FINITARY UPPER LOGICISM

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Abstract. This paper proposes and partially defends a novel philosophy of arithmetic—
FINITARY UPPER LOGICISM. According to it, the natural numbers are finite cardinalities—
conceived of as properties of properties—and arithmetic is nothing but higher-order modal
logic. FINITARY UPPER LOGICISM is furthermore essentially committed to the logicality of
FINITARY PLENITUDE, the principle according to which every finite cardinality could have been
instantiated. Among other things, it is proved in the paper that second-order Peano arithmetic
is interpretable, on the basis of the finite cardinalities' conception of the natural numbers, in
a weak modal type theory consisting of the modal logic K, negative free quantified logic, a
contingentist-friendly comprehension principle, and FINITARY PLENITUDE. By replacing FINITARY
PLENITUDE for the AXIOM OF INFINITY this result constitutes a significant improvement on Russell
and Whitehead's interpretation of second-order Peano arithmetic, itself based on the finite
cardinalities' conception of the natural numbers.

§1. Introduction. Russell and Whitehead (henceforth, R&W) adopted a conception of the natural numbers akin to the following:[1]

Numbers as Cardinalities (NaC): The natural numbers are the finite cardinalities.

A property of properties is a finite cardinality just in case it is the property of having (exactly) n instances, for some natural number n. Thus, according to NaC, the natural numbers are properties of a special kind, rather than individuals. Through NaC and related theses R&W [80] offered characterizations of the arithmetical primitives in their Ramified Type Theory (RTT) and famously interpreted Second-Order Peano Arithmetic (PA$_2$) in it.[2] Since RTT's primitives all seem to have some claim to being purely logical notions, as they consist of Boolean connectives and quantifiers, all typically regarded as logical, R&W's result promised to establish the logical truth of all of PA$_2$'s theorems, and that arithmetic is a part of logic.[3]

[1] Why just 'akin to'? While R&W conceived of the natural numbers as 'higher type' entities, as is done here, they took these entities to be *extensionally* individuated, contrary to what will be assumed in the paper.

[2] We will be especially interested in PA$_2$ throughout the paper. Its philosophical interest no doubt stems, at least in part, from Dedekind's categoricity theorem [13, sec. 132].

[3] Those opposed to higher-order quantification's logicality will resist the claim that all of RTT's primitives are logical. Alas, a defense of the logicality of higher-order logic goes beyond paper's scope. I hope to address it elsewhere. A different issue concerns R&W's *ramification* of types. There is good reason to reject it, and the paper does not adopt it.

R&W's interpretation of PA$_2$ in RTT essentially appealed to the following principle:[4]

Axiom of Infinity (InfAx): There are infinitely many individuals.

So, their result shows that PA$_2$'s theorems are all logically true only if INFAX is itself logically true. Yet, arguably, and as Russell [61, p. 141] himself acknowledged, INFAX is neither *necessarily* true nor *a priori* knowable.[5] Hence, on common conceptions of *logical truths* as *necessary* (or *a priori* knowable), INFAX arguably fails to count as logically true. For this reason it is nowadays thought that R&W's type-theoretic interpretation of PA$_2$ has failed to establish arithmetic's logicality.

Here, my main aim is to present and partially defend FINITARY UPPER ARITHMETICAL LOGICISM (UPPER), a novel form of logicism about arithmetic. In a nutshell, according to UPPER arithmetical entities and relations are all higher-type entities—and, in particular, NAC is true—and arithmetic is reducible to higher-order modal logic under a higher-orderist interpretation (I'll explain what is meant by 'higher-orderist interpretation' in Section 2). This is why UPPER is 'upper.' A further essential commitment of UPPER is to the necessary truth (and, indeed, the logicality) of the following thesis:

Finitary plenitude(FP): All finite cardinalities of properties of individuals could have been instantiated.

FINITARY PLENITUDE is a *principle of plenitude* in the sense that it is directly concerned with what is *possible*.[6] Notably, UPPER is 'finitary' in that it is not committed to INFAX, or even to its possible truth. So, even if UPPER is itself committed to NAC, it is significantly different from R&W's original approach. A further difference is that UPPER makes essential use of *modal* resources, which were absent from R&W's theory. These, it will be argued in Section 2, go naturally with a higher-orderist interpretation of type theory.

The partial defense of UPPER here offered will be premised on the following results:

[Int] PA$_2$ is interpretable in the very weak system KQL$_C^{FP}$—an extension of the modal logic K with a negative free type theory, a *contingentist*-friendly comprehension principle, and principle FP.

[Nec] It is provable in S5PQL$_C$—a plural as well as modal extension of KQL$_C^{FP}$ (with the axioms of the modal logic S5), but without FP—that the truths of pure arithmetic are all metaphysically necessary.

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[4] Here, 'infinite' is being used in the sense of not-finite (see Definition 25 in Appendix E), and not in the sense of *Dedekind-infinite* (where a property is Dedekind-infinite if and only if there is a bijection between it and at least one of its proper subproperties).

[5] A referee asks whether *a prioricity* is here understood as *weak* or as *strong* a prioricity (see [20]). I won't engage here in the scholarly question concerning which of these notions Russell had in mind. For my part, what I have in mind is *weak a prioricity*. So, the claim is that one (metaphysically) could not have entitlement which was *independent* of one's past and present experience and on which *knowledge* was based, that there are infinitely many individuals. A defense of the claim that INFAX is not weakly a priori lies, however, outside the paper's scope. Suffice it to say that *even* if INFAX were a priori knowable and necessary, an interpretation of PA$_2$ relying on FP rather than INFAX is nonetheless of interest, as the latter is a stronger assumption.

[6] See, e.g., [6].

Owing to FP's platitudinous character, which contrasts with INFAX's controversiality, [INT] in particular constitutes a significant improvement on R&W's interpretability result. It promises to deliver a sound and NAC-based logicist foundation for PA$_2$.

UPPER's core theses are presented in Section 2. In Section 3 are characterized the languages and deductive systems of relevance for the remainder of the paper. The proofs of [INT] and [NEC] are outlined in, respectively, Sections 4 and 5. In Section 6 is defended the metaphysical necessity of both FP and all of S5PQL$_{SA,C}$'s theorems—an extension of S5PQL$_C$ with instances of SERIOUS ACTUALISM, the thesis that standing in a relation implies being something.[7] In Section 7 UPPER is briefly compared to prima facie related philosophies of arithmetic—specifically, Hellman's [31] MODAL STRUCTURALISM, Hodes's [32, 33] ALTERNATIVE THEORY, Roeper's [60] VINDICATION OF LOGICISM, and Linnebo and Shapiro's [46] ARITHMETICAL POTENTIALISM—and Section 8 concludes.

[INT] is proven in Appendix A, the necessity of FP in Appendix B, and [NEC] in Appendix C. In Appendix D is shown that, in the deductive systems adopted in the paper, the rules of necessitation and universal generalization can be dispensed with if the necessitation and the universal generalization of any axiom is itself an axiom. In Appendix E is shown that GENERATION, a watered-down version of Lewis's [43] principle of RECOMBINATION, implies FP, and some results concerning the relationship between INFAX and FP established. And in Appendix F is characterized a model theory fitting UPPER, and a structure is presented which witnesses the joint consistency of the system S5PQL$_{SA,C}$ (the result of adding SERIOUS ACTUALISM to S5PQL$_C$) with FP, CONTINGENTISM at all orders, and the necessary falsehood of INFAX.

§2. FINITARY UPPER ARITHMETICAL LOGICISM. INFAX's sole role in R&W interpretation of PA$_2$ is to ensure SIC's derivability in RTT:

Successor is cardinality injective (SIC): Distinct finite cardinalities have distinct successors.

Here, the *successor* of a property of properties *n* is the property of properties **S**[*n*] of having exactly one more instance than any instance of property *n*. The importance of R&W's result is that SIC is their interpretation of the following axiom of PA$_2$:

Successor is injective (SI): Distinct natural numbers have distinct successors.

SIC is not derivable in RTT – INFAX (that system which results from removing INFAX from RTT's axioms). Further, and partly due to R&W's commitment to the identity of coextensional properties, it is derivable in RTT – INFAX that SIC implies INFAX.

As it turns out, R&W's interpretability result can be improved by abandoning the principle that coextensional properties are identical—after all, some coextensive properties *could have* failed to have been coextensive—and replacing INFAX by the weaker principle FP, which also guarantees SIC's derivability. Since FP's platitudinous character makes it reasonable to think that it is logically true, the improved interpretability result paves the way for a defense of UPPER.

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[7]   For the most part, the results here offered do not presuppose the truth of SERIOUS ACTUALISM. The exceptions concern a couple of results—Lemmas 36 and 40—on the relationship between FP and seemingly related thesis.

In this section will be presented UPPER's core theses. While not all of them will be defended in the paper, they are nonetheless relevant as they provide a fuller picture of the sort of logicist view that the results here presented can be seen as supporting. Two of UPPER's component theses are the following:

Necessity of FP: FP is necessarily true.

Higher-orderism: Predicates have a semantic role distinct from those of individual and of plural terms. Quantification into predicate position is legitimate and irreducible to first-order singular or to plural quantification.

Some reasons in favor of NECESSITY OF FP will be given in Section 6. As to HIGHER-ORDERISM, it specifies UPPER's intended interpretation of the type-theoretic language. Prima facie, if individual and plural terms had the same semantic role as that of predicates, then they would be substitutable for predicates in meaningful sentences without loss of meaningfulness. But they are not. For instance, 'John runs' is meaningful, whereas 'John Mary' is meaningless. Similarly, predicates are not substitutable for individual or plural terms without loss of meaningfulness. For instance, 'John runs' is meaningful, but 'swims runs' is meaningless.[8] Accordingly, from the standpoint of higher-orderists, while individual and plural terms refer (respectively, singularly and plurally), this is not the role of predicates. Following Trueman's [79, chaps. 2–7] gloss (see also [9, sec. 5.3]), the semantic role of predicates may be said to be that of 'saying things' of what they are predicated of. For instance, 'swims' says of any individual that it swims, and 'everything' says of anything said of individuals that it is had by all of them.[9]

Moreover, quantification into predicate position is to be understood on its own terms—not as restricted first-order singular or plural quantification over individuals, sets, classes, etc. While individual variables take subject position, are bound by first-order quantifiers, and have a semantic role akin to that of individual terms, predicate variables take predicate position, are bound by higher-order quantifiers, and have a semantic role akin to that of predicates.

What has been offered here is nothing but a very brief gloss on the higher-orderist interpretation of typed languages. Indeed, from the standpoint of higher-orderists, an appropriate semantic theory for a typed language is to be formulated not in set theory (by assigning sets as the semantic values of predicates), but rather in a (more expressive) type-theoretic language, by assigning entities of distinct types as the semantic values of predicates of distinct types (see, e.g., [81, pp. 453–454]).

According to higher-orderists natural language talk of 'properties' and 'relations' is to be understood with a grain of salt since expressions such as 'the property of swimming' and 'the membership relation' take subject rather than predicate position. For this reason, in this paper I engage in such talk merely for the sake of readability.

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[8] See [79, sec. 2] for a developed argument to the effect that the semantic role of predicates is different from that of individual and plural terms owing to the existence of substitutions which fail to preserve meaningfulness.

[9] I am here are adopting the Fregean view that 'everything' is a higher-type predicate. While in our official typed language quantifiers are treated syncategorematically, this language also possesses a higher-type predicate that 'says' of anything 'said' of individuals that it is had by all of them, specifically, the complex predicate '$\lambda x_{\langle e \rangle}(\forall y_e(x(y)))$'.

Defenses of HIGHER-ORDERISM can be found in, among others, [23, 55, 59, 84], [47, sec. 19.2.2], [81, pp. 254–261], and [39, 79].[10] Here is not the place to further defend it.

The adoption of HIGHER-ORDERISM naturally leads to distinctive views on the interaction between modality and higher-order quantification. While sets and pluralities are commonly thought to (i) have their instances *essentially*, and (ii) be *extensionally* individuated, in general the values of higher-order variables have their instances inessentially and are not extensionally individuated.[11] Indeed, in Appendix F is offered a model theory for modal type theory on which, in general, the values of higher-order variables fail to satisfy (i) and (ii).

Since INFAX is derivable from the conjunction of FP with any of (i) and (ii) already in somewhat weak modal type-theoretic systems,[12] there would be no real gain in adopting FP rather than INFAX, unless it was thought that the values of higher-order variables did not satisfy (i) nor (ii). HIGHER-ORDERISM is thus an important aspect of UPPER as it makes it reasonable to think that the values of higher type variables do not satisfy (i) nor (ii), and so that a commitment to FP does not collapse in a commitment to INFAX.

UPPER's third and fourth core theses are the following:

Logicality: The metaphysically necessary truths formulated in a pure modal and type-theoretic language are all logically true.

Higher-type arithmetical ontology: Natural numbers are finite cardinalities, and the Russellian characterization of arithmetic's primitives—*natural number*, *zero* and *successor*—are all true.[13]

LOGICALITY is a joint consequence of (i) the popular view (see, e.g., [64, 65]) that the *logical truths* include the necessary truths formulated solely in terms of logical expressions; and (ii) that among the latter are all of pure modal type theory's primitives. Notoriously, these include the *necessity operator*, which did not figure in the logicist theories of Frege, Whitehead, and Russell, and Hale and Wright.[14] Alas, since a defense of the metaphysical necessity operator's logicality, and so of LOGICALITY, would require separate treatment, it will be left for another occasion.[15]

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[10] See also [68] for an overview of higher-orderist metaphysics, and [27] for a collection on the topic.

[11] E.g., (i) it is not essential to the property of being the Earth's satellite that it is had by the Moon, as something else might have been the Earth's satellite; and (ii) while the property of being the Earth's satellite is coextensive with the property of being the Moon, the two are distinct (since nothing but the Moon could have been identical to it).

[12] For instance, adding any one of these theses to the system $\mathsf{S5QL_C}$, to be presented later on, results in a system in which INFAX is derivable. This is shown in Lemmas 40 and 41 in Appendix E.

[13] In rigor, UPPER is committed to NAC and to ARITHMETICAL PRIMITIVES, a thesis to be characterized in Section 3.

[14] By contrast with Frege, Whitehead, and Russell, and Hale and Wright, higher-order identity predicates are primitives of the theories here considered. Notwithstanding, this difference is not of much significance. For higher-order identity could just as well have been defined in terms of indiscernibility.

[15] On this issue, see, e.g., [8, 48, 51, 66, 77, 82]. Still on metaphysical necessity's logicality, I am inclined towards the view that metaphysical necessity is truly characterizable along the following lines: for it to be the case that $\Box p$ just is for p to be identical to $p \rightarrow p$ (for similar

HIGHER TYPE ARITHMETICAL ONTOLOGY's conception of the arithmetical entities is directly connected to what has been called the *applications constraint* and, more specifically, to *Frege's constraint*. Roughly, according to *Frege's constraint* the canonical applications of mathematical entities to the characterization of the world must be contained within their nature.[16] Arguably, the application of properties and relations to the characterization of the world *is* contained within their nature. That is, part of what it is to be a property is to have application conditions—conditions under which entities instantiate or fail to instantiate it—and those conditions will be essential to the properties that they are application conditions of.[17]

Now, conceiving of the natural numbers as properties of properties does not, on its own, imply the satisfaction of *Frege's constraint*. For satisfaction of *Frege's constraint* requires that the *canonical* applications of *natural numbers* be contained within the properties claimed to be natural numbers. Still, NAC accounts for the application of natural numbers to *counting*—arguably, their *canonical* application—and it does so in the most straightforward way. For the result of counting is the attribution of a cardinality property. So, it would appear that HIGHER-TYPE ARITHMETICAL ONTOLOGY does imply the satisfaction of *Frege's constraint*.[18]

Relatedly, NAC accounts for our use of natural number expressions as *determiners*—as in, e.g., 'Mars has *two* moons'—as these are commonly thought to consist of properties of (and relations between) properties. Now, a number of philosophers (perhaps most notably, Frege) have held that the use of number expressions in subject position requires viewing numbers as *individuals* rather than properties. Unsurprisingly, I reject that this is so, though a sustained defense of the view that natural numbers are properties rather than individuals lies beyond the scope of the present paper. Suffice it to say for the moment that proponents of the view that natural numbers are individuals likewise face the challenge of accounting for the use of natural number expression as determiners, and that NAC *might* afford the means of accounting for our natural

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views, see [11], [76, sec. 7], [15, pp. 68–70] and [57, chap. 5]). So, modality need not be a primitive of a logical theory having amongst its theorems the logicist interpretations of all of PA₂'s theorems, provided that the logicality of propositional quantification and propositional identity are accepted. But this stance is, of course, compatible with modality being a primitive of a different logical theory which also has amongst its theorems the logicist interpretations of all of PA₂'s theorems—which is what I here argue for.

[16] Wright [83, p. 324] formulated the *applications constraint* as the constraint that '... a satisfactory foundation for a mathematical theory must somehow build its applications, actual and potential, into ... the content it ascribes to the statements of the theory ...'. For insightful discussion on the *applications constraint*, the specifics of Wright's and Frege's views, and the role played by *Frege's constraint* in his philosophy of mathematics, see [52].

[17] More explicitly, it is natural to cash out as follows the idea that φ are the application conditions of a relation $X_{\langle \tau^1, ..., \tau^n \rangle}$, where $z^1_{\tau^1}$, ..., $z^n_{\tau^n}$ occur free in φ: φ are $X_{\langle \tau^1, ..., \tau^n \rangle}$'s application conditions if and only if $\Box \forall z^1_{\tau^1} ... \Box \forall z^n_{\tau^n} \Box (X(z^1, ..., z^n) \leftrightarrow \varphi)$. It is then S4-derivable that *necessarily*, if relation $X_{\langle \tau^1, ..., \tau^n \rangle}$ exists, then $\Box \forall z^1_{\tau^1} ... \Box \forall z^n_{\tau^n} \Box (X(z^1, ..., z^n) \leftrightarrow \varphi)$. So, a relation's application conditions are essential to it at least in the sense that, necessarily, if the relation exists, then it has those application conditions.

[18] A full development of a defense of HIGHER-TYPE ARITHMETICAL ONTOLOGY via *Frege's constraint* will, however, have to be left for another occasion.

language use of natural number expressions in *subject* position provided that this use is derived from their use as determiners—a view argued for in, e.g., [36, chap. 2].[19]

Finally, the appeal to modal resources fits naturally with HIGHER-TYPE ARITHMETICAL ONTOLOGY and the canonical application of the natural numbers to counting. For besides considering what finite cardinality a given property actually falls under, we also commonly consider those that it *could* have fallen under. For instance, it is not only true that Mars has two moons, but also that it *could have had* more.

