Delaunay ends of constant mean curvature surfaces

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Abstract

The generalized Weierstrass representation is used to analyze the asymptotic behavior of a constant mean curvature surface that arises locally from an ordinary differential equation (ODE) with a regular singularity. We prove that a holomorphic perturbation of an ODE that represents a Delaunay surface generates a constant mean curvature surface which has a properly immersed end that is asymptotically Delaunay. Furthermore, that end is embedded if the Delaunay surface is unduloidal.

Introduction

Delaunay surfaces play a prominent role in the theory of non-compact complete constant mean curvature (CMC) surfaces because they constitute the simplest possible end behavior. A famous result by Korevaar *et al.* [KKS89], building on results of Meeks [Mee88], asserts that a properly embedded annular end of a CMC surface is a Delaunay end. The study of Delaunay ends by the conjugate surface methods of Große-Brauckmann *et al.* [GKS03, GKS05] requires the additional assumption of Alexandrov embeddedness, and is restricted to embedded (unduloidal) Delaunay ends. The gluing techniques of Mazzeo and Pacard [MP01, MP02] are limited to attaching Delaunay ends with small asymptotic necksizes.

The methods used in this paper, based on the generalized Weierstrass representation of Dorfmeister *et al.* [DPW98], provide a means to study both embedded unduloidal and non-Alexandrov-embedded nodoidal type ends of arbitrary asymptotic necksize. The results presented here were initially motivated by recent examples of CMC surfaces, such as the three-parameter families of trinoids in the three space forms [SKKR], the n-noids [RS06, Sch07a], the dressed Delaunay surfaces (bubbletons) [PS93, SW93], the n-noid bubbletons [KKS89] and perturbed Delaunay surfaces [KMS00]. On the evidence of computer experiments [Sch07b, Sch02b], it has been long thought that the ends of these surfaces are asymptotic to half unduloids and half nodoids. We prove this asymptotic behavior here.

The generalized Weierstrass representation describes CMC immersions locally via potentials. The relation between the potential and the immersion involves a loop group valued differential equation, a loop group factorization and a Sym–Bobenko type formula [Bob91b, Sym85]. Hence, the method provides only an indirect relation between the geometric properties of the induced immersion and its potential. Nevertheless, the method has been useful in proving the existence of many new classes of non-compact CMC surfaces with non-trivial topology. Particular progress has been made when the surface is homeomorphic to an n-punctured Riemann sphere [KMS00, KSS04, Sch07a, SKKR]. In this case, the punctures correspond to poles of the potential. We prove that an immersion whose

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potential has a simple pole with an appropriate residue has, at that pole, an end which is asymptotic to a half-Delaunay surface.

More specifically, given a potential $A\,dz/z$ of a Delaunay surface, consider a holomorphic perturbation $\xi = A\,dz/z + \mathrm{O}(z^0)\,dz$. Our main result, stated precisely as Theorem 3.6, and generalized in Theorem 5.9, is as follows.

THEOREM. *An annular CMC immersion induced by a holomorphic perturbation of a Delaunay potential is C^∞-asymptotic to a half-Delaunay surface. In particular, it is properly immersed. Moreover, if the underlying half-Delaunay surface is embedded, then the end of the immersion is properly embedded.*

The surface induced by a perturbed Delaunay potential may gain topology or geometric complexity; see, for example, the n-noids [RS06, Sch07a, SKKR], as shown in Figure 1, and higher genus examples with ends [KKRS05]. Since these surfaces are obtained from potentials which at each end are perturbations of Delaunay potentials, all of their ends are asymptotically Delaunay.

The convergence of the surfaces is obtained by showing that their moving frames and metrics converge. More specifically, let Φ_0 and Φ be the respective solutions to the ordinary differential equations (ODEs) $d\Phi_0 = \Phi_0 A\,dz/z$ and its perturbation $d\Phi = \Phi\xi$. The convergence of the ratio of Φ to Φ_0 is shown using the holomorphic gauge relating them at the regular singularity $z = 0$. The periodicity of the Delaunay surface provides growth rate estimates on the positive part of Φ_0 by Floquet analysis. This leads to the convergence of their unitary and positive factors, in turn implying C^1-convergence of the surfaces. A bootstrap argument strengthens this to C^∞-convergence.

In Part II we deal with the situation in which the initial condition to the ODE does not extend holomorphically from the r-circle to the unit circle, even though the monodromy of the solution is still unitary. We show that in this setting the solution has acquired singularities that arise from Bianchi–Bäcklund transforms. Thus, the second part accommodates the additional singularities that appear from dressing by simple factors and proves that adding bubbles to a surface with a Delaunay end preserves this Delaunay end.

The generalized Weierstrass representation

The r-Iwasawa factorization

We use the following subsets of $\mathbb{C}$:

$$\mathbb{S}^1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\} \quad \text{and} \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\},$$
$$\mathcal{C}_r = \{\lambda \in \mathbb{C} \,|\, |\lambda| = r\}, \quad r \in (0, 1],$$
$$\mathcal{D}_r = \{\lambda \in \mathbb{C} \,|\, |\lambda| < r\}, \quad \mathcal{D}_r^* = \mathcal{D}_r \setminus \{0\}, \quad r \in (0, 1],$$
$$\mathcal{A}_{s,r} = \{\lambda \in \mathbb{C} \,|\, s < |\lambda| < r\}, \quad 0 < s < r,$$
$$\mathcal{A}_r = \mathcal{A}_{r,1/r}, \quad r \in (0, 1).$$

Given a domain $\mathcal{U} \subset \mathbb{C} \setminus \{0\}$ which is invariant under the map $\lambda \mapsto 1/\overline{\lambda}$, and a holomorphic map $X : \mathcal{U} \to \mathrm{M}_{2\times 2}\mathbb{C}$, define the holomorphic map $X^* : \mathcal{U} \to \mathrm{M}_{2\times 2}\mathbb{C}$ by

$$X^*(\lambda) := \overline{X(1/\overline{\lambda})}^t.$$

We use the following loop groups.

- For $r \in (0, 1]$, $\Lambda_r \mathrm{SL}_2\mathbb{C}$ is the group of analytic maps $\mathcal{C}_r \to \mathrm{SL}_2\mathbb{C}$.
- For $r \in (0, 1)$, $\Lambda_r^* \mathrm{SL}_2\mathbb{C} \subset \Lambda_r \mathrm{SL}_2\mathbb{C}$ is the subgroup of loops each of which is the boundary of a holomorphic map $Y : \mathcal{A}_r \to \mathrm{SL}_2\mathbb{C}$ satisfying the condition $Y^* = Y^{-1}$. The subgroup $\Lambda_1^* \mathrm{SL}_2\mathbb{C} \subset \Lambda_1 \mathrm{SL}_2\mathbb{C}$ is the subgroup of loops X satisfying the condition $X^* = X^{-1}$.
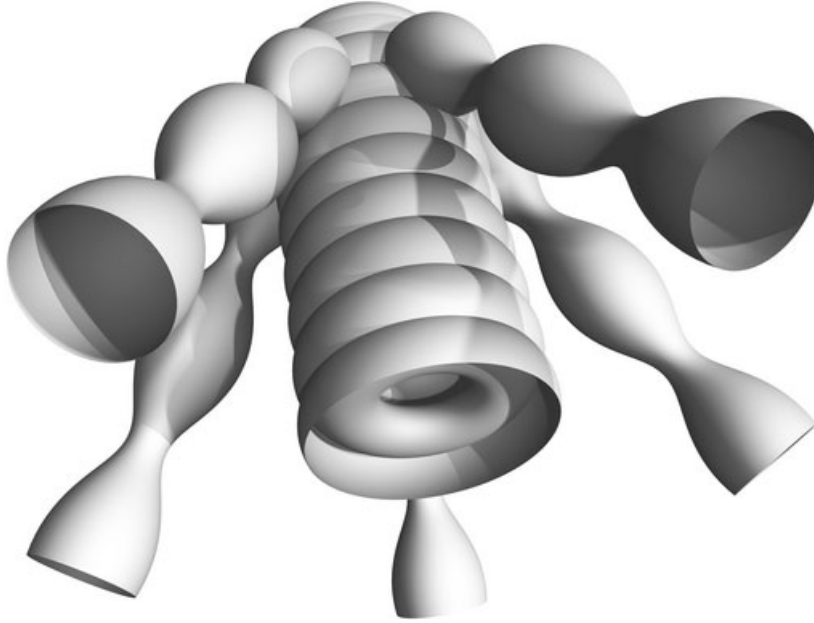
Figure 1. A CMC immersion of the six-punctured sphere with asymptotically Delaunay ends and pyramidal symmetry [Sch07a]. Five of the six ends are unduloidal; the sixth is nodoidal with large negative weight. This image was created with CMCLab [Sch02a], a freely available software implementation of the generalized Weierstrass representation.

- For $r \in (0,1]$, $\Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C} \subset \Lambda_r\mathrm{SL}_2\mathbb{C}$ is the subgroup of loops for which each loop is the boundary of a holomorphic map $\mathcal{D}_r \to \mathrm{SL}_2\mathbb{C}$.

- Let $\mathcal{T} \subset \mathrm{SL}_2\mathbb{C}$ denote the group of upper triangular matrices whose diagonal elements are in $\mathbb{R}_{>0}$. For $r \in (0,1]$, $\Lambda_r^+\mathrm{SL}_2\mathbb{C} \subset \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$ is the subgroup of loops X such that $X(0) \in \mathcal{T}$.

- For $r \in (0,1)$, $\Lambda_r^{\uparrow}\mathrm{SL}_2\mathbb{C} \subset \Lambda_r\mathrm{SL}_2\mathbb{C}$ is the group of analytic maps $X \in \Lambda_r\mathrm{SL}_2\mathbb{C}$ such that X is the boundary of a holomorphic map $\mathcal{A}_{r,1} \to \mathrm{SL}_2\mathbb{C}$.

- For $r \in (0,1)$, $\Lambda_r^{*\mathrm{M}}\mathrm{SL}_2\mathbb{C} \subset \Lambda_r\mathrm{SL}_2\mathbb{C}$ is the group of analytic maps $X \in \Lambda_r\mathrm{SL}_2\mathbb{C}$ such that X is the boundary of a meromorphic map $\mathcal{A}_r \to \mathrm{SL}_2\mathbb{C}$ satisfying $X^* = X^{-1}$ away from its poles. For $r = 1$, $\Lambda_1^{*\mathrm{M}}\mathrm{SL}_2\mathbb{C} = \Lambda_1^*\mathrm{SL}_2\mathbb{C}$.

For $k \in \mathbb{Z}_{\geqslant 0}$, we define the C^k-topology on each of these loop groups with respect to the loop parameter λ in their respective domains $\mathcal{C}_r$, $\mathcal{A}_r$ or $\mathcal{D}_r$. The C^∞-topology is the intersection of the C^k-topologies. The multiplication map

$$\Lambda_r^*\mathrm{SL}_2\mathbb{C} \times \Lambda_r^+\mathrm{SL}_2\mathbb{C} \to \Lambda_r\mathrm{SL}_2\mathbb{C}$$

is a C^∞ diffeomorphism [BP95, Mci94]. The unique factorization of a loop $\Phi \in \Lambda_r\mathrm{SL}_2\mathbb{C}$ into

$$\Phi = \mathsf{Uni}_r[\Phi] \cdot \mathsf{Pos}_r[\Phi]$$

is the *r-Iwasawa factorization* of Φ. We call $\mathsf{Uni}_r[\Phi]$ the *r-unitary factor* and $\mathsf{Pos}_r[\Phi]$ the *r-positive factor* of Φ. The *QR-factorization* is the Iwasawa factorization of constant loops $\mathrm{SL}_2\mathbb{C} \to \mathrm{SU}_2 \times \mathcal{T}$.

Dressing a loop $X \in \Lambda_r\mathrm{SL}_2\mathbb{C}$ by $C \in \Lambda_r\mathrm{SL}_2\mathbb{C}$ is left-multiplication of X by C, followed by projection of the r-Iwasawa factorization to the unitary group. We denote the dressed loop by $C\#_r X$ or $\mathsf{Uni}_r[CX]$.

188

The generalized Weierstrass representation

The generalized Weierstrass representation [DPW98] represents harmonic maps in terms of certain holomorphic 1-forms with values in a loop algebra (*holomorphic potentials*). This representation constructs all CMC surfaces in the three-dimensional Euclidean, spherical and hyperbolic space-forms [SKKR], and is as follows for Euclidean 3-space.

1. Let Σ be a Riemann surface. With $r \in (0, 1]$, let ξ be an *r-potential*, that is, a $\Lambda_r \mathrm{sl}_2 \mathbb{C}$-valued meromorphic 1-form on Σ which is the boundary of a meromorphic differential on $\mathcal{D}_r$, with a pole only at $\lambda = 0$, which is simple and is only in the upper-right entry. To avoid branch points in the induced surface, assume also that the coefficient of λ^{-1} in the series expansion of the upper-right entry of ξ in λ at $\lambda = 0$ is never zero on Σ.

2. Let Φ be a solution to the ODE $d\Phi = \Phi\xi$ on the universal cover $\widetilde{\Sigma}$ of Σ. We call Φ the *holomorphic frame*.

3. Then $F = \mathsf{Uni}_r[\Phi]$ is the extended frame of some CMC immersion $f : \widetilde{\Sigma} \to \mathbb{R}^3$.

4. The Sym formula
$$f = \mathsf{Sym}_r[\Phi] = -2H^{-1}F'F^{-1}$$
computes the associate family of CMC immersion f with CMC $H \in \mathbb{R}^*$ from the frame F, with associate family parameter $\lambda \in \mathbb{S}^1$. Here, the prime denotes differentiation with respect to θ, where $\lambda = e^{i\theta} \in \mathbb{S}^1$.

The immersion f of $\widetilde{\Sigma}$ descends to an immersion of Σ at $\lambda = 1$ if and only if every element M_F of the monodromy group of the extended frame F satisfies

$$M_F(1) = \pm\mathrm{I} \quad \text{and} \quad M_F'(1) = 0.$$

Given $d\Phi = \Phi\xi$ and an analytic *gauge* $g = g(z, \lambda) : \Sigma \to \Lambda_r \mathrm{SL}_2\mathbb{C}$, then $\Psi := \Phi g$ satisfies the equation $d\Psi = \Psi\eta$, where

$$\eta = \xi.g := g^{-1}\xi g + g^{-1}\,dg.$$

Part I. Delaunay asymptotics

In Part I we show that under the assumption of unitary monodromy, an immersion constructed from a perturbed Delaunay potential is asymptotic to a Delaunay immersion. In Part II we generalize this result to the setting of $\Lambda_r \mathrm{SL}_2\mathbb{C}$ for arbitrary r.

Outline of results

In §1 we discuss the construction of the family of Delaunay immersions via the generalized Weierstrass representation. The generalized Weierstrass potential for the Delaunay immersion is of the form $A\,dz/z$, where A is an $\mathrm{sl}_2\mathbb{C}$-valued *Delaunay residue* given in Proposition 1.1. We compute the Iwasawa factorization of the holomorphic Delaunay frame $\exp(A \log z)$ in Theorem 1.5. This is an extension of a result in [SKKR].

These factors are used in §1.4 to compute the growth rate of the positive Iwasawa factor and in §4 to compute asymptotics of dressed Delaunay frames. We also estimate the growth rate τ of the positive Iwasawa factor of $\exp(A \log z)$ as $z \to 0$ (Theorem 1.13). This growth rate result is used in

189

§ 2.4 to show the following: given a holomorphic perturbation

$$\xi = A\frac{dz}{z} + \mathrm{O}(z^0)\,dz$$

of the Delaunay potential $A\,dz/z$ producing a closed once-wrapped Delaunay surface, let Φ satisfy $d\Phi = \Phi\xi$ on an r-circle with unitary monodromy around $z = 0$. Then we show that the unitary and positive factors of Φ are asymptotic to those of a holomorphic Delaunay frame (Theorem 2.11).

In § 3.2, we use this convergence to obtain C^∞-convergence of the CMC end to a Delaunay surface. The convergence of the positive part implies that of the metric, and together with the frame convergence, a bootstrap argument on the Gauss equation gives the C^∞-convergence. If the base Delaunay surface is embedded, then the asymptotic end is embedded and has exponential convergence. These results are summarized in Theorem 3.6.

In Part II we generalize these results to the case of dressed holomorphic frames.

1. The Delaunay frame and its growth

1.1 The Delaunay residue

A Delaunay surface is described by a holomorphic potential on $\mathbb{C}^*$ of a very simple kind. Our description of these potentials is in the setting of [SKKR].

PROPOSITION 1.1. *Let $A : \mathbb{S}^1 \to \mathrm{sl}_2\mathbb{C}$ be analytic. Then the following are equivalent.*

(i) *We have $A^* = A$, and A is the boundary of a meromorphic map $A : \mathcal{D}_1 \to \mathrm{sl}_2\mathbb{C}$ such that A is holomorphic on $\mathcal{D}_1 \setminus \{0\}$, the upper-right entry of A has a simple pole or no pole at zero and the other entries of A do not have poles at zero.*

(ii) *There exist $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$ such that*

$$A = \begin{pmatrix} c & a\lambda^{-1} + \overline{b} \\ b + \overline{a}\lambda & -c \end{pmatrix}. \tag{1}$$

Proof. If A is of the form (1), then part (i) clearly holds.

Conversely, suppose that part (i) holds. Since A extends meromorphically to $\mathcal{D}_1$, and $A^* = A$, then A extends meromorphically to $\mathbb{CP}^1$. The entries of A are then meromorphic functions on $\mathbb{CP}^1$, from which it follows that A must be of the form (1). □

After a rigid motion, the harmonic Gauss map of any Delaunay surface is framed by the r-unitary part of the solution $\Phi(z) = \exp(zA)$ of the ODE $d\Phi = \Phi A\,dz$, for some A as in (1) (see [BK06, SKKR]). This prompts us to make the following definition.

DEFINITION 1.2. A *Delaunay residue* is a meromorphic $\mathrm{sl}_2\mathbb{C}$-valued matrix map as in (1), with $a, b \in \mathbb{C}^*$ and $c \in \mathbb{R}$. Let $\mu : \mathbb{C}^* \to \mathbb{C}$ be an eigenvalue of A satisfying $\mathrm{Re}\,\mu \geqslant 0$.

It can be shown [BK06, KMS00] that up to rigid motions, any Delaunay surface in $\mathbb{R}^3$ can be obtained by an off-diagonal Delaunay residue with real non-zero parameters $a, b \in \mathbb{R}^*$ satisfying the closing condition $a + b = 1/2$, and that the resulting *necksize* of the surface depends on the product ab: when $ab > 0$, the resulting surface is an unduloid, when $ab < 0$, it is a nodoid, and when $a = b$, the resulting surface is a round cylinder (see [Kil04, SKKR]).

For $a, b \in \mathbb{C}^*$, we denote as the *vacuum* the case $|a| = |b|$ and $c = 0$.

1.2 The Delaunay frame

The unitary frames for all CMC tori are computed in [Bob91a, Bob91b] in terms of theta functions. In the case of spectral genus 1, the Iwasawa factorization of $\exp((x + iy)A))$ can be expressed in
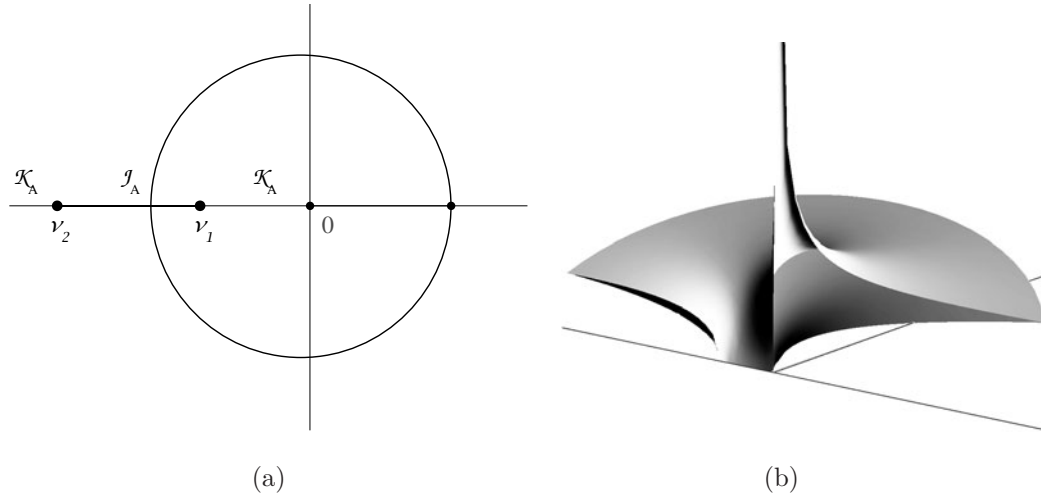
190

(a)   (b)

FIGURE 2. (a) The λ-plane with the zeros ν_1 and $\nu_2 = 1/\overline{\nu_1}$ of the eigenvalues of a typical Delaunay residue. See Notation 1.3 and Lemma 1.7. (b) Graphs of $\operatorname{Re}\mu$ (above) and τ (below) over a half-disk for a typical Delaunay frame. Here $\operatorname{Re}\mu$ represents the growth of $\exp(A\log z)$, τ the growth of its positive factor, and their difference the growth of its unitary factor.

terms of elliptic functions and elliptic integrals, and has a Floquet form whose period is that of the induced Delaunay surface. The computation of this factorization in Theorem 1.5 requires the following sets and functions.

