a* 1*-norming* M*-basis whenever* $\|\cdot\|^*$ *is dual to an* LUR *norm* $\|\cdot\|$ *on* JL.

We refer the reader *e.g.* to [DGZ], Chapter VII for the existence of an LUR norm on JL.

PROOF. Suppose $\{x_i, f_i\}_{i\in I}$ is a 1-norming M-basis on $(X, \|\cdot\|^*)$. Let $Y = \overline{\mathrm{span}}(\{f_i\}_{i\in I})$. Given $x \in S_{\mathrm{JL}}$, we choose $x^* \in S_X$ such that $x^*(x) = 1$. Because Y is norming one can select $x_n \in S_Y$ such that $x^*(x_n) \longrightarrow 1$. Now $\|x_n + x\| \longrightarrow 2$ and thus $\|x_n - x\| \longrightarrow 0$ because $\|\cdot\|$ is LUR. From this it follows that $\mathrm{JL} \subset Y$. The space Y admits a 1-norming M-basis and thus it has a PRI $\{P_\alpha\}_{\omega_0 \leq \alpha \leq \omega_1}$ with $P_\alpha Y$ separable for each α (see *e.g.* Theorem 2.1). By a standard countable exhaustion argument $c_0(\mathbb{N}) \subset P_\alpha Y$ for some α, since $c_0(\mathbb{N}) \subset \mathrm{JL}$. Thus by Sobczyk's theorem, $c_0(\mathbb{N})$ is complementable in $P_\alpha Y$ and thus in JL. This is not possible since JL is not isomorphic to $l_2(\aleph_1) \oplus c_0(\mathbb{N})$. ∎

REMARK 4.5. Let $(X, \|\cdot\|^*)$ be as in Theorem 4.4. Then there is no WLD space $Y \subset (X^*, \|\cdot\|^{**})$ such that Y would be 1-norming.

Note that every $f \in Y$ is countably supported on any M-basis of X since X is WCG. Thus the remark follows from Remark 4.3.

The following improves the heredity result of [JZ, Proposition 6].

PROPOSITION 4.6. *If* X *admits a* 1*-norming* M*-basis and* $Z \subset X$ *is* WLD, *then* Z *admits a* 1*-norming* M*-basis.*

PROOF. Let $\{x_i, f_i\}_{i\in I}$ be a 1-norming M-basis on X. Let $g_i = f_i|_Z$. Re-indexing to avoid duplicates, we have $Y = \overline{\mathrm{span}}(\{g_j : j \in J\})$ is 1-norming on Z and $|\{j : g_j(z) \neq 0\}| \leq \aleph_0$ for each $z \in Z$. Let $\{z_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ be an M-basis of Z, then $|\{\gamma : g_j(z_\gamma) \neq 0\}| \leq \aleph_0$ for each $j \in J$ (Theorem 1.1). By Theorem 4.1, Z admits a 1-norming M-basis. ∎

COROLLARY 4.7. *There is an Eberlein compact* K *such that* $C(K)$ *in its sup norm does not admit a* 1*-norming* M*-basis.*

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PROOF. Let $X = \mathrm{JL}^*$ be a equipped with a norm that is dual to an LUR norm on JL and let $K = B_{X^*}$. Since X is a WLD subspace of $C(K)$, it follows from Proposition 4.6 and Theorem 4.4 that $C(K)$ admits no 1-norming M-basis. ∎

Recently Gilles Godefroy has used Proposition 4.6 to obtain a much sharper result than that in Corollary 4.7. However, the following questions are still open.

QUESTIONS 4.8. Does every WCG space or every WLD space admit a norming Markuševič basis? If γ is an ordinal, does the space $C([0,\gamma])$ admit a norming Markuševič basis?

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