Importantly, it will be a commitment of HIGHER TYPE ARITHMETICAL ONTOLOGY that natural number-talk is *ambiguous* across types. For each type τ, there is a *zero*$_{\langle\langle\tau\rangle\rangle}$ which applies to properties of type τ-entities.[20] *Mutatis mutandis* for talk of 'successor' and 'natural number'. Proponents of UPPER thus regard PA_2 as ambiguous between different theories $PA_{2,\tau}$, for each type τ, whose apparently first-order quantifiers in fact range over properties of properties of type τ entities.

This kind of type ambiguity fits naturally with HIGHER-ORDERISM. From a higher-orderist standpoint, talk of *identity*, *equinumerosity*,[21] and *cardinality* is itself ambiguous across types. To give just one example, there is not only a cardinality property of being instantiated by exactly three individuals, which applies to properties of individuals, but also, for each type, the property of being instantiated by exactly three entities of that type, a property which applies to properties applicable to entities of that type. The ambiguity of natural number talk is thus of a kind already familiar to higher-orderists. And it naturally follows, given HIGHER-ORDERISM, from a conception of natural numbers as finite cardinalities.

UPPER's remaining core theses are the following:

Interpretability of Arithmetic: There is a deductive system formulated in a pure modal and type-theoretic language and including FP among its axioms whose theorems are necessary truths, and include the HIGHER TYPE ARITHMETICAL ONTOLOGY-respecting translations of all theorems of PA_2.[22]

Necessity of Arithmetic: The purely arithmetical truths expressible in the language of PA_2 are all metaphysically necessary truths also expressible in a pure modal type theory.

INTERPRETABILITY OF ARITHMETIC—alongside with HIGHER TYPE ARITHMETICAL ONTOLOGY and LOGICALITY—establishes that every theorem of $PA_{2,\tau}$ is *logically true*, for every type τ. This is a pleasing result from a logicist standpoint insofar as PA_2 is a canonical second-order system for arithmetic.

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[19] Some further remarks on this issue will be offered in Section 7.2, in the context of discussing Hodes's ALTERNATIVE THEORY.

[20] Here, the subscript indicates the type of the expression in question. For each type ρ, $\langle\rho\rangle$ is the type of predicates applicable to type ρ expressions, and so $\langle\langle\tau\rangle\rangle$ is the type of predicates applicable to type $\langle\tau\rangle$ expressions, for every type τ. Thus, 'zero$_{\langle\langle\tau\rangle\rangle}$' is used not as a singular term referring to individuals, but as a predicate applicable to type $\langle\tau\rangle$ predicates, and so 'standing for' a higher-type property. The typing will be further explained in Section 3.

[21] Where two properties are equinumerous if and only if there is some 1–1 function from one of them onto the other. For a rigorous characterization of the notion in the adopted type-theoretic setting, see Definition 14 in Appendix A.

[22] These are the *Russellian translations*, characterized in Definition 8 in Section 4.

Now, Gödel's incompleteness theorems appear to pose an important challenge to any logicist project as they establish that *no* (effectively axiomatizable and consistent) deductive system contains as theorems all arithmetical truths expressible in the language of PA_2. As a consequence, INTERPRETABILITY OF ARITHMETIC, HIGHER TYPE ARITHMETICAL ONTOLOGY and LOGICALITY fail to imply (on their own) that all arithmetical *truths* expressible in the language of PA_2 are logically true.

In my view, logicists should see Gödel's incompleteness theorems as revealing not that some arithmetical truths are not logically true, but rather that *no* (effectively axiomatizable) deductive system contains as theorems *all* logical truths. For this reason, a successful logicist account of a mathematical field *F* will ideally involve not only an interpretability result showing how to derive a canonical (true) deductive theory of *F* by purely logical means, but also a sound argument for the logical truth of the logicist interpretation, in purely logical terms, of every truth of *F*. In the case of UPPER, this second component consists in a defense of NECESSITY OF ARITHMETIC.[23] For NECESSITY OF ARITHMETIC, HIGHER-TYPE ARITHMETICAL ONTOLOGY and LOGICALITY do jointly imply that *all* arithmetical truths expressible in the language of PA_2 are logically true. Summing up, the following are all of UPPER's core theses.

Upper: The conjunction of (i) NECESSITY OF FP; (ii) HIGHER-ORDERISM; (iii) LOGICALITY; (iv) HIGHER-TYPE ARITHMETICAL ONTOLOGY; (v) INTERPRETABILITY OF ARITHMETIC; and (vi) NECESSITY OF ARITHMETIC.

The partial defense here offered of UPPER will only be concerned with the theses of NECESSITY OF FP, INTERPRETABILITY OF ARITHMETIC and NECESSITY OF ARITHMETIC. I hope to offer defenses of LOGICALITY and HIGHER-TYPE ARITHMETICAL ONTOLOGY in another occasion.

§3. Typed modal systems. The typed modal systems of interest in the remainder of the paper are formulated in either the language MT, UPPER's official pure typed logical language, or else in PMT, an extension of MT with plural quantification. The set of types is the following:

DEFINITION 1 (Types). *The set $\mathcal{T}$ of types is the smallest set such that* e *is a type, and, for every positive integer* n, *if $\tau^1, \ldots, \tau^n$ are all types, then $\langle \tau^1, \ldots, \tau^n \rangle$ is a type.*

The variables of PMT come in two kinds: singular variables and plural variables.

DEFINITION 2 (Variables). *For each type τ, Var_τ^s, the set of* singular *type τ-variables, is the set $\{x_\tau^i : i \in \mathbb{N}\}$, and Var_τ^p, the set of* plural *type τ-variables, is the set $\{xx_\tau^i : i \in \mathbb{N}\}$, for each type τ.*

Terms and *formulae* of PMT are simultaneously defined as follows:

DEFINITION 3 (Terms and formulae of PMT). *For each type τ, the sets PMT_τ^s and $\mathrm{PMT}_{\mathrm{wff}}$ of, respectively, singular type τ-terms and formulae of PMT, are the smallest sets such that*:

    – Atomic singular terms: (i) $\mathrm{Var}_\tau^s \subseteq \mathrm{PMT}_\tau^s$; (ii) $\ulcorner =_\tau \urcorner \in \mathrm{PMT}_\tau^s$ *whenever τ is the type $\langle \sigma, \sigma \rangle$, for any type σ.*

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[23] For a defense of the claim that every arithmetical truth is *determinately* so, see [30].

– Formulae: *For all types* $\tau^1, \dots, \tau^n$ *and* σ: (*iii*) *if* $\beta \in \text{PMT}^s_{\langle \tau^1, \dots, \tau^n \rangle}$ *and* $\alpha^i \in$ $\text{PMT}^s_{\tau^i}$, *for each* i *such that* $1 \leq i \leq n$, *then* $\ulcorner \beta(\alpha^1, \dots, \alpha^n) \urcorner \in \text{PMT}_{\text{wff}}$; (*iv*) *if* $\{\varphi, \psi\} \subseteq \text{PMT}_{\text{wff}}$, *then* $\{\ulcorner \neg \varphi \urcorner, \ulcorner \varphi \wedge \psi \urcorner, \ulcorner \Box \varphi \urcorner\} \subseteq \text{PMT}_{\text{wff}}$; (*v*) *if* $v \in \text{Var}^s(\sigma)$ *and* $\varphi \in \text{PMT}_{\text{wff}}$, *then* $\ulcorner \forall v \varphi \urcorner \in \text{PMT}_{\text{wff}}$; (*vi*) *If* $vv \in \text{Var}^p_\sigma$ *and* $\varphi \in \text{PMT}_{\text{wff}}$, *then* $\ulcorner \forall vv \varphi \urcorner \in \text{PMT}_{\text{wff}}$; (*vii*) *if* $\alpha \in \text{PMT}^s_\sigma$ *and* $vv \in \text{Var}^p_\sigma$, *then* $\ulcorner \alpha \prec vv \urcorner \in \text{PMT}_{\text{wff}}$; (*viii*) *if* $vv, uu \in \text{Var}^p_\sigma$, *then* $\ulcorner vv = uu \urcorner \in \text{PMT}_{\text{wff}}$.

– Complex predicates: (*ix*) *If* τ *is the type* $\langle \tau^1, \dots, \tau^n \rangle$, $v^1 \in \text{Var}^s_{\tau^1}, \dots, v^n \in \text{Var}^s_{\tau^n}$ *are distinct variables, and* $\varphi \in \text{PMT}_{\text{wff}}$, *then* $\ulcorner \lambda v^1 \dots v^n \varphi \urcorner \in \text{PMT}^s_\tau$, *for all types* $\tau^1, \dots, \tau^n$.

DEFINITION 4 (Variables, terms and formulae of MT). *For each type* τ, MT*'s type* τ*-variables are* PMT*'s singular type* τ*-variables,* MT*'s type* τ*-terms are those type* τ*-terms of* PMT *in which no plural variables occur, and* MT*'s formulae are those formulae of* PMT *in which no plural variables occur.*

Loosely, an entity will be said to be of a given type when the terms that stand for entities of its kind are of that type. Type *e*-singular terms stand for individuals—entities of which it is *nonsensical* to say that they are instantiated as well as that they are not.[24] Roughly, type $\langle \tau^1, \dots, \tau^n \rangle$-singular terms stand for relations between entities of, respectively, types τ^1, ..., τ^n. Moreover, whereas a type τ-singular variable ranges over type τ-entities, a type τ-plural variable ranges over pluralities of type τ-entities.[25]

The Boolean connectives '$\neg$' and '$\wedge$' are given their standard reading, '$\Box$' expresses metaphysical necessity, and each complex predicate $\ulcorner \lambda v^1 \dots v^n \varphi \urcorner$ stands for the type $\langle \tau^1, \dots, \tau^n \rangle$ relation of being v^1, ..., v^n such that φ. There are no complex terms standing for pluralities. Intuitively, $\ulcorner \alpha \prec vv \urcorner$ states that α is one of the vv, and $\ulcorner vv = uu \urcorner$ states that the vv are nothing but the uu. The remaining Boolean connectives, the existential quantifier and the possibility operator are all contextually defined in the usual manner. Further, $\alpha \neq \beta := \neg \alpha = \beta$, and for each type τ and natural number n, $\langle \tau \rangle_n := \underbrace{\langle \tau, \dots, \tau \rangle}_{\times n}$.

Note that '$=$' as it occurs in clause (viii) of Definition 3, is not to be confused with $\ulcorner =_{\langle \tau, \tau \rangle} \urcorner$, for any type τ. Rather, it is a syncategorematic expression. A different option would have been to have taken formulae of the form $\ulcorner vv = uu \urcorner$ to be abbreviations of, e.g., *plural coextensiveness* (i.e., $yy_\tau = zz_\tau := \forall x_\tau(x \prec yy_\tau \leftrightarrow x \prec zz_\tau)$), or *plural coextensiveness* and *existence* (i.e., $yy_\tau = zz_\tau := \exists ww_\tau \forall x_\tau(x \prec yy_\tau \leftrightarrow x \prec ww) \wedge \forall x_\tau(x \prec yy_\tau \leftrightarrow x \prec zz_\tau))$. None of these options has been pursued here in order not to presuppose any particular view on whether the identity of pluralities requires their existence, or whether it suffices that they be coextensive.

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[24] That is, it is nonsensical to say that they are not instantiated except in a *derivative* sense (e.g., when their haecceities are not instantiated). For a defense of the related view that it is nonsensical to say both that attributes have and that they lack spatiotemporal location, see [39].

[25] Language MT is quite close to Gallin's [28] modal type-theoretic language. One difference is that Gallin's relational types also include the type $\langle \rangle$—the type of formulae—with some variables thus taking sentence position. Quantification into sentence position would, however, bring nothing new to our results, and leaving it out simplifies the discussion to some extent. Also, while MT includes amongst its primitives the 'λ' operator and higher-type identity predicates, these are absent from Gallin's original language. Finally, Gallin's language does not include plural quantification.

We will also make use of the following defined expressions:

DEFINITION 5 (Existence). *For any type τ, type τ-singular term or plural variable ζ, and where* u *is the first type τ variable* (*singular if ζ is singular, plural if ζ is plural*) *that occurs nowhere in ζ: $E[\zeta] := \exists u(u = \zeta)$.*

Regimenting FINITARY PLENITUDE requires defining what it is to be a *finite cardinality*. This and related notions are defined as follows (in a manner akin to Frege's own definitions):

DEFINITION 6. *Where, for every type τ:* (*i*) α *is a type $\langle\tau\rangle$-singular term;* (*ii*) β *is a type τ-singular term;* (*iii*) ξ *is a type $\langle\langle\tau\rangle\rangle$-singular term;* (*iv*) v *is the first type $\langle\tau\rangle$-singular variable that occurs nowhere free in α, β or ξ;* (*v*) w *is the first type τ-singular variable that occurs nowhere free in ξ;* (*vi*) γ *is a type $\langle\langle\langle\tau\rangle\rangle\rangle$-singular term,* (*vii*) r *and* t *are, respectively, the first and the second type $\langle\langle\tau\rangle\rangle$-singular variables that occur nowhere free in γ:*

(a) Nothing: $\mathbf{0}_{\langle\langle\tau\rangle\rangle} =_{df} \lambda x_{\langle\tau\rangle}(\neg\exists y_{\tau}(x(y)))$;

(b) Minus: $\alpha - \beta := \lambda v(\alpha(v) \wedge v \neq \beta)$;

(c) Successor of: $\mathbf{S}_{\langle\langle\langle\tau\rangle\rangle\rangle_2}[\xi] := \lambda v(\exists w(v(w) \wedge \xi(v - w)))$;

(d) Successor: $\mathbf{S}_{\langle\langle\langle\tau\rangle\rangle\rangle_2} =_{df} \lambda x_{\langle\tau\rangle} y_{\langle\tau\rangle}(y = \boldsymbol{S}_{\langle\langle\langle\tau\rangle\rangle,\langle\langle\tau\rangle\rangle}[x])$;

(e) Hereditary: $Her[\gamma] := \forall r \forall t((\gamma(r) \wedge \boldsymbol{S}(r,t)) \to \gamma(t))$;

(f) Finite cardinality: $\mathbf{N}_{\langle\langle\langle\tau\rangle\rangle\rangle} =_{df} \lambda x_{\langle\langle\tau\rangle\rangle}(\forall y_{\langle\langle\langle\tau\rangle\rangle\rangle}((y(\boldsymbol{0}) \wedge Her[y]) \to y(x)))$.

For each type τ, FINITARY PLENITUDE$_\tau$ is then regimented as follows, where FP is just FPe:

$$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y))). \tag{FP$_\tau$}$$

Moreover, according to HIGHER-TYPE ARITHMETICAL ONTOLOGY:

Arithmetical primitives: For every type τ:

(a) To be the natural number $zero_{\langle\langle\tau\rangle\rangle}$ just is to be $\mathbf{0}_{\langle\langle\tau\rangle\rangle}$—i.e., the property of having no instances.

(b) To be the *successor relation*$_{\langle\langle\langle\tau\rangle\rangle\rangle_2}$ just is to be $\mathbf{S}_{\langle\langle\langle\tau\rangle\rangle\rangle_2}$—i.e., the relation of being $x_{\langle\langle\tau\rangle\rangle}$ and $y_{\langle\langle\tau\rangle\rangle}$ such that $y_{\langle\langle\tau\rangle\rangle} = \mathbf{S}_{\langle\langle\langle\tau\rangle\rangle\rangle_2}[x_{\langle\langle\tau\rangle\rangle}]$, where $\mathbf{S}_{\langle\langle\langle\tau\rangle\rangle\rangle_2}[x_{\langle\langle\tau\rangle\rangle}]$ is the property of being a $z_{\langle\tau\rangle}$ such that some u_τ falls under $z_{\langle\tau\rangle}$ while the property of being a $z_{\langle\tau\rangle}$ distinct from u_τ falls under $x_{\langle\langle\tau\rangle\rangle}$.

(c) To be the property of being a *natural number*$_{\langle\langle\langle\tau\rangle\rangle\rangle}$ just is to be $\mathbf{N}_{\langle\langle\langle\tau\rangle\rangle\rangle}$—i.e., the property of having all of $\mathbf{0}_{\langle\langle\tau\rangle\rangle}$'s hereditary properties, where a property $p_{\langle\langle\langle\tau\rangle\rangle\rangle}$ is *hereditary* just in case $x_{\langle\langle\tau\rangle\rangle}$ has it only if $x_{\langle\langle\tau\rangle\rangle}$'s successor has it.

The base deductive systems of interest in the remainder of the paper are characterized in terms of the schemata and inference rules in Table 1.

In Table 1 φ and ψ range over PMT-formulae. In addition, for every type τ, v ranges over singular as well as plural variables of type τ, α and γ range over type τ-singular terms as well as over type τ-plural variables, u is a singular type τ-variable, and uu and tt are plural type τ-variables.

Furthermore, for all $n \in \mathbb{N}^+$ and i such that $1 \leq i \leq n$ and every type τ^i: (i) δ^i ranges over type τ^i-singular terms; (ii) $\bar{\delta}$ abbreviates the sequence $\delta^1, \ldots, \delta^n$; (iii) u^i ranges over type τ^i-variables; (iv) $\bar{u}$ abbreviates the sequence $u^1 \ldots u^n$; (v) β ranges over type

Table 1. *Axiom schemata and inference rules*

Axiom schemata

(PL)	Propositional tautologies	(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(T)	$\Box\varphi \to \varphi$	(5)	$\Diamond\varphi \to \Box\Diamond\varphi$
($\forall$1)	$\forall v(\varphi \to \psi) \to (\forall v\varphi \to \forall v\psi)$	($\forall$2)	$\varphi \to \forall v\varphi$[a]
($\forall$E)	$E[\alpha] \to (\forall v\varphi \to \varphi_\alpha^v)$[b]	($\forall$=)	$\forall v(v = v \wedge E[v])$
(Ind)	$\alpha = \gamma \to (\varphi \to \varphi')$[c]	(SA)	$\beta(\overline{\delta}) \to E[\delta^i]$
(Ab1)	$\lambda\overline{u}\varphi(\overline{\delta}) \to \varphi_{\overline{\delta}}^{\overline{u}}$	(Ab2)	$(\varphi_{\overline{\delta}}^{\overline{u}} \wedge \bigwedge_{1\le i \le n} E[\delta^i] \wedge$ $\wedge E[\lambda\overline{u}\varphi]) \to \lambda\overline{u}\varphi(\overline{\alpha})$
(Comp$_\mathrm{C}$)	$E_\beta \to E[\beta]$[d]	($\prec$ I)	$u \prec uu \to \Box(E[uu]$ $\to (u \prec uu \wedge E[u]))$
(=I$_\mathrm{P}$)	$(E[uu] \wedge \forall u(u \prec uu \leftrightarrow u \prec tt))$ $\to uu = tt$	(PluComp)	$\exists u\varphi \to \exists uu\forall u(u \prec uu \leftrightarrow \varphi)$
($\Box$E$_\mathrm{P}$)	$(E[uu] \wedge \forall u(u \prec uu \to \Box E[u]))$ $\to \Box E[uu]$		

Inference rules

(MP)	$(\vdash \varphi$ & $\vdash \varphi \to \psi) \Rightarrow \vdash \psi$	(Nec)	$\vdash \varphi \Rightarrow \vdash \Box\varphi$	(Gen)	$\vdash \varphi \Rightarrow \vdash \forall v\varphi$

[a] Provided that v occurs free nowhere in φ.
[b] Where α is free for v in φ and φ_α^v is a formula that results from substituting α for all free occurrences of v in φ.
[c] Where φ' is a formula which results from φ by having γ occur at some places where α occurs, re-lettering bound variables to ensure that no variables free in $\alpha = \gamma$ are bound in φ or in φ'.
[d] Given a list $\mu^1, ..., \mu^n$ of all atomic terms or plural variables (except $\ulcorner=_{\langle\tau,\tau\rangle}\urcorner$, for all types τ) free in β, $E_\beta := E[\mu^1] \wedge \cdots \wedge E[\mu^n]$.