Notation 1.3. Given a Delaunay residue A, let $\nu_1, \nu_2 \in \mathbb{C}$ be the zeros of $\det A$, with $|\nu_1| \leqslant |\nu_2|$. Let $p \in \mathbb{S}^1$ be the point on the straight line segment with endpoints ν_1 and ν_2, or $p = \nu_1 = \nu_2$ in the case of the vacuum. Let $\alpha = -p$. Define the following subsets of $\mathbb{C}$ (see Figure 2(a)):

$$\mathcal{I}_A = \{rp \in \mathbb{C} \,|\, 0 \leqslant r < \infty\}, \quad \mathcal{J}_A = \{rp \in \mathbb{C} \,|\, |\nu_1| \leqslant r \leqslant |\nu_2|\}, \quad \mathcal{K}_A = (\mathcal{I}_A \setminus \mathcal{J}_A) \cup \{\nu_1, \nu_2\}.$$

The set of *resonance points* for A is

$$\mathcal{S}_A = \{\lambda \in \mathbb{C}^* \,|\, \mu(\lambda) \in \tfrac{1}{2}\mathbb{Z}^*\}. \tag{2}$$

Notation 1.4. Let A be a Delaunay residue and let a, b, c be its coefficients as in Proposition 1.1. Define $v : \mathbb{R} \to \mathbb{R}_{>0}$ as the elliptic function satisfying

$$(v')^2 = -v^4 + 4(|a|^2 + |b|^2 + c^2)v^2 - 16|ab|^2, \quad v(0) = 2|b|, \tag{3}$$

taking the non-constant solution except in the case of the vacuum. When $c \neq 0$, $v'(0)$ is taken to have the same sign as $-c$. The function v is the restriction to $\mathbb{R}$ of an elliptic function on $\mathbb{C}$ with a real and a pure imaginary period; let $\rho \in \mathbb{R}_+$ be its real period.

Define the elliptic integral of the third kind $\psi : \mathbb{R} \times (\mathbb{C} \setminus \mathcal{J}_A) \to \mathbb{C}$ by

$$\psi(x, \lambda) = \int_0^x \frac{2\,dt}{1 + (4\overline{a}\overline{b}\lambda)^{-1}v^2(t)}, \tag{4}$$

and define $\sigma : \mathbb{C} \setminus \mathcal{J}_A \to \mathbb{C}$ by $\sigma(\lambda) = \psi(\rho, \lambda)$.

THEOREM 1.5. *Let A be a Delaunay residue and let $\Phi = \exp((x + iy)A)$. Then there exists an analytic map $R : \mathbb{R} \times (\mathbb{C} \setminus \mathcal{J}_A) \to \mathrm{SL}_2\mathbb{C}$ satisfying $R(x + \rho, \lambda) = R(x, \lambda)$, such that, restricting to $\mathbb{C} \setminus \mathcal{J}_A$,*

$$\mathsf{Uni}_1[\Phi] = \exp((x + iy - \rho^{-1}\sigma x)A)R, \tag{5a}$$

$$\mathsf{Pos}_1[\Phi] = R^{-1}\exp(\rho^{-1}\sigma x A). \tag{5b}$$

191

Proof. Let prime denote the derivative with respect to x. With $h = \mathrm{diag}((b/|b|)^{1/2}, (b/|b|)^{-1/2})$, define $R(x,\lambda) : \mathbb{R} \times (\mathbb{C} \setminus \mathcal{J}_A) \to \mathrm{SL}_2\mathbb{C}$ by

$$R(x,\lambda) = \exp((\rho^{-1}\sigma(\lambda)x - \psi(x,\lambda))A)S(x,\lambda),$$

$$S_1(x,\lambda) = \begin{pmatrix} v^2 + 4\overline{a}\overline{b}\lambda & v' + 2cv \\ 0 & 2(b+\overline{a}\lambda)v \end{pmatrix} h, \quad S(x,\lambda) = (\det S_1)^{-1/2} S_1.$$

The square root $(\det S_1)^{1/2}$ can be taken to be a single-valued analytic function in λ on $\mathbb{R} \times (\mathbb{C}\setminus\mathcal{J}_A)$, its sign chosen so that $S(0,\lambda) = \mathrm{I}$. Then R is periodic in x with period ρ because each of the two factors defining it are.

On $\mathbb{C}\setminus\mathcal{J}_A$, define $F = \exp((1-\rho^{-1}\sigma)xA)R$ and $B = R^{-1}\exp(\rho^{-1}\sigma xA)$. Then

$$F^{-1}F' = ((1-\psi')A).S = \begin{pmatrix} 0 & -\dfrac{v}{2} + \dfrac{2ab}{\lambda v} \\ \dfrac{v}{2} - \dfrac{2\overline{a}\overline{b}\lambda}{v} & 0 \end{pmatrix}.h =: \theta, \quad F(0,\lambda) = \mathrm{I}, \tag{6a}$$

$$B'B^{-1} = -(-\psi'A).S = \begin{pmatrix} -\dfrac{v'}{2v} & v \\ \dfrac{4\overline{a}\overline{b}\lambda}{v} & \dfrac{v'}{2v} \end{pmatrix}.h =: \eta, \quad B(0,\lambda) = \mathrm{I}. \tag{6b}$$

Since θ is analytic in λ on $\mathbb{C}^*$, and $\theta^* = -\theta$, then F extends to a map $\mathbb{R} \to \Lambda_1^*\mathrm{SL}_2\mathbb{C}$. Similarly, η is analytic in λ on $\mathbb{C}$, and $\eta|_{\lambda=0}$ is upper-triangular with real diagonal entries, so B extends to a map $\mathbb{R} \to \Lambda_1^+\mathrm{SL}_2\mathbb{C}$. Hence, $\Phi = \exp(iyA)F \cdot B$ is the 1-Iwasawa factorization of Φ. □

COROLLARY 1.6. *Let A be a Delaunay residue and let σ be as in Notation 1.4 for this A. Let $\exp((x+iy)A) = F \cdot B$ be the 1-Iwasawa factorization. Then $\exp(\sigma A)$ extends to a holomorphic function of λ on $\mathbb{C}$, and the following quasiperiodicity formulas hold for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$:*

$$F(x+n\rho, y) = \exp(n(\rho-\sigma)A)F(x,y), \quad \lambda \in \mathbb{C}^*, \tag{7a}$$

$$B(x+n\rho) = B(x)\exp(n\sigma A), \quad \lambda \in \mathbb{C}. \tag{7b}$$

Proof. Since $\Phi = FB$, and Φ and B are holomorphic in λ on $\mathcal{D}_1^*$, then F on $\mathbb{S}^1$ extends holomorphically in λ to $\mathcal{D}_1^*$. Since $F^* = F^{-1}$, then F extends holomorphically in λ to $\mathbb{C}^*$. Since Φ is holomorphic in λ on $\mathbb{C}^*$, then B extends holomorphically in λ on $\mathbb{C}$. (This can also be seen from the ODE (6b) satisfied by B.) However, by (5b), $B(\rho)$ and $\exp(\sigma A)$ are equal on $\mathbb{C}\setminus\mathcal{J}_A$. The quasiperiodicity formulas for F and B follow by (5). □

1.3 The Delaunay growth exponent

Our proof of the convergence of the r-Iwasawa factors of a holomorphic Delaunay frame $\Phi = \exp(A\log z)$ and a perturbation of it requires growth bounds on $\mathsf{Pos}_r[\Phi]$. Being an exponential, Φ grows exponentially, and its growth rate is the absolute value of the real part of an eigenvalue μ of the Delaunay residue A. On $\mathbb{S}^1$ ($r=1$), the factor $\mathsf{Pos}_r[\Phi]$ has the same growth behavior as Φ, because $\mathsf{Uni}_r[\Phi]$ does not grow. For $r < 1$, the Floquet behavior of $\mathsf{Pos}_r[\Phi]$ again implies exponential growth, its rate determined by the eigenvalues of its value after one period.

We begin by studying the real part τ of these eigenvalues, showing it is less than the real part of the eigenvalue μ of A.

LEMMA 1.7. *Let A be a Delaunay residue and let μ be its eigenvalue as in Definition 1.2. Let σ be as in Notation 1.4 for this A. Then the function $\tau = \rho^{-1}\mathrm{Re}\,\mu\sigma$ extends to a single-valued continuous function on $\mathbb{C}$, which is real analytic and harmonic on $\mathbb{C}\setminus\mathcal{K}_A$. Moreover, with $\mathcal{K}_A$ as Notation 1.3, $\tau = 0$ on $\mathcal{K}_A$, $\tau > 0$ on $\mathcal{D}_1\setminus\mathcal{K}_A$, and $\tau = \mu$ on $\mathbb{S}^1$.*

192

Proof. We first consider the nonvacuum case.

Step 1. We show that σ extends holomorphically as a function of λ along any curve in $\mathbb{C} \setminus \{\nu_1, \nu_2\}$. Let $J(t, \lambda) \, dt$ be the integrand defining ψ in (4). Then J can be extended to a meromorphic function of t on $\mathbb{C}$, by considering v as an elliptic function on $\mathbb{C}$. Let γ be a curve with endpoints 0 and ρ which is homotopic to the straight line $[0, \rho]$ in $\mathbb{C} \setminus \text{sing}(v)$. Let $\Lambda = \{\lambda \in \mathbb{C} \mid J(t, \lambda) \neq \infty \text{ along } \gamma\}$. Then we can define $\tilde{\sigma}(\lambda)$ on Λ as the integral of J along γ. Since $\tilde{\sigma}$ is analytic, and is equal to σ on the intersections of their domains $\Lambda \cap (\mathbb{C} \setminus \mathcal{J}_A)$, then $\tilde{\sigma}$ is an analytic extension of σ. This provides a construction for analytically extending σ along any curve in $\mathbb{C} \setminus \{\nu_1, \nu_2\}$.

Step 2. Since for all $x \in \mathbb{R}$, $B(x, \lambda) = \text{Pos}_1[\exp(xA(\lambda))]$ is a holomorphic function of λ on $\mathbb{C}$, then $B(\rho, \lambda)$ is holomorphic in λ. Hence, $\cosh(\mu\sigma) = \frac{1}{2} \text{tr} \, B(\rho, \lambda)$ is analytic on $\mathbb{C}$. An argument shows that at $\lambda \in \mathbb{C}$, if $\cosh(\mu\sigma) \notin \{\pm 1\}$, $\mu\sigma$ is analytic at λ, and if $\cosh(\mu\sigma) \in \{\pm 1\}$, $\mu\sigma$ is of the form $\pi i k + \mu g$ for some $k \in \mathbb{Z}$ and some holomorphic function g near λ.

Step 3. Near $\lambda = 0$, σ is analytic. Hence, if any branch of $\mu\sigma$ is analytically extended along a closed once-wrapped curve around 0, with respective values p_0 and p_1 at the beginning and end of the curve, then $p_1 = -p_0$. Likewise, if any branch of $\mu\sigma$ is analytically extended along a closed curve around ν_1, with respective values q_0 and q_1 at the beginning and end of the curve, it follows from step 2 that $q_1 \equiv -q_0 \bmod 2\pi i$.

Putting these together, if $\mu\sigma$ is analytically extended along a closed once-wrapped curve around 0 and ν_1, with respective values r_0 and r_1 at the beginning and end of the curve, then $r_1 \equiv r_0 \bmod 2\pi i$. Hence, τ is a single-valued real analytic function on $\mathbb{C} \setminus \mathcal{K}_A$. It is harmonic there because it is locally the real part of an analytic function.

Step 4. To show that τ is continuous on $\mathcal{K}_A \setminus \{\nu_1, \nu_2\}$ with value 0, write μ and σ in terms of their real and imaginary parts $\mu = \mu_1 + i\mu_2$ and $\sigma = \sigma_1 + i\sigma_2$. Then $\text{Re} \, \mu\sigma = \mu_1\sigma_1 - \mu_2\sigma_2$. Since σ_2 and μ_1 are 0 on $\mathcal{K}_A$, τ is continuous on $\mathcal{K}_A \setminus \{\nu_1\nu_2\}$ with value 0 there.

Step 5. We now show that τ is continuous at ν_1 and ν_2 with value 0. By step 2, with $k \in \{1, 2\}$, we have $\tau = \text{Re}(c + \mu g) = \text{Re}(\mu g)$ near ν_k, for some $c \in \pi i \mathbb{Z}$ and holomorphic function g. Since $\mu(\nu_k)$ is continuous at ν_k and $\mu(\nu_k) = 0$, then τ extends continuously to ν_k with value 0 there.

Step 6. To show that $\tau = \mu$ on $\mathbb{S}^1$, since F in (7a) takes values in SU_2 on $\mathbb{S}^1$, then by (7a), so does $\exp((\rho - \sigma)A)$. Since A is tracefree and is hermitian on $\mathbb{S}^1$, then $\rho - \sigma$ is pure imaginary on $\mathbb{S}^1$. Hence, $\tau = \mu$ on $\mathbb{S}^1$.

Step 7. Since τ is continuous on $\overline{\mathcal{D}_1}$ and harmonic on $\mathcal{D}_1 \setminus \mathcal{K}_A$, then by the maximum principle for harmonic functions, τ on $\mathcal{D}_1$ attains its minimum on the boundary $\mathbb{S}^1 \cup \mathcal{K}_A$ of $\overline{\mathcal{D}_1} \setminus \mathcal{K}_A$. Since τ is 0 on $\mathcal{K}_A$, then τ is strictly positive on $\mathcal{D}_1 \setminus \mathcal{K}_A$.

For the vacuum case, let a, b be the coefficients of the Delaunay residue as in Definition 1.2, and let α be as in Notation 1.3. Then $\tau = \text{Re} \, \rho^{-1}\mu\sigma$ satisfies all of the properties of the theorem, where

$$\mu = |b|\alpha^{-1/2}\lambda^{-1/2}(\lambda + \alpha) \quad \text{and} \quad \rho^{-1}\mu\sigma = 2|b|\alpha^{-1/2}\lambda^{1/2}. \tag{8}$$

$\square$

LEMMA 1.8. *Let A be a Delaunay residue, let μ be its eigenvalue as in Definition 1.2, and let τ be as in Lemma 1.7 for this A. Then $\tau \in (0, \text{Re} \, \mu]$ on $\mathcal{D}_1 \setminus \mathcal{K}_A$.*

Proof. In the case of the vacuum, the result follows by (8); hence, we assume the nonvacuum case. (See Figure 2(b) for the graphs of $\text{Re}(\mu)$ and τ for a typical Delaunay residue.)

We first show that $\text{Re}(\mu) - \tau$ is nonnegative on $\mathcal{D}_r^*$ for some r near zero, and then apply the maximum principle for harmonic functions to conclude it is strictly positive on $\mathcal{D}_1 \setminus \mathcal{K}_A$.

Step 1. We prove the following claim: Let $r > 0$ and let $f : \mathcal{D}_r \to \mathbb{C}$ be a holomorphic function. If f has the symmetry $f(\overline{\lambda}) = \overline{f(\lambda)}$, and $f(0) \in \mathbb{R}_+$, then there exists $r_1 \in (0, r]$ such that $g := \text{Re}(\lambda^{-1/2}f)$ is nonnegative on $\mathcal{D}_{r_1}^*$, where the square root is chosen to have nonnegative real part.

To prove the claim, let $z = x + iy = \lambda^{1/2}$, and let

$$h := \frac{g}{\operatorname{Re}\lambda^{-1/2}} = \operatorname{Re} f(z^2) + \frac{y}{x}\operatorname{Im} f(z^2).$$

The function $k(x,y) = \operatorname{Im} f(z^2)$ is real analytic in x and y. By the symmetry of f, $f(-y^2)$ is real, so for all y, $k(0,y) = \operatorname{Im} f(-y^2) = 0$. It follows that $k(x,y)/x$ is real analytic in x and y. Then $\lim_{z\to 0} yk(x,y)/x = 0$, so $\lim_{z\to 0} h = f(0) \in \mathbb{R}_+$. Hence, there exists $r_1 \in (0,r]$ such that h is strictly positive in $\mathcal{D}_{r_1}$. Since $\operatorname{Re}\lambda^{-1/2}$ is strictly positive on $\mathcal{D}_{r_1} \setminus \mathbb{R}_{\leqslant 0}$, then so is g. However, g is 0 on $\mathbb{R}_-$, so g is nonnegative on $\mathcal{D}_{r_1}^*$.

Step 2. Let σ be as in Lemma 1.7 on $\mathcal{D}_{|\nu_1|}$, so $\tau = \operatorname{Re}\sigma$. With α as in Notation 1.3, define $f : \mathcal{D}_{|\nu_1|} \to \mathbb{C}$ by

$$f = \lambda^{1/2}(\mu(\alpha\lambda) - \sigma(\alpha\lambda)).$$

Then f is holomorphic on $\mathcal{D}_{|\nu_1|}$, $f(\overline{\lambda}) = \overline{f(\lambda)}$ and $f(0) \in \mathbb{R}_+$. By the above claim applied to f, there exists $r_1 \in (0, |\nu_1|]$ such that $\operatorname{Re}\lambda^{-1/2}f = \operatorname{Re}(\mu(\alpha\lambda)) - \tau(\alpha\lambda)$ is nonnegative on $\mathcal{D}_{r_1}^*$. Hence, $\operatorname{Re}(\mu) - \tau$ is nonnegative $\mathcal{D}_{r_1}^*$.

Step 3. For any $s \in (0, r_1]$, define V_s as the union of $\mathcal{C}_s$ and the straight line segment along $\mathcal{K}_A$ from ν_1 to this circle. Then for all $s < r_1$, $\operatorname{Re}(\mu) - \tau$ is harmonic on the open region R_s between V_s and $\mathbb{S}^1$, is 0 on $\mathbb{S}^1$ and $V_s \cap \mathcal{K}_A$, and is nonnegative on $\mathcal{C}_s$ by step 2. By the maximum principle for harmonic functions, $\operatorname{Re}(\mu) - \tau$ is strictly positive on R_s. Since this is true for any $s \in (0, r_1)$, then $\operatorname{Re}(\mu) - \tau$ is strictly positive on $\mathcal{D}_1 \setminus \mathcal{K}_A$. $\qquad\square$

1.4 Growth of the Delaunay positive part

Above we investigated the function τ, the real part of the eigenvalue of the value of $\mathsf{Pos}_r[\exp(A\log z)]$ after one period, showing that it is less than the real part of the eigenvalue μ of the corresponding Delaunay residue. The Floquet behavior of $\mathsf{Pos}_r[\exp(A\log z)]$, detailed in Theorem 1.5, implies that τ, and hence μ, bounds the exponential growth of $\mathsf{Pos}_r[\exp(A\log z)]$ (Lemma 1.11).

Notation 1.9. For $v \in \mathbb{C}^2$, we denote the *vector norm* by $|v| = \sqrt{\overline{v}^t v}$, and for $M \in \mathrm{M}_{2\times 2}(\mathbb{C})$, we set

$$\|M\| = \max_{|v|=1} |Mv|.$$

For a map $M : \mathcal{R} \to \mathrm{M}_{2\times 2}(\mathbb{C})$ on a subset $\mathcal{R} \subset \mathbb{C}$ we set

$$\|M\|_{\mathcal{R}} = \sup_{\lambda\in\mathcal{R}} \|M(\lambda)\|.$$

Given $r \in (0,1]$, a subset $\mathcal{R} \subset \mathbb{C}$ containing $\mathcal{C}_r$, and a loop $X \in \Lambda_r\mathrm{SL}_2\mathbb{C}$ which extends to a map $Y : \mathcal{R} \to \mathrm{SL}_2\mathbb{C}$, by an abuse of notation we write $\|X\|_{\mathcal{R}}$ for $\|Y\|_{\mathcal{R}}$.