Table 2. *Base systems of modal and quantified logic*

System	Axioms	Inference rules
PL	All theorems of classical propositional logic	MP
K	Axioms of PL+ all instances of K	MP, Nec
T	Axioms of K+ all instances of T	MP, Nec
S5	Axioms of T+ all instances of 5	MP, Nec
QL	Axioms of PL+ all instances of $\forall$1, $\forall$2, $\forall$E, $\forall$=, Ind, SA, Ab1, Ab2	MP, Gen
PQL	Axioms of QL+ all instances of $\Box$E$_\mathrm{P}$, $\prec$ I, PluComp, =I$_\mathrm{P}$	MP, Gen

$\langle\tau^1, ..., \tau^n\rangle$-singular terms; and (vi) $\varphi_{\overline{\delta}}^{\overline{u}}$ is the result of simultaneously substituting each free occurrence of u^i in φ by δ^i, where each δ^i is free for u^i in φ.

The present interest is in the deductive systems in Table 2 and their combinations characterized in Definition 7.

DEFINITION 7 (Systems of modal type theory). *If $A \in Mod := \{\mathrm{K}, \mathrm{T}, \mathrm{S5}\}$, $B \in Qu := \{\mathrm{QL}, \mathrm{PQL}\}$, then AB is that system whose axioms and inference rules result from putting together, respectively, the axioms and inferences of A and B. If B is* QL, *then it and AB*

are formulated in MT. *If it is* PQL *then it and* AB *are formulated in* PMT. *Further, each one of* B_{SA}, B_C, $B_{SA,C}$, AB_{SA}, AB_C, $AB_{SA,C}$ *is the system that results from adding to* B, *or* AB, *respectively, all instances of* SA, *all instances of* Comp_C, *and all instances of both* SA *and* Comp_C.

Moreover, for each one of the defined deductive systems S, $\ulcorner \vdash_S \urcorner$ *stands for derivability in* S, '$\vdash$' *stands for* $\vdash_{\mathrm{KQL}_C}$, *and* S^{FP} *is that system whose theorems are those formulae* φ *such that* $\vdash_S \mathrm{FP} \to \varphi$. *Finally, and as usual, for each system* S, $\Gamma \vdash_S \varphi$ *if and only if there are* $\gamma^1, \dots, \gamma^n \in \Gamma$ *such that* $\vdash_S \gamma^1 \to (\gamma^2 \to (\dots \to (\gamma^n \to \varphi) \dots))$.

FP does not figure as an extra axiom of any of the systems characterized in Definition 7 is so that its *metaphysical necessity* not be presupposed without argument—which it would otherwise be, owing to the presence of the Nec rule. While, I'll later argue for NECESSITY OF FP, the system $\mathrm{KQL}_C^{\mathrm{FP}}$ which will be shown to interpret PA_2 does not have FP's metaphysical necessity as a theorem.

§4. Interpretability of PA_2 in $\mathrm{KQL}_C^{\mathrm{FP}}$. Recall that PA_2 is formulated in a second-order language (without λ-terms), PA_2, whose only non-logical primitives are the individual constant '0', the unary predicate 'N' ('is a natural number') and the binary predicate 'S' ('is succeeded by'). PA_2 will be shown to be interpretable in $\mathrm{KQL}_C^{\mathrm{FP}}$ via *Russellian translations*—so called owing to their alignment with Russell and Whitehead's take on arithmetic's subject matter.

DEFINITION 8 (Russellian translations). *For each type* τ, *the* Russellian translation$_\tau$ *from* PA_2 *to* MT, $(\cdot)_\tau^{\mathcal{R}}$ *is defined as follows, where* φ *and* ψ *are* PA_2-*formulae,* v *is an individual or predicate variable of* PA_2, $\ulcorner x^i \urcorner$ *is* PA_2's i*th individual variable,* $\ulcorner x_{\langle\langle\tau\rangle\rangle}^i \urcorner$ *is* MT*'s i*th type $\langle\langle\tau\rangle\rangle$-variable,* $\ulcorner X_n^i \urcorner$ *is* PA_2's i*th n-ary predicate variable, and* $\ulcorner x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}^i \urcorner$ *is* MT*'s i*th type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-variable*:

(a) $('0')_\tau^{\mathcal{R}} = \ulcorner \boldsymbol{0}_{\langle\langle\tau\rangle\rangle} \urcorner$;

(b) $('N')_\tau^{\mathcal{R}} = \ulcorner \boldsymbol{N}_{\langle\langle\langle\tau\rangle\rangle\rangle} \urcorner$;

(c) $('S')_\tau^{\mathcal{R}} = \ulcorner \boldsymbol{S}_{\langle\langle\langle\tau\rangle\rangle\rangle_2} \urcorner$;

(d) $('=')_\tau^{\mathcal{R}} = \ulcorner =_{\langle\langle\langle\tau\rangle\rangle\rangle_2} \urcorner$;

(e) $(\ulcorner x^i \urcorner)_\tau^{\mathcal{R}} = \ulcorner x_{\langle\langle\tau\rangle\rangle}^i \urcorner$;

(f) $(\ulcorner X_n^i \urcorner)_\tau^{\mathcal{R}} = \ulcorner x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}^i \urcorner$;

(g) $(\ulcorner \beta(\alpha^1, \dots, \alpha^n) \urcorner)_\tau^{\mathcal{R}} = \ulcorner (\beta)_\tau^{\mathcal{R}}((\alpha^1)_\tau^{\mathcal{R}}, \dots, (\alpha^n)_\tau^{\mathcal{R}}) \urcorner$;

(h) $(\ulcorner \neg\varphi \urcorner)_\tau^{\mathcal{R}} = \ulcorner \neg(\varphi)_\tau^{\mathcal{R}} \urcorner$;

(i) $(\ulcorner \varphi \wedge \psi \urcorner)_\tau^{\mathcal{R}} = \ulcorner (\varphi)_\tau^{\mathcal{R}} \wedge (\psi)_\tau^{\mathcal{R}} \urcorner$;

(j) $(\ulcorner \forall v \varphi \urcorner)_\tau^{\mathcal{R}} = \ulcorner \forall (v)_\tau^{\mathcal{R}} (\varphi)_\tau^{\mathcal{R}} \urcorner$.

In Definition 8 is characterized a family of translations, one for each type, so as to reflect UPPER's commitment to the systematic ambiguity across types of the arithmetical vocabulary. The following is the paper's main result (the numbering of the theorems and lemmas stated in the main text reflect the order by which they are proven in Appendixes A–F):

THEOREM 8 (Interpretability). $\mathrm{KQL}_C^{\mathrm{FP}}$ *interprets* PA_2 *via the* Russellian translation $(\cdot)_\tau^{\mathcal{R}}$, *for each type* τ. *More precisely, for every closed theorem* φ *of* PA_2, $(\varphi)_\tau^{\mathcal{R}}$ *is a theorem of* $\mathrm{KQL}_C^{\mathrm{FP}}$, *for every type* τ.

In Definition 11 (Appendix A) is given an axiomatization of PA_2. The proofs of the Russellian translations of all of PA_2's axioms are routine (they will be given in Appendix A)—except for SIC_τ, the Russellian translation of SUCCESSOR IS INJECTIVE:

$$\forall x_{\langle\langle\tau\rangle\rangle}\forall y_{\langle\langle\tau\rangle\rangle}\forall z_{\langle\langle\tau\rangle\rangle}(((\mathbf{N}(x) \wedge \mathbf{N}(y)) \wedge (\mathbf{S}(x,z) \wedge \mathbf{S}(y,z))) \rightarrow x = y). \qquad (\mathbf{SIC}_\tau)$$

In the remainder of this section I'll go through the main ideas involved in the proof of SIC_τ from FP. I leave the details for Appendix A, where a full proof of Theorem 8 can be found.

A crucial result appealed to in R&W's proof of SIC_τ from INFAX is the *equinumerosity lemma*, according to which, for every natural number, if a property $y_{\langle\tau\rangle}$ has it, then any property $z_{\langle\tau\rangle}$ has it if and only if $y_{\langle\tau\rangle}$ and $z_{\langle\tau\rangle}$ are equinumerous. In the context KQL_C it is possible to prove the following strengthening of this lemma—one of the main results on which the KQL_C^{FP}-derivation of SIC_τ relies:

LEMMA 6 (Modal equinumerosity lemma). $\vdash \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \Box\forall y_{\langle\tau\rangle}(x(y) \rightarrow \forall z_{\langle\tau\rangle}(x(z) \leftrightarrow y \approx z)))$.

Two other important results are the following:

LEMMA 7 (Impossible sharing). $\vdash \forall x_{\langle\langle\tau\rangle\rangle}\forall y_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(x) \wedge \mathbf{N}(y) \wedge \Diamond\exists z_{\langle\tau\rangle}(x(z) \wedge y(z))) \rightarrow x = y)$.

LEMMA 8. $\vdash FP_e \rightarrow FP_\tau$.

Lemmas 6, 7 and 10 are proved in Appendix A. On their basis, SIC_τ can be derived in KQL_C^{FP}. To see this, assume, for arbitrarily existing $x_{\langle\langle\tau\rangle\rangle}$ and $y_{\langle\langle\tau\rangle\rangle}$, that they are natural numbers, and that $z_{\langle\langle\tau\rangle\rangle}$ is a successor of both. Then, $z_{\langle\langle\tau\rangle\rangle}$ is a natural number, by the arithmetical axiom according to which the successor of a natural number is itself a natural number. So, FP_τ implies that it could have been that $z_{\langle\langle\tau\rangle\rangle}$ was instantiated by some $w_{\langle\tau\rangle}$. So, by the definition of *successor*, it could have been that there were s_τ and t_τ such that s_τ and t_τ both instantiated $w_{\langle\tau\rangle}$, $w_{\langle\tau\rangle} - s_\tau$ (i.e., the property of being a $w_{\langle\tau\rangle}$ distinct from s_τ) instantiated $x_{\langle\langle\tau\rangle\rangle}$, and $w_{\langle\tau\rangle} - t_\tau$ (i.e., the property of being a $w_{\langle\tau\rangle}$ distinct from t_τ) instantiated $y_{\langle\langle\tau\rangle\rangle}$. In such a case, $w_{\langle\tau\rangle} - s_\tau$ would have been equinumerous with $w_{\langle\tau\rangle} - t_\tau$. So it could have been that there were $P_{\langle\tau\rangle}$ and $Q_{\langle\tau\rangle}$ such that $P_{\langle\tau\rangle}$ instantiated $x_{\langle\langle\tau\rangle\rangle}$, $Q_{\langle\tau\rangle}$ instantiated $y_{\langle\langle\tau\rangle\rangle}$, and $P_{\langle\tau\rangle}$ was equinumerous with $Q_{\langle\tau\rangle}$. So, by the *modal equinumerosity* lemma, it could have been that there was a $P_{\langle\tau\rangle}$ such that $P_{\langle\tau\rangle}$ instantiated both $x_{\langle\langle\tau\rangle\rangle}$ and $y_{\langle\langle\tau\rangle\rangle}$. Hence, $x_{\langle\langle\tau\rangle\rangle} = y_{\langle\langle\tau\rangle\rangle}$, by the *impossible sharing lemma*. So, if FP_τ, then any natural numbers $x_{\langle\langle\tau\rangle\rangle}$ and $y_{\langle\langle\tau\rangle\rangle}$ having the same successor are identical. Hence, SIC_τ, by Lemma 10.

§5. Necessity in $S5PQL_C$ of the true purely arithmetical formulae of MT. The paper's second main result—that the truths of pure arithmetic are all metaphysically necessary—requires getting clearer on what formulae of the typed modal language MT count, from the standpoint of UPPER, as *purely* arithmetical.

A first suggestion is that the purely arithmetical formulae are those formulae of MT which are Russellian translations of formulae of PA_2. Alas, this option is unsatisfactory. For instance, it implies that the Russellian translations of all formulae of *pure second-order logic* (formulae without any distinctively arithmetical vocabulary) express purely arithmetical facts. Among these is, for instance, a formula that intuitively expresses that (i) there is some type $\langle\langle\langle e\rangle\rangle\rangle$-property X with a cardinality greater than some infinite

cardinality; (ii) there is some type $\langle\langle\langle e\rangle\rangle\rangle$-property Y with a cardinality greater than X; and (iii) there is some type $\langle\langle\langle e\rangle\rangle\rangle$-property Z with a cardinality greater than Y. But it is not reasonable to think that, even if true, the conjunction of (i)–(iii) is a *purely arithmetical* fact. After all, this claim seems to have little to do with what is true about the type $\langle\langle e\rangle\rangle$-*natural numbers*. In general, questions concerning how many different infinite cardinalities are instantiated are not under the purview of arithmetic.

Likewise, a type-theoretic version of the *continuum hypothesis* can be formulated in pure second-order logic.[26] But even if the *continuum hypothesis* turns out to be neither necessarily true nor necessarily false, that does not show that some truths of pure arithmetic are not necessarily true. For, arguably, neither the *continuum hypothesis* is a truth of pure arithmetic nor its negation is.

A better option arises once PA_2's first-order quantifiers are interpreted as implicitly *restricted* to the *natural numbers*, as is already customary to do in arithmetical theories. Consider the following class of translations from PA_2 to MT:

DEFINITION 9 (Arithmetical translations). *For each type τ, the* arithmetical translation$_\tau$ *from PA_2 to MT, $(\cdot)_\tau^{\mathcal{A}}$, is defined exactly as the* Russellian translation$_\tau$, *except that* $(\ulcorner \forall v \varphi \urcorner)_\tau^{\mathcal{A}} = \ulcorner \forall (v)_\tau^{\mathcal{A}} (\boldsymbol{N}((v)_\tau^{\mathcal{A}}) \to (\varphi)_\tau^{\mathcal{A}}) \urcorner$.

The proposal is then that, for each type τ, the formulae of MT which express purely arithmetical facts, as they concern the type $\langle\langle\tau\rangle\rangle$-natural numbers, are the images of the *arithmetical translation$_\tau$*:

DEFINITION 10 (τ-Arithmetical formulae). *A formula φ of MT is a closed τ-arithmetical formula if and only if $\varphi = (\psi)_\tau^{\mathcal{A}}$, for some closed formula ψ of PA_2.*

Each τ-arithmetical formula is obtained from some PA_2-formula φ by indexing φ's variables with the type $\langle\langle\tau\rangle\rangle$, substituting the primitives of PA_2 occurring in φ by their Russellian characterizations, and restricting the type $\langle\langle\tau\rangle\rangle$-quantifiers to natural numbers.

We are now able to state the paper's second main result:

THEOREM 9. $\vdash_{\text{S5PQL}_C} (\varphi)_\tau^{\mathcal{A}} \to \Box(\varphi)_\tau^{\mathcal{A}}$, *for every closed formula φ of PA_2 and every type τ.*

Theorem 9 is a corollary of a more general result, Lemma 28 (both proven in Appendix C). Say that a relation is *arithmetical* if and only if it exists necessarily and necessarily, whatever it holds of, those things are natural numbers and it necessarily holds of them. Then, Lemma 28, proven by induction on the complexity of formulae, establishes that if the parameters occurring in $(\varphi)_\tau^{\mathcal{A}}$ are all natural numbers or arithmetical relations, then $(\varphi)_\tau^{\mathcal{A}} \to \Box(\varphi)_\tau^{\mathcal{A}}$. Theorem 9 then follows.

For most cases, Lemma 28's proof is straightforward, owing especially to the following facts: (i) natural numbers are necessarily natural numbers; (ii) if a natural number is a successor of another, then it is necessarily so; (iii) identical natural numbers are necessarily identical; and (iv) if an arithmetical relation holds of some natural numbers, then it necessarily holds of them.

The one tricky case is when φ is a universal quantification into predicate position. The proof of this case relies on the following observation: that if some n-ary relation R is such that $(\varphi)_\tau^{\mathcal{A}}$, then there is some n-ary *arithmetical* relation U such that $((\varphi)_\tau^{\mathcal{A}})_U^R$. The witness arithmetical relation U is characterized in terms of the plurality uu of

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[26] See [63, sec. 5.1.2].

those n-tuples (understood as haecceitistic relations) of natural numbers of which R obtains. It is the relation of being $x_1, ..., x_n$ such that this n-tuple is one of the n-tuples in uu.

Owing to pluralities' rigidity, and to the natural numbers' necessary existence, U can be shown to be an arithmetical relation. And from this observation it is shown that if every relation R is such that $(\varphi)_\tau^{\mathcal{A}}$, then, necessarily, every relation R *whatsoever* (and not just every *arithmetical* relation) is such that $((\varphi)_\tau^{\mathcal{A}})$, thus concluding Lemma 28's proof.

Once Theorem 9 is conjoined with the view that the truths of pure arithmetic are those truths expressible by closed τ-arithmetical formulae, it implies that all truths of pure arithmetic are necessary truths expressible in a pure modal type theory, from which it follows, given LOGICALITY, that all truths of arithmetic are logically true—thus vindicating the form of logicism encapsulated in UPPER.

Theorem 9's proof, in S5PQL$_C$, involves principles of the logic of *plurals*. So, should a proponent of UPPER also be committed to plural logic really being *logic*? I am sympathetic to this view, and anticipate that purely logical, neoRussellian characterizations of the primitives of other mathematical domains might have to rely on plural resources. Notwithstanding, UPPER's proponents need not be committed to the view in order to have available a successful defense of the logical truth of all truths of pure arithmetic premised on Theorem 9. It suffices that plural logic's principles be (plainly) true. For, if they are, then every truth of pure arithmetic is a necessary truth expressible by a formula of pure modal type theory. Even if plural logic isn't really *logic*, it can still be used to show that some truths are logically true.[27]

§6. **In favor of the necessary truth of every theorem of** S5PQL$_{SA,C}^{FP}$. Given Theorem 9, and provided that Definition 10 pins down the purely arithmetical formulae, NECESSITY OF ARITHMETIC will have been established if all theorems of S5PQL$_C$ can be shown to be true. Further, given Theorem 8, NECESSITY OF FP and INTERPRETABILITY OF ARITHMETIC will have been established if it can be shown that KQL$_C^{FP}$'s theorems are all metaphysically necessary. And a couple of auxiliary results (Lemmas 36 and 40, proven in Appendix E) presuppose schema SA.[28]

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[27] A reviewer observes that a *higher-order* version of FINITE HUME'S PRINCIPLE—according to which if properties F or G have no more than finitely many instances, then the number of Fs is identical to the number of Gs if and only if the Fs are equinumerous with the Gs—is provable in the type-theoretic system here adopted. This is indeed so. For instance, the 'number of'-relation (relating properties to the finite cardinalities which are their number) can be characterized as follows: $\# := \lambda x_{\langle\langle\tau\rangle\rangle} y_{\langle\tau\rangle}(\mathbf{N}(x) \land x(y))$. Then, the following is one way of formalizing the principle, where $\approx$ stands for the equinumerosity relation (characterized in Definition 14 in Appendix A), and $Inf[\alpha]$ states that α has infinitely many instances (a notion characterized in Definition 25 in Appendix E):

Higher-order finite Hume's principle$_\tau$: $\forall x_{\langle\tau\rangle}\forall y_{\langle\tau\rangle}(\neg(Inf[x] \land Inf[y]) \to$
$(\exists n_{\langle\langle\tau\rangle\rangle}(\#(n,x) \land \#(n,y)) \leftrightarrow x \approx y))$.

Since HIGHER-ORDER FINITE HUME'S PRINCIPLE is already provable in system QL$_C$ (characterized in Section 3), a system without any modal resources, its proof will be left to the reader.

[28] Several proofs in a previous draft relied on SA. I thank an anonymous reviewer for inquiring whether those uses of SA were essential to the results. In all but a couple of cases, they were not.

Accordingly, in this section I'll start by offering some reasons for thinking that the result of prefixing any instance of any axiom of $\text{S5PQL}_{\text{SA,C}}$ with any sequence of necessity operators and universal quantifiers, in any order, is true. From this it will follow that every theorem of $\text{S5PQL}_{\text{SA,C}}$ is true, assuming the truth-preservingness of *modus ponens*—as shown in Lemma 30 in Appendix D. The truth of all theorems of S5PQL_{C} and $\text{KQL}_{\text{C}}^{\text{FP}}$, and of all instances of SA, is then obtained as a corollary.

I'll then offer offering some considerations in favor of FP. Since it is a theorem of $\text{S5PQL}_{\text{SA,C}}$ that if FP is true, then the result of prefixing it with any sequence of necessity operators and universal quantifiers, in any order, is also true (a straightforward corollary of Lemma 15, proven in Appendix B), this will establish the necessary truth of all theorems of $\text{S5PQL}_{\text{SA,C}}^{\text{FP}}$ (owing to $\text{S5PQL}_{\text{SA,C}}$'s Nec rule and axiom K).

6.1. Necessity of $\text{S5PQL}_{\text{SA,C}}$. System K constitutes the *weakest* normal modal logic. It is sound for metaphysical necessity. Indeed, and citing Williamson [81, p. 44], '(...) most metaphysicians accept S5 [a system stronger than K] as the propositional modal logic of metaphysical modality (...)'.[29] The PMT-instances of schemata $\forall 1$, $\forall 2$ and $\forall E$ are all commonly taken to be metaphysically necessary. Schema SA encodes SERIOUS ACTUALISM (so called by Plantinga [53], called 'being constraint' by Williamson [81]), a principle according to which falling under a (genuine) predicate implies being identical to something. Since I have little to add to the defense of SERIOUS ACTUALISM offered in [38], I won't here further argue for this principle.[30]

Axiom schemata $\forall =$ and Ind are commonly thought to be among the principles governing *identity*: everything is self-identical, and identicals are indiscernible. Notwithstanding, some might want to take issue with Ind in the present context. For Ind is known to have false instances when at least one of the expressions flanking the identity predicate is a nonrigid designator, and some might worry that *complex predicates* are nonrigid designators.

As an example of a failure of Ind, consider the sentence 'if Joe Biden is the US president, then if it is necessary that, if he exists, then Joe Biden is Joe Biden, then it is necessary that, if he exists, Joe Biden is the US president'—'$b = f(a) \rightarrow (\Box(\exists x(x = b) \rightarrow b = b) \rightarrow \Box(\exists x(x = b) \rightarrow b = f(a)))$'—where '$b$' and '$a$' are individual constants, assumed to be rigid designators, standing for, respectively, Biden and the US, and 'f' is a functional term standing for 'the president of'-function, and which I'll assume for the sake of exposition is a nonrigid designator.

At each world of evaluation w, '$f(a)$' picks out whoever is the US president at w. Since the US could have had different presidents, there will be worlds of evaluation at which '$f(a)$' picks out distinct individuals. Now, '$b = f(a)$' and '$\Box(\exists x(x = b) \rightarrow b = b)$' are both true at the actual world. Yet, '$\Box(\exists x(x = b) \rightarrow b = f(a))$' is not true at the actual world, since there is at least one world of evaluation w such that Joe Biden exists at w and '$f(a)$' designates Kamala Harris at w. So, '$\exists x(x = b) \rightarrow b = f(a)$' is not true at w. So, Ind has false instances when the identity signed is flanked by some nonrigid designators.

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[29] Though see [16, 62] for dissenting voices.

[30] Though see [24] for the tension between SERIOUS ACTUALISM and CONTINGENTISM at all orders, and for discussion of a principle somewhat similar to SERIOUS ACTUALISM, the APPLICATIVE BEING CONSTRAINT, according to which if a relation obtains between things, then the relation exists. Since this is a more controversial principle (for instance, it is rejected by Stalnaker [72–74]), and none of the results here presented rely on it, I've opted to leave it out.

The worry presently under consideration is thus that complex predicates semantically function in a manner akin to functional terms, being nonrigid designators themselves. But this worry is unfounded. On the present conception, complex predicates *are* rigid designators—provided that they designate anything at all. Their presence in the language does not lead to there being false instances of Ind.[31] Briefly, and roughly, in my view the semantic value of each complex predicate $\ulcorner \lambda u^1 \ldots u^n \varphi \urcorner$ consists in a relation R compositionally determined in terms of the *propositional function* that is formula φ's semantic value; R obtains, at each world w, of $u^1, \ldots, u^n$, just in case the semantic value of the formula φ maps $u^1, \ldots, u^n$ to a proposition which is true at w. So, $\ulcorner \lambda u^1 \ldots u^n \varphi = \alpha \urcorner$ is true at a world w if, at w, *this relation R* is identical to the denotation of α, and is not true at w otherwise. There is no world-relative denotation involved.

This is not, however, the place to further develop this view on the semantics of complex predicates.[32] For the present purposes, suffice it to say that in Appendix F is presented a standard model-theoretic treatment of complex predication on which the value of a complex predicate in a model is not world-dependent, and which validates Ind. At the very least, this shows the *coherence* of the view that complex predicates are rigid designators, if they designate anything at all.

Turning now to schemata Ab1 and Ab2, Ab1 is fairly uncontroversial. If a predication involving a complex predicate is true, then the corresponding formula is also true. Ab2 is also fairly uncontroversial, as it is a weakening of more common principles such as (i) that a complex predication is true if the corresponding formula is true, and (ii) that a complex predication is true if the corresponding formula is true *and* the entities of which the complex predicate is predicated all exist.

One reason to prefer Ab2 to both (i) and (ii) concerns the following theses—one for each type τ:

Contingentism$_\tau$: There could have been some x_τ that could have been nothing. $\Diamond \exists x_\tau \Diamond \neg \exists y_\tau (x = y)$.

CONTINGENTISM—i.e., CONTINGENTISM$_e$—is seemingly plausible. For instance, prima facie, Joe Biden could have been nothing. In addition, some philosophers find CONTINGENTISM$_\tau$ attractive also whenever $\tau \neq e$.[33] But principle (ii) (as well as (i)) is inconsistent with the following theses: (a) the truth of a predication implies the being of the relation that the predicate stands for; (b) the property of being distinct from a given x_τ exists only if that particular x_τ exists;[34] and (c) CONTINGENTISM$_\tau$. To see why, assume, for instance, that Biden is a contingent being (and so, is a witness

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[31] If a complex predicate fails to have a designation, then if it occurs in the equation which is the antecedent of the relevant instance of Ind, then that instance is vacuously true, as it will then have a false antecedent.

[32] For a model-theoretic development of this idea, see [72, Appendix B]. See also [58] for a possible role of propositional functions in problem-solving in modal metaphysics.

[33] See, e.g., [1, 53, 54, 70]. The view that CONTINGENTISM$_\tau$ is true whenever $\tau \neq e$ is called 'higher-order contingentism'.

[34] This is an instance of a general view advocated by some contingentists according to which the being of some higher-type entities depends on the being of lower type entities. For simplicity, I focus on *anti-haecceities* as higher-type entities whose being is dependent on that of lower type entities—where, for each x_τ, x_τ's anti-haecceity is the property of being distinct from x_τ.

to (c)). Then, and where 'h' stands for Kamala Harris, '$h \neq b$' will be true at any world w at which Kamala Harris exists but Joe Biden doesn't. Then, by (ii), '$\lambda x(x \neq b)(h)$' is true at w. So, by (a), $\lambda x(x \neq b)$ exists at w. But this conflicts with (b), since Joe Biden doesn't exist at w. By contrast with principle (ii), Ab2 leads to no contradiction when conjoined with (a)–(c). To conclude, then, Ab1 and Ab2 are extremely plausible.

According to Comp_C, there is a relation that a complex predicate stands for provided that the values of the parameters occurring in it all exist. One rationale in favor of the metaphysical necessity of its instances goes via a principle of 'semantic determination' according to which (i) the semantic value of a complex predicate $\lambda \overline{u} \varphi$ with parameters $t^1, ..., t^n$ exists at world w if the propositional function which is the semantic value of φ relative to an assignment of values to $t^1, ..., t^n$ exists at w, and (ii) this propositional function exists at w just in case the semantic values of the terms occurring in φ, alongside with the values of the parameters $t^1, ..., t^n$, all exist at w. But this is just what is required by Comp_C, assuming that the semantic values of MT's identity predicates are all necessary beings.[35]

Principle Comp_C is jointly consistent both with all instances of CONTINGENTISM$_\tau$ and with all instances of NECESSITISM$_\tau$ where, for each type τ, NECESSITISM$_\tau$ is CONTINGENTISM$_\tau$'s contradictory—the thesis that necessarily, every x_τ is necessarily something.[36] The subscripted 'C' in 'Comp_C' signals its compatibility with a contingentist world-view. Williamson [81] ultimately argued that contingentists and necessitists alike should find Comp_C to be too *weak* for some of the development of modal mathematics. Be that as it may, Theorem 8 shows that Comp_C suffices for the interpretability of arithmetic in modal type theory.[37]

The instances of the plural principles $\prec$ I, PluComp and $=$I$_P$ are all (typed versions of) theorems of standard (free) modal plural logic.[38] Concerning $\Box$E$_P$, given a conception of pluralities as, in some sense, being nothing over and above their members, it is an uncontroversial thesis that necessarily, it was once the case that necessarily, if the *uu* exist and every one of the *uu* existed then, then the *uu* also existed then.[39] And it is a consequence of this thesis that if the *vv* exist and all the entities which are among the *vv* necessarily exist, then the *vv* necessarily exist.

So, it is reasonable to think not only that all of S5PQL$_{SA,C}$'s axioms are metaphysically necessary, but also that the results of prefixing them with any sequences of necessity operators and universal quantifiers, in any order, are all true. But then, all theorems of S5PQL$_{SA,C}$ are true—assuming that MP is truth-preserving, as is all but reasonable to assume—owing to Lemma 30 (as already mentioned). So, *all* theorems of S5PQL$_{SA,C}$ are metaphysically necessary, due to the S5PQL$_{SA,C}$'s Nec rule. This establishes NECESSITY OF ARITHMETIC on the assumption that Definition 10 does pin down the purely arithmetical formulae.

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[35] For further considerations in favor of Comp_C, see, e.g., [37, 73] and [26, sec. 2].

[36] NECESSITISM$_e$ is known in the literature simply as 'necessitism'. 'HIGHER-ORDER NECESSITISM' is the view that NECESSITISM$_\tau$ is true whenever $\tau \neq e$.

[37] For a reply, on behalf of (higher-order) contingentists, to Williamson's arguments, see [37]. Another notorious view available is 'hybrid contingentism', the conjunction of CONTINGENTISM$_e$ with higher-order necessitism. For an overview on the NECESSITISM vs. CONTINGENTISM debate, and its higher-order correlates, see, e.g., [67].

[38] See, e.g., [21].

[39] This thesis can be regimented by resorting to the Vlach operators '↑' ('once') and '↓' ('then') in the following way: $\Box \uparrow \Box ((E[uu] \land \forall u(u \prec uu \rightarrow \downarrow E[u])) \rightarrow \downarrow E[uu])$.

6.2. The truth of FINITARY PLENITUDE. As it turns out, FP implies its metaphysical necessity, by S5QL$_C$-reasoning (see Appendix B):

LEMMA 15. $\vdash_{S5QL_C} FP_\tau \to \Box FP_\tau$.

So, given what has been argued so far, INTERPRETABILITY OF ARITHMETIC will have been established provided that FP can be shown to be true. Accordingly, in the remainder of this section I'll put forward a couple of considerations in favor of FP.

The first of these—a minor adaptation of an argument of Williamson's [81, p. 144]—goes as follows. It is plausible to think that there could have been *n* donkeys, for every finite cardinality *n*. The contradictory hypothesis—that the metaphysical laws imply the existence of an upper bound on the finite cardinality that possibly numbers *being a donkey*—is presumably *absurd*. So, every finite cardinality *n* could have been instantiated. Hence, FP is true.

Even if 'easy,' the argument is compelling. In addition, anyone resisting it must also be committed to the metaphysical laws being such that, for every property whatsoever, there is an upper bound on the finite cardinality that numbers it: there is a bound on the finite cardinality that possibly numbers *being a quark*, *being a spatiotemporal location*, *being one of someone's hairs*, etc. But such a view is surely absurd. It is absurd to think that there is a natural number *n* such that no property whatsoever could have had more than *n*-many instances. In any case, at this point the onus should be on the opponent of FP to explain why it should be rejected, not on FP's proponent.

FINITARY PLENITUDE is also a consequence of prominent *principles of plenitude* (indeed, a failure to imply FP would presumably reveal the weakness of any such principle). In what follows I'll focus on Lewis's [43] popular principle of RECOMBINATION,[40] according to which, for any objects in any worlds, there is a world that contains any number of duplicates of all those objects, 'size and shape permitting.'

There has been much discussion concerning how best to formulate a more precise version of RECOMBINATION, and even whether the principle is consistent.[41] Regardless, those attracted to at least its spirit, if not its letter, should find the following, much weaker principle close to a triviality:

Generation: There is an attribute of individuals numbered by some finite cardinality which, necessarily, no matter what finite cardinality numbers it, its successor could have numbered it.

$$\exists x_{\langle e\rangle}\exists y_{\langle\langle e\rangle\rangle}((\mathbf{N}(y)\wedge y(x))\wedge\forall w_{\langle\langle e\rangle\rangle}\forall z_{\langle\langle e\rangle\rangle}((\mathbf{N}(w)\wedge\mathbf{S}(w,z))\to\Box(w(x)\to\Diamond z(x)))).$$

GENERATION is a consequence of the claim that, no matter how finitely many duplicates of a chosen individual there are, there could have been one more. Arguably, it will be amongst RECOMBINATION's consequences independently of what its best regimentation is, and while being substantially weaker than it. But FP is derivable from GENERATION in S5QL$_{SA,C}$ (a proof of this result can be found in Appendix E):

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[40] A different principle of plenitude in the philosophy of mathematics is Balaguer's [3, 4] PLENITUDINOUS PLATONISM, according to which every consistent set of mathematical axioms describes a universe of mathematical objects. This principle is, however, at odds with UPPER, as it seemingly postulates the *actual* existence of the natural numbers as *individuals*. Further, it implies the *actual* infinitude of the individuals, even though proponents of UPPER wish to remain neutral on this issue.

[41] See, e.g., [7, 12, 17, 22, 50, and 56].

LEMMA 36. $\vdash_{\text{S5QL}_{\text{SA,C}}}$ GENERATION $\rightarrow$ FP.

Thus, those attracted to the spirit, if not the letter, of Lewis's principle of RECOMBINATION—so that they advocate at least GENERATION—should also accept FINITARY PLENITUDE.

Hence, FP is not only *minimally demanding*—the conclusion of 'easy' but *compelling* arguments—but also a *consequence* of GENERATION, and so of a popular, general, conception of what is possible. There are, then, good reasons for thinking that FP is true, and so metaphysically necessary.

To those that remain unconvinced, I would just like to point out that FP is indeed significantly weaker than INFAX. On the one hand, FP is consistent with the necessary falsehood of INFAX even in the context of the deductive system $\text{S5PQL}_{\text{SA,N}}$—a system just like $\text{S5PQL}_{\text{SA,C}}$ except that it has among its axioms all instances of the *necessitist* comprehension principle Comp_{N}:

$$E[\beta]^{42} \qquad\qquad (\text{Comp}_{\text{N}})$$

Comp_{N} is a *necessitist* comprehension principle for modal type theory insofar as NECESSITISM_τ is derivable in KQL_{N}, for every type τ.[43] A model witnessing the consistency of $\text{S5PQL}_{\text{SA,N}}^{\text{FP}}$ with $\square\neg\text{INFAX}$ is given in Lemma 45 in Appendix F. In this context, it should also be mentioned that $\text{S5PQL}_{\text{SA,C}}^{\text{FP}}$ is consistent with $\text{CONTINGENTISM}_\tau$, for all types τ, and $\square\neg\text{INFAX}$. This is shown in Lemma 44 in Appendix F. So, FP is furthermore consistent with a thoroughly contingentist metaphysics.