LEMMA 1.10. *Let* $X : \mathcal{R} \to \mathrm{sl}_2\mathbb{C}$ *be a continuous map on a domain* $\mathcal{R} \subseteq \mathbb{C}$*, and let* $\mu : \mathcal{R} \to \mathbb{C}$ *be any eigenvalue function of* X*. Then there exists a continuous function* $c : \mathcal{R} \to \mathbb{R}_+$ *such that* $\|\exp X\| \leqslant ce^{|\operatorname{Re}\mu|}$*.*

Proof. The result follows from the formula $\exp X = \cosh(\mu)\,\mathrm{I} + \mu^{-1}\sinh(\mu)X$ and the estimates $|\cosh\mu| \leqslant e^{|\operatorname{Re}\mu|}$ and $|\mu^{-1}\sinh\mu| \leqslant e^{|\operatorname{Re}\mu|}$. $\qquad\square$

LEMMA 1.11. *Let* A *be a Delaunay residue, and let* μ *be its eigenvalue as in Definition 1.2. Let* τ *be as in Lemma 1.7. Then there exists a continuous function* $c : \overline{\mathcal{D}_1} \setminus \{0\} \to \mathbb{R}_+$ *such that for all* $(z,\lambda) \in \{0 < |z| < 1\} \times \overline{\mathcal{D}_1} \setminus \{0\}$*,*

$$\|\mathsf{Pos}_1[\exp(A\log z)]\| \leqslant c|z|^{-\tau} \leqslant c|z|^{-\operatorname{Re}\mu}. \tag{9}$$

Proof. For $x + iy \in \mathbb{C}$, define $B(x+iy, \lambda) = \mathsf{Pos}_1[\exp((x+iy)A)]$. With ρ as in Corollary 1.6, define $x_0 : \mathbb{R} \to [0, \rho)$ and $n : \mathbb{R} \to \mathbb{Z}$ as the unique functions such that $x = x_0 + n\rho$. Let σ be as in

Notation 1.4 for A. By Corollary 1.6,

$$B(x + iy) = B(x_0 + n\rho) = B(x_0)\exp(n\sigma A). \tag{10}$$

This quasiperiodicity of B determines its growth rate as follows.

Define the continuous function $c_1 : \mathbb{C}^* \to \mathbb{R}_+$ by $c_1(\lambda) = \max_{x \in [0,\rho)} \|B(x, \lambda)\|$. Then $\|B(x_0, \lambda)\| \leqslant c_1(\lambda)$ on $[0, \rho) \times \mathbb{C}^*$. With τ as in Lemma 1.7 for A, by Lemma 1.10 there exists a continuous function $c_2 : \mathbb{C}^* \to \mathbb{R}_+$ such that on $\mathbb{C}^*$,

$$\|\exp(n\psi(\rho, \lambda)A(\lambda))\| \leqslant c_2 e^{|n\rho\tau(\lambda)|} \leqslant c_2 e^{|-x_0\tau(\lambda)|}e^{|x\tau(\lambda)|}.$$

Define the continuous function $c_3 : \mathbb{C}^* \to \mathbb{R}_+$ by $c_3(\lambda) = \max_{x \in [0,\rho)} e^{|x\tau(\lambda)|}$. Then

$$\|\exp(n\psi(\rho, \lambda)A(\lambda))\| \leqslant c_2(\lambda)c_3(\lambda)e^{|\tau(\lambda)x|}. \tag{11}$$

The quasiperiodicity (10), the choice of c_1, and (11) yield the estimate

$$\|B(x + iy, \lambda)\| \leqslant c_1(\lambda)c_2(\lambda)c_3(\lambda)e^{|\tau(\lambda)x|}.$$

Since $0 < |z| < 1$, then $x < 0$. However, $|z| = e^x$, so $|z|^{-\tau} = e^{-\tau x} = e^{|\tau x|}$. The growth estimate (9) follows with $c = c_1 c_2 c_3$.

The second inequality in (9) follows by Lemma 1.8. $\qquad\square$

1.5 Dressed Delaunay frames

The growth bound on the positive r-Iwasawa factor of the holomorphic Delaunay frame $\Phi = \exp(A \log z)$, computed above in Lemma 1.11, is preserved by dressing (right-multiplying) Φ by an r-loop C, provided that the unitarity of its monodromy is preserved. In the special case that C can be extended analytically to $\mathcal{A}_{r,1}$, the resulting frame $C\Phi$ again induces a Delaunay immersion (Lemma 1.12), and the growth of its positive factor is bounded by the same bound as that for $\mathsf{Pos}_r[\Phi]$ (Theorem 1.13).

The general case of dressing by an arbitrary loop C which preserves the unitarity of the monodromy is characterized in § 4.2, where it is shown that $C\Phi$ induces a multibubbleton, a Bianchi–Bäcklund transformed Delaunay immersion. The same growth bound applies to the positive part of $C\Phi$ for this larger class of dressing matrices.

LEMMA 1.12. *Let A be a Delaunay residue. Let $r \in (0, 1]$. Let $C \in \Lambda_r^{\uparrow}\mathrm{SL}_2\mathbb{C}$, and assume $C\exp(2\pi i A)C^{-1} \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$. Let $C = C_u \cdot C_+$ be the r-Iwasawa factorization of C. Then:*

(i) *$C_+ A C_+^{-1}$ extends meromorphically to $\mathbb{CP}^1$ and is a Delaunay residue;*

(ii) *$\mathsf{Sym}_r[C\exp(A\log z)]$ and $\mathsf{Sym}_r[\exp(A\log z)]$ are Delaunay surfaces differing by a rigid motion.*

Proof. Let $M = \exp(2\pi i A)$ and let $A_1 = C_+ A C_+^{-1}$. Since $CMC^{-1} \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$, then $C_+ M C_+^{-1} = \exp(2\pi i A_1) \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$. With μ a local analytic eigenvalue of A or A_1, the formula

$$\exp(2\pi i A_1) = \cos(2\pi\mu)\,\mathrm{I} + \mu^{-1}\sin(2\pi\mu)A_1$$

shows that A_1 extends meromorphically to $\mathbb{S}^1$. Since $\exp(2\pi i A_1) \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$, then $A_1^* = A_1$ away from its poles. Write

$$A_1 = \begin{pmatrix} x & y \\ y^* & -x \end{pmatrix}$$

for some meromorphic functions x and y in a neighborhood of $\mathbb{S}^1$ satisfying $x = x^*$. Since A is holomorphic on $\mathbb{S}^1$, then so is

$$-\det A = -\det A_1 = xx^* + yy^*.$$

195

Hence, $xx^* + yy^*$ is bounded on $\mathbb{S}^1$. Since xx^* and yy^* are each nonnegative on $\mathbb{S}^1$, then each is bounded on $\mathbb{S}^1$, so each is holomorphic on $\mathbb{S}^1$. It follows that x, y and y^* are holomorphic on $\mathbb{S}^1$, so A_1 is holomorphic on $\mathbb{S}^1$. Since A_1 is holomorphic on $\mathcal{D}_1$ and satisfies $A_1^* = A_1$, then A_1 is a Delaunay residue by Proposition 1.1.

To show that $C \exp(A \log z)$ induces a Delaunay immersion, note that

$$C \exp(A \log z) = C_u \exp(A_1 \log z) C_+,$$

so

$$\mathsf{Uni}_r[C \exp(A \log z)] = C_u \mathsf{Uni}_r[\exp(A_1 \log z)].$$

Hence, $\mathsf{Sym}_r[C \exp(A \log z)]$ and $\mathsf{Sym}_r[\exp(A_1 \log z)]$ differ by a rigid motion. By [SKKR, Lemma 6], $\mathsf{Sym}_r[\exp(A_1 \log z)]$ and $\mathsf{Sym}_r[\exp(A \log z)]$ are Delaunay surfaces differing by a rigid motion. □

THEOREM 1.13. *Let A be a Delaunay residue and let μ be its eigenvalue as in Definition 1.2. Let τ be as in Lemma 1.7. Let $r \in (0,1]$. Let $C \in \Lambda_r^{\uparrow}\mathrm{SL}_2\mathbb{C}$ be such that the dressed monodromy $C \exp(2\pi i A) C^{-1}$ around $z = 0$ is in $\Lambda_r^*\mathrm{SL}_2\mathbb{C}$. Then there exists a continuous function $c : \mathcal{D}_1^* \to \mathbb{R}_+$ such that for all $(z, \lambda) \in \{0 < |z| < 1\} \times \mathcal{D}_1^*$,*

$$\|\mathsf{Pos}_r[C \exp(A \log z)]\| \leqslant c|z|^{-\tau} \leqslant c|z|^{-\operatorname{Re}\mu}.$$

Proof. Let $C = C_u \cdot C_+$ be the r-Iwasawa factorization of C. By Lemma 1.12, $A_1 = C_+ A C_+^{-1}$ is a Delaunay residue. Then

$$C \exp(A \log z) = C_u \exp(A_1 \log z) C_+,$$

so

$$\mathsf{Pos}_r[C \exp(A \log z)] = \mathsf{Pos}_r[\exp(A_1 \log z)] C_+. \tag{12}$$

It follows from $\det A = \det A_1$ that the function τ in Lemma 1.7 for A is the same as that for A_1. By Lemma 1.11, there exists a continuous function $c_1 : \overline{\mathcal{D}_1} \setminus \{0\} \to \mathbb{R}_+$ such that on $\{0 < |z| < 1\} \times \mathcal{D}_1^*$,

$$\|\mathsf{Pos}_r[\exp(A_1 \log z)]\| \leqslant c_1|z|^{-\tau} \leqslant c_1|z|^{-\operatorname{Re}\mu}. \tag{13}$$

Let $c_2 = \|C_+\|$ on $\mathcal{D}_1$. Then by (12) and (13),

$$\|\mathsf{Pos}_r[C \exp(A \log z)]\| \leqslant c_1 c_2 \|\mathsf{Pos}_r[\exp(A_1 \log z)]\| \leqslant c_1 c_2 |z|^{-\tau} \leqslant c_1 c_2 |z|^{-\operatorname{Re}\mu}.$$

The result follows with $c = c_1 c_2$. □

2. The perturbed Delaunay frame

We use the following notation throughout the next several sections. Let A be a Delaunay residue and let μ be an eigenvalue of A with nonnegative real part. Let $\mathcal{S}_A$ be the set of resonance points for A, defined in (2). Let $\Sigma \subset \mathbb{C}$ be a neighborhood of $0 \in \mathbb{C}$ and let $\Sigma^* = \Sigma \setminus \{0\}$. Choose $r \in (0, 1]$.

DEFINITION 2.1. We make the following definitions.

(i) A *perturbed Delaunay r-potential* is an r-potential ξ on Σ^* of the form

$$\xi = A z^{-1} \, dz + \mathrm{O}(z^0) \, dz.$$

(ii) An *r-gauge* is an analytic map $\Sigma \to \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$.

(iii) Given neighborhoods $\Sigma, \Sigma' \subset \mathbb{C}$ of $0 \in \mathbb{C}$, a *coordinate change* ϑ is a holomorphic map $\vartheta : \Sigma' \to \Sigma$ which satisfies $\vartheta(0) = 0$ and has a holomorphic inverse $\vartheta(\Sigma) \to \Sigma'$.

2.1 The $z^A P$ lemma

Given a linear matrix ODE $d\Phi = \Phi\xi$ for which ξ has a simple pole at $z = 0$ and residue A, a standard result in the theory of regular singularities [Har82] states that under certain conditions on

196

the eigenvalues of A, there exists a solution of the form $\Phi = z^A P = \exp(A \log z)P$, where P extends holomorphically to $z = 0$. Lemma 2.3 summarizes these results for our context, in which ξ depends analytically on a parameter λ. We call the decomposition (16) the $z^A P$ *decomposition*.

The coefficients of the $z^A P$ gauge P can be computed in terms of a linear map $\mathscr{L}_n$, whose definition and properties are given in the next lemma.

LEMMA 2.2. *Let $A \in \mathrm{gl}_2\mathbb{C}$ and let $\mu_1, \mu_2 \in \mathbb{C}$ be the eigenvalues of A. For $n \in \mathbb{Z}_{\geqslant 0}$, define the linear map $\mathscr{L}_n : \mathrm{gl}_2\mathbb{C} \to \mathrm{gl}_2\mathbb{C}$ by*

$$\mathscr{L}_n(X) = nX + [A, X].$$

Then we have the following.

   (i) *The eigenvalues of $\mathscr{L}_n$ are n, n, $n + \mu_1 - \mu_2$, $n - \mu_1 + \mu_2$. Hence, $\mathscr{L}_n$ is invertible if and only if $n \neq 0$ and $\mu_1 - \mu_2 \notin \{n, -n\}$.*
   (ii) *Suppose $n \in \mathbb{Z}_+$. Let $R = n\,\mathrm{I} + A - \widehat{A}$, where $\widehat{A}$ denotes the adjugate of A. Then $\mathscr{L}_n$ is invertible if and only if R is invertible, and in this case, $\mathscr{L}_n^{-1}$ is given by*

$$n\mathscr{L}_n^{-1}(X) = X - R^{-1}[A, X]. \tag{14}$$

   (iii) *For any $X \in \mathrm{gl}_2\mathbb{C}$, $\mathrm{tr}(\mathscr{L}_n X) = n\,\mathrm{tr}\,X$. Hence, if $\mathscr{L}_n$ is invertible, then for any $Y \in \mathrm{gl}_2\mathbb{C}$, $\mathrm{tr}\,Y = n\,\mathrm{tr}(\mathscr{L}_n^{-1}Y)$.*
   (iv) *Let $n \in \mathbb{Z}_+$. If A is holomorphic (respectively meromorphic) on some domain $\mathcal{R}$, then $\mathscr{L}_n$ is holomorphic (respectively meromorphic) on $\mathcal{R}$. In this case, if $\mu_1 - \mu_2$ is not identically n or $-n$, then $\mathscr{L}_n^{-1}$ extends meromorphically to $\mathcal{R}$.*

Proof. Statement (i) follows from the fact that the eigenvalues of ad_A are $0, 0, \mu_1 - \mu_2, -\mu_1 + \mu_2$.

To prove statement (ii), since the eigenvalues of R are $n + \mu_1 - \mu_2$, $n - \mu_1 + \mu_2$, then by statement (i), R is invertible if and only if $\mathscr{L}_n$ is invertible. In this case,

$$\begin{aligned}
\mathscr{L}_n(X - R^{-1}[A, X]) &= nX + (\mathrm{I} - nR^{-1} - AR^{-1})[A, X] + R^{-1}[A, X]A \\
&= nX + R^{-1}((R - n\,\mathrm{I} - RAR^{-1})[A, X] + [A, X]A) \\
&= nX + R^{-1}(-\widehat{A}[A, X] + [A, X]A) = nX.
\end{aligned}$$

Statement (iii) is clear from the definition of $\mathscr{L}_n$ and the fact that $\mathrm{tr}[A, X] = 0$.

Identifying $\mathrm{gl}_2\mathbb{C}$ with $\mathbb{C}^4$, the entries of the 4×4 matrices for $\mathscr{L}_n$ and $\mathscr{L}_n^{-1}$ are rational functions of the entries of A, and hence are meromorphic. This proves statement (iv). $\qquad\square$

LEMMA 2.3. *Let $\mathcal{R} = \mathbb{C}^* \setminus \mathcal{S}_A$.*

   (i) *There exists a holomorphic solution $P : \Sigma \times \mathcal{R} \to \mathrm{SL}_2\mathbb{C}$ to the gauge equation*

$$(Az^{-1}dz).P = \xi, \quad P(0, \lambda) = \mathrm{I}. \tag{15}$$

   (ii) *Let $r \in (0, 1]$ and assume $\mathcal{C}_r \cap \mathcal{S}_A = \emptyset$. Let $\Phi : \widetilde{\Sigma^*} \to \Lambda_r \mathrm{SL}_2\mathbb{C}$ be a holomorphic solution to the equation $d\Phi = \Phi\xi$ on the universal cover $\widetilde{\Sigma^*} \to \Sigma^*$ of Σ^*. Then there exists $C \in \Lambda_r \mathrm{SL}_2\mathbb{C}$ such that*

$$\Phi(z, \lambda) = C(\lambda) \exp(A(\lambda) \log z) P(z, \lambda). \tag{16}$$

   (iii) *Moreover, if ξ satisfies $\xi = Az^{-1}\,dz + \mathrm{O}(z^n)\,dz$, then P satisfies $P = \mathrm{I} + \mathrm{O}(z^{n+1})$.*

Proof. (i) By the pointwise version of this lemma (see [Har82, Theorem 10.1]), at each $\lambda_0 \in \mathcal{R}$, there exists a unique solution $P(z, \lambda_0)$ to the gauge equation (15). To show that P is holomorphic in λ on $\mathcal{R}$, let

$$\xi = Az^{-1}\,dz + \sum_{k=0}^{\infty} B_k z^k\,dz$$

197

and

$$P = \sum_{k=0}^{\infty} P_k z^k, \quad P_0 = \mathrm{I}$$

be the respective series expansions for ξ and P in z at $z = 0$. Then for all $k \in \mathbb{Z}_+$, $\mathscr{L}_k$ is invertible at λ_0 by Lemma 2.2(i) and the assumption that $\mathcal{R} \cap \mathcal{S}_A = \emptyset$. The coefficients P_k (see [Har82]) are given by $P_k = \mathscr{L}_k^{-1}(C_k)$, where

$$C_k = \sum_{i+j=k-1} P_i B_j, \quad k \in \mathbb{Z}_+.$$

By Lemma 2.2(iv), each P_k is holomorphic in λ at λ_0. It follows by the absolute convergence of power series that P is holomorphic in λ at λ_0.

(ii) Since P satisfies (15), then the map $\Phi_0 : \widetilde{\Sigma^*} \times \mathcal{C}_r \to \mathrm{SL}_2\mathbb{C}$ defined by

$$\Phi_0(z, \lambda) = \exp(A(\lambda) \log z) P(z, \lambda)$$

satisfies $d\Phi_0 = \Phi_0 \xi$. Since Φ satisfies the same linear ODE as Φ_0, then the map $C = \Phi \Phi_0^{-1}$ is z-independent, and is an element of $\Lambda_r \mathrm{SL}_2 \mathbb{C}$.

(iii) For any $k \in \{1, \ldots, n-1\}$ we have $B_0 = \cdots = B_{k-1} = 0$, so $C_k = 0$. Since $\mathscr{L}_k$ is invertible on $\mathcal{R}$, then by the formula for P_k in Lemma 2.3, $P_k = \mathscr{L}_k^{-1}(C_k) = \mathscr{L}_k^{-1}(0) = 0$. $\quad\square$

2.2 Gauging the potential

With A a Delaunay residue, let ξ be a perturbed Delaunay r-potential

$$A z^{-1}\, dz + \mathrm{O}(z^0)\, dz.$$

Let $n \in \mathbb{Z}_+$ be such that

$$\frac{n}{2} < \max_{\mathcal{C}_r} \mathrm{Re}\, \mu < \frac{n+1}{2}.$$

The left inequality, by Theorem 2.7, guarantees that ξ can be gauged to the form

$$A z^{-1}\, dz + \mathrm{O}(z^n)\, dz.$$

The right inequality, by Theorem 2.11, implies the convergence of the unitary and positive factors of $\exp(\log z A)$ and its perturbation.

The following preliminary lemma determines a necessary and sufficient condition on a map X such that $\mathcal{L}_n^{-1}(X)$ extends holomorphically to the pole of A. This condition is used to insure that the gauge constructed in Lemma 2.5 is a positive gauge, that is, that it extends holomorphically to $\lambda = 0$. The entries of a matrix $X \in \mathrm{M}_{2 \times 2}(\mathbb{C})$ are denoted by X_{ij} with $i, j \in \{1, 2\}$.

Lemma 2.4. *Let $\mathcal{R} \subset \mathbb{C}$ be a neighborhood of $p \in \mathbb{C}$. Let $A : \mathcal{R} \to \mathrm{gl}_2\mathbb{C}$ and $X : \mathcal{R} \to \mathrm{gl}_2\mathbb{C}$ be meromorphic on $\mathcal{R}$, with orders of entries*

$$\operatorname*{ord}_p A_{11} \geqslant 0, \quad \operatorname*{ord}_p A_{12} = -1, \quad \operatorname*{ord}_p A_{21} = 0, \quad \operatorname*{ord}_p A_{22} \geqslant 0,$$
$$\operatorname*{ord}_p X_{11} \geqslant 0, \quad \operatorname*{ord}_p X_{12} \geqslant -1, \quad \operatorname*{ord}_p X_{21} \geqslant 0, \quad \operatorname*{ord}_p X_{22} \geqslant 0. \tag{17}$$

Let $n \in \mathbb{Z}_+$ and let $\mathscr{L}_n$ as in Lemma 2.2 defined with respect to A.

Then $\mathscr{L}_n^{-1}(X)$, which extends meromorphically to p by Lemma 2.2(iv), is holomorphic at p if and only if $A_{12}X_{21} + A_{21}X_{12}$ is holomorphic at p.