On the other hand, FP is derivable from INFAX already in TQL_{C}, the result of augmenting KQL_{C} with the modal axiom T (see Lemma 42 in Appendix E). So, PA_2's interpretability in $\text{KQL}_{\text{C}}^{\text{FP}}$ makes the case for UPPER stronger than the case for a logicism along Russellian lines has arguably ever been.

§7. **Other approaches.** By way of conclusion, I will briefly compare and contrast UPPER with four seemingly related philosophies of arithmetic: Hellman's [31] MODAL STRUCTURALISM, Hodes's [33] ALTERNATIVE THEORY, Roeper's [60] VINDICATION OF LOGICISM, and Linnebo and Shapiro's [46] ARITHMETICAL POTENTIALISM.

7.1. MODAL STRUCTURALISM. Hellman's [31] MODAL STRUCTURALISM is sometimes viewed as a logicist programme couched in *modal* and *higher-order* resources. Applied to arithmetic, it is the view that there are no natural numbers, and that the real content of each arithmetical theorem φ is given by the following formula of pure second-order modal logic (where (i) $Ax(\text{PA}_2)$ is the conjunction of PA_2's axioms, (ii) 'X' is a unary predicate variable, 'S' is a binary predicate variable and 'z' is an individual variable, and (iii) $Ax(\text{PA}_2)_{X\ R\ z}^{N\ S\ 0}$ is the result of replacing every occurrence of 'N', 'S' and '0' in $Ax(\text{PA}_2)$ by, respectively, 'X', 'R' and 'z', and likewise for $\varphi_{X\ R\ z}^{N\ S\ 0}$): $\Diamond\exists X\exists R\exists z(Ax(\text{PA}_2)_{X\ R\ z}^{N\ S\ 0}) \wedge \square\forall X\forall R\forall z(Ax(\text{PA}_2)_{X\ R\ z}^{N\ S\ 0} \rightarrow \varphi_{X\ R\ z}^{N\ S\ 0})$. That is, it is possible that some property, relation and individual 'play the role' of *being a natural number*, *successor*, and *zero* with respect to PA_2's axioms, and necessarily, whatever plays those roles satisfies $\varphi_{X\ R\ z}^{N\ S\ 0}$.

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[42] Where β ranges over type $\langle\tau^1,\dots,\tau^n\rangle$-terms, for every positive integer n and all types $\tau^1,\dots,\tau^n$.

[43] See [81, chap. 6] and [38] for defenses of Comp_{N}.

MODAL STRUCTURALISM thus shares with UPPER the views that the real content of each arithmetical statement can be expressed by a formula of a pure higher-order modal language, and that logic includes higher-order modal resources. Notwithstanding, a respect in which UPPER and MODAL STRUCTURALISM clearly differ is that modal structuralists *reject* the existence of natural numbers. By contrast, UPPER is committed to their existence. Another significant difference is that MODAL STRUCTURALISM is presumably not committed to *higher-orderism*. For Hellman [31] subscribes to Quine's view that 'some predication relation is hidden in the notation '$F(x)$', once we admit F as a quantifiable variable', and speaks of the values of second-order variables as 'classes' and 'extensions', taking them to be identical if coextensional.

Concerning MODAL STRUCTURALISM's logicist character, under Hellman's specific way of interpreting its theorems PA_2 is interpretable in second-order modal logic only if it is the case that $\Diamond \exists X \exists R \exists z (Ax(PA_2)_X^N {}_R^S {}_z^0)$, a claim true only if $\Diamond INFAX$ is. So, MODAL STRUCTURALISM's interpretability result affords a reduction of arithmetic to logic only provided that $\Diamond INFAX$ is logically true. For this reason, UPPER seemingly does at least as well as MODAL STRUCTURALISM vis-à-vis the two theories' respective claims to logicality. For, at least in the context of $S5PQL_{SA,N}$ and weaker systems, FP is consistent with both $\Diamond INFAX$ and $\neg \Diamond INFAX$. By contrast, $\Diamond INFAX$ implies FP already in the context of $S5QL_C$ (this is a straightforward consequence of Lemma 15 in Appendix B, and Lemma 42 in Appendix E). We can thus conclude that if $\Diamond INFAX$ is logically true, so is FP, whereas the logical truth of FP does not appear to imply $\Diamond INFAX$'s logical truth.[44]

In Hellman's [29] more recent development of MODAL STRUCTURALISM, to which I'll refer to as 'MODAL STRUCTURALISM+', higher-order quantification is replaced by a combination of *plural quantification* and *mereology*, owing to Hellman's qualms about the nominalist pedigree of classes (which, recall, he takes to be the values of second-order variables). The differences between UPPER and MODAL STRUCTURALISM+ are even greater than those between UPPER and MODAL STRUCTURALISM. In particular, it is unreasonable to think that MODAL STRUCTURALISM+ affords a reduction of arithmetic to logic. For *parthood* is commonly regarded as a *non*logical notion. Relatedly, MODAL STRUCTURALISM+'s adopted mereological theory is *classical extensional mereology*, which includes the strong UNRESTRICTED FUSION schema according to which for any satisfied condition, there is a fusion of those things that satisfy it. Yet, there arguably is no such thing as the fusion of my nose with the Eiffel Tower, in which case the UNRESTRICTED FUSION schema has false instances. Further, even if the UNRESTRICTED FUSION SCHEMA turns out to be *possibly true*, I for one find it unreasonable to think that it is necessarily true. In any case, it arguably isn't a logical truth, and so MODAL STRUCTURALISM+ does not found arithmetic in logic.[45]

Still by way of comparison, it will be useful to consider how MODAL STRUCTURALISM and MODAL STRUCTURALISM+ account for arithmetic's *applicability*. For instance,

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[44] MODAL STRUCTURALISM's account of real analysis and set theory requires postulating the truth of INFAX, not just of $\Diamond INFAX$. Since the present discussion concerns only the prospects of MODAL STRUCTURALISM as a form of logicism about *arithmetic*, not about real analysis or set theory, I have not taken issue with this aspect of the view.

[45] If MODAL STRUCTURALISM is chosen over MODAL STRUCTURALISM+, then why not avail oneself of languages of higher-orders? If one does so, then the interpretability result here offered will be available. But then, why not take the natural numbers to be the finite cardinalities that one's theory will then be committed to—thus embracing UPPER and leaving behind MODAL STRUCTURALISM?

according to Hellman, to attribute the number 2 to the property *is a moon of Mars* really consists in taking something like (1) to be true:

$$\Box\forall X\forall R\forall z\forall x((\text{the actually true material facts are true} \land Ax(\mathtt{PA_2})^{N\ S\ 0}_{X\ R\ z} \land R(z,x))$$
$$\rightarrow \exists Y(Bij[Y,\lambda u(u=z \lor u=x),\lambda u(\text{Moon of Mars}(u))]))). \tag{1}$$

According to (1), necessarily, for every property X, relation R and individuals z and x, if the actually true material facts are true, as well as $Ax(\mathtt{PA_2})^{N\ S\ 0}_{X\ R\ z}$, and x is 'the successor' of z, then there is a bijection between the properties *is identical to z or to x* and *is a moon of Mars*.

The condition that the *actually true material facts be true* is required for a number of reasons. For instance, if there is some possibility at which Mars has *three* moons rather than *two* and there are some individuals arranged in an ω-sequence, then (2) will be false:

$$\Box\forall X\forall R\forall z\forall x((Ax(\mathtt{PA_2})^{N\ S\ 0}_{X\ R\ z} \land R(z,x)) \tag{2}$$
$$\rightarrow \exists Y(Bij[Y,\lambda u(u=z \lor u=x),\lambda u(\text{Moon of Mars}(u))]))).$$

But if it is furthermore required that the actually true material facts be true in the possibility in question, then the case where Mars has three moons rather than two is ruled out as a counterexample.

The *nonvacuous* true of (1) requires that it be possible for both the actually true material facts to be true and for some individuals to be arranged in an ω-sequence—a possibility postulated by Hellman. The nonvacuous truth of possibility claims of number (e.g., that Mars could have had three moons) likewise requires that, for every possibility, it be possible for the material facts true at *that* possibility to be composible with there being some individuals arranged in an ω-sequence. But while some might be eased into thinking that $\Diamond\textsc{InfAx}$ is (logically) true, the claim that for every possibility there is a 'shadow possibility' in which the material facts are the same and yet there are also infinitely many individuals is highly controversial. As I see it, this claim is almost (if not exactly as) controversial as the claim that $\textsc{InfAx}$ is *necessarily* true.

So, with respect to how to account for the applicability of arithmetic, MODAL STRUCTURALISM's contrast with UPPER couldn't be greater. From the standpoint of UPPER's proponents, for Mars to have two moons just is for the property $2_{\langle\langle e\rangle\rangle}$ to be had by the property *is a moon of Mars*. Indeed, it is the recognition of the natural numbers as finite cardinalities—the main contrasting point between UPPER and MODAL STRUCTURALISM—that makes it so easy for UPPER's proponents to accommodate the applicability of arithmetic in what appears to be a straightforward manner.[46]

7.2. The ALTERNATIVE THEORY. According to Hodes [32, 33], arithmetical sentences have truth-conditions which concern what holds of finite cardinality (type $\langle\langle e\rangle\rangle$)

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[46] Hellman holds that the modality in terms of which MODAL STRUCTURALISM is formulated is *logical modality* rather than *metaphysical modality*. This might appear to make it more palatable to accept that there are the 'shadow possibilities' postulated by Hellman. Yet, it is rather unclear what is meant by 'logical possibility', this being a major difficulty confronting the modal structuralist project, as acknowledged by Hellman numerous times. Thus, with respect to how to account for the applicability of arithmetic, the finitary upper logicist story seems much cleaner and more plausible.

properties. These truth-conditions are expressible solely in terms of logical expressions, and the ones which obtain are expressible by logical truths. But since arithmetic itself is formulated via numerals, which appear to be individual (type e) terms, a semantic story is required explaining how arithmetical sentences get to have truth-conditions concerned not with individuals but rather with what is true of the finite cardinality properties.

Hodes proposes a semantic account formulated via a ternary *designation relation*. Let a *representor* be a 1–1 function from finite cardinality properties to individuals. Then, the ternary *designation relation* relates representors and expressions to worldly entities. Relative to different representors, the same expression—in particular, the same arithmetical expression—may designate different entities.

Specifically, relative to each representor the designation of 'N'—the 'is a natural number' predicate—is the representor's range. As to the designation of numerals, Hodes takes each numeral to be shorthand for a singular term formulated in terms of '#'—the *number of*-operator. '0' is shorthand for '#$(\lambda x_e(\bot))$, and $\ulcorner S(n) \urcorner$ is shorthand for $\ulcorner \#(\lambda x_e(x_e \leq n) \vee \neg N(n)) \urcorner$. The designation of '$\leq$'—the 'is less-than-or-equal to' predicate (applicable to individuals)—relative to each representor is specified in terms of the *less-than-or-equal-to* relation between properties of properties of individuals: '$\leq$' designates, relative to each representor f, that relation that obtains between x_e and y_e if and only if there are cardinality properties $Q_{\langle\langle e \rangle\rangle}$ and $R_{\langle\langle e \rangle\rangle}$ such that $f(Q_{\langle\langle e \rangle\rangle}) = x_e$, $f(R_{\langle\langle e \rangle\rangle}) = y_e$, and $Q_{\langle\langle e \rangle\rangle}$ is *less-than-or-equal-to* $R_{\langle\langle e \rangle\rangle}$. The number of-operator # designates, relative to each representor f, a function from properties of individuals to individuals which maps each property $X_{\langle e \rangle}$ to an individual y_e if and only if there is some cardinality property $Q_{\langle\langle e \rangle\rangle}$ such that $f(Q_{\langle\langle e \rangle\rangle}) = y_e$ and $X_{\langle e \rangle}$ falls under $Q_{\langle\langle e \rangle\rangle}$.

Truth relative to a representor is defined in a standard manner. Relative to each representor, the designation of a numeral can then be seen as representing a particular finite cardinality property, and arithmetical sentences are true or false in virtue of facts about individuals which can be seen as 'encoding' facts about finite cardinality properties.

The crux of Hodes's semantic account is that plain truth consists of truth relative to *every* representor and plain falsehood of falsehood relative to *every representor*. This supervaluationist conception of truth thus 'abstracts away' from any contribution that specific individuals and relations between them may make to the truth of arithmetical sentences. What remains are truth-conditions concerned solely with the finite cardinality properties. Thus, a sentence such as '$7 + 5 = 12$' is true owing not to any particular *individuals* standing in any particular relation, but instead owing to the finite cardinality properties *exactly seven*$_{\langle\langle e \rangle\rangle}$, *exactly five*$_{\langle\langle e \rangle\rangle}$ and *exactly twelve*$_{\langle\langle e \rangle\rangle}$ standing in the *addition* relation.

Among other things, Hodes's semantic account yields a prima facie elegant solution to the Caesar problem for Fregean and NeoFregean logicists. While relative to each representor '7' designates a particular individual, and 'Caesar = 7' will be true relative to those representors assigning Caesar to *exactly seven*$_{\langle\langle e \rangle\rangle}$, and false relative to the remaining ones, this sentence is neither true nor false.

Yet, and as acknowledged by Hodes [32], this semantic account presupposes INFAx's truth. For instance, if there are only finitely many individuals, then there are *no representors*, owing to there being no 1–1 function mapping the finite cardinality properties to the individuals. After all, if there are only finitely many individuals, then even if the finite cardinality properties are themselves finitely many, they will be more

numerous than the individuals (if there are only *n*-many individuals, there are at least $(n + 1)$-many finite cardinality properties, since *having zero instances* is a cardinality property). So, since truth is truth relative to every representor, if there are only finitely many individuals, then Hodes's simple account of the semantics of the language of arithmetic will imply the truth of every arithmetical sentence whatsoever—including obvious falsehoods such as '0 = 1' and 'every natural number is identical to 0'.

Even though the semantic account offered in [32] presupposes the truth of INFAx, Hodes [32] is of the opinion that 'Arithmetic should be able to face boldly the dreadful chance that in the actual world there are only finitely many objects'. Further, and as has previously been discussed, it is difficult to see how the INFAx can count as a logical truth.

In the last couple of paragraphs of [32] it is suggested to replace INFAx by FP. Indeed, Hodes concludes by advocating the view that 'mathematics is higher-order *modal* [our emphasis] logic', with the development and defense of this view being left for another occasion. It is thus not unreasonable to take a view along the lines of UPPER to be among those advocated in [32]. Notwithstanding, no *arguments* for the claim that 'mathematics is higher-order modal logic', or *logico-mathematical results* potentially figuring in such arguments, are offered in [32].

In [33] is undertaken the task of developing a model theory for the language of arithmetic which is both neutral on the truth of the INFAx and capable of accounting for how we are able to speak as if numbers were individuals, given a commitment to the truth-conditions of arithmetical sentences concerning what is true of the finite cardinality properties. In this model theory, which presupposes the truth and logicality of FP, representors may map finite cardinality properties to individuals not all of which need actually exist (i.e., not all of which need be in the domain of the 'actual world' of a model). The underlying idea is that arithmetical have truth-conditions which concern finite cardinality properties in virtue of these being represented by *possibly* existing individuals.

Hodes shows that *potentialist* versions of some arithmetical truths are true in every model, where the potentialist version of a sentence is obtained by prefixing each existential quantifier occurring in the original sentence with a possibility operator.[47] He also shows that *possibilist* versions of more arithmetical truths are true in every model, where the *possibilist* version of a sentence is obtained by substituting each existential quantifier occurring in the original sentence with '$\overset{'}{\exists}$', the *possibilist existential quantifier*, where $\overset{'}{\exists}x\varphi$ is true at a world w if and only if some possible individual satisfies φ at w. Still, as Hodes also shows this alternative semantic account of the language of arithmetic, while consistent with $\Box\neg$INFAx, will not count the potentialist or the possibilist versions of all arithmetical truths as true in every model.[48]

Owing to the mismatch between what is true of finite cardinality properties and the truth of arithmetical sentences formulated in terms of numerals, understood by Hodes as individual (type *e*) terms, Hodes sees the project of developing an account of the

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47   With the universal quantifier being defined as the dual of the existential quantifier.
48   Hodes attributes this failure to the fact that the 'number of' operator '#' may pick out different individuals at different worlds. But the results in [69], especially Theorem 1.9, make it reasonable to think that even with a rigid 'number of' operator some potentialist versions of some arithmetical truths would fail to count as true in every model.

semantics of the arithmetical language vindicating logicism and NAC, not committed to the AXIOM OF INFINITY, and consistent with CONTINGENTISM as having failed. As he put it [33, p. 391],

> 'I tentatively conclude that an Individual-Actualist who accepts the Alternative theory does best to accept an actual infinitude [of individuals]'.

Roughly, Hodes's ALTERNATIVE THEORY consists of the conjunction of NAC with the claim that the semantics of the arithmetical language is to be accounted for by 'supervaluating' over *representors*, and that what makes arithmetical sentences true or false are facts about finite cardinality properties expressible by, respectively, logical true and logically false sentences of higher-order logic.

The subsequent discussion in [33] proceeds on the assumption of INFAX's truth, with the semantic account in terms of merely possible individuals being abandoned. This suggests that in [33] a view along the lines of UPPER was no longer endorsed.

In my view, contingentist *serious actualists* should already be suspicious of a semantic theory formulated in terms of representors having in their range merely possible individuals. For representors are functions. So, and assuming functions are relations, by SERIOUS ACTUALISM there are no representors having in their (actual) range merely possible individuals. Relatedly, some representors may map *exactly zero*$_{\langle\langle e\rangle\rangle}$ and *exactly one*$_{\langle\langle e\rangle\rangle}$ to *incompossible* individuals. But in such a case contingentist *serious actualists* would reject the claim that these incompossible individuals possibly stand in any relation. So, '$\diamond(0 \leq 1)$' will be false relative to some representors. But then, '$\diamond(0 \leq 1)$' is not logically true—i.e., not true in all models (satisfying FP) relative to all worlds and all representors—though it is true (in all models satisfying FP, and relative to all worlds and all representors) that *exactly zero*$_{\langle\langle e\rangle\rangle}$ is *less than exactly one*$_{\langle\langle e\rangle\rangle}$.