Proof. By (17), we have $\operatorname{ord}_p \det A = -1$. A calculation using (14) shows that the orders of the entries of $Y = \mathscr{L}_n^{-1}(X)$ satisfy

$$\operatorname*{ord}_p Y_{11} \geqslant 0, \quad \operatorname*{ord}_p Y_{12} \geqslant -1, \quad \operatorname*{ord}_p Y_{21} \geqslant 0, \quad \operatorname*{ord}_p Y_{22} \geqslant 0.$$

198

Hence, $\mathscr{L}_n^{-1}(X)$ is holomorphic at p if and only if its upper-right entry Y_{12} is holomorphic at p. A calculation shows

$$\operatorname*{ord}_p Y_{12} = \operatorname*{ord}_p (A_{12} X_{21} + A_{21} X_{12}).$$

Hence, Y is holomorphic at p if and only if $A_{12} X_{21} + A_{21} X_{12}$ is. $\qquad\square$

LEMMA 2.5. *Let ξ be a perturbed Delaunay r-potential*

$$\xi = A z^{-1}\, dz + \mathrm{O}(z^{n-1})\, dz$$

for some $n \in \mathbb{Z}_+$, and suppose

$$\min_{\lambda \in \mathcal{C}_r} \operatorname{Re} \mu(\lambda) \leqslant \frac{n}{2} < \max_{\lambda \in \mathcal{C}_r} \operatorname{Re} \mu(\lambda). \tag{18}$$

Then there exists a neighborhood $\Sigma' \subset \mathbb{C}$ of $0 \in \mathbb{C}$, an r-gauge $g : \Sigma' \to \Lambda_r^{\mathrm{pos}} \mathrm{SL}_2\mathbb{C}$, and $\kappa \in \mathbb{C}$ such that

$$\xi.g = A z^{-1}\, dz + \kappa A z^{n-1}\, dz + O(z^n)\, dz. \tag{19}$$

Proof. Let $\mathscr{L}_n$ be as in Lemma 2.2, defined with respect to A on $\mathcal{D}_r^*$. The inequalities (18) imply $\mathcal{D}_r^* \cap \{\lambda \in \mathbb{C}^* \,|\, \mu(\lambda) = n/2\} = \emptyset$. Hence, by Lemma 2.2(i), $\mathscr{L}_n$ is invertible on $\mathcal{D}_r^*$.

Define B by

$$\xi = A z^{-1}\, dz + B z^{n-1}\, dz + \mathrm{O}(z^n)\, dz.$$

Then

$$\operatorname*{ord}_{\lambda=0} A_{12} = -1, \quad \operatorname*{ord}_{\lambda=0} A_{21} = 0, \quad \operatorname*{ord}_{\lambda=0} B_{12} \geqslant -1, \quad \operatorname*{ord}_{\lambda=0} B_{21} \geqslant 0,$$

so

$$\operatorname*{ord}_{\lambda=0} \frac{B_{12}}{A_{12}} \geqslant 0, \quad \operatorname*{ord}_{\lambda=0} \frac{B_{21}}{A_{21}} \geqslant 0.$$

Hence, the limit

$$\kappa := \frac{1}{2} \lim_{\lambda \to 0} \left(\frac{B_{12}}{A_{12}} + \frac{B_{21}}{A_{21}} \right)$$

exists in $\mathbb{C}$.

Let $X = \kappa A - B$. A calculation shows that with this choice of κ, $A_{12} X_{21} + A_{21} X_{12}$ is holomorphic at 0. Hence, by Lemma 2.4, $\mathscr{L}_n^{-1}(\kappa A - B)$ on $\mathcal{D}_r^*$ extends holomorphically to $\lambda = 0$. Hence, we have the holomorphic map $C : \mathcal{D}_r \to \mathrm{SL}_2\mathbb{C}$ defined by

$$C = \mathscr{L}_n^{-1}(\kappa A - B). \tag{20}$$

Define the holomorphic map $h : \Sigma \times \mathcal{D}_r \to \mathrm{GL}_2\mathbb{C}$ by $h = \mathrm{I} + C z^n$. Since $\det h$ will not in general be identically 1, we define $g = h + \mathrm{O}(z^{n+1})$ with $\det g = 1$ as follows. Since $h_{22} = C_{22} z^n$, and $\overline{\mathcal{D}_r}$ is bounded away from the poles of C_{22}, then we have the uniform convergence $h_{22} \to 1$ on $\overline{\mathcal{D}_r}$ as $z \to 0$. Hence, there exists a neighborhood $\Sigma' \subset \Sigma$ of $0 \in \Sigma$ on which $h_{22} > 0$. Define the holomorphic map $g : \Sigma' \to \Lambda_r^{\mathrm{pos}} \mathrm{SL}_2\mathbb{C}$ by

$$g = h + \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \frac{1 - \det h}{h_{22}}.$$

A calculation shows $\det g = 1$.

In order to show that g satisfies the gauge equation (19), we show that $g = h + \mathrm{O}(z^{n+1})$. We have $\det h = 1 + \operatorname{tr}(C) z^n + \mathrm{O}(z^{n+1})$. Since $\phi A - B$ is tracefree, then C is tracefree by Lemma 2.2(iii). Hence, $\det h = 1 + \mathrm{O}(z^{n+1})$. Since $h_{22} = 1 + \mathrm{O}(z^n)$, then $q = \mathrm{O}(z^{n+1})$. Hence, $g = h + \mathrm{O}(z^{n+1})$.

<center>199</center>

To complete the proof, we use the following formula. For any $n \in \mathbb{Z}_+$, r-gauge $g = \mathrm{I} + g_n z^n + \mathrm{O}(z^{n+1})$ and r-potential $\xi = A z^{-1}\, dz + B_{n-1} z^{n-1} + \mathrm{O}(z^n)$, we have

$$\xi.g = A z^{-1}\, dz + (\mathscr{L}_n(g_n) + B_{n-1}) z^{n-1}\, dz + \mathrm{O}(z^n)\, dz. \tag{21}$$

The formula can be shown by expanding $\xi.g = g^{-1}\xi g + g^{-1}\, dg$ in z.

The result (19) follows from (20), together with formula (21) replacing g_n with C and B_{n-1} with B. $\qquad\square$

LEMMA 2.6. *Let ξ be a perturbed Delaunay r-potential (Definition 2.1)*

$$\xi = A z^{-1}\, dz + \kappa A z^{n-1}\, dz + \mathrm{O}(z^n)\, dz \tag{22}$$

for some $n \in \mathbb{Z}_+$ and $\kappa \in \mathbb{C}$. Then there exists a coordinate change $\vartheta : \Sigma' \to \mathbb{C}$ on a neighborhood $\Sigma' \subset \mathbb{C}$ of $0 \in \mathbb{C}$ with conformal coordinate w, such that on $\Sigma' \setminus \{0\}$,

$$\vartheta^* \xi = A w^{-1}\, dw + \mathrm{O}(w^n)\, dw. \tag{23}$$

Proof. Let $\sigma : \Sigma \to \mathbb{C}$ be any analytic function satisfying

$$\sigma(z) = z + (\kappa/n) z^{n+1} + O(z^{n+2}).$$

Then $\sigma(0) = 0$ and $\sigma'(0) = 1$, so there exists a coordinate change $\vartheta : \Sigma' \to \Sigma$ on a neighborhood $\Sigma' \subset \mathbb{C}$ of 0 which is the inverse of σ. Because ϑ is a conformal diffeomorphism, for any $j \in \mathbb{Z}$ and any differential $\omega = \mathrm{O}(z^j)\, dz$, we have $\vartheta^* \omega = \mathrm{O}(w^j)\, dw$. A calculation shows that on $\vartheta(\Sigma')$,

$$(z^{-1} + \kappa z^{n-1})\, dz = (\sigma^{-1} + \mathrm{O}(\sigma^n))\, d\sigma.$$

Hence,

$$\vartheta^*((z^{-1} + \kappa z^{n-1})\, dz) = w^{-1}\, dw + \mathrm{O}(w^n)\, dw.$$

This with (22) implies the result (23). $\qquad\square$

We now apply Lemmas 2.5 and 2.6 iteratively to transform ξ to the form $A w^{-1}\, dw + O(w^n)\, dw$.

THEOREM 2.7. *Let ξ be a perturbed Delaunay r-potential (Definition 2.1)*

$$\xi = A z^{-1}\, dz + \mathrm{O}(z^0)\, dz.$$

Assume for some $n \in \mathbb{Z}_+$ that

$$\min_{\lambda \in \mathcal{C}_r} \mathrm{Re}\,\mu(\lambda) \leqslant \frac{1}{2} \quad \text{and} \quad \frac{n}{2} < \max_{\lambda \in \mathcal{C}_r} \mathrm{Re}\,\mu(\lambda). \tag{24}$$

Then there exists a coordinate change $\vartheta : \Sigma' \to \Sigma$ on a neighborhood $\Sigma' \subset \mathbb{C}$ of $0 \in \mathbb{C}$ with coordinate w, and an r-gauge $g : \Sigma' \to \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$, such that

$$\vartheta^*(\xi.g) = A w^{-1}\, dw + O(w^n)\, dw.$$

Moreover, let Φ satisfy $d\Phi = \Phi\xi$. Then $\Psi = \vartheta^*(\Phi g)$ satisfies $d\Psi = \Psi(\vartheta^*(\xi.g)))$, and the monodromies of Φ and Ψ on a loop around $z = 0$ are equal.

Proof. For any conformal maps $\vartheta_1 : \Sigma' \to \Sigma$, $\vartheta_2 : \Sigma'' \to \Sigma'$ and r-gauges $g_1 : \Sigma \to \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$, $g_2 : \Sigma' \to \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$, we have the composition formula

$$\vartheta_2^*((\vartheta_1^*(\xi_1.g_1)).g_2) = \theta^*(\xi.g), \tag{25}$$

where $\vartheta : \Sigma'' \to \Sigma$ is the conformal map and $g : \Sigma \to \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$ is the r-gauge defined by

$$\vartheta = \vartheta_1 \circ \vartheta_2 \quad \text{and} \quad g = g_1 \cdot ((\vartheta_1^{-1})^* g_2).$$

The result then follows using the composition formula (25) by induction on n, applying Lemmas 2.5 and 2.6 repeatedly. The conditions (24) insure that condition (18) holds for each step. $\qquad\square$

200

LEMMA 2.8. *Let $r \in (0, 1]$, $X \in \Lambda_r \mathrm{SL}_2\mathbb{C}$, and $Y \in \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$. Then we have the following.*

(i) *There exists $U \in \mathrm{SU}_2$ such that*

$$\mathsf{Uni}_r[XY] = \mathsf{Uni}_r[X]U \quad and \quad \mathsf{Pos}_r[XY] = U^{-1}\mathsf{Pos}_r[X]Y. \qquad (26)$$

(ii) *Let ξ be an r-potential on a Riemann surface Σ. Let Φ satisfy $d\Phi = \Phi\xi$. Let $g : \Sigma \to \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$ be a gauge. Then $\mathsf{Sym}_r[\Phi] = \mathsf{Sym}_r[\Phi g]$.*

Proof. To prove part (i), let $Y_+ = \mathsf{Pos}_r[Y]$ and let U_1 be the unitary factor in the QR-decomposition of $Y(0)$. Then $Y = U_1 Y_+$. Let $X_u = \mathsf{Uni}_r[X]$ and $X_+ = \mathsf{Pos}_r[X]$, so $X = X_u X_+$. Let U be the unitary factor in the QR-decomposition of $X_+(0)U_1$. Then $X_+ U_1 = U\mathsf{Pos}_r[X_+ U_1]$. Hence,

$$\mathsf{Pos}_r[XY] = \mathsf{Pos}_r[X_+ Y] = \mathsf{Pos}_r[X_+ U_1 Y_+] = \mathsf{Pos}_r[X_+ U_1] Y_+ = U^{-1} X_+ U_1 Y_+ = U^{-1} X_+ Y.$$

This proves the second equality in (26), and the first equality follows from it.

To prove part (ii), by part (i) there exists $U \in \mathrm{SU}_2$ such that $\mathsf{Uni}_r[\Phi g] = \mathsf{Uni}_r[\Phi]U$. It follows that $\mathsf{Sym}_r[\Phi] = \mathsf{Sym}_r[\Phi g]$. $\qquad\square$

2.3 Cauchy integral formula for vector-valued maps

The following technical lemma shows from the convergence of a family of holomorphic functions the convergence of all of its derivatives in a strictly contained subdomain. The proof is standard and uses the Cauchy integral formula. We use the notation $\mathcal{R}' \subset\subset \mathcal{R}$ to mean that $\mathcal{R}'$ is bounded away from $\partial\mathcal{R}$. Let $X^{(n)}$ denote the nth derivative of X with respect to λ.

LEMMA 2.9. *Let V be a finite-dimensional vector space over $\mathbb{C}$ and let $|\cdot|$ be a vector norm on V. With $\Sigma \subset \mathbb{C}$ a punctured neighborhood of $0 \in \mathbb{C}$ and $\mathcal{R} \subset \mathbb{C}$ a bounded domain with smooth boundary, let $X = X(z, \lambda) : \Sigma \times \mathcal{R} \to V$ be continuous in z and holomorphic in λ. Suppose $\lim_{z\to 0} \sup_{\lambda \in \mathcal{R}} |X| = 0$. Then for all $n \in \mathbb{Z}_{\geqslant 0}$, and every subset $\mathcal{R}' \subset\subset \mathcal{R}$,*

$$\lim_{z\to 0} \sup_{\lambda \in \mathcal{R}'} |X^{(n)}| = 0.$$

Proof. With $n = \dim V$, we may assume that $V = \mathbb{C}^n$. Since all vector norms on a finite-dimensional vector space are equivalent, we may assume that the norm $|\cdot|$ is given by $|(Y_1, \ldots, Y_n)| = \max_j |Y_j|$. Let $\mathcal{R}' \subset\subset \mathcal{R}$. By the Cauchy integral formula, for all $n \in \mathbb{Z}_{\geqslant 0}$ and all $\lambda \in \mathcal{R}'$,

$$X^{(n)}(z, \lambda) = \frac{n!}{2\pi i} \int_{\partial\mathcal{R}} \frac{X(z, \nu)}{(\nu - \lambda)^{n+1}} \, d\nu,$$

so

$$|X^{(n)}(z, \lambda)| \leqslant \frac{n!}{2\pi} \max_j \int_{\partial\mathcal{R}} \frac{|X_j(z, \nu)|}{|\nu - \lambda|^{n+1}} |d\nu| \leqslant \frac{n!}{2\pi} \int_{\partial\mathcal{R}} \frac{|X(z, \nu)|}{|\nu - \lambda|^{n+1}} |d\nu|.$$

Since $\lambda \in \mathcal{R}'$ and $\mathcal{R}'$ is bounded away from $\partial\mathcal{R}$, then there exists $c_1 \in \mathbb{R}_{>0}$ such that for all $\nu \in \partial\mathcal{R}$, $|\nu - \lambda|^{-(n+1)} < c_1$. Hence, with $c_2 = c_1 n!/(2\pi)$,

$$|X^{(n)}(z, \lambda)| \leqslant c_2 \int_{\partial\mathcal{R}} |X(z, \nu)| |d\nu|.$$

Since this holds for all $\lambda \in \mathcal{R}'$, and the right-hand side is independent of λ, so

$$\sup_{\lambda \in \mathcal{R}'} |X^{(n)}(z, \lambda)| \leqslant c_2 \int_{\partial\mathcal{R}} |X(z, \nu)| |d\nu|.$$

The result follows by taking limits. $\qquad\square$

2.4 Asymptotics of the perturbed Delaunay frame

We now have the tools to show that the r-Iwasawa factors of a holomorphic frame constructed from a perturbed Delaunay potential converge to the respective factors of a holomorphic frame constructed from a Delaunay potential (Theorem 2.11). This convergence implies that the immersions and their metrics converge (Theorem 3.2). A bootstrap argument in §3 strengthens this to C^∞-convergence of the immersions.

LEMMA 2.10. *Let ξ be a perturbed Delaunay potential, let $d\Phi = \Phi\xi$ and assume that the monodromy M of Φ around $z = 0$ satisfies $M \in \Lambda_r^* \mathrm{SL}_2\mathbb{C}$. With $\Phi_0 = \exp(A \log z)$, let $\Phi = C\Phi_0 P$ be the decomposition as in Lemma 2.3. Let $C\Phi_0 = F_0 \cdot B_0$ and $\Phi = F \cdot B$ be the respective r-Iwasawa factorizations on $\widetilde{\Sigma^*}$. Then $F_0^{-1}F$, B and B_0 are lifts of unique single-valued maps on Σ^*.*

Proof. Let $\tau : \widetilde{\Sigma^*} \to \widetilde{\Sigma^*}$ be a deck transformation corresponding to the curve defining the monodromy M of Φ. Then by the definition of M, for all $z \in \widetilde{\Sigma^*}$ we have $\tau^*\Phi = M\Phi$, so

$$(\tau^*F)(\tau^*B) = MFB,$$

or

$$(\tau^*B)B^{-1} = (\tau^*F)^{-1}MF. \tag{27}$$

The left-hand side of (27) is in $\Lambda_r^+ \mathrm{SL}_2\mathbb{C}$ and since, by hypothesis, $M \in \Lambda_r^* \mathrm{SL}_2\mathbb{C}$, the right-hand side of (27) is in $\Lambda_r^* \mathrm{SL}_2\mathbb{C}$. Hence, each side is equal to I, so $\tau^*B = B$, and B is the lift of a single-valued map on Σ^*.

Since $\Phi = C\exp(A\log z)P$, and $\tau^*P = P$, then

$$M = C\exp(2\pi iA)C^{-1}.$$

Since $\Phi_0 = C\exp(A\log z)$, then M is also the monodromy of Φ_0. Hence, $\tau^*\Phi_0 = M\Phi_0$. The same argument as above, with F_0 and B_0 replacing F and B, respectively, shows that $\tau^*B_0 = B_0$, so B_0 is the lift of a single-valued map on Σ^*.

Since

$$F_0^{-1}F = B_0PB^{-1}. \tag{28}$$

and the right-hand side of (28) is invariant under the action of τ^*, then so is the left-hand side of (28), so $F_0^{-1}F^{-1}$ is the lift of a single-valued map on Σ^*. $\square$

THEOREM 2.11. *Let $r \in (0,1)$ and assume $\mathcal{C}_r \cap \mathcal{S}_A = \emptyset$. Let ξ be a perturbed Delaunay r-potential*

$$\xi = Az^{-1}\, dz + \mathrm{O}(z^n)\, dz. \tag{29}$$

Suppose

$$\max_{\lambda \in \mathcal{C}_r} \mathrm{Re}\,\mu(\lambda) < \frac{n+1}{2}. \tag{30}$$

Let $\Phi : \widetilde{\Sigma^*} \to \Lambda_r\mathrm{SL}_2\mathbb{C}$ satisfy $d\Phi = \Phi\xi$ on the universal cover $\widetilde{\Sigma^*} \to \Sigma^*$ of Σ^*, and assume that the monodromy M of Φ around $z = 0$ satisfies $M \in \Lambda_r^* \mathrm{SL}_2\mathbb{C}$. Let $\Phi_0 = \exp(A\log z)$ and let $\Phi = C\Phi_0 P$ be the $z^A P$-decomposition of Φ as in Lemma 2.3. Assume for some $0 < s_1 < r < s_2 < 1$ there exists a continuous function $c : \mathcal{A}_{s_1,s_2} \to \mathbb{R}_+$ such that $\mathsf{Pos}_r[C\Phi_0]$ extends analytically to $\mathcal{A}_{s_1,s_2}$ and

$$\|\mathsf{Pos}_r[C\Phi_0]\|_{\mathcal{A}_{s_1,s_2}} \leqslant c|z|^{-\mathrm{Re}\,\mu}. \tag{31}$$

Then Φ and $C\Phi_0$ have the convergence

$$\lim_{z \to 0} \|(\mathsf{Uni}_r[C\Phi_0])^{-1}\mathsf{Uni}_r[\Phi] - \mathrm{I}\|_{\mathcal{A}_r} = 0,$$
$$\lim_{z \to 0} \|\mathsf{Pos}_r[\Phi](\mathsf{Pos}_r[C\Phi_0])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{32}$$

Proof. *Step 1.* Let $v_1 = \max_{\lambda \in \mathcal{C}_r} \operatorname{Re} \mu(\lambda)$. There exist $r_1, r_2 \in \mathbb{R}_+$ such that $s_1 \leqslant r_1 < r < r_2 \leqslant s_2$, $\mathcal{A}_{r_1,r_2} \cap \mathcal{S}_A = \emptyset$ and $\operatorname{Pos}_r[C\Phi_0]$ extends analytically to $\mathcal{A} := \mathcal{A}_{r_1,r_2}$. By (31),

$$\|\operatorname{Pos}_r[C\Phi_0]\|_{\overline{\mathcal{A}}} \leqslant c|z|^{-v_1}. \tag{33}$$

Step 2. Let $C\Phi_0 = F_0 \cdot B_0$ and $\Phi = C\Phi_0 P = F \cdot B$ and be the respective r-Iwasawa factorizations on $\widetilde{\Sigma^*}$. Since by hypothesis $M \in \Lambda_r^* \mathrm{SL}_2\mathbb{C}$, by Lemma 2.10, B, B_0 and $F_0^{-1}F$ on $\widetilde{\Sigma^*}$ descend to single-valued analytic maps on Σ^*.

Let $m = n + 1$. By Lemma 2.3(iii), P has the expansion on Σ'

$$P(z, \lambda) = I + \sum_{k=m}^{\infty} P_k(\lambda) z^k.$$

Then

$$B_0 P B_0^{-1} - I = \sum_{k=m}^{\infty} B_0 P_k B_0^{-1} z^k.$$

By (33),

$$\|B_0\|_{\mathcal{A}} \leqslant c|z|^{-v_1} \quad \text{and} \quad \|B_0^{-1}\|_{\mathcal{A}} \leqslant c|z|^{-v_1},$$

so

$$\|B_0 P B_0^{-1} - I\|_{\mathcal{A}} \leqslant \sum_{k=m}^{\infty} \|B_0\|_{\mathcal{A}} \|P_k\|_{\mathcal{A}} \|B_0^{-1}\|_{\mathcal{A}} |z|^k \leqslant c^2 \sum_{k=m}^{\infty} \|P_k\|_{\mathcal{A}} |z|^{k-2v_1}.$$

Since by (43), $v_1 < m/2$, then $k - 2v_1 > 0$ for all $k \in \mathbb{Z}_{\geqslant m}$. Hence,

$$\lim_{z \to 0} \|B_0 P B_0^{-1} - I\|_{\mathcal{A}} = 0.$$

By Lemma 2.9, $B_0 P B_0^{-1} \to I$ as $z \to 0$ as a map in $C^\infty(\mathcal{C}_r, \mathrm{SL}_2\mathbb{C})$.

Step 3. We have the r-Iwasawa factorization

$$B_0 P B_0^{-1} = (F_0^{-1}F) \cdot (BB_0^{-1}).$$

As noted in the preliminary section, the r-Iwasawa factorization is a homeomorphism in the C^∞-topologies on its domain and target spaces. The result (32) follows. $\square$

3. Asymptotics of immersions

3.1 Convergence of geometric data

In Theorem 2.11 we showed that the unitary and positive part of a holomorphic frame constructed from a perturbed Delaunay potential converge to those of a Delaunay holomorphic frame. From this we obtain convergence of the immersions, metrics, and moving frames and normals. This convergence is strengthened to C^∞-convergence in § 3.2.

LEMMA 3.1. *Let $\Sigma^* \subset \mathbb{C}$ be a punctured neighborhood of $0 \in \mathbb{C}$ with coordinate z. Let $r \in (0, 1]$. Let ξ_1 and ξ_2 be r-potentials. Let α_k be the coefficient of λ^{-1} in the upper-right entry ξ_k in its λ expansion, and suppose*

$$\lim_{z \to 0}(\alpha_1/\alpha_2) = 1. \tag{34}$$

For $k \in \{1, 2\}$, let Φ_k satisfy $d\Phi_k = \Phi_k \xi_k$. Let $\mathcal{R} \subset \mathbb{C}$ be a neighborhood of $\lambda_0 \in \mathbb{S}^1$. Suppose

$$\lim_{z \to 0} \|\operatorname{Uni}_r[\Phi_1]^{-1}\operatorname{Uni}_r[\Phi_2] - I\|_{\mathcal{R}} = 0,$$
$$\lim_{z \to 0} \operatorname{Pos}_r[\Phi_1](\operatorname{Pos}_r[\Phi_2])^{-1}|_{\lambda=0} = I. \tag{35}$$

203

Let $f_k = \mathsf{Sym}_r[\Phi_k]$ be the immersions obtained from Φ_k, and let $v_k^2|dz|^2$ be the respective metrics of f_k. Then:

(i) the immersions converge, that is, $\lim_{z\to 0}(f_2 - f_1)|_{\lambda_0} = 0$;

(ii) the metrics converge in ratio, that is, $\lim_{z\to 0}(v_2/v_1) = 1$;

(iii) the moving frames G_k for f_k converge, that is, $\lim_{z\to 0}(G_1^{-1}G_2)|_{\lambda_0} = \mathrm{I}$;

(iv) the difference of the normals of f_1 and f_2 converges to zero as $z \to 0$.

Moreover, let $x + iy = \log z$ and let $w_k^2(dx^2 + dy^2)$ be the metric for f_k in the x and y coordinates. If w_1 is bounded and bounded away from zero on Σ, then the immersions have C^1-convergence in the coordinates x and y; that is,

$$\lim_{z\to 0}\left(\frac{d}{dx}f_2 - \frac{d}{dx}f_1\right)\Big|_{\lambda_0} = 0 \quad \text{and} \quad \lim_{z\to 0}\left(\frac{d}{dy}f_2 - \frac{d}{dy}f_1\right)\Big|_{\lambda_0} = 0. \tag{36}$$

Proof. For $k \in \{1, 2\}$, let $F_k = \mathsf{Uni}_r[\Phi_k]$ and $B_k = \mathsf{Pos}_r[\Phi_k]$. Let prime denote differentiation with respect to θ, where $\lambda = e^{i\theta}$.

Step 1. To show the convergence of the immersions, by the Sym formula for $\mathbb{R}^3$ in the preliminary section, we have

$$f_k = -2H^{-1}F_k' F_k^{-1}, \quad k \in \{1, 2\},$$

where $H \in \mathbb{R}_+$ is the constant mean curvature. Hence,

$$f_2 - f_1 = -2H^{-1}F_1(F_1^{-1}F_2)'F_2^{-1}. \tag{37}$$

Let $\mathcal{R}' \subset\subset \mathcal{R}$ be a neighborhood of λ_0. Since $F_1^{-1}F_2 - \mathrm{I}$ is holomorphic on $\mathcal{R}$, by Lemma 2.9,

$$\lim_{z\to 0}\|(F_1^{-1}F_2)'\|_{\mathcal{R}'} = 0. \tag{38}$$

The convergence of the immersions $\lim_{z\to 0}(f_2 - f_1)|_{\lambda_0} = 0$ follows from (37) and (38) and the fact that $\|F_1(\lambda_0)\| = 1 = \|F_2^{-1}(\lambda_0)\|$.

Step 2. To show the convergence of the metrics, since $F_k B_k = \Phi_k$, then B_k satisfies the gauge equation $(F_k^{-1}\,dF_k).B_k = \xi_k$. An examination of the coefficient of λ^{-1} in this gauge equation above shows that

$$v_k = 2|H^{-1}\alpha_k|\rho_k^2, \tag{39}$$

where ρ_k is the constant term of the upper-right entry of B_k in its λ expansion. Equation (34) implies $\lim_{z\to 0}(\rho_1/\rho_2) = 1$, and the result follows.

Step 3. The moving frames G_k for f_k are defined by the equations

$$(f_k)_x = v_k G_k \begin{pmatrix} 0 & dz \\ -d\overline{z} & 0 \end{pmatrix} G_k^{-1}, \quad k \in \{1, 2\}. \tag{40}$$

The extended frames F_k and the moving frames G_k are related by a gauge g_k defined by

$$G_k = F_k g_k, \quad g_k = \mathrm{diag}(p_k, p_k^{-1}), \quad p_k^2 = iH\alpha_k/(\lambda|H\alpha_k|) \quad k \in \{1, 2\}.$$

Then

$$G_1^{-1}G_2 - \mathrm{I} = g_1^{-1}(F_1^{-1}F_2 - g_1 g_2^{-1})g_2. \tag{41}$$

By (34), it follows that $\lim_{z\to 0} g_1(\lambda_0)g_2^{-1}(\lambda_0) = 1$. Since $\|g_k(\lambda_0)\|$ is finite, the convergence of the moving frames, part (iii), follows from (41). The convergence of the normals follows from the convergence of the moving frames.

Step 4. To show the convergence of the derivatives of the immersions (36), let

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

204

Writing $x = x_1$ and $y = x_2$, by (40) we have $(f_k)_{x_j} = w_k G_k e_j G_k^{-1}$. Hence,

$$\frac{d}{dx_j}(f_2 - f_1) = (w_2 - w_1) G_2 e_j G_2^{-1} + w_1 G_1 [G_1^{-1} G_2, e_j] G_2^{-1}.$$

Since $\lim_{z \to 0}(v_2/v_1) = 1$ and $v_k = w_k e^{-x/2}$, then $\lim_{z \to 0}(w_2/w_1) = 1$. Since by assumption, w_1 or w_2 is bounded away from zero, then $\lim_{z \to 0}(w_2 - w_1) = 0$. The first equation of (36) follows, since $\|G_1(\lambda_0) e_j G_1^{-1}(\lambda_0)\|$ is finite, w_1 is bounded, $\|G_1(\lambda_0)\|$ and $\|G_2^{-1}(\lambda_0)\|$ are finite, and $\lim_{z \to 0} \|G_1^{-1}(\lambda_0) G_2(\lambda_0)\| = 0$ by (38). $\qquad\square$

THEOREM 3.2. *Let A be a Delaunay residue satisfying*

$$\min_{\lambda \in \mathcal{C}_r} \operatorname{Re} \mu(\lambda) \leqslant \tfrac{1}{2}. \tag{42}$$

Let ξ be a Delaunay r-perturbation of A. Let Φ satisfy $d\Phi = \Phi\xi$ with unitary monodromy at $z = 0$. Assume Φ is in $\Lambda_r^{\uparrow} \mathrm{SL}_2\mathbb{C}$, and $(\mathcal{C}_r \cup \mathcal{A}_{r,1}) \cap \mathcal{S}_A = \emptyset$. Let f be the CMC immersion induced by Φ. Then there exists a Delaunay immersion f_0 with the same necksize as that induced by $\Phi_0 := \exp(A \log z)$, such that as $z \to 0$, in the coordinates $x + iy = \log z$, the ratio of their metrics converges to one and $f - f_0$ converges C^1 to zero.

Proof. We shall require the following fact: as functions of r, $\max_{\lambda \in \mathcal{C}_r} \operatorname{Re} \mu$ is strictly decreasing on $(0, 1]$ and $\min_{\lambda \in \mathcal{C}_r} \operatorname{Re} \mu$ is zero on $(0, |\nu_1|]$ and is strictly increasing on $[|\nu_1|, 1]$. (The boundary of the upper graph in Figure 2(b) illustrates this behavior.)

Let $\upsilon_0 = \min_{\lambda \in \mathbb{C}_r} \operatorname{Re} \mu(\lambda)$ and $\upsilon_1 = \max_{\lambda \in \mathbb{C}_r} \operatorname{Re} \mu(\lambda)$. Since $\upsilon_1 \geqslant \upsilon_0 \geqslant 1/2$, there exists $n \in \mathbb{Z}_+$ such that

$$\frac{n}{2} \leqslant \upsilon_1 < \frac{n+1}{2}.$$

If $\upsilon_1 = n/2$, by the above fact, r can be replaced with a smaller value and υ_0 and υ_1 by the corresponding values so that this new ν_1 satisfies

$$\frac{n}{2} < \upsilon_1 < \frac{n+1}{2}. \tag{43}$$

Then υ_0 is still at most $1/2$ by the above fact. By Theorem 2.7 and Lemma 2.8(ii), we may assume after a gauge and coordinate change that ξ has the form (29).

Since Φ is in $\Lambda_r^{\uparrow} \mathrm{SL}_2\mathbb{C}$, we have the $z^A P$ decomposition $\Phi = C \exp(A \log z) P$ on $\mathcal{A}_{r,1} \setminus \mathcal{S}_A$. Since by assumption $(\mathcal{C}_r \cup \mathcal{A}_{r,1}) \cap \mathcal{S}_A = \emptyset$, then C and P extend to $\mathcal{C}_r \cup \mathcal{A}_{r,1}$.

By Theorem 1.13, there exists a continuous function $c : \mathcal{D}_1^* \to \mathbb{R}_+$ such that for all $(z, \lambda) \in \{0 < |z| < 1\} \times \mathcal{D}_1^*$,

$$\|\mathsf{Pos}_r[C\Phi_0]\| \leqslant c|z|^{-\operatorname{Re}\mu}.$$

By Theorem 2.11, Φ and $C\Phi_0$ have the convergence (32). Hence, taking $\Phi_1 = \Phi_0$ and $\Phi_2 = \Phi$ in Lemma 3.1, condition (35) holds. By Lemma 1.12, $C\Phi_0$ induces a Delaunay immersion which differs from Φ_0 by a rigid motion. Taking $\xi_1 = A\,dz/z$ and $\xi_2 = \xi$ in Lemma 3.1, condition (34) of Lemma 3.1 holds because ξ_2 is a holomorphic perturbation of ξ_1.

With w_1 and w_2 as in Lemma 3.1, a computation shows that with v as in Theorem 1.5, by (39), the metric of the Delaunay immersion is $w_1 = 4|abH^{-1}|v^{-1}$. Hence, w_1 is periodic and nonzero in the coordinate $x = \log |z|$, and is hence bounded and bounded away from zero on Σ. The assertion follows by Lemma 3.1. $\qquad\square$

Remark 3.3. In the case when the surface is a nodoid, and under suitable normalization $\lambda = -1$ is a resonance point, or equivalently a double point of the corresponding spectral curve (that is, the eigenvalue of $A(-1)$ is $n/2$, $n \in \mathbb{Z}_+$ in the above proof), then it is possible to deform the nodoid

into a spectral genus two CMC cylinder. These are precisely the bifurcating nodoids of Mazzeo–Pacard [MP02] (see also [Gro07]). An easy computation using the formula for the necksize in terms of the parameters of the Delaunay residue (see e.g. [Kil04]) shows that this occurs for nodoids with necksizes $\frac{1}{2}|H|^{-1}n$.

3.2 Asymptotics of conformal immersions

Now that we have shown that the frames and metrics of the immersion produced by a perturbed Delaunay potential converge to those of a Delaunay immersion, we apply a bootstrap argument on the Gauss equation to obtain C^∞-convergence of the immersions up to rigid motion. For details and formulas, see [Bob91b]. The notation $f \xrightarrow{C^n} g$ means $\lim_{z \to 0} \|f - g\|_{C^n} = 0$, where the C^n norm is taken over some implied compact domain.

THEOREM 3.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $\Omega' \subset\subset \Omega$ be a strictly contained subset. Let $f : \overline{\Omega} \to \mathbb{R}^3$ and the sequence $f_j : \overline{\Omega} \to \mathbb{R}^3$, $j \in \mathbb{Z}_+$ be a sequence of conformal immersions in $C^\infty(\Omega) \cap C^0(\overline{\Omega})$. Let $e^u, H, Q\,dz^2$ and $e_j^u, H_j, Q_j\,dz^2$ be the respective conformal factors, mean curvature functions, and Hopf differential of f and f_j. Suppose that $f_j \xrightarrow{C^1} f$, $H_j \xrightarrow{C^0} H$, $Q_j \xrightarrow{C^\infty} Q$ and $u_j \xrightarrow{C^0} u$ on Ω. Then $f_j \xrightarrow{C^\infty} f$ on Ω'.*

Proof. We use the following gradient estimate for Poisson's equation [GT77]. Let Σ be a bounded domain in $\mathbb{R}^2$ and $\Sigma' \subset\subset \Sigma$. Then there exists $c \in \mathbb{R}_+$ depending only on $\mathrm{dist}(\Sigma', \partial\Sigma)$ such that for all $w \in C^2(\Sigma) \cap C^0(\overline{\Sigma})$,

$$\sup_{\Sigma'} |\partial_x w| + \sup_{\Sigma'} |\partial_y w| \leqslant c\Big(\sup_{\Sigma} |w| + \sup_{\Sigma} |\Delta w| \Big). \tag{44}$$

Fix $k \in \mathbb{N}_{\geqslant 1}$ and choose domains $\Omega_1, \ldots, \Omega_k$ with $\Omega' = \Omega_k \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega$.

Step 1. We apply the estimate (44) k times to show that $u_j \xrightarrow{C^k} u$ and $H_j \xrightarrow{C^{k-1}} H$ on Ω'

The first iteration is as follows. For the functions u, H, Q, define

$$\Psi[u, H, Q] = -\tfrac{1}{2}H^2 e^u + 2|Q|^2 e^{-u}.$$

Then the Gauss equations for f and the f_j are, respectively,

$$\Delta u = \Psi[u, H, Q], \quad \Delta u_j = \Psi[u_j, H_j, Q_j].$$

Hence,

$$\Delta(u_j - u) = \Phi[u_j, H_j, Q_j] - \Phi[u, H, Q].$$

Since $u_j \xrightarrow{C_0} u$, $H_j \xrightarrow{C_0} H$ and $Q_j \xrightarrow{C_0} Q$ on Ω, then

$$\Phi[u_j, H_j, Q_j] - \Phi[u, H, Q] \xrightarrow{C_0} 0,$$

so $\Delta(u_j - u) \xrightarrow{C_0} 0$ on Ω. Applying the estimate (44) with $\Sigma = \Omega$, $\Sigma' = \Omega_1$ and $w = u_j - u$, we obtain $u_j \xrightarrow{C_1} u$ on Ω_1.

The Codazzi equations for f and the f_j are, respectively,

$$H_z = 2e^{-u}Q_{\overline{z}}, \quad (H_j)_z = 2e^{-u_j}(Q_j)_{\overline{z}}.$$

Since $u_j \xrightarrow{C^0} u$ and $Q_j \xrightarrow{C^1} Q$ on Ω, it follows that $H_j \xrightarrow{C^1} H$ on Ω.

Using the Gauss equation again in the same way, since $u_j \xrightarrow{C^1} u$, $H_j \xrightarrow{C_1} H$ and $Q \xrightarrow{C_1} Q$ on Ω_1, then $\Delta(u_j - u) \xrightarrow{C_1} 0$. Hence, $\Delta \partial_x(u_j - u) \xrightarrow{C_0} 0$ and $\Delta \partial_y(u_j - u) \xrightarrow{C_0} 0$. Applying the estimate (44) again, with $\Sigma = \Omega_1$, $\Sigma' = \Omega_2$ and w take to be first $\partial_x(u_j - u)$ and then $\partial_y(u_j - u)$, we obtain $u_j \xrightarrow{C_2} u$ on Ω_2.

206

Repeating the argument $k-2$ more times shows that $u_j \overset{C_k}{\to} u$ and $H_j \overset{C^{k-1}}{\to} H$ on $\Omega_k = \Omega'$. Since k is arbitrary, the above argument shows $u_j \overset{C^\infty}{\to} u$ and $H_j \overset{C^\infty}{\to} H$ on Ω'.

Step 2. Let e_1, e_2, e_3 be a positively oriented orthonormal basis for $\mathfrak{su}_2$. Let F and F_j be the respective moving frames for f and f_j with values in SU_2, defined respectively by

$$\partial_x f = e^u F e_1 F^{-1}, \quad \partial_y f = e^u F e_2 F^{-1} \quad \text{and} \quad \partial_x f_j = e^u F_j e_1 F_j^{-1}, \quad \partial_y f_j = e^u F_j e_2 F_j^{-1}. \quad (45)$$

By assumption, $f_j \overset{C^1}{\to} f$, from which it follows that $F_j \overset{C^0}{\to} F$.

Write the Lax pairs for F and F_j respectively

$$\partial_x F = FU, \quad \partial_y F = FV \quad \text{and} \quad \partial_x F_j = F_j U_j, \quad \partial_y F_j = F_j V_j.$$

Since $u_j \overset{C^\infty}{\to} u$, $H_j \overset{C^\infty}{\to} H$ and $Q_j \overset{C^\infty}{\to} Q$, then $U_j \overset{C^\infty}{\to} U$ and $V_j \overset{C^\infty}{\to} V$. Since the Lax pair equations are linear and $F_k \overset{C^0}{\to} F$, it follows that $F_j \overset{C^\infty}{\to} F$ by smooth dependence of solutions on parameters. By (45), we obtain the convergence $f_j \overset{C^\infty}{\to} f$. □

3.3 Embedded ends

Under appropriate assumptions, if an immersion f of a cylinder converges C^2 to an embedding of the cylinder, then f is embedded.

Let Σ_{Cyl} be the cylinder

$$\Sigma_{\mathrm{Cyl}} = \{(x, y) \in \mathbb{R}^2 \,|\, x > 0 \text{ and } 0 \leqslant y \leqslant 2\pi\}, \quad (46)$$

with the boundary lines $y = 0$ and $y = 2\pi$ identified.

LEMMA 3.5. *With Σ_{Cyl} as in (46), let $f : \Sigma_{\mathrm{Cyl}} \to \mathbb{R}^3$ be a properly embedded surface with bounded curvature which satisfies the following condition:*

For all $\delta > 0$ there exists $\epsilon > 0$ such that for all $p, q \in \Sigma_{\mathrm{Cyl}}$,

$$\text{if } |p - q|_{\Sigma_{\mathrm{Cyl}}} > \delta, \text{ then } |f(p) - f(q)|_{\mathbb{R}^3} > \epsilon. \quad (47)$$

Let $g : \Sigma_{\mathrm{Cyl}} \to \mathbb{R}^3$ be an immersion which converges to f as $x \to +\infty$, in the C^2-topology of Σ_{Cyl}. Then there exists $x_0 \in \mathbb{R}_+$ such that g restricted to $\{x > x_0\}$ is properly embedded.

Proof. The C^0-convergence of g to f and the properness of f implies that f is proper.