Furthermore, even if higher-orderist proponents of NAC owe *some* explanation for seemingly true talk of numbers as individuals, there is reason to think that appealing to a semantic theory for the language of arithmetic along the lines of Hodes's is wrong-headed. As an example, Hodes is concerned with how to account for the occurrence of number-expressions in sentences such as the following:

$$\text{'The number of students is three'.} \tag{3}$$

As it occurs in (3), 'the number of students' appears to be functioning as an individual term. If so, then 'three' would appear to itself be functioning in (3) as an individual term (assuming (3)'s truth).

Yet, from higher-orderists' standpoint, seemingly true talk of numbers as individuals is arguably part of a much more general phenomenon. For instance, and assuming its truth, in the sentence

$$\text{'The colour of the sky is blue',} \tag{4}$$

the expressions 'the colour of the sky' and 'blue' appear to be functioning as individual terms. Yet, *being blue* is arguably a paradigmatic case of a *property*, not of an individual. Similarly, as they occur in (5),

$$\text{'The identity relation is the smallest equivalence relation',} \tag{5}$$

'the smallest equivalence relation' and 'the identity relation' appear to be functioning as individual terms (assuming the truth of (5)). Yet, the *identity relation* is a paradigmatic case of a *relation*, not of an individual.

Now, the view that, *in general*, talk of properties as individuals is to be semantically accounted for by taking properties to have individuals representing them seems unmotivated at best. Indeed, it faces a general version of the problem afflicting Hodes's strategy for semantically accounting for talk of numbers as individuals. Just as there is no one-to-one function from the finite cardinality properties to the (actually existing) individuals if there are only finitely many individuals, there is also no one-to-one function mapping properties to individuals, by a higher-order version of Cantor's theorem. And in this case assuming the truth of INFAX does not solve the problem.

Since seemingly true talk of properties as individuals would, in general, appear *not* to be correctly accounted for by bringing in *representors*, it is worth considering a different strategy for accounting for seemingly true talk of properties as individuals.[49] Such an explanation would be of interest not only to those committed to NAC, but also to other projects in higher-order metaphysics. My 'two cents' on the matter are that such talk is explained by the fact that natural language quantifier-expressions are flexible with respect to the type of the quantifier they pick at a particular context. For this reason they are flexible with respect to what lies in their 'domains of quantification' at different contexts, and so sometimes *straddle* across type-distinctions. Developing this view is, however, outside the scope of this paper.[50] For the present purposes, the relevant point is that higher-orderist proponents of NAC are under little pressure to accept Hodes's specific semantic account of the language of arithmetic. For they would then still be faced with explaining seemingly true talk of properties as individuals in general. Conversely, a general, higher-orderist explanation of seemingly true talk of properties as individuals—which would not go along the lines of Hodes's semantic theory, for the reasons just mentioned—would *ipso facto* deliver an explanation for seemingly true talk of natural numbers as individuals.

Notwithstanding, there are important points of agreement between UPPER and Hodes's views on arithmetic: (a) the truth-conditions of arithmetical sentences concern what is true of the finite cardinality properties; (b) these truth-conditions can ultimately be expressed in purely logical terms; (c) the *true* sentences of higher-order logic expressing them are logical truths.

In addition, Hodes did acknowledge the difficulty in establishing, in higher-order logic, certain true facts about finite cardinalities—e.g., that $eleven_{\langle\langle e\rangle\rangle}$ is not the sum of $seven_{\langle\langle e\rangle\rangle}$ and $five_{\langle\langle e\rangle\rangle}$—unless INFAX's logicality was assumed, despite the unreasonableness of this assumption. A further point of agreement between the views

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[49] For discussion of strategies rejecting the view that in sentences like (3) 'three' is functioning as a singular term and the 'is' is that of identity, see e.g., [18, 34, 35, 49].

[50] One immediate worry for this proposal is how to account for quantifiers having in their range, e.g., both individuals and properties. I think that the worry can be addressed by acknowledging the legitimacy of *cumulative* type theory—though, again, this is not the place to develop the view. See, e.g., [45], [81, chap. 5, sec. 7], and [9, 41, 40] for discussion on the meaningfulness of cumulative type theory. For a different, *fictionalist* strategy for accommodating seemingly true talk of attributes as individuals, see [10].

in [32] and UPPER is thus the solution found to this difficulty: to recognize that 'necessity' and 'possibility' are themselves logical expressions, and that FP is a logical truth.

But Hodes has not proceeded to show that all the theorems of a natural theory of the finite cardinality properties—encapsulated in the Russellian translation of PA$_2$—are logically derivable, or that all the relevant truths about the finite cardinality properties—encapsulated by those truths which are arithmetical translations of formulae of PA$_2$—are logically necessarily. Instead, in [33] he gave up on replacing INFAX with FP, owing to the failure to provide a satisfactory semantic account of the arithmetical language in terms of representors mapping finite cardinality properties to (perhaps merely) possible individuals. By doing so the account of arithmetic in [33] ends up relying on a nonlogical principle.

From the standpoint of UPPER's proponents, Hodes drew the wrong conclusion from this failure. It is the semantic account in terms of representors that must ultimately be dismissed, not the logicism about arithmetic. The arithmetical facts about the finite cardinality properties are all logical truths, and the logical principle FP plays a crucial role in the derivation of some of those facts. Even if he changed his mind later on, Hodes was right in claiming, in [32], that mathematics (well, at least arithmetic) is higher-order modal logic.

7.3. A VINDICATION OF LOGICISM. Roeper [60] aims to 'vindicate logicism' by improving both Frege's logicist account of arithmetic—via a consistent logicist characterization of cardinal number—and the neoFregean's account of arithmetic—via a characterization of cardinal number from which HUME'S PRINCIPLE can be logically derived.

According to Roeper, to be a cardinal number is to be a cardinality property—a property such as *one*, 255587, *countably many*, etc. Further, Roeper takes cardinality properties to be applicable to *pluralities of individuals*—e.g., the Beatles were *four*, and there are *seven* Daltons. Roeper's characterization of cardinality properties is the following (I'll use variables indexed with '$\langle ee \rangle$' for variables ranging over properties of pluralities). A property of pluralities $X_{\langle ee \rangle}$ of individuals is a *cardinality property* just in case $X_{\langle ee \rangle}$ is: (i) *Strongly rigid*: for every plurality yy_e, the yy are necessarily $X_{\langle ee \rangle}$s, or the yy are necessarily not $X_{\langle ee \rangle}$s (i.e., $\forall yy_e(\Box X_{\langle ee \rangle}(yy) \vee \neg\Box X_{\langle ee \rangle}(yy)))$;[51] (ii) *Strictly congruent*: for all yy_e and zz_e, necessarily, if the yy are $X_{\langle ee \rangle}$s, then the yy and the zz are equinumerous if and only if the zz are $X_{\langle ee \rangle}$s (i.e., $\forall yy_e \forall zz_e \Box(X(yy) \rightarrow (yy \approx zz \leftrightarrow X(zz))))$; and (iii) *Consistent*: $X_{\langle ee \rangle}$ could have been instantiated (i.e., $\Diamond \exists yy_e(X_{\langle ee \rangle}(yy)))$.

Insofar as the natural numbers are cardinal numbers, it is to be expected that it is a consequence of Roeper's account of cardinality properties that every natural number property (understood as a property of pluralities) could have been instantiated, and so that it is committed to FP, owing to cardinality properties being *consistent*.[52] This is an important respect in which UPPER and Roeper's views are in agreement.

The disagreement concerning whether the natural numbers are properties of pluralities or instead properties of properties also seems to be minor. For, given

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[51] Roeper calls such properties *logical*. I've instead called them 'strongly rigid', since 'logical property' is also as a label for those properties which are the semantic values of logical expressions.

[52] As we will see, Roeper's formal theory does not appeal to modal resources, and so FP is not derivable from Roeper's account of cardinalities as it is formulated in the formal theory's language.

a conception of the natural numbers as properties of pluralities, one can define corresponding properties of properties, and vice versa.[53] So, Roeper and UPPER's proponents can presumably agree that, in some sense, talk of natural numbers as properties of properties is meaningful, as is talk of natural numbers as properties of pluralities.

Now, a more significant difference between the two views is that Roeper includes in the *characterization* of what it is to be a natural number that it is a cardinality property, and so possibly instantiated. By contrast, UPPER's characterization does not include that condition. According to UPPER, to be a natural number is simply to have all the hereditary properties (under the *successor relation*) had by *zero*.

This difference turns out to be of significance. While proponents of UPPER and Roeper are both committed to the logical truth of FP, without the adoption of commitments beyond those of upper logicists Roeper would not have been able to derive the principle that the successor of a natural number is itself a natural number. After all, to show, this, Roeper would have to first establish that the successor of a natural number is a cardinality property, and so that the successor of a natural number is possibly instantiated. By contrast that every natural number has a successor is a straightforward consequence of upper logicists' characterization of the property of being a natural number.

Indeed, and as I will go on to explain, in order to be able to derive the principle that every natural number has a successor natural number Roeper commits to the logical truth of POSTULATE—a principle (characterized below) which, at least syntactically, has strong affinities to Frege's infamous BASIC LAW V, according to which two properties have the same extension just in case they are equinumerous. And like in the case of BASIC LAW V, there are excellent reasons for thinking that POSTULATE is false.

A further commitment of Roeper's is to *properties* also belonging, secondarily, to the category of *individuals*, owing to our seemingly being able to refer to them to via individual (type e) terms. The *postulate*, as Roeper calls it, which governs the move from expressing a property via a predicate to referring to that property via a singular term is the following (where φ and ψ are open formulae in which xx_e occurs free):

Postulate: Properties $\rho xx_e(\varphi)$ and $\rho xx_e(\psi)$ are identical if and only if for all xx_e, necessarily, φ if and only if ψ.

$$\rho xx_e(\varphi) = \rho xx_e(\psi) \leftrightarrow \forall xx_e(\Box(\varphi \leftrightarrow \psi)).$$

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[53] Let '$N^{pl}_{\langle\langle ee\rangle\rangle}$' be a predicate for the property of being a natural number, qua property of pluralities. Then, the property $N_{\langle\langle\langle e\rangle\rangle\rangle}$ of being a natural number, qua property of properties, is definable as follows: $N_{\langle\langle\langle e\rangle\rangle\rangle}(X_{\langle\langle e\rangle\rangle}) := \exists Y_{\langle ee\rangle}(N^{pl}_{\langle\langle ee\rangle\rangle}(Y) \wedge X = \lambda Z_{\langle e\rangle}(\exists zz_e(\forall z_e(z \prec zz \leftrightarrow Z(z)) \wedge Y(zz))))$. That is, for $X_{\langle\langle e\rangle\rangle}$ to be a property-applicable natural number just is for $X_{\langle\langle e\rangle\rangle}$ to be the property of being a $Z_{\langle e\rangle}$ such that those things which fall under $Z_{\langle e\rangle}$ fall under some plurality-applicable natural number.

Similarly, the property $N^{pl}_{\langle\langle ee\rangle\rangle}$ of being a natural number, qua property of pluralities, is definable as follows: $N^{pl}_{\langle\langle ee\rangle\rangle}(X_{\langle ee\rangle}) := \exists Y_{\langle\langle e\rangle\rangle}(N_{\langle\langle\langle e\rangle\rangle\rangle}(Y) \wedge X = \lambda zz_e(Y(\lambda z_e(z \prec zz))))$. That is, for $X_{\langle ee\rangle}$ to be a plurality-applicable natural number just is for $X_{\langle ee\rangle}$ to be the property of being zz_e such that the property of being one of the zz_e falls under some property-applicable natural number.

The ρ-operator, as used in the formulation of POSTULATE, is a variable binding operator which binds plural variables. It is 'introduced by stipulation; its role is to change a predicate expressing a property into a singular term for that property' [60, sec. 2.2, p. 366].[54]

Since 'Any attempt to formulate the sameness in question will be ungrammatical', Roeper holds that POSTULATE 'is not a definition, but it records the effect of the stipulation [concerning the role of ρ] to the extent that it can be expressed in the symbolism' [60, sec. 2.2, p. 367]. Importantly, and by contrast with the neoFregeans' take on the relationship between HUME'S PRINCIPLE and the 'number of' operator, Roeper does not take POSTULATE to be a *definition* of ρ; ρ's meaning is already fixed when used in the formulation of POSTULATE.

Roeper seemingly takes POSTULATE to be logically true, and the system in which he offers a (purportedly logicist) derivation of PA$_2$ has a weakening of POSTULATE amongst its axioms. Yet, not only does POSTULATE have strong affinities with BASIC LAW V, there is also reason to think that, far from being logically true, POSTULATE is itself false, since a form of Russell's paradox can be derived from it. Consider the following predicate:

$$xx_e \sqsubset yy_e := \exists X_{\langle ee \rangle}(\rho zz_e(X(zz)) \prec yy \wedge X(xx)).$$

Assume, for reductio, that POSTULATE is true. Then, by *plural comprehension* we have that $\exists uu_e(\forall u_e(u \prec uu \leftrightarrow u = \rho zz_e(\neg zz \sqsubset zz)))$—since $\forall zz_e(\Box(\neg zz \sqsubset zz \leftrightarrow \neg zz \sqsubset zz))$, and so $\rho zz_e(\neg zz \sqsubset zz) = \rho zz_e(\neg zz \sqsubset zz)$, by POSTULATE, and therefore $\exists x_e(x = \rho zz_e(\neg zz \sqsubset zz))$. Let rr_e—the *Russell plurality*—be one such plurality uu_e such that $\forall u_e(u \prec uu_e \leftrightarrow u = \rho zz_e(\neg zz \sqsubset zz))$.

Suppose that $rr_e \sqsubset rr_e$. Then, $\rho zz_e(X(zz)) \prec rr_e \wedge X(rr_e)$, for some $X_{\langle ee \rangle}$. So, $\rho zz_e(X_{\langle ee \rangle}(zz)) = \rho zz_e(\neg zz \sqsubset zz)$. Hence, $\forall zz_e \Box(X_{\langle ee \rangle}(zz) \leftrightarrow \neg zz \sqsubset zz)$, by POSTULATE. Since $X_{\langle ee \rangle}(rr_e)$, we have that $\neg rr_e \sqsubset rr_e$. Suppose now that $\neg rr_e \sqsubset rr_e$. We have that $\rho zz_e(\neg zz \sqsubset zz) \prec rr_e$. So, $\exists X_{\langle ee \rangle}(\rho zz_e(X(zz)) \prec rr_e \wedge X(rr_e))$, by a version of a principle of comprehension (and indeed, of Comp$_C$) for properties of pluralities. So, $rr_e \sqsubset rr_e$. Contradiction. So, POSTULATE is false.

Now, the *formal theory* in which Roeper offers a derivation of PA$_2$ has as an axiom not POSTULATE but rather C-POSTULATE, a weakening of POSTULATE. This theory is formulated in a language stripped off of modal operators.[55] The C-POSTULATE is formulated in terms of the operator 'C'. This operator, which binds plural variables, is defined as follows in terms of conditions arising from the formulations of *strict congruency* and *consistency* by removing the modal operators occurring in them (where xx_e occurs free in φ, and yy_e and zz_e are free for xx_e in φ):

$$Cxx_e(\varphi) := \forall yy_e \forall zz_e(\varphi_{yy}^{xx_e} \rightarrow (yy \approx zz \leftrightarrow \varphi_{zz}^{xx_e})) \wedge \exists xx_e(\varphi).$$

Intuitively, $Cxx_e(\varphi)$ states that $\lambda xx_e(\varphi)$ is a cardinality property. The C-POSTULATE is formulated as follows (where xx_e occurs free in both φ and ψ):

C-postulate: If $Cxx_e(\varphi)$ and $Cxx_e(\psi)$, then properties $\rho xx_e(\varphi)$ and $\rho xx_e(\psi)$ are identical if and only if for all xx_e, φ if and only if ψ.

$$(Cxx_e(\varphi) \wedge Cxx_e(\psi)) \rightarrow (\rho xx_e(\varphi) = \rho xx_e(\psi) \leftrightarrow \forall xx_e(\varphi \leftrightarrow \psi)).$$

Roeper goes on to lay down some further principles relating talk of natural numbers as properties to talk of natural numbers as individuals, and defines the plurality of natural numbers as the smallest plurality of *cardinalities* (qua individuals) which includes *zero* and is closed under succession.

Roeper then shows that the axioms of PA₂ are derivable in his theory, given his characterization of the plurality of natural numbers.[56] Importantly, while Roeper is able to straightforwardly derive the result that different natural numbers have different successors on the basis of his definition of cardinality properties as (strictly congruent and) *instantiated* (rather than just *possibly* instantiated) properties, his proof that every natural number has a successor natural number makes essential use of the C-POSTULATE. Indeed, and in Fregean fashion, Roeper's derivation relies on the observation that the plurality of numbers smaller than or equal to *n* has as its natural number *the successor of n*.

An important difference between Roeper's logicist account of arithmetic and UPPER's is thus that while Roeper's initial characterization of cardinality properties is formulated in modal terms, and seemingly implies FP, the characterization of cardinality properties and natural numbers which is actually used in the derivation of PA₂'s axioms presupposes that every cardinality property, and so also every natural number property, is *actually*—rather than just *possibly*—instantiated. Further, and by contrast with Theorem 8's proof, Roeper's proof that there are infinitely many natural numbers essentially appeals to C-POSTULATE and proceeds not via the observation that modal space can be sufficiently rich so as to have all natural numbers, *qua* properties, be possibly instantiated just in case they are infinitely many, but instead via the observation that if the natural numbers are *individuals*, then the plurality of natural numbers smaller than or equal to *n* will have as its natural number *n*'s successor. Thus, the result is proven in a way akin to how *Fregeans* demonstrate the infinitude of the natural numbers.

Now, how satisfactory is Roeper's logicist account of arithmetic? As has been shown, the principle which is the linchpin of Roeper's account, POSTULATE, is presumably false, and perhaps even logically contradictory. But then, what reason is there for thinking that its weakening, C-POSTULATE, is true? That is, what reason is there for thinking that while, in general, properties of pluralities do not have reifications, some properties of pluralities—specifically, the cardinality properties—do? What makes them special?

Strikingly, NEOFREGEANISM may offer the best way of defending the analyticity, if not the logicality, of C-POSTULATE. For, the ρ-operator may be defined in a natural manner in terms of the neoFregeans' 'number-of' operator # (applied to pluralities rather than properties), where φ is a predicate in which xx_e occurs free and in which y_e does not occur free, and ι is a variable-binding, definite-description operator:

$$\rho xx_e(\varphi) := \iota y_e((Cxx_e\varphi \wedge \exists xx_e(\varphi \wedge \#(xx) = y_e)) $$
$$\vee (\neg Cxx_e\varphi \wedge \exists zz_e(\forall z_e(z \prec zz \leftrightarrow z = z) \wedge \#(zz) = y_e))).$$

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[56] While Roeper does not state what is the underlying logic of his formal theory, he seems to be relying on classical second-order logic together with an extension of standard plural logic (with a principle of plural comprehension allowing for the existence of a plurality of no things) allowing for quantification over properties of pluralities.