Since f has bounded curvature, there exists $\delta > 0$ such that for any two points $p, q \in \mathbb{R}^2$, if $|p - q|_{\Sigma_{\mathrm{Cyl}}} < \delta$, then the total curvature of the image under f of the shortest straight line from p to q in Σ_{Cyl} is at most $\pi/4$. By the C^2-convergence of g to f, we can choose $x_1 > 0$ so that the total curvature of the image under g of those straight lines is less than $\pi/2$ when the x coordinates of p and q are greater than x_1.

With the above choice of δ, let ϵ be as in (47). By the C^0-convergence of g to f, we can choose an x_2 so that $|f - g|_{\mathbb{R}^3} < \epsilon/4$ at any point for which $x > x_2$. Set $x_0 = \max\{x_1, x_2\}$.

Now suppose that g is not embedded on $\Sigma_{\mathrm{Cyl}} \cap \{x > x_0\}$, so there exist distinct $p, q \in \Sigma_{\mathrm{Cyl}} \cap \{x > x_0\}$ such that $g(p) = g(q)$. Then the proof divides into two cases.

Case 1: $|p - q|_{\Sigma_{\mathrm{Cyl}}} \geqslant \delta$. Then $|f(p) - f(q)|_{\mathbb{R}^3} > \epsilon$. Thus $|g(p) - g(q)|_{\mathbb{R}^3} > \epsilon/2$, which contradicts $g(p) = g(q)$.

Case 2: $|p - q|_{\Sigma_{\mathrm{Cyl}}} < \delta$. Let $\gamma : [0, 1] \to \Sigma_{\mathrm{Cyl}}$ be the parametrized straight line from p to q. Then $g \circ \gamma$ is a closed curve in $\mathbb{R}^3$ which is smooth except at one point. If $g \circ \gamma$ is smoothed out at that point, it has total curvature at least 2π. However, the turning angle of the tangent at the non-smooth point is at most π, hence the total curvature of the smoothed out $g \circ \gamma$ is $\leqslant 3\pi/2$, a contradiction. □

207

3.4 Asymptotics of Delaunay immersions

We conclude Part I with its culminating theorem, showing that a CMC end obtained from a perturbed Delaunay potential is asymptotic to a half-Delaunay surface. This theorem restricts itself to the case that the holomorphic frame extends analytically to $\mathcal{A}_{r,1}$.

THEOREM 3.6. *Let A be a Delaunay residue as in (1) satisfying condition (42). Let $r \in (0,1]$ and assume $\mathcal{A}_{r,1} \cap \mathcal{S}_A = \emptyset$. On the punctured unit disk $\Sigma^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, let*

$$\xi = A\frac{dz}{z} + \mathrm{O}(z^0)\,dz$$

be a perturbed Delaunay r-potential. Let f_0 and f be the immersions of Σ^* induced by the generalized Weierstrass representation at $\lambda = 1$ by $A\,dz/z$ and ξ, respectively, so f_0 is a Delaunay immersion. Assume that f is obtained from a holomorphic r-frame with values in $\Lambda_r^{\uparrow}\mathrm{SL}_2\mathbb{C}$ whose monodromy at $z = 0$ is in $\Lambda_r^*\mathrm{SL}_2\mathbb{C}$.

With Σ_{Cyl} as in (46), let $\phi : \Sigma_{\mathrm{Cyl}} \to \Sigma^*$ be the map $\phi(x,y) = e^{x+iy}$. Then some rigid motion of $f \circ \phi$ converges to $f_0 \circ \phi$ in the C^∞-topology of Σ_{Cyl} as $x \to -\infty$. Furthermore, if f_0 is embedded, then f is properly embedded.

Proof. Let ρ be the period of the Delaunay surface f_0. Let $\Sigma_{\mathrm{Rect}} = [0,\rho] \times [0,2\pi] \subset \mathbb{R}^2$. Fix a small $\epsilon > 0$. Then $f \circ \phi$ and $f_0 \circ \phi$ can be lifted to the ϵ-neighborhood $\mathcal{N}_\epsilon(\Sigma_{\mathrm{Rect}})$ of Σ_{Rect}. We define a sequence g_k of C^∞ functions on Ω by

$$g_k(x,y) = (f \circ \phi)((x + k\rho) + iy)$$

on Σ_{Rect} and extend g_k real analytically to $\mathcal{N}_\epsilon(\Sigma_{\mathrm{Rect}})$.

Let u, H, Q and u_k, H_k, Q_k be the geometric data for f_0 and the g_k, respectively. Let Φ_0 and Φ be the respective solutions to $d\Phi_0 = \Phi A\,dz/z$ and $d\Phi = \Phi\xi$ which respectively induce f_0 and f via the generalized Weierstrass representation. By Theorem 3.2, the metrics are $v = e^u$ and $v_k = e^{u_k}$, we have C^1-convergence of g_k to some rigid motion of $f_0 \circ \phi$ on $\mathcal{N}_\epsilon(\Sigma_{\mathrm{Rect}})$ as $x \to \infty$, and $v/v_k \overset{C^0}{\to} 1$ as $x \to \infty$. Hence, $u_k \overset{C^0}{\to} u$ on $\mathcal{N}_\epsilon(\Sigma_{\mathrm{Rect}})$. We have that $Q_k \overset{C^\infty}{\to} Q$, and $H_k = H$ for all k, so Theorem 3.4 and Lemma 3.5 imply the theorem. $\qquad\square$

Remark 3.7. When f_0 is embedded, the rigid motion of f converges exponentially to f_0 in the sense of [KKS89], as proven there.

Remark 3.8. The above Delaunay asymptotics result for Euclidean space $\mathbb{R}^3$ most likely also holds for the spaceform $\mathbb{S}^3$, with appropriate changes in the Sym and moving frame formulas. For hyperbolic space $\mathbb{H}^3$, the value λ_0 in the Sym formula is in the interior of the unit disk, so if the asymptotics result is to hold, its proof would require the results of Part II.

Part II. Dressed Delaunay asymptotics

Part II generalizes the results in Part I to the setting of $\Lambda_r\mathrm{SL}_2\mathbb{C}$ for arbitrary r. It shows for this larger class of frames that an immersion constructed from a perturbed Delaunay r-potential via r-dressing is asymptotic to a Delaunay surface.

Outline of results

Dressing by simple factors performs a Bianchi–Bäcklund transformation on a CMC surface. The formula for dressing by simple factors is known explicitly [TU00]. In §4.2, we show that a dressed Delaunay frame $C\exp(A\log z)$ which has unitary monodromy is a multibubbleton frame, that is, a Delaunay frame dressed by a finite product of simple factors (Theorem 4.4). This result is used in §5.4.

In §5.1, we show that dressing by a finite product G of simple factors with distinct singularities preserves asymptotics (Theorem 5.2).

In §5, we show that if an immersion f converges to a Delaunay immersion f_0, then f dressed by a product of simple factors with distinct singularities also converges to a Delaunay immersion with the same necksize as f_0 (Theorem 5.6). Together with the methods of §3, this implies that dressing by finitely many simple factors preserves convergence to a Delaunay surface (Theorem 5.7).

Given a holomorphic perturbation

$$\xi = A\frac{dz}{z} + \mathrm{O}(z^0)\, dz$$

of the Delaunay potential $A\, dz/z$ which represents a closed once-wrapped Delaunay surface, let Φ satisfy $d\Phi = \Phi\xi$ with unitary monodromy at $z = 0$. In §5.4 we show that the unitary factor of Φ is asymptotic to a Delaunay frame and the metric of the surface induced by Φ is asymptotic to the corresponding Delaunay metric (Theorem 5.8).

These asymptotic frames results are applied to construct CMC surfaces in $\mathbb{R}^3$. The surface induced by Φ is asymptotic to a half-Delaunay surface induced by $A\, dz/z$ (Theorem 5.9).

4. Simple factor dressing

4.1 Simple factors

We recall the notion of a *simple factor* [TU00].

DEFINITION 4.1. Given $r \in (0, 1]$ and $\lambda_0 \in \mathcal{A}_{r,1}$, let f be the unique rational map on $\mathbb{CP}^1$ with degree one whose pole is at λ_0, satisfying $f^* = f^{-1}$ and $f(1) = 1$. (Here, f^{-1} denotes the multiplicative inverse.) Let $L \in \mathbb{CP}^1$. An *unnormalized simple factor* $\mathfrak{g}^\circ[\lambda_0, L] \in \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$ is a loop

$$\mathfrak{g}^\circ[\lambda_0,\, L] = f^{1/2}\pi_L + f^{-1/2}\pi_{L^\perp},$$

where π_L denotes the orthogonal projection to L. A *normalized simple factor* $\mathfrak{g}[\lambda_0, L] \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ is a map of the form $\mathfrak{g}[\lambda_0, L] = U^{-1}\mathfrak{g}^\circ[\lambda_0, L]$, where U is the unitary factor of the QR-decomposition of $\mathfrak{g}^\circ[\lambda_0, L]|_{\lambda=0}$. A *general simple factor* is a map of the form $Ug \in \Lambda_r^{\mathrm{pos}}\mathrm{SL}_2\mathbb{C}$, where $U \in \mathrm{SU}_2$ and g is a normalized simple factor.

By [TU00, Proposition 4.2], dressing by simple factors is explicit: given $r \in (0, 1)$, an extended r-unitary frame $F(z, \lambda)$ and a normalized simple factor $\mathfrak{g}[\lambda_0, L]$ with $\lambda_0 \in \mathcal{A}_{r,1}$, we have the formula [TU00] (see also [KSS04, Kob04])

$$\mathsf{Uni}_r[\mathfrak{g}[\lambda_0,\, L]F] = \mathfrak{g}[\lambda_0,\, L]F\mathfrak{g}[\lambda_0,\, \overline{F(z,\lambda_0)}^t L]^{-1}. \tag{48}$$

While simple factors are positive r-loops, the product of two simple factors with the same singularity extends to a meromorphic map on $\mathbb{CP}^1$.

LEMMA 4.2. For $k \in \{1, 2\}$, let $g_k = U_k^{-1}\mathfrak{g}^\circ[\lambda_0, L_k]$ be general simple factors, where $\lambda_0 \in \mathcal{D}_1^*$, $L_k \in \mathbb{CP}^1$ and $U_k \in \mathrm{SU}_2$. Let $X : \mathcal{R} \to \mathrm{M}_{2\times2}\mathbb{C}$ be a meromorphic map on a domain $\mathcal{R} \subset \mathbb{C}$. Then $g_1 X g_2^{-1}$ extends meromorphically to a map Y on $\mathcal{R}$. Moreover, suppose $\mathcal{R}$ is invariant under the map $\lambda \to 1/\overline{\lambda}$. Then:

(i) *if $g_1 = g_2$ and $X^* = X$, then $Y^* = Y$ away from its poles;*

(ii) *if X is invertible and $X^* = X^{-1}$, then $Y^* = Y^{-1}$ away from its poles.*

Proof. By the definition of simple factors (Definition 4.1), for $k \in \{1, 2\}$ we can write $g_k = U_k f^{-1/2} h_k$, where $U_k \in \mathrm{SU}_2$, $h_k = f \pi_{L_k} + \pi_{L_k^\perp}$, f has a simple pole at λ_0, $f^* = f^{-1}$, and $h_k^* = h_k^{-1}$. Then $g_1 X g_2^{-1} = U_1 h_1 X h_2^{-1} U_2^{-1}$ extends meromorphically to $\mathcal{R}$ because all of its components do. Statements (i) and (ii) follow from the symmetries $f^* = f^{-1}$, $h_k^* = h_k^{-1}$ and $U_k^* = U_k^{-1}$. $\square$

4.2 The dressed Delaunay frame

We shall prove that if a dressing matrix preserves the unitarity of a Delaunay monodromy, then the dressing matrix is a product of simple factors and a loop V which conjugates the Delaunay residue to a Delaunay residue. The effect of V is a coordinate change and a rigid motion of the surface. Thus, for Delaunay surfaces, the class of dressing matrices that preserve unitarity of monodromy coincide with the Bianchi–Bäcklund transformations.

LEMMA 4.3. *Let A be a Delaunay residue and let $M = \exp(2\pi i A)$. With $r \in (0, 1)$, let $C_+ \in \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$ and suppose $C_+ M C_+^{-1} \in \Lambda_r^* \mathrm{SL}_2 \mathbb{C}$. Then $C_+ A C_+^{-1}$ extends to a rational map on $\mathbb{CP}^1$ all of whose poles are simple and lie in $(\mathcal{S}_A \cap (\mathcal{A}_r \setminus \mathbb{S}^1)) \cup \{0, \infty\}$.*

Proof. Let μ be an eigenvalue of A as in Definition 1.2. Let $x, y : \mathbb{C}^* \to \mathbb{C}$ be the holomorphic functions $x = \cos(2\pi\mu)$ and $y = \mu^{-1} \sin(2\pi\mu)$. Note that x and y are always holomorphic on $\mathbb{C}^*$, even if μ is not. Then $M = x\,\mathrm{I} + iy A$. Writing $A_1 = C_+ A C_+^{-1}$, on $\mathcal{D}_r^*$ we have

$$C_+ M C_+^{-1} = x\,\mathrm{I} + iy A_1.$$

Since by hypothesis, $C_+ M C_+^{-1}$ extends holomorphically to $\mathbb{C}^*$, and since $x\,\mathrm{I}$ is holomorphic on $\mathbb{C}^*$, then $y A_1$ extends holomorphically to $\mathbb{C}^*$. Since y is holomorphic on $\mathbb{C}^*$, then A_1 extends meromorphically to $\mathbb{C}^*$, and for all $\lambda \in \mathbb{C}^*$

$$\operatorname*{ord}_\lambda A_1 \geqslant -\operatorname*{ord}_\lambda y. \tag{49}$$

We compute the orders of the poles of A_1 on $\mathbb{C}^*$. These occur only at the zeros of y, all of which are in $\mathcal{S}_A$. Let $\lambda_0 \in \mathbb{C}^*$ be a zero of y. Then $\mu(\lambda_0) \in \frac{1}{2}\mathbb{Z}^*$.

Case 1. If $\lambda_0 = 0$, then C_+ is holomorphic and A has a simple pole, so A_1 is meromorphic with at worst a simple pole at zero.

Case 2. If $0 < |\lambda_0| \leqslant r$, then A_1 is holomorphic at λ_0 because C_+ and A are.

Case 3. $r < |\lambda_0| < 1$. It can be shown that $\{\lambda \in \mathbb{C}^* \mid \mu'(\lambda) = 0\} \subset \mathbb{S}^1$. Hence, $\mu'(\lambda_0) \neq 0$, from which it follows that $\operatorname{ord}_{\lambda_0} y = 1$. By (49) $\operatorname{ord}_{\lambda_0} A_1 \geqslant -1$, so A_1 has at most a simple pole at λ_0.

Case 4. We have $|\lambda_0| = 1$. In this case, A_1 is holomorphic at λ_0 as in the proof of Lemma 1.12.

Since $\exp(2\pi i A_1)$ takes values in SU_2 on $\mathbb{S}^1$, and A_1 is tracefree, then A_1 has the hermitian symmetry $A_1^* = A_1$. Hence, the poles of A_1 on $\mathbb{C}^*$ lie in $(\mathcal{A}_r \setminus \mathbb{S}^1) \cap \mathcal{S}_A$. $\square$

THEOREM 4.4. *Let A be a Delaunay residue and let $M = \exp(2\pi i A)$. With $r \in (0, 1)$, let $C_+ \in \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$ and suppose $C_+ M C_+^{-1} \in \Lambda_r^* \mathrm{SL}_2 \mathbb{C}$. Then there exists a loop $G \in \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$ which is a product of normalized simple factors (or $G = \mathrm{I}$), the singularities of the simple factors are distinct and lie in $\mathcal{S}_A \cap (\mathcal{A}_r \setminus \mathbb{S}^1)$ and, with $V = G^{-1} C_+$, $A_1 = V A V^{-1}$ is a Delaunay residue.*

Proof. The proof is by induction on the number of poles of $C_+ A C_+^{-1}$ in $\mathcal{A}_{r,1}$. We prove the following induction step, which constructs a single simple factor g.

With A, M, r and C_+ as in the statement of the theorem, there exists a normalized simple factor $g \in \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$ such that, with $V = g^{-1} C_+ \in \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$, we have $V M V^{-1} \in \Lambda_r^* \mathrm{SL}_2 \mathbb{C}$ and $V A V^{-1}$ has one fewer pole in $\mathcal{D}_1$ than $C_+ A C_+^{-1}$.

Proof of the induction step. By Lemma 4.3, $C_+ A C_+^{-1}$ extends to a rational map on $\mathbb{CP}^1$ all of whose poles are simple and lie in $(\mathcal{S}_A \cap (\mathcal{A}_r \setminus \mathbb{S}^1)) \cup \{0, \infty\}$. Let

$$M_1 = C_+ M C_+^{-1} \quad \text{and} \quad A_1 = C_+ A C_+^{-1}.$$

Let $x = \cos(2\pi\mu)$ and $y = \mu^{-1}\sin(2\pi\mu)$ be the holomorphic functions on $\mathbb{C}^*$ as in the proof of Lemma 4.3, so that $M = x\,\mathrm{I} + iyA$. Then

$$C_+ M C_+^{-1} = x\,\mathrm{I} + iyA_1. \tag{50}$$

Choose a pole $\lambda_0 \in \mathcal{S}_A \cap (\mathcal{A}_r \setminus \mathbb{S}^1)$ of A_1, so $\mathrm{ord}_{\lambda_0} A_1 = -1$. Since $\lambda_0 \in \mathcal{S}_A$, then $\mu(\lambda_0) \in \frac{1}{2}\mathbb{Z}^*$, so $\iota := x(\lambda_0) \in \{1, -1\}$. As in case 3 of the proof of Lemma 4.3, $\mathrm{ord}_{\lambda_0} y = 1$. Hence, $\mathrm{ord}_{\lambda_0} yA_1 = 0$. Then $M_1(\lambda_0) \neq \iota\,\mathrm{I}$, by (50). Hence, $M_1(\lambda_0)$ is not semisimple, so the eigenspace of $M_1(\lambda_0)$ associated to the eigenvalue ι is one-dimensional. Let $v \in \mathbb{C}^2 \setminus \{0\}$ be an eigenline of $M_1(\lambda_0)$ and let $L = [v] \in \mathbb{CP}^1$. Define the unnormalized simple factor $h = \mathfrak{g}^\circ[\lambda_0, L]$ on $\mathcal{D}_r$.

We show that $h^{-1}A_1 h$ is holomorphic at λ_0. Let f be as in Definition 4.1 for h, so $h = f^{1/2}\pi_L + f^{-1/2}\pi_{L^\perp}$ and $\mathrm{ord}_{\lambda_0} f = -1$. Working in the basis $\{v, v^\perp\}$, define the functions a, b, c, d by $A_1 v = av + cv^\perp$ and $A_1 v^\perp = bv + dv^\perp$. Since $\mathrm{span}_{\mathbb{C}}\{v\}$ is the kernel and image of $yA_1|_{\lambda_0}$, then

$$\mathrm{ord}_{\lambda_0} a \geqslant 0, \quad \mathrm{ord}_{\lambda_0} b = -1, \quad \mathrm{ord}_{\lambda_0} c \geqslant 0, \quad \mathrm{ord}_{\lambda_0} d \geqslant 0.$$

Then $\det A_1 = ad - bc$, so $\mathrm{ord}_{\lambda_0}(ad - bc) \geqslant 0$. Hence, $\mathrm{ord}_{\lambda_0} c \geqslant 1$. On the other hand,

$$h^{-1}A_1 hv = av + fcv^\perp \quad \text{and} \quad h^{-1}A_1 hv^\perp = f^{-1}bv + dv^\perp.$$

Hence, $h^{-1}A_1 hv$ and $h^{-1}A_1 hv^\perp$ are holomorphic at λ_0, so $h^{-1}A_1 h$ is holomorphic at λ_0.

Let $U \in \mathrm{SU}_2$ be the unitary factor in the QR-decomposition of $h(0)$ and define $g = hU^{-1}$. Then $g \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ is a normalized simple factor. Define $V = g^{-1}C_+$. Then C_+ and V have the same poles on $\mathbb{CP}^1 \setminus \{\lambda_0, 1/\overline{\lambda_0}\}$. Since $C_+ A C_+^{-1}$ has a pole at λ_0, and as shown above, VAV^{-1} does not, then VAV^{-1} has one fewer pole that $C_+ A C_+^{-1}$ in $\mathcal{D}_1$.

Since A_1 is hermitian away from its poles, then so is $h^{-1}A_1 h$ by Lemma 4.2(i). By conjugating (50) by h^{-1}, we obtain that $h^{-1}M_1 h \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$. Then $VMV^{-1} = Uh^{-1}M_1 hU^{-1} \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$. This proves the induction step.