That is, the property of being xx_e such that φ is either the number of some things such that φ, provided that being such that φ is a cardinality property, or else is the number of the plurality of all things.[57]

Once the ρ-operator is so defined, the C-POSTULATE is derivable from PLURAL HUME'S PRINCIPLE, the plural version of HUME'S PRINCIPLE according to which the number of the xx_e is identical to the number of the yy_e if and only if the xx_e are equinumerous with the yy_e $(\forall xx_e \forall yy_e (\#(xx) = \#(yy) \leftrightarrow xx \approx yy))$.

I leave the exercise of deriving this result to the reader. What it shows is that neoFregeans *can* take the ρ-operator, as it occurs in C-POSTULATE, as an independently interpreted operator, and not take C-POSTULATE to be a *definition* of ρ. Presumably, there is even room for them to adhere to the idea that the natural numbers are not only individuals but also themselves properties of pluralities, if they nonetheless stick to Roeper's type-restrictions in their formal theory.

So, NEOFREGEANISM may constitute the best shot at defending Roeper's C-POSTULATE. Of course, such a defense will only be as successful as the neoFregeans' defense of the truth and analyticity of PLURAL HUME'S PRINCIPLE. As Roeper himself remarks, that defense is plagued with difficulties—e.g., *the Caesar problem* and *bad company*-style objections.[58] Arguably, UPPER is not faced with objections of this sort.

7.4. ARITHMETICAL POTENTIALISM.   According to ARITHMETICAL POTENTIALISM [46], there could be no completed infinitude of natural numbers. Rather, only the following weaker thesis is true:

Potential infinity: Necessarily, every natural number could have had some successor. $\Box \forall x (N(x) \rightarrow \Diamond \exists y (S(x, y)))$.

ARITHMETICAL POTENTIALISM and UPPER are both formulated in terms of modal resources. Further, they agree that, for each natural number n, there could have been at least n individuals, and none of these views is committed to $\Diamond$INFAX. Still, the two views differ in a number of respects.[59]

First, potentialists take natural numbers to be *individuals*, a view rejected by UPPER's proponents. An arguably more significant difference is that potentialists *reject* the possible existence of all possible natural numbers, while arithmetical upper logicists advocate the necessary existence of all possible natural numbers. Further, while some arithmetical potentialists might be attracted to logicist projects, ARITHMETICAL POTENTIALISM is not directly concerned with the question whether arithmetic is reducible to logic.[60]

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[57] The second disjunct may be omitted. It is there just to ensure that $\rho xx_e(\varphi)$ is defined even if being such that φ is not a cardinality property. *Lambert's law* [42], according to which $\forall u(u = \iota v(\varphi) \leftrightarrow \forall v(\varphi \leftrightarrow v = u))$ (where v and u are distinct variables of the same type, and v but not u occurs free in φ) may be adopted as the logical axiom-schema governing the definite description operator ι. Alternatively, the ι-operator itself may be explained away *à la* Russell.

[58] See, e.g. [78] for more on these and other difficulties.

[59] As a reviewer notes, a salient difference between UPPER and ARITHMETICAL POTENTIALISM concerns the logic of their respective modalities. Whereas potentialists advocate S4 or S4.2 as the logic of their modality, here we take S5 to be the logic of metaphysical modality, and the interpretation of PA$_2$ here offered relies on the weaker system K.

[60] Linnebo's [44] potentialism is wedded to a *thin* conception of mathematical entities, a view with some affinities with the logicist take on them. Notwithstanding, a thin conception need

To mention just one difficulty for the potentialist programme, it would appear that mathematical entities, qua abstract entities, necessarily exist if they possibly do, so that all possible numbers actually exist after all. Yet, the actual existence of all natural numbers is inconsistent with ARITHMETICAL POTENTIALISM. One reply available to potentialists is to formulate their view in terms of a modality other than metaphysical necessity while acknowledging the metaphysically necessary existence of all possible natural numbers. They then face the challenge of explaining what this modality consists of.

Focusing on just one alternative interpretation, some potentialists have interpreted the modality in question as *interpretational* modality (see, e.g., [44]). On this understanding, '$\diamond\varphi$' means that 'we can abstract so as to make it the case that φ' [44, chap. 3, p. 61], while '$\Box\varphi$' means that 'no matter how we abstract and thereby shift the meaning of the language, φ' [44, chap. 12, p. 205]. As Linnebo [44, chaps. 3, pp. 61–62] puts it, the 'interpretational' interpretation of the modal operators is 'concerned with how the language is interpreted, not with how reality is. In particular, every interpretational possibility is compatible with the metaphysically actual world'.

According to Linnebo [44, chap. 8], abstraction is something made by a community. For a community *to abstract* with respect to an equivalence relation $\sim$ is for it to extend its language and the meanings of the expressions occurring in it so that the meaning of, e.g., the quantifiers and the identity predicate shift so as to make it the case that, e.g., '$\exists w \exists v(Abs_\sim(w, y) \land Abs_\sim(v, z) \land w = v)$' is true whenever $y \sim z$ for some y and z, where '$Abs_\sim$' is a predicate that applies to any a and b whenever a is the *abstract* of b with respect to the equivalence relation $\sim$. This requires that the community uses their language in accordance with appropriate assertibility conditions (specified in [44, chap. 8]). Once they do so, the quantifiers' meanings become 'more expansive'. Whereas before y's and z's respective abstracts were not within the language's quantifiers' range, after the meaning shifts they come to be.

In the arithmetical context, one salient option is to abstract via HUME'S PRINCIPLE.[61] Let '$\bar{0}$' abbreviate 'number of the property of being self-distinct' and '1' abbreviate 'number of the property of being identical to $\bar{0}$'. Since the property of being self-distinct is equinumerous with itself, we can abstract on equinumerosity, thereby shifting the meanings of our language's expressions so as to make '$\bar{0}$ is identical to something' true. Then, 'the property of being identical to $\bar{0}$ is equinumerous with itself' will be true under such an extension. And in such a case one can further shift the meanings of our language's expressions so as to make true '$\bar{1}$ is identical to something'. After this second step of abstraction it will also be true that '$\bar{1}$ is the successor of $\bar{0}$'. Since

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not imply a form of logicism. According to Linnebo's views on 'thin objects', the obtaining of specific equivalence relations—which may be nonlogical—between objects—which may be of a *concrete* kind—is *sufficient for* the identity and existence of mathematical entities. But logic is not immediately concerned with any truths *specific* to kinds of concrete objects. Borrowing an often used slogan, *logic is topic neutral*—which is why it is so important for UPPER's prospects that arithmetic be reduced to *pure* modal type theory.

[61] Linnebo's [44] account of arithmetic is not in terms of interpretational modality, and does not appeal to HUME'S PRINCIPLE. In the main text I focus on HUME'S PRINCIPLE merely for expository purposes, as it affords a natural development of ARITHMETICAL POTENTIALISM formulated in terms of interpretational modality. Relatedly, such an account would presumably require a *predicative* version of HUME'S PRINCIPLE (see [44, chap. 6]), though here I won't go through these complications.

whenever one abstracts on equinumerosity the meanings of our language's expressions will shift so as to make it the case that 'something is x's successor' is satisfied by some 'previously given number', potentialists will contend POTENTIAL INFINITY comes out true.

Now, it appears that arithmetical potentialists must take '$\diamond$' to express a *propositional* rather than a *sentential* operator, so that it acts not on *formulae* but rather on their *semantic values*. For instance, potentialists take *de re* modal constructions to be well formed, and several of the theses advocated by them involve quantification into modal contexts. But it is notoriously difficult to make sense of quantification into modal contexts when modal operators are understood as acting on formulae rather than propositions. For example, how are we to make sense of the 'previously given number' which will satisfy 'something is x's successor' on a permissible shift of the language's meanings, after a previous permissible shift of the language's meanings? After all, we have no number, or even possible number, being given. What we have are *language extensions*.

But if '$\square$' and '$\diamond$' express *propositional* operators, then presumably '$\square\forall x(N(x) \rightarrow \diamond\exists y(S(x,y)))$' expresses that *no matter how we abstract so as to shift the meaning of our expressions, for every natural number x, we can abstract so as to shift the meaning of our expressions so that something is x's successor*, with 'every', 'some', 'number', 'successor' and so on being used with their *actual* meanings, not with the meanings they might have had. If that is right, it would appear that '$\square\forall x(N(x) \rightarrow \diamond\exists y(S(x,y)))$' expresses no more than that *for every natural number, something is its successor*. The modal operators occurring it are *vacuous*.

This observation is reminiscent of the riddle *how many legs does a dog have if you call his tail a leg?* Just as the right answer is *four* since calling a tail a leg does not *make it* a leg, so it is that being able to abstract so as to make it the case that our language's expressions mean something different from what they actually mean does not make it the case that they actually mean something different. So, if the potentialists' modal operators act on semantic values rather than formulae, then it is the semantic values that formulae *in fact have*, not the ones that they would have, which are relevant to the truth of modalized claims.

Yet, this is not how '$\diamond$' is supposed to function from the standpoint of potentialists. According to them, the truth of '$\square\forall x(N(x) \rightarrow \diamond\exists y(S(x,y)))$' should depend not on the meaning that the expressions occurring in it actually have, but rather on the meanings that they would have under some 'legitimate abstraction'. There is thus an important tension in the potentialists' *interpretational* understanding of the modal operators.[62] By contrast, UPPER is formulated in terms of plain old, familiar metaphysical modality, and so does not immediately face this sort of difficulty (which is not to say that the question of how to understand metaphysical modality in other terms is not worth exploring).

Let us bracket the concern with the coherence of the arithmetical potentialists' use of interpretational modality. What would be *clarified* by understanding ARITHMETICAL POTENTIALISM in such a way? A possible response (based on [44]) is that appealing to interpretational modality promises to account for the sense in which the natural

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[62] Bacon [2] likewise finds the potentialists' *interpretational modality* suspicious. For objections to the project of elucidating the *iterative conception of set* via SET-THEORETIC POTENTIALISM, see [19, sec. 4].

numbers' existence is 'thin'. It is thin because it is a product of little more but a shift in the meaning of the language's quantifiers. Barely no contribution from the world is needed.

The idea is the following. In our *present* language we do quantify over all the infinitely many natural numbers. But ARITHMETICAL POTENTIALISM allows us to understand why their existence is ontologically thin. On the basis of the property of being self-distinct's equinumerosity with itself the meanings of our quantifiers have shifted, so that '0 exists' is now true, with the equinumerosity of the property of being self-distinct grounding 0's existence. And once that shift occurred, further meaning shifts were made available—without any substantive change in the underlying facts. The present truth of 'there are infinitely many natural numbers' is ultimately grounded in those shifts and in equinumerosity facts which come 'for free'—specifically, that the property of being self-distinct is equinumerous with itself, as is the property of being less than or equal to n, for each natural number n. From this standpoint, the potentiality of the natural numbers encapsulates the permissibility of abstraction-based meaning change through which our quantifiers ended up ranging over infinitely many natural numbers.

UPPER's proponents will tell a different story. Given adherence to a logical theory as weak as $\mathtt{KQL}_{\mathtt{C}}^{\mathtt{FP}}$, $\mathtt{PA}_2$ can be seen to be true by taking the natural numbers to be *finite cardinalities*. Logic on its own will then imply not only the actual but also the necessary existence of the infinitely many natural numbers. The natural numbers will then be thin at least insofar as their necessary existence is implied by principles of an extremely general nature—indeed, by purely logical principles.

Which story is more satisfactory? Well, the potentialist explanation arguably relies on a *creation myth*: the myth of a language created by a community of speakers coordinating on its use so that the meanings of its quantifiers expanded further and further, with the consequence that they have ended up ranging over the infinitely many numbers. I know of no empirical data supporting the claim that the language of any actual community has been created in this way. Furthermore, it would appear that no language of any human community *could* have been created in this way. After all, in order for our quantifiers to range over infinitely many natural numbers there would have had to have been infinitely many rounds of coordinated meaning-expansion. But surely, no human community could have gone through so many rounds of coordinated meaning-expansion.[63]

So, how can a story along *these* lines be more satisfactory, vis-à-vis the natural numbers' 'thinness', than the one put forward by UPPER's proponents? I, for one, don't think it is.

§8. Conclusion. The paper's main aim was to introduce UPPER, a novel, neoRussellian philosophy of arithmetic, and to offer a partial defense of this view. UPPER was characterized in Section 2, while its defense took place in Sections 3–6. After having laid out UPPER's ground type-theoretic modal language (in Section 3), it was shown that $\mathtt{KQL}_{\mathtt{C}}^{\mathtt{FP}}$ interprets $\mathtt{PA}_2$ via a translation that respects NaC and the Russellian conception of the arithmetical primitives (in Section 4), and that $\mathtt{S5PQL}_{\mathtt{C}}$ implies that every purely arithmetical formula of MT is necessarily true (in Section 5). Then, a defense was offered in Section 6 of the metaphysical necessity of all theorems of $\mathtt{S5PQL}_{\mathtt{SA,C}}^{\mathtt{FP}}$. Finally,

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[63] For a related objection, and further concerns with Linnebo's [44] *abstractionism*, see [75].

UPPER was compared (in Section 7) with Hellman's MODAL STRUCTURALISM, Hodes's ALTERNATIVE THEORY, Roeper's VINDICATION OF LOGICISM, and Linnebo and Shapiro's ARITHMETICAL POTENTIALISM.

By way of conclusion, UPPER is a clear instance of the following, more general logicist view:

Finitary Upper Mathematical Logicism (FUML): the conjunction of NECESSITY OF FP, HIGHER-ORDERISM and LOGICALITY with the following theses:

 – HIGHER-TYPE MATHEMATICAL ONTOLOGY: The mathematical entities are all of higher types, and mathematics's primitives are all truly characterizable as higher type-entities definable in pure modal type theory;
 – INTERPRETABILITY OF MATHEMATICS: For every standard, true deductive theory T_f of any mathematical field f, there is a deductive system formulated in a pure *modal* and type-theoretic language, and that includes FP among its axioms, whose theorems: (i) are all metaphysically necessary truths; and (ii) include the MATHEMATICAL PRIMITIVES-respecting translations of all theorems of T_f;
 – NECESSITY OF MATHEMATICS: The mathematical truths are all metaphysically necessary truths expressible in a pure modal type theory.

Arguably, FUML affords an attractive picture of the mathematical realm and the relationship between mathematics and logic. According to FUML, 'mathematical entities are higher type-entities, mathematics is higher-order modal logic, and FP is true'. Despite FUML's attractiveness, defending it presumably amounts to a never-ending project as the process of formulating novel mathematical theories is seemingly never-ending. But that need not count against FUML. Indeed, the development of an upper logicist philosophy of mathematics would presumably amount to 'mathematical philosophy', in Russell's sense of the expression, at its best: a programme of continuously revealing mathematics's underlying foundations.

The present paper's aim has been to offer a defense of UPPER, not of FUML. Notwithstanding, a cogent defense of the former presumably constitutes a 'proof of concept' with respect to the latter view's prospects: if the finitary upper logicist picture is true of arithmetic, then it might cogently extend to other mathematical fields, perhaps all. For instance, in my view recent work on the relationship between set theory and type theory [14, 45] can be deployed at the service of a defense of a form of FINITARY UPPER SET-THEORETICAL LOGICISM. Alas, the development of such a defense will have be to be left for future work.

§A. Interpretability of PA_2 in KQL_C^{FP} via Russellian translations. In this section is proven Theorem 8. In order to do so, it will be helpful to start by briefly presenting the axiomatization of PA_2 that will be considered:

DEFINITION 11 (PA_2). *PA_2 is the deductive system in language* PA_2 *whose inference rules are* MP *and* Gen *and whose axioms are all instances of the following axiom schemata* (*Table 3*):

The proofs to be offered in what follows will, for the most part, be presented as natural deduction proofs. Suffice it to say that when a line where φ occurs is labelled as an assumption, it is really an instance of the PL-theorem $\varphi \rightarrow \varphi$. We will also be frequently appealing to the following derived rules:

Table 3. *Axioms of* PA$_2$

Axiom schemata of classical second-order logic		
(PL) Propositional tautologies	(C∨E)	$\forall v\varphi \rightarrow \varphi^v_\alpha$
	(∀1)	$\forall v(\varphi \rightarrow \psi) \rightarrow (\forall v\varphi \rightarrow \forall v\psi)$ (∀2) $\varphi \rightarrow \forall v\varphi$
(=I) $v = v$	(Ind)	$\alpha = \gamma \rightarrow (\varphi \rightarrow \varphi')$
	(Comp)	$\exists U \forall v^1 ... \forall v^n (U(v^1, ..., v^n) \leftrightarrow \varphi)^a$
Arithmetical axioms		
ZERO IS A NUMBER (ZN)	$N(0)$	
NUMBER IS HEREDITARY (NH)	$\forall x \forall y(((N(x) \wedge S(x, y)) \rightarrow N(y))$	
SUCCESSOR IS FUNCTIONAL (SF)	$\forall x \forall y \forall z((N(x) \wedge S(x, y) \wedge S(x, z)) \rightarrow y = z)$	
ZERO SUCCEEDS NOTHING (ZSN)	$\neg \exists x(S(x, 0))$	
SUCCESSION IS ENDLESS (SE)	$\forall x(N(x) \rightarrow \exists y(S(x, y)))$	
SUCCESSOR IS INJECTIVE (SI)	$\forall x \forall y \forall z(((N(x) \wedge N(y)) \wedge (S(x, z) \wedge S(y, z)))$ $\rightarrow x = y)$	
MATHEMATICAL INDUCTION (MI)	$\forall Y((Y(0) \wedge \forall x \forall z((Y(x) \wedge S(x, z)) \rightarrow Y(z)))$ $\rightarrow \forall x(N(x) \rightarrow Y(x)))$	

[a]Where, U ranges over *n*-ary predicate variables, $v^1, ..., v^n$ range over individual variables, and U occurs nowhere free in φ.

– (QGen) $\Gamma \vdash_S E[v] \rightarrow \varphi \Rightarrow \Gamma \vdash_S \forall v\varphi$, where *v* occurs nowhere free in Γ;
– (∃E) $(\Gamma \vdash_S \exists v\varphi \,\&\, \Delta \vdash_S \varphi^v_u \rightarrow \psi) \Rightarrow \Gamma, \Delta \vdash_S \psi$, where *u* occurs nowhere free in Γ, Δ or ψ, and the usual provisos preventing recapture of variables apply.

It is a routine exercise to show that these are derived rules of QL and all its extensions considered in the paper. Finally, the justifications provided for our lines of deduction will be given in terms of the previous lines from which that line is derived, and the name of the system sanctioning the reasoning involved.

LEMMA 1. $\vdash_{\text{QL}_C} E[\mathbf{0}], \vdash_{\text{QL}_C} E[\mathbf{S}], \vdash_{\text{QL}_C} E[\mathbf{N}]$.

Proof of Lemma 1. A straightforward consequence of Comp$_C$, by PL-reasoning. □

THEOREM 1 (Axiom PA$_2$: (ZN)$^{\mathcal{R}}_\tau$). $\vdash_{\text{QL}_C} \mathbf{N}(\mathbf{0})$.

THEOREM 2 (Axiom PA$_2$: (NH)$^{\mathcal{R}}_\tau$). $\vdash_{\text{QL}_C} \forall x_{\langle\langle\tau\rangle\rangle} \forall y_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(x) \wedge \mathbf{S}(x, y)) \rightarrow \mathbf{N}(y))$.

Proof of Theorems 1 *and* 2. Straightforward consequences of Lemma 1, by QL-reasoning. □

THEOREM 3 (Axiom PA$_2$: (SF)$^{\mathcal{R}}_\tau$). $\vdash_{\text{QL}} \forall x_{\langle\langle\tau\rangle\rangle} \forall y_{\langle\langle\tau\rangle\rangle} \forall z_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(x) \wedge \mathbf{S}(x, y) \wedge \mathbf{S}(x, z)) \rightarrow y = z)$.

Proof of Theorem 3. Provable by straightforward QL-reasoning. □

Before we turn to the proof of the arithmetical axiom according to which *zero* has no successor, it will be useful to prove an auxiliary lemma about *empty* relations, which are defined as follows:

DEFINITION 12 (Empty relations). *For every positive integer n and all types* $\tau^1, ..., \tau^n$:

$$\Lambda_{\langle\tau^1,...,\tau^n\rangle} := \lambda x^1_{\tau^1} ... x^n_{\tau^n} \bigwedge_{1 \leq i \leq n}(x^i = x^i \wedge \neg x^i = x^i).$$

So, an empty relation is one that obtains between whatever things are both self-identical and not self-identical. The following lemma will be useful later on. As it turns out empty relations exist as a matter of logic, and, unsurprisingly, obtain between no things:

LEMMA 2. $\vdash_{\mathrm{QL_C}} E[\Lambda]$ *and* $\vdash_{\mathrm{QL}} \neg\Lambda(y^1,\dots,y^n)$.

Proof of Lemma 2. $E[\Lambda]$ is provable from Comp$_C$, and $\neg\Lambda(y^1,\dots,y^n)$ is provable by QL-reasoning. □

THEOREM 4 (Axiom PA$_2$: $(\mathrm{ZSN})^{\mathcal{R}}_\tau$). $\vdash_{\mathrm{QL_C}} \neg\exists x_{\langle\langle\tau\rangle\rangle}(\boldsymbol{S}(x,\boldsymbol{0}))$.

Proof of Theorem 4. Provable from Lemmas 1 and 2 by straightforward QL-reasoning. □

THEOREM 5 (Axiom PA$_2$: $(\mathrm{SE})^{\mathcal{R}}_\tau$). $\vdash_{\mathrm{QL_C}} \forall x_{\langle\langle\tau\rangle\rangle}(\boldsymbol{N}(x) \to \exists y_{\langle\langle\tau\rangle\rangle}(\boldsymbol{S}(x,y)))$.

Proof of Theorem 5. Provable from Lemma 1 by straightforward QL$_C$-reasoning. □

THEOREM 6 (Axiom PA$_2$: $(\mathrm{MI})^{\mathcal{R}}_\tau$). $\vdash_{\mathrm{QL}} \forall y_{\langle\langle\langle\tau\rangle\rangle\rangle}((y(\boldsymbol{0}) \wedge Her[y]) \to \forall x_{\langle\langle\tau\rangle\rangle}(\boldsymbol{N}(x) \to y(x)))$.

Proof of Theorem 6. Provable by straightforward QL-reasoning. □

We will now work our way towards the proof that *successor* is an injective relation. In order to do so, a number of auxiliary lemmas will be required—starting with a familiar one:

LEMMA 3. $\vdash_{\mathrm{QL_C}} \forall x_{\langle\langle\tau\rangle\rangle}(\boldsymbol{N}(x) \to (x \neq \boldsymbol{0} \to \exists y_{\langle\langle\tau\rangle\rangle}(\boldsymbol{S}(y,x) \wedge \boldsymbol{N}(y))))$.

Proof of Lemma 3. Follows from Lemma 1 and Theorems 1, 2 and 6, by QL$_C$-reasoning. □

Our next auxiliary result, the *equinumerosity lemma*, is directly concerned with numbers qua cardinalities. It plays a crucial role in Russell's original derivation of the injectivity of the successor relation, and it is also proved in the Appendix of [5]. As its name registers, it is formulated in terms of the equinumerosity relation. So let me quickly define it before getting on to stating the lemma.

DEFINITION 13 (Subrelation). *For all types* $\tau_1, \dots, \tau_n$ *and positive integers* n, *and provided that none of* $v^1, \dots, v^n$ *occurs free in* α *nor* β:
  (a) $\alpha_{\langle\tau^1,\dots,\tau^n\rangle} \subseteq \beta_{\langle\tau^1,\dots,\tau^n\rangle} := \forall v^1 \dots \forall v^n(\alpha(v^1,\dots,v^n) \leftrightarrow \beta(v^1,\dots,v^n))$;
  (b) $\alpha_{\langle\tau^1,\dots,\tau^n\rangle} \subset \beta_{\langle\tau^1,\dots,\tau^n\rangle} := \alpha \subseteq \beta \wedge \neg\beta \subseteq \alpha$.

DEFINITION 14 (Equinumerosity). *Where, for all types* τ *and* ρ, α *is a type* $\langle\tau\rangle$*-singular term,* β *is any type* $\langle\rho\rangle$*-singular term,* π *is any type* $\langle\langle\tau\rangle,\langle\rho\rangle\rangle$*-singular term,* v *and* s *are, respectively, the first and second type* τ*-singular variables that occur nowhere free in* α, β *or* π, u *and* t *are, respectively, the first and second type* ρ*-singular variables that occur nowhere free in* α, β *or* π *(or the third and fourth, if* $\tau = \rho$):
  (a) 1–$1[\pi] := \forall v\forall s\forall u((\pi(v,u) \wedge \pi(s,u)) \to v = s)$; (b) $Dom[\pi,\alpha] := \forall v(\alpha(v) \leftrightarrow \exists u(\pi(v,u)))$; (c) $Func[\pi] := \forall v\forall u\forall t((\pi(v,u) \wedge \pi(v,t)) \to u = t)$; (d) $Img[\pi,\beta] := \forall u(\beta(u) \leftrightarrow \exists v(\pi(v,u)))$; (e) $Inj_{\langle\langle\langle\tau\rangle,\langle\rho\rangle\rangle,\langle\tau\rangle,\langle\rho\rangle\rangle} =_{\mathrm{df}} \lambda x_{\langle\langle\tau\rangle,\langle\rho\rangle\rangle} y_{\langle\tau\rangle} z_{\langle\rho\rangle}(Func[x] \wedge 1-1[x] \wedge Dom[x,y] \wedge \exists w_{\langle\rho\rangle}(Img[x,w] \wedge w \subseteq z))$; (f) $Bij_{\langle\langle\langle\tau\rangle,\langle\rho\rangle\rangle,\langle\tau\rangle,\langle\rho\rangle\rangle} =_{\mathrm{df}} \lambda x_{\langle\langle\tau\rangle,\langle\rho\rangle\rangle} y_{\langle\tau\rangle} z_{\langle\rho\rangle}(Func[x] \wedge 1-1[x] \wedge Dom[x,y] \wedge Img[x,z])$; (e) $\approx_{\langle\langle\tau\rangle,\langle\rho\rangle\rangle} =_{\mathrm{df}} \lambda y_{\langle\tau\rangle} z_{\langle\rho\rangle} \exists x_{\langle\langle\tau\rangle,\langle\rho\rangle\rangle}(Bij(x,y,z))$.

LEMMA 4 (Equinumerosity lemma). $\vdash_{\mathtt{QL_C}} \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \forall y_{\langle\tau\rangle}(x(y) \rightarrow \forall z_{\langle\tau\rangle}(x(z) \leftrightarrow y \approx z)))$.

Proof of Lemma 4. A consequence of Lemma 1 and Theorem 6, by $\mathtt{QL_C}$-reasoning. □

Our remaining auxiliary lemmas go beyond the previous results by integrating modality. For this reason we make the reasoning explicit.

LEMMA 5. $\vdash \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow (\Box\mathbf{N}(x) \wedge \Box E[x]))$.

Proof of Lemma 5. The proof is by induction.

Base case:

(1)		$\Box E[\mathbf{0}]$	Lem. 1; K
(2)		$\Box\mathbf{N}(\mathbf{0})$	Theo. 1; K
(3)		$E[\lambda x_{\langle\langle\tau\rangle\rangle}(\Box\mathbf{N}(x) \wedge \Box E[x])]$	$\mathtt{QL_C}$
(4)		$\lambda x_{\langle\langle\tau\rangle\rangle}(\Box\mathbf{N}(x) \wedge \Box E[x])(\mathbf{0})$	1; 2; 3; Lem. 1; QL

Inductive case:

5	(5)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp.(QGen)
6	(6)	$E[y_{\langle\langle\tau\rangle\rangle}]$	Asmp.(QGen)
7	(7)	$\lambda x_{\langle\langle\tau\rangle\rangle}(\Box\mathbf{N}(x) \wedge \Box E[x])(x_{\langle\langle\tau\rangle\rangle})$ $\wedge\mathbf{S}(x_{\langle\langle\tau\rangle\rangle}, y_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\rightarrow$ I)
7	(8)	$\Box E[x_{\langle\langle\tau\rangle\rangle}]$	7; QL
7	(9)	$y_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]$	7; QL
	(10)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow \Box E[\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]]$	$\mathrm{Comp_C}$; K
7	(11)	$\Box E[\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]]$	8; 10; PL
7	(12)	$\Box E[y_{\langle\langle\tau\rangle\rangle}]$	9; 11; QL
	(13)	$\Box\forall x_{\langle\langle\tau\rangle\rangle}(x = x) \rightarrow$ $(\Box E[\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]] \rightarrow \Box\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}] = \mathbf{S}[x_{\langle\langle\tau\rangle\rangle}])$	KQL
7	(14)	$\Box\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}] = \mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]$	11; 13;
7	(15)	$\Box y_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]$	9; 14; QL
	(16)	$\Box(E[\mathbf{S}] \wedge E[x_{\langle\langle\tau\rangle\rangle}] \wedge E[y_{\langle\langle\tau\rangle\rangle}] \wedge y_{\langle\langle\tau\rangle\rangle}$ $= \mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]) \rightarrow \Box\mathbf{S}(x_{\langle\langle\tau\rangle\rangle}, y_{\langle\langle\tau\rangle\rangle})$	KQL
7	(17)	$\Box(E[\mathbf{S}] \wedge E[x_{\langle\langle\tau\rangle\rangle}] \wedge E[y_{\langle\langle\tau\rangle\rangle}]$ $\wedge y_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[x_{\langle\langle\tau\rangle\rangle}])$	8; 12; 15; Lem. 1; K
	(18)	$\Box\mathbf{S}(x_{\langle\langle\tau\rangle\rangle}, y_{\langle\langle\tau\rangle\rangle})$	16; 17; PL
	(19)	$\Box(E[x_{\langle\langle\tau\rangle\rangle}] \wedge E[y_{\langle\langle\tau\rangle\rangle}]) \rightarrow (\Box(\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$ $\wedge\mathbf{S}(x_{\langle\langle\tau\rangle\rangle}, y_{\langle\langle\tau\rangle\rangle})) \rightarrow \Box\mathbf{N}(y_{\langle\langle\tau\rangle\rangle}))$	Theo. 2; KQL
7	(20)	$\Box\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$	7; QL
7	(21)	$\Box\mathbf{N}(y_{\langle\langle\tau\rangle\rangle})$	8; 12; 18; 19; 20; K
6, 7	(22)	$\lambda x_{\langle\langle\tau\rangle\rangle}(\Box\mathbf{N}(x) \wedge \Box E[x])(y_{\langle\langle\tau\rangle\rangle})$	3; 6; 12; 21; QL
	(23)	$Her[\lambda x_{\langle\langle\tau\rangle\rangle}(\Box\mathbf{N}(x) \wedge \Box E[x])]$	5; 6; 7; 22; QL
	(24)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow (\Box\mathbf{N}(x) \wedge \Box E[x]))$	3; 4; 23; Theo. 6; $\mathtt{QL_C}$

□

Our next result is a modal strengthening of the *equinumerosity lemma*:

LEMMA 6 (Modal equinumerosity lemma). $\vdash \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Box\forall y_{\langle\tau\rangle}(x(y) \to \forall z_{\langle\tau\rangle}(x(z) \leftrightarrow y \approx z)))$.

Proof of Lemma 6.

1	(1)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
2	(2)	$\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$ I)
	(3)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \to (\Box\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to$ $\Box\forall y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y) \to \forall z_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(z) \leftrightarrow y \approx z)))$	Lem. 4; KQL
1,2	(4)	$\Box\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \wedge \Box E[x_{\langle\langle\tau\rangle\rangle}]$	1; 2; Lem. 5; QL
1,2	(5)	$\Box\forall y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y) \to \forall z_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(z) \leftrightarrow y \approx z))$	3; 4; PL
	(6)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Box\forall y_{\langle\tau\rangle}(x(y) \to \forall z_{\langle\tau\rangle}(x(z) \leftrightarrow y \approx z)))$	1; 2; 5; QL

$\Box$

The next result is arguably the crux of our proof that SIC is a theorem of KQL_C^{FP}:

LEMMA 7 (Impossible sharing). $\vdash \forall x_{\langle\langle\tau\rangle\rangle}\forall y_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(x) \wedge \mathbf{N}(y) \wedge \Diamond\exists z_{\langle\tau\rangle}(x(z) \wedge y(z))) \to x = y)$.

Proof of Lemma 7. The proof is by induction.
Base case:

1	(1)	$E[y_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
2	(2)	$\mathbf{N}(y_{\langle\langle\tau\rangle\rangle}) \wedge \Diamond\exists z_{\langle\tau\rangle}(\mathbf{0}(z) \wedge y_{\langle\langle\tau\rangle\rangle}(z))$	Asmp. ($\to$ I)
3	(3)	$\mathbf{0} \neq y_{\langle\langle\tau\rangle\rangle}$	Asmp. (*reduct.*)
1,2,3	(4)	$\exists n_{\langle\langle\tau\rangle\rangle}(\mathbf{S}(n, y_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{N}(n))$	1; 2; 3; Lem. 3; QL
5	(5)	$\mathbf{S}(n_{\langle\langle\tau\rangle\rangle}, y_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{N}(n_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\exists$E)
6	(6)	$E[n_{\langle\langle\tau\rangle\rangle}]$	Asmp. ($\exists$E)
5	(7)	$y_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[n_{\langle\langle\tau\rangle\rangle}]$	5; QL
8	(8)	$E[z_{\langle\tau\rangle}]$	Asmp. (QGen)
	(9)	$\mathbf{S}[n_{\langle\tau\rangle}](z_{\langle\tau\rangle}) \to \exists w_\tau(z(w))$	QL
	(10)	$\mathbf{0}(z_{\langle\tau\rangle}) \to \neg\exists w_\tau(z_{\langle\tau\rangle}(w))$	QL
	(11)	$\neg(\mathbf{0}(z_{\langle\tau\rangle}) \wedge \mathbf{S}[n_{\langle\tau\rangle}](z_{\langle\tau\rangle}))$	9; 10; PL
	(12)	$\neg\exists z_{\langle\tau\rangle}(\mathbf{0}(z_{\langle\tau\rangle}) \wedge \mathbf{S}[n_{\langle\tau\rangle}](z_{\langle\tau\rangle}))$	8; 11; QL
	(13)	$\Box\neg\exists z_{\langle\tau\rangle}(\mathbf{0}(z_{\langle\tau\rangle}) \wedge \mathbf{S}[n_{\langle\tau\rangle}](z_{\langle\tau\rangle}))$	12; K
5	(14)	$\Box\neg\exists z_{\langle\tau\rangle}(\mathbf{0}(z_{\langle\tau\rangle}) \wedge y_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle}))$	7; 13; QL
1,2,3	(15)	$\Box\neg\exists z_{\langle\tau\rangle}(\mathbf{0}(z_{\langle\tau\rangle}) \wedge y_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle}))$	2; 14; QL
1,2	(16)	$\mathbf{0} = y_{\langle\langle\tau\rangle\rangle}$	2; 3; 15; K
1	(17)	$(\mathbf{N}(y_{\langle\langle\tau\rangle\rangle}) \wedge \Diamond\exists z_{\langle\tau\rangle}(\mathbf{0}(z) \wedge y_{\langle\langle\tau\rangle\rangle}(z))) \to \mathbf{0} = y_{\langle\langle\tau\rangle\rangle}$	2; 16; PL
	(18)	$\forall y_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(y) \wedge \Diamond\exists z_{\langle\tau\rangle}(\mathbf{0}(z) \wedge y(z))) \to \mathbf{0} = y)$	1; 17; QL
	(19)	$\mathbf{N}(\mathbf{0}) \wedge \forall y_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(y) \wedge \Diamond\exists z_{\langle\tau\rangle}(\mathbf{0}(z) \wedge y(z))) \to \mathbf{0} = y)$	18; Theo. 1; PL
	(20)	$E[\lambda x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \wedge \forall y_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(y)$ $\wedge \Diamond\exists z_{\langle\tau\rangle}(x(z) \wedge y(z))) \to x = y))]$	QL$_C$
	(21)	$\lambda x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \wedge \forall y_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(y)$ $\wedge \Diamond\exists z_{\langle\tau\rangle}(x(z) \wedge y(z))) \to x = y))(\mathbf{0})$	19; 20; Lem. 1; QL

Inductive case:

22	(22)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
23	(23)	$E[y_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
24	(24)	$\lambda x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \wedge \forall n_{\langle\langle\tau\rangle\rangle}((\mathbf{N}(n)$ $\wedge \Diamond