To prove the theorem, let n be the number of poles of $C_+ A C_+^{-1}$ in $\mathcal{A}_{r,1}$. If $n = 0$, then by Lemma 1.12, $C_+ A C_+^{-1}$ is a Delaunay residue and the theorem follows with $G = \mathrm{I}$ and $V = C_+$. Otherwise, repeated application of the induction step produces n normalized simple factors $g_1, \ldots, g_n \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ with distinct singularities in $\mathcal{S}_A \cap (\mathcal{A}_r \setminus \mathbb{S}^1)$, such that, with $G = g_1 \cdots g_n$ and $V = G^{-1}C_+ \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$, VAV^{-1} has no poles in $\mathcal{A}_r$, and hence in $\mathcal{D}_1$, and $VMV^{-1} \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$. By Lemma 1.12, VAV^{-1} is a Delaunay residue. This proves the theorem. $\qquad\square$

4.3 Dressed Delaunay growth

Lemma 1.11 showed that under the assumption of unitary monodromy, dressing a Delaunay holomorphic frame $\exp(A \log z)$ by a loop $C \in \Lambda_r\mathrm{SL}_2\mathbb{C}$ does not affect the growth behavior of its positive factor, provided that $C \in \Lambda_r^\uparrow\mathrm{SL}_2\mathbb{C}$. Theorem 4.6 generalizes this result by removing this provision.

LEMMA 4.5. *Let $h(z) = U(z)^{-1}\mathfrak{g}^\circ[\lambda_0, L(z)]$ be a map from a domain $\Sigma \subset \mathbb{C}$ into the space of general simple factors. Then for every set $\mathcal{R} \subset \mathbb{CP}^1$ bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$, $\|h(z)\|_{\mathcal{R}}$ is bounded as a function of z.*

Proof. With f and U as in Definition 4.1, write

$$h = U^{-1}(f(\lambda)^{1/2}\pi_{L(z)} + f(\lambda)^{-1/2}\pi_{L(z)^\perp}).$$

211

Then $|h| = \max\{|f^{1/2}|, |f^{-1/2}|\}$. On any set $\mathcal{R} \subset \mathbb{CP}^1$ bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$, the function $|f|$ is bounded away from zero and infinity, and so $|f^{1/2}|$ and $|f^{-1/2}|$ are bounded on $\mathcal{R}$. Hence, for all $z \in \Sigma$,

$$\|h(z)\|_{\mathcal{R}} \leqslant \max\{\|f^{1/2}\|_{\mathcal{R}}, \|f^{-1/2}\|_{\mathcal{R}}\}. \qquad \square$$

Theorem 4.6. *Let A be a Delaunay residue, and let μ be its eigenvalue as in Definition 1.2. Let τ be as in Lemma 1.7. Let $r \in (0,1)$, and assume $\mathcal{C}_r \cap \mathcal{S}_A = \emptyset$, where $\mathcal{S}_A$ is defined by (2). Let $C \in \Lambda_r \mathrm{SL}_2\mathbb{C}$ and assume the monodromy $C \exp(2\pi i A)C^{-1}$ is in $\Lambda_r^* \mathrm{SL}_2\mathbb{C}$. Then there exists r_1, r_2 with $0 < r_1 < r < r_2 < 1$, and a continuous function $c : \mathcal{A}_{r_1,r_2} \to \mathbb{R}_+$, such that for all $(z,\lambda) \in \{0 < |z| < 1\} \times \mathcal{A}_{r_1,r_2}$,*

$$\|\mathsf{Pos}_r[C \exp(A \log z)]\| \leqslant c|z|^{-\tau} \leqslant c|z|^{-\mathrm{Re}\,\mu}.$$

Proof. Step 1. Let $\Sigma^* = \{0 < |z| < 1\}$. Let $C = C_u \cdot C_+$ be the r-Iwasawa factorization of C. By Theorem 4.4, there exist loops $G, V \in \Lambda_r^+ \mathrm{SL}_2\mathbb{C}$ such that $C_+ = GV$, G is a product of normalized simple factors (or $G = \mathrm{I}$), $A_1 := VAV^{-1}$ is a Delaunay residue, and

$$X := C \exp(A \log z) = C_u G(\exp A_1 \log z)V.$$

Step 2. Since $\mathcal{C}_r \cap \mathcal{S}_A = \emptyset$, and the points of $\mathcal{S}_A$ are isolated on $\mathbb{C}^*$, there exist r_1 and r_2' satisfying $0 < r_1 < r < r_2' < 1$ such that $\mathcal{A}_{r_1,r_2'} \cap \mathcal{S}_A = \emptyset$, and V extends holomorphically to $\mathcal{D}_{r_2'}$. Let $r_2 \in (r, r_2')$ and let $\mathcal{A} := \mathcal{A}_{r_1,r_2}$. Since V extends holomorphically to $\mathcal{D}_{r_2'}$, then $c_1 := \|V\|_{\mathcal{A}}$ is finite on $\Sigma^* \times \mathcal{A}$, so

$$\|\mathsf{Pos}_r[X]\| = c_1\|\mathsf{Pos}_r[G \exp(A_1 \log z)]\|. \tag{51}$$

Step 3. Write $G = g_n \cdots g_1 \cdot \mathrm{I}$ as a product of normalized simple factors and for $k \in \{1, \ldots, n\}$, write $g_k = \mathfrak{g}[\lambda_k, L_k]$, with $\lambda_k \in \mathcal{A}_{r,1}$ and $L_k \in \mathbb{CP}^1$. By repeated use of the simple factor dressing formula (48), we obtain maps $h_1, \ldots, h_n : \{0 < |z| < 1\} \to \Lambda_r^+ \mathrm{SL}_2\mathbb{C}$ defined by

$$h_k = \mathfrak{g}[\lambda_k, \overline{F_k(\lambda_k)}^t L_k], \quad F_k = \mathsf{Uni}_r[g_{k-1} \ldots g_1 X]$$

such that

$$\mathsf{Pos}_r[g_n \ldots g_1 X] = h_n \cdots h_1 \mathsf{Pos}_r[X].$$

Since the singularities of G are in $\mathcal{S}_A$, which is bounded away from $\mathcal{A}$, then by Lemma 4.5, $\|h_k\|_{\mathcal{A}}$ is finite for each $k \in \{1, \ldots, n\}$, so let c_2 be their product. Then using (51), we have on $\Sigma^* \times \mathcal{A}$,

$$\|\mathsf{Pos}_r[X]\| \leqslant c_1 c_2\|\mathsf{Pos}_r[\exp(A_1 \log z)]\|. \tag{52}$$

Step 4. It follows from $\det A = \det A_1$ that the function τ in Lemma 1.7 for A is the same as that for A_1. By Lemma 1.11, there exists a continuous function $c_3 : \mathcal{A} \to \mathbb{R}_+$ such that on $\{0 < |z| < 1\} \times \mathcal{A}$,

$$\|\mathsf{Pos}_r[\exp(A_1 \log z)]\| \leqslant c_3|z|^{-\tau} \leqslant c_3|z|^{-\mathrm{Re}\,\mu}. \tag{53}$$

The result follows from (52) and (53) with $c = c_1 c_2 c_3$. $\qquad \square$

5. Delaunay dressing asymptotics

5.1 Simple factor asymptotics

The next preliminary lemma shows a basic convergence property of simple factors: two simple factors with the same fixed singularities converge if the lines defining them converge.

Lemma 5.1. *For $k \in \{1,2\}$, let $h_k = \mathfrak{g}[\lambda_0, L_k]$, where $\lambda_0 \in \mathcal{D}_1^*$ and $L_k : \Sigma^* \to \mathbb{CP}^1$ are continuous maps on a punctured neighborhood $\Sigma^* \subset \mathbb{C}$ of $0 \in \mathbb{C}$. Note that the map $h_1 h_2^{-1}$ extends meromorphically to $\mathbb{CP}^1$ (see Lemma 4.2) and this extension is holomorphic on $\mathbb{CP}^1 \setminus \{\lambda_0, 1/\overline{\lambda_0}\}$. If L_1 and*

L_2 *converge as $z \to 0$, then on every region $\mathcal{R} \subseteq \mathbb{CP}^1$ bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$,*

$$\lim_{z \to 0} \|h_1 h_2^{-1} - \mathrm{I}\|_{\mathcal{R}} = 0. \tag{54}$$

Proof. Let $\mathcal{R} \subset \mathbb{CP}^1$ be a region bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$.

Step 1. For $k \in \{1, 2\}$, let $h_k = U_k^{-1} \mathfrak{g}^\circ[\lambda_0, L_k]$. Since h_1 and h_2 share the same singularity λ_0, we have f as in Definition 4.1 such that

$$h_k^\circ = f^{1/2} \pi_{L_k} + f^{-1/2} \pi_{L_k^\perp}, \quad k \in \{1, 2\}.$$

We show that $h_1^\circ h_2^{\circ -1} \to \mathrm{I}$. We have

$$h_1^\circ h_2^{\circ -1} - \mathrm{I} = (\pi_{L_1} \pi_{L_2} + \pi_{L_1^\perp} \pi_{L_2^\perp} - \mathrm{I}) + f \pi_{L_1} \pi_{L_2^\perp} + f^{-1} \pi_{L_1^\perp} \pi_{L_2}. \tag{55}$$

Then

$$\lim_{z \to 0} (\pi_{L_1} \pi_{L_2} + \pi_{L_1^\perp} \pi_{L_2^\perp} - \mathrm{I}) = 0, \quad \lim_{z \to 0} \pi_{L_1} \pi_{L_2^\perp} = 0, \quad \lim_{z \to 0} \pi_{L_1^\perp} \pi_{L_2} = 0. \tag{56}$$

Moreover, since f and f^{-1} are holomorphic on $\mathcal{R}$, which is bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$, then f and f^{-1} are bounded on $\mathcal{R}$. Then (55) and (56) imply

$$\lim_{z \to 0} \|h_1^\circ h_2^{\circ -1} - \mathrm{I}\|_{\mathcal{R}} = 0. \tag{57}$$

Step 2. We show that $h_1 h_2^{-1} \to \mathrm{I}$. We have

$$h_1 h_2^{-1} - \mathrm{I} = U_1^{-1} h_1^\circ h_2^{\circ -1} U_2 - \mathrm{I} = U_1^{-1} (h_1^\circ h_2^{\circ -1} - U_1 U_2^{-1}) U_2,$$

so

$$\|h_1 h_2^{-1} - \mathrm{I}\|_{\mathcal{R}} \leqslant \|h_1^\circ h_2^{\circ -1} - U_1 U_2^{-1}\|_{\mathcal{R}} \leqslant \|h_1^\circ h_2^{\circ -1} - \mathrm{I}\|_{\mathcal{R}} + \|U_1 U_2^{-1} - \mathrm{I}\|. \tag{58}$$

By (57), the first term of the right-hand side of (58) converges to zero as $z \to 0$.

We have $U_1 U_2^{-1} = h_1^\circ(0) h_2^\circ(0)^{-1}$. Since $\{0\}$ is bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$, then by step 1, (57) holds at zero. Hence,

$$\lim_{z \to 0} \|h_1^\circ(x, 0) h_2^\circ(x, 0)^{-1} - \mathrm{I}\| = 0.$$

Hence, the second term of the right-hand side of (58) converges to zero as $z \to 0$. This implies the result (54). $\qquad \square$

The following theorem shows that dressing by a simple factor preserves convergence. For a meromorphic map U, the set of poles of U is denoted by $\mathrm{sing}(U)$.

THEOREM 5.2. *With $r \in (0, 1)$, let $\Phi_1, \Phi_2 : \widetilde{\Sigma^*} \to \Lambda_r \mathrm{SL}_2 \mathbb{C}$ be analytic maps on the universal cover $\widetilde{\Sigma^*} \to \Sigma^*$ of a punctured neighborhood $\Sigma^* \subset \mathbb{C}$ of $0 \in \mathbb{C}$. Let $U \in \Lambda_r^{*M} \mathrm{SL}_2 \mathbb{C}$ and $V \in \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$. Suppose that for every region $\mathcal{A} \subseteq \mathcal{A}_r$ bounded away from $\mathrm{sing}(U)$, Φ_1 and Φ_2 have the convergence*

$$\lim_{z \to 0} \|(\mathsf{Uni}_r[\Phi_1])^{-1} U \mathsf{Uni}_r[\Phi_2] - \mathrm{I}\|_{\mathcal{A}} = 0, \tag{59a}$$

$$\lim_{z \to 0} \|\mathsf{Pos}_r[\Phi_2] V^{-1} (\mathsf{Pos}_r[\Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{59b}$$

For $k \in \{1, 2\}$, let $g_k = W \mathfrak{g}[\lambda_0, L_k]$ be general simple factors, where $W \in \mathrm{SU}_2$, $\lambda_0 \in \mathcal{A}_{r,1} \setminus \mathrm{sing}(U)$ and $L_1, L_2 \in \mathbb{CP}^1$ are related by $L_2 = \overline{U(\lambda_0)}^t L_1$. Then for every region $\mathcal{A}' \subseteq \mathcal{A}_r$ bounded away from $\mathrm{sing}(U) \cup \{\lambda_0, 1/\overline{\lambda_0}\}$, $g_1 \Phi_1$ and $g_2 \Phi_2$ have the convergence

$$\lim_{z \to 0} \|(\mathsf{Uni}_r[g_1 \Phi_1])^{-1} (g_1 U g_2^{-1}) \mathsf{Uni}_r[g_2 \Phi_2] - \mathrm{I}\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|\mathsf{Pos}_r[g_2 \Phi_2] V^{-1} (\mathsf{Pos}_r[g_1 \Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{60}$$

Proof. For $k \in \{1, 2\}$, let $F_k = \mathsf{Uni}_r[\Phi_k]$ and $B_k = \mathsf{Pos}_r[\Phi_k]$. For $k \in \{1, 2\}$ we have by the simple factor formula (48)

$$\mathsf{Uni}_r[g_k \Phi_k] = g_k F_k h_k^{-1} \quad \text{and} \quad \mathsf{Pos}_r[g_k \Phi_k] = h_k B_k, \quad k \in \{1, 2\},$$

where the $h_k : \Sigma^* \to \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$ are defined by

$$h_k = \mathfrak{g}[\lambda_0, \overline{F_k(\lambda_0)}^t L_k], \quad k \in \{1, 2\}.$$

Let $\mathcal{A}' \subset \mathcal{A}_r$ be a region bounded away from $\mathrm{sing}(U) \cup \{\lambda_0, 1/\overline{\lambda_0}\}$. Since $\mathcal{A}'$ is bounded away from $\mathrm{sing}(U)$, then (59) hold for $\mathcal{A}'$:

$$\lim_{z \to 0} \|F_1^{-1} U F_2 - \mathrm{I}\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|B_2 V^{-1} B_1^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{61}$$

By Lemma 4.2, $h_1 h_2^{-1}$ extends to a map $\Sigma^* \times \mathcal{A}_r' \to \mathrm{SL}_2 \mathbb{C}$ which is meromorphic in the second variable and $h_2 h_1^{-1}$ extends to a holomorphic map $\Sigma^* \to \Lambda_r^+ \mathrm{SL}_2 \mathbb{C}$. For $k \in \{1, 2\}$, define $M_k = \overline{F_k(\lambda_0)}^t$ and define $U_0 = \overline{U(\lambda_0)}^t$. Since $\lambda_0 \notin \mathrm{sing}(U)$, by (59a) we have $\lim_{z \to 0} M_2 U_0 M_1^{-1} = \mathrm{I}$. Then

$$M_2 L_2 = (M_2 U_0 M_1^{-1}) M_1 (U_0^{-1} L_2) = (M_2 U_0 M_1^{-1}) M_1 L_1,$$

so $M_1 L_1$ and $M_2 L_2$ converge as $z \to 0$. Since $\mathcal{A}'$ is bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$, we have by Lemma 5.1,

$$\lim_{z \to 0} \|h_1 h_2^{-1} - \mathrm{I}\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|h_2 h_1^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{62}$$

Since $\mathcal{A}'$ and $\mathcal{D}_r$ are bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$, then by Lemma 4.5, each of h_1, h_1^{-1}, h_2 and h_2^{-1} is bounded on $\Sigma^* \times \mathcal{A}'$ and on $\Sigma^* \times \mathcal{D}_r$. By (61) and (62) it follows that

$$\lim_{z \to 0} \|h_1 (F_1^{-1} U F_2 - \mathrm{I}) h_2^{-1} + (h_1 h_2^{-1} - \mathrm{I})\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|h_2 (B_2 V^{-1} B_1^{-1} - \mathrm{I}) h_1^{-1} + (h_2 h_1^{-1} - \mathrm{I})\|_{\mathcal{D}_r} = 0.$$

This gives the result (60). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5.2 Bubbleton asymptotics

In this section, we show that if an immersion f converges to a Delaunay immersion f_0, then f dressed by a product of simple factors with distinct singularities converges to a Delaunay immersion with the same necksize as f_0 (Theorem 5.6). The proof of Theorem 5.6 is outlined as follows.

For each singularity λ_0 there is a family of simple factors sharing this singularity. Given a Delaunay residue A, dressing the Delaunay frame $F = \mathsf{Uni}[\exp((x + iy)A)]$ by this family generically produces a bubbleton frame $g\#F$. However, for one special member g_0 of this family of simple factors, $g_0\#F$ is not a bubbleton frame as expected, but rather a Delaunay frame (Lemma 5.5).

On the other hand, any two Delaunay frames dressed by simple factors are asymptotic (Lemma 5.4). In particular, a generic bubbleton frame $g\#F$ constructed by dressing a Delaunay frame F by a simple factor g, is asymptotic to the corresponding special Delaunay frame $g_0\#F$. This implies that any bubbleton frame is asymptotic to a Delaunay frame.

By the general result in § 5.1, that dressing by simple factors preserves end convergence (Theorem 5.2), step 3 above implies that dressing by a finite product of simple factors with distinct singularities preserves Delaunay asymptotics (Theorem 5.6).

214

The following lemma shows that given a unitary Delaunay frame F and simple factors g_1 and g_2 with the same singularity, then, generically, $g_1 \# F$ and $g_2 \# F$ are asymptotic modulo a unitary factor. A similar result holds for the positive factors.

LEMMA 5.3. *Let $A \in \mathrm{sl}_2\mathbb{C}$, let μ be an eigenvalue of A and suppose $\mu \in \mathbb{R}_+$. Let $(\mu, L_+), (-\mu, L_-) \in \mathbb{R}^* \times \mathbb{CP}^1$ be the eigenvalue–eigenline pairs for A. Then for all $L \in \mathbb{CP}^1 \setminus \{L_+\}$, $\exp(xA)L \to L_-$ as $x \to -\infty$.*

LEMMA 5.4. *Let A be a Delaunay residue and let μ be its eigenvalue as in Definition 1.2. With $r \in (0,1)$, let $\lambda_0 \in \mathcal{A}_{r,1} \cap \{\mu \in \mathbb{R}_+\}$ and let E be the eigenline of $\overline{A(\lambda_0)}^t$ corresponding to the eigenvalue $\overline{\mu(\lambda_0)}$. For $k \in \{1,2\}$, let $L_k \in \mathbb{CP}^1 \setminus \{E\}$ and let $g_k = \mathfrak{g}[\lambda_0, L_k]$ be normalized simple factors. Let $\Phi_0 = \exp(A \log z)$. Then for every region $\mathcal{A} \subseteq \mathcal{A}_r$ bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$, $g_1\Phi_0$ and $g_2\Phi_0$ have the convergence*

$$\lim_{z \to 0} \|(\mathsf{Uni}_r[g_1\Phi_0])^{-1}(g_1 g_2^{-1})\mathsf{Uni}_r[g_2\Phi_0] - \mathrm{I}\|_{\mathcal{A}} = 0,$$
$$\lim_{z \to 0} \|\mathsf{Pos}_r[g_2\Phi_0](\mathsf{Pos}_r[g_1\Phi_0])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{63}$$

Proof. Let $F = \mathsf{Uni}_r[\Phi_0]$ and $B = \mathsf{Pos}_r[\Phi_0]$. By the simple factor dressing formula (48) applied to g_1 and g_2, we have

$$\mathsf{Uni}_r[g_k\Phi_0] = g_k F h_k^{-1} \quad \text{and} \quad \mathsf{Pos}_r[g_k\Phi_0] = h_k B, \quad k \in \{1,2\},$$

where

$$h_k = \mathsf{Pos}_r[g_k F] = \mathfrak{g}[\lambda_0, \mathrm{GL}_k], \quad G = \overline{F(\lambda_0)}^t.$$

By Lemma 4.2, $h_1 h_2^{-1}$ and $h_2 h_1^{-1}$ are holomorphic on $\mathbb{CP}^1 \setminus \{\lambda_0, 1/\overline{\lambda_0}\}$. We compute in the coordinates $x + iy = \log z$. We show that the lines GL_1 and GL_2 converge as $x \to -\infty$. By (7a),

$$F(x + n\rho, y) = C^n F(x,y), \quad C = \exp((\rho - \sigma)A).$$

Let $C_0 = \overline{C(\lambda_0)}^t$. By Lemma 5.3, for any $L \in \mathbb{CP}^1 \setminus \{E\}$, the sequence $C_0^n L$ converges to L_0 as $n \to \infty$. Let $\mathcal{F}$ be the space of continuous functions $[0,\rho] \to \mathbb{CP}^1$ with the C^0-norm. Define the map $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ by $f(x) \mapsto \overline{F(x,y,\lambda_0))}^t f(x)$. Then $\mathcal{P}$ is continuous, so $\overline{F(x,y,\lambda_0))}^t C_0^n L$ converges to $\overline{F(x,y,\lambda_0))}^t L_0$ in $\mathcal{F}$ as $n \to \infty$. Hence,

$$\lim_{x \to \infty} \mathrm{dist}(\overline{F(x,y,\lambda_0)}^t L_1, \overline{F(x,y,\lambda_0)}^t L_0) = 0.$$

Hence, GL_1 and GL_2 converge as $x \to -\infty$. By Lemma 5.1, for every region $\mathcal{A} \subseteq \mathcal{A}_r$ bounded away from $\{\lambda_0, 1/\overline{\lambda_0}\}$,

$$\lim_{z \to 0} \|h_1 h_2^{-1} - \mathrm{I}\|_{\mathcal{A}} = 0,$$
$$\lim_{z \to 0} \|h_2 h_1^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{64}$$

Since

$$(\mathsf{Uni}_r[g_1\Phi_0])^{-1}(g_1 g_2^{-1})\mathsf{Uni}_r[g_2\Phi_0] = (g_1 F h_1^{-1})^{-1}(g_1 g_2^{-1})(g_2 F h_2^{-1}) = h_1 h_2^{-1},$$
$$\mathsf{Pos}_r[g_2\Phi_0](\mathsf{Pos}_r[g_1\Phi_0])^{-1} = h_2 B(h_1 B)^{-1} = h_2 h_1^{-1}, \tag{65}$$

the result (63) follows from (64) and (65). $\qquad\square$

5.3 Delaunay dressing asymptotics

We show that for a certain special simple factor g, the dressed Delaunay frame $g\# \exp((x + iy)A)$ is a unitary Delaunay frame, rather than a multibubbleton frame as is generically the case. For each choice of λ_0, there are generically two special Delaunay dressings, one for each eigenline of $\overline{A(\lambda_0)}^t$.

215

LEMMA 5.5. *With A a Delaunay residue, let $g = \mathfrak{g}[\lambda_0, L]$ be the normalized simple factor defined with $\lambda_0 \in \mathcal{D}_1^*$ and L an eigenline of $\overline{A(\lambda_0)}^t$. Then gAg^{-1} is a Delaunay residue.*

Proof. By Lemma 4.2(i), gAg^{-1} extends meromorphically to $\mathbb{C}^*$ and this extension satisfies $(gAg^{-1})^* = gAg^{-1}$ away from its poles.

We show that gAg^{-1} is holomorphic at λ_0. As in Definition 4.1, let $U \in \mathrm{SU}_2$ and f with $\mathrm{ord}_{\lambda_0} f = -1$ be such that $g = U^{-1}(f^{1/2}\pi_L + f^{-1/2}\pi_{L^\perp})$. Then

$$gAg^{-1} = U^{-1}(f^{1/2}\pi_L + f^{-1/2}\pi_{L^\perp})A(f^{-1/2}\pi_L + f^{1/2}\pi_{L^\perp})U$$
$$= U^{-1}(\pi_L A\pi_L + f\pi_L A\pi_{L^\perp} + f^{-1}\pi_{L^\perp}A\pi_L + \pi_{L^\perp}A\pi_{L^\perp})U. \tag{66}$$

At λ_0, the image of $\pi_{L^\perp}$ is $L^\perp$. Since L is an eigenline of $\overline{A(\lambda_0)}^t$, it follows that $L^\perp$ is an eigenline of $A(\lambda_0)$. Hence, the image of $L^\perp$ under $A(\lambda_0)$ is $L^\perp$. Since the image of $L^\perp$ under π_L is zero, then $\mathrm{ord}_{\lambda_0} \pi_L A\pi_{L^\perp} \geqslant 1$. Hence, $f\pi_L A\pi_{L^\perp}$ is holomorphic at λ_0. Since none of the other three terms in (66) has a pole at λ_0, then gAg^{-1} is holomorphic at λ_0. By its hermitian symmetry, gAg^{-1} is holomorphic at $1/\overline{\lambda_0}$, and hence is holomorphic on $\mathbb{C}^*$.

We consider gAg^{-1} at $\lambda = 0$. Since g is holomorphic and upper-triangular at $\lambda = 0$ and the only pole of A at $\lambda = 0$ appears in the upper-right entry, with order -1, the same is true of gAg^{-1}. Hence, gAg^{-1} is a Delaunay residue by Proposition 1.1. $\qquad\square$

Lemma 5.4 and Theorem 5.2 come together in the following theorem, which shows that if the Iwasawa factors of a holomorphic frame Φ converge to those of a holomorphic Delaunay frame respectively, then the same holds after dressing Φ by a finite product of simple factors.

THEOREM 5.6. *With $r \in (0,1)$, let $\Phi : \Sigma^* \to \Lambda_r\mathrm{SL}_2\mathbb{C}$ be an analytic map on a punctured neighborhood $\Sigma^* \subset \mathbb{C}$ of $0 \in \mathbb{C}$. Let A be a Delaunay residue. Suppose that Φ and $\Phi_1 = \exp(A \log z)$ have the convergence*

$$\lim_{z \to 0} \|(\mathsf{Uni}_r[\Phi_1])^{-1}\mathsf{Uni}_r[\Phi] - \mathrm{I}\|_{\mathcal{A}_r} = 0,$$
$$\lim_{z \to 0} \|\mathsf{Pos}_r[\Phi](\mathsf{Pos}_r[\Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{67}$$

Let $G \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ be a product of general simple factors (or $G = \mathrm{I}$) with distinct singularities in $\mathcal{A}_{r,1} \cap \{\mu \in \mathbb{R}_+\}$. Then there exist:

- *a loop $U_2 \in \Lambda_r^{*M}\mathrm{SL}_2\mathbb{C}$ satisfying $\mathrm{sing}(U_2) = \mathrm{sing}(G)$; and*
- *a loop $V_2 \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ for which $A_2 := V_2AV_2^{-1}$ is a Delaunay residue;*

such that, for every region $\mathcal{A} \subseteq \mathcal{A}_r$ bounded away from $\mathrm{sing}(U_2)$, $G\Phi$ and $\Phi_2 := \exp(A_2 \log z)$ have the convergence

$$\lim_{z \to 0} \|(\mathsf{Uni}_r[\Phi_2])^{-1}U_2\mathsf{Uni}_r[G\Phi] - \mathrm{I}\|_{\mathcal{A}} = 0,$$
$$\lim_{z \to 0} \|\mathsf{Pos}_r[G\Phi]V_2^{-1}(\mathsf{Pos}_r[\Phi_2])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0.$$

Proof. The proof is by induction on the factors of G. We first prove the following induction step, which shows convergence after dressing by a single simple factor g.

INDUCTION STEP. *With $r \in (0,1)$, let $\Phi : \widetilde{\Sigma^*} \to \Lambda_r\mathrm{SL}_2\mathbb{C}$ be an analytic map on a universal cover $\widetilde{\Sigma^*} \to \Sigma^*$ of a punctured neighborhood $\Sigma^* \subset \mathbb{C}$ of $0 \in \mathbb{C}$. Let A be a Delaunay residue and let μ be its eigenvalue as in Definition 1.2.*

- *Let $U_1 \in \Lambda_r^{*M}\mathrm{SL}_2\mathbb{C}$.*
- *Let $V_1 \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ for which $A_1 := V_1AV_1^{-1}$ is a Delaunay residue.*

216

Suppose that for every region $\mathcal{A} \subseteq \mathcal{A}_r$ bounded away from $\mathrm{sing}(U_1)$, Φ and $\Phi_1 = \exp(A_1 \log z)$ have the convergence

$$\lim_{z \to 0} \|(\mathrm{Uni}_r[\Phi_1])^{-1} U_1 \mathrm{Uni}_r[\Phi] - \mathrm{I}\|_{\mathcal{A}} = 0,$$
$$\lim_{z \to 0} \|\mathrm{Pos}_r[\Phi] V_1^{-1} (\mathrm{Pos}_r[\Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{68}$$

Let g be a general simple factor with singularity $\lambda_0 \in \mathcal{A}_{r,1} \cap \{\mu \in \mathbb{R}^+\}$. Then there exist:

- *a loop $U_2 \in \Lambda_r^{*\mathrm{M}} \mathrm{SL}_2\mathbb{C}$ satisfying $\mathrm{sing}(U_2) = \mathrm{sing}(U_1) \cup \{\lambda_0, 1/\overline{\lambda_0}\}$; and*
- *a loop $V_2 \in \Lambda_r^+ \mathrm{SL}_2\mathbb{C}$ for which $A_2 := V_2 A V_2^{-1}$ is a Delaunay residue;*

such that for every region $\mathcal{A}' \subseteq \mathcal{A}_r$ bounded away from $\mathrm{sing}(U_2)$, $g\Phi$ and $\Phi_2 := \exp(A_2 \log z)$ have the convergence

$$\lim_{z \to 0} \|(\mathrm{Uni}_r[\Phi_2])^{-1} U_2 \mathrm{Uni}_r[g\Phi] - \mathrm{I}\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|\mathrm{Pos}_r[g\Phi] V_2^{-1} (\mathrm{Pos}_r[\Phi_2])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{69}$$

Proof of the induction step. Let $\mathcal{A}' \subset \mathcal{A}_r$ be a region bounded away from $\mathrm{sing}(U_1) \cup \{\lambda_0, 1/\overline{\lambda_0}\}$.

Step 1. Write $g = W\mathfrak{g}[\lambda_0, L]$ and let $\widehat{g} = W\mathfrak{g}[\lambda_0, \overline{U(\lambda_0)}^{t^{-1}} L]$. By Theorem 5.2 applied to (68), $g\Phi$ and $\widehat{g}\Phi_1$ have the convergence

$$\lim_{z \to 0} \|(\mathrm{Uni}_r[\widehat{g}\Phi_1])^{-1} (\widehat{g} U_1 g^{-1}) \mathrm{Uni}_r[g\Phi] - \mathrm{I}\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|\mathrm{Pos}_r[g\Phi] V_1^{-1} (\mathrm{Pos}_r[\widehat{g}\Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{70}$$

Step 2. Let μ be the eigenvalue function of A as in Definition 1.2. Let $(\overline{\mu(\lambda_0)}, E_+)$ and $(-\overline{\mu(\lambda_0)}, E_-)$ be the eigenvalue–eigenline pairs of $\overline{A_1(\lambda_0)}^t$. Let $g' = \mathfrak{g}[\lambda_0, E_-]$ be the normalized simple factor as in Lemma 5.5. By that lemma, $A_2 := g' A_1 g'^{-1}$ is a Delaunay residue. Since $\mu(\lambda_0) \neq 0$, then $E_- \neq E_+$. Hence, by Lemma 5.4 applied to g' and $\widehat{g}$, the loops $\widehat{g}\Phi_1$ and $g'\Phi_1$ have the convergence

$$\lim_{z \to 0} \|(\mathrm{Uni}_r[g'\Phi_1])^{-1} g' \widehat{g}^{-1} \mathrm{Uni}_r[\widehat{g}\Phi_1] - \mathrm{I}\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|\mathrm{Pos}_r[\widehat{g}\Phi_1] (\mathrm{Pos}_r[g'\Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{71}$$

Let $\Phi_2 = \exp(A_2 \log z)$. Then $g'\Phi_1 = \Phi_2 g'$, so by (71), $\widehat{g}\Phi_1$ and Φ_2 have the convergence

$$\lim_{z \to 0} \|(\mathrm{Uni}_r[\Phi_2])^{-1} g' \widehat{g}^{-1} \mathrm{Uni}_r[\widehat{g}\Phi_1] - \mathrm{I}\|_{\mathcal{A}'} = 0,$$
$$\lim_{z \to 0} \|\mathrm{Pos}_r[\widehat{g}\Phi_1] g'^{-1} (\mathrm{Pos}_r[\Phi_2])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{72}$$

Step 3. By Lemma 4.2(ii), $U_2 := g' U_1 g^{-1}$ is an element of $\Lambda_r^{*\mathrm{M}} \mathrm{SL}_2\mathbb{C}$ and $\mathrm{sing}(U_2) = \mathrm{sing}(U_1) \cup \{\lambda_0, 1/\overline{\lambda_0}\}$. The loop $V_2 := g' V_1$ is an element of $\Lambda_r^+ \mathrm{SL}_2\mathbb{C}$ and $V_2 A V_2^{-2} = g' A_1 g'^{-1}$ is a Delaunay residue. By (70) and (72), $g\Phi$ and Φ_2 have the convergence given by (69). This proves the induction step. $\square$

To prove the theorem, first note that if $G = \mathrm{I}$, the theorem is trivially true with $U_2 = V_2 = \mathrm{I}$. Otherwise, write $G = g_n \cdots g_1$ as a product of simple factors, and starting with $U_1 = V_1 = \mathrm{I}$, apply the induction step to $g_1, \ldots, g_n$ in turn. $\square$

By the methods of §3, Theorem 5.6 implies the following (see also [Kob]).

THEOREM 5.7. *If a CMC surface f converges to a half Delaunay surface f_0, then the surface produced from f by dressing by finitely many simple factors converges to a rigid motion of f_0.*

5.4 Dressed Delaunay asymptotics

We show that the r-Iwasawa factors of a holomorphic frame obtained from a perturbed Delaunay potential converges to those of a Delaunay frame. This theorem is a generalization of Theorem 2.11, removing the restriction that the holomorphic frame extend to $\mathcal{A}_{r,1}$.

THEOREM 5.8. *Let* $r \in (0,1)$ *and assume* $\mathcal{C}_r \cap \mathcal{S}_A = \emptyset$. *Let* ξ *be a perturbed Delaunay* r-*potential*

$$\xi = Az^{-1}\,dz + \mathrm{O}(z^n)\,dz. \tag{73}$$

Suppose

$$\max_{\lambda \in \mathcal{C}_r} \mathrm{Re}\,\mu(\lambda) < (n+1)/2. \tag{74}$$

Let $\Phi : \widetilde{\Sigma^*} \to \Lambda_r \mathrm{SL}_2\mathbb{C}$ *satisfy* $d\Phi = \Phi\xi$ *on the universal cover* $\widetilde{\Sigma^*} \to \Sigma^*$ *of* Σ^*, *and assume that the monodromy* M *of* Φ *around* $z = 0$ *satisfies* $M \in \Lambda_r^* \mathrm{SL}_2\mathbb{C}$.

Then there exists a loop $U \in \Lambda_r^{*\mathrm{M}}\mathrm{SL}_2\mathbb{C}$ *with* $\mathrm{sing}(U) \subset \mathcal{A}_{r,1} \cap \mathcal{S}_A$ *and only simple poles, and a loop* $V \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ *for which* $A_1 := VAV^{-1}$ *is a Delaunay residue, such that for every region* $\mathcal{A} \subseteq \mathcal{A}_r$ *bounded away from* $\mathrm{sing}(U)$, Φ *and* $\Phi_1 := \exp(A_1 \log z)$ *have the convergence*

$$\lim_{z\to 0} \|(\mathsf{Uni}_r[\Phi_1])^{-1} U \mathsf{Uni}_r[\Phi] - \mathrm{I}\|_{\mathcal{A}} = 0,$$
$$\lim_{z\to 0} \|\mathsf{Pos}_r[\Phi] V^{-1} (\mathsf{Pos}_r[\Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{75}$$

Proof. Step 1. Let $\Phi_0 = \exp(A \log z)$ and let $\Phi = C\Phi_0 P$ be the $z^A P$-decomposition of Φ. By Theorem 4.6, there exist $0 < s_1 < r < s_2 < 1$ and a continuous function $c : \mathcal{A}_{s_1,s_2} \to \mathbb{R}_+$ such that $\mathsf{Pos}_r[C\Phi_0]$ extends continuously to $\mathcal{A}_{s_1,s_2}$ and

$$\|\mathsf{Pos}_r[C\Phi_0]\|_{\mathcal{A}_{s_1,s_2}} \leqslant c|z|^{-\mathrm{Re}\,\mu}.$$

Step 2. By Theorem 2.11, Φ and $C\Phi_0$ have the convergence

$$\lim_{z\to 0} \|(\mathsf{Uni}_r[C\Phi_0])^{-1}\mathsf{Uni}_r[\Phi] - \mathrm{I}\|_{\mathcal{A}_r} = 0,$$
$$\lim_{z\to 0} \|\mathsf{Pos}_r[\Phi](\mathsf{Pos}_r[C\Phi_0])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{76}$$

Step 3. Let $C = C_u \cdot C_+$ be the r-Iwasawa factorization of C. Using that $C\exp(2\pi i A)C^{-1} \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$, by Theorem 4.4, there exist loops $G, V_1 \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ such that $C_+ = GV_1$, G is a product of normalized simple factors (or $G = \mathrm{I}$), $A_0 := V_1 A V_1^{-1}$ is a Delaunay residue, and the singularities of the simple factors are distinct and in $\mathcal{S}_A \cap (\mathcal{A}_r \setminus \mathbb{S}^1)$. Then, with $\Phi_0 = \exp(A_0 \log z)$,

$$C\Phi_0 = C_u G V_1 \exp(A \log z) = C_u G \exp(A_0 \log z) V_1 = C_u G \Phi_0 V_1. \tag{77}$$

By (76) and (77), Φ and $G\Phi_0$ have the convergence

$$\lim_{z\to 0} \|(\mathsf{Uni}_r[G\Phi_0])^{-1} C_u^{-1} \mathsf{Uni}_r[\Phi] - \mathrm{I}\|_{\mathcal{A}_r} = 0,$$
$$\lim_{z\to 0} \|\mathsf{Pos}_r[\Phi] V_1^{-1} (\mathsf{Pos}_r[G\Phi_0])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{78}$$

Step 4. The singularities of G in $\mathcal{D}_1$ are distinct and lie in $\mathcal{A}_{r,1} \cap \{\mu \in \mathbb{R}_+\}$. By Theorem 5.6, where in (67), the Φ and Φ_0 are both replaced by Φ_0, so that (67) holds vacuously, there exists a loop $V_2 \in \Lambda_r^+\mathrm{SL}_2\mathbb{C}$ for which $A_1 := V_2 A V_2^{-1}$ is a Delaunay residue, and a loop $U_2 \in \Lambda_r^{*\mathrm{M}}\mathrm{SL}_2\mathbb{C}$ such that for every region $\mathcal{A} \subset \mathcal{A}_r$ bounded away from $\mathrm{sing}(U_2)$, $G\Phi_0$ and $\Phi_1 := \exp(A_1 \log z)$ have the convergence

$$\lim_{z\to 0} \|(\mathsf{Uni}_r[\Phi_1])^{-1} U_2 \mathsf{Uni}_r[G\Phi_0] - \mathrm{I}\|_{\mathcal{A}} = 0,$$
$$\lim_{z\to 0} \|\mathsf{Pos}_r[G\Phi_0] V_2^{-1} (\mathsf{Pos}_r[\Phi_1])^{-1} - \mathrm{I}\|_{\mathcal{D}_r} = 0. \tag{79}$$

Step 5. The result (75) follows from (78) and (79), with $U = U_2 C_u^{-1}$ and $V = V_2 V_1$. $\square$

218

We conclude with the main theorem of the paper, showing that a CMC end obtained from a perturbed Delaunay potential is asymptotic to a half-Delaunay surface. This theorem is a generalization of Theorem 3.6, removing the conditions $\mathcal{A}_{r,1} \cap \mathcal{S}_A = \emptyset$ and $\Phi \in \Lambda_r^{\uparrow}\mathrm{SL}_2\mathbb{C}$. Hence, it applies to surfaces obtained by r-dressing, such as the n-noids with bubbles constructed in [KSS04]. The proof is similar to that of Theorem 3.6, taking the appearance of the loops U and V into consideration. In the case of embedded ends, we obtain exponential convergence (see Remark 3.7).

THEOREM 5.9. *Let A be a Delaunay residue as in (1) satisfying condition (30). On the punctured unit disk $\Sigma^* = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$, let*

$$\xi = A\frac{dz}{z} + \mathrm{O}(z^0)\,dz$$

be a perturbed Delaunay r-potential, $r \in (0,1]$. Let f_0 and f be the immersions of Σ^* induced by the generalized Weierstrass representation at $\lambda = 1$ by $A\,dz/z$ and ξ, respectively, so f_0 is a Delaunay immersion. Assume that f is obtained from a holomorphic r-frame whose monodromy at $z = 0$ is in $\Lambda_r^*\mathrm{SL}_2\mathbb{C}$.

With Σ_{Cyl} as in (46), let $\phi : \Sigma_{\mathrm{Cyl}} \to \Sigma^*$ be the map $\phi(x,y) = e^{x+iy}$. Then some rigid motion of $f \circ \phi$ converges to $f_0 \circ \phi$ in the C^∞-topology of Σ_{Cyl} as $x \to -\infty$. Furthermore, if f_0 is embedded, then f is properly embedded.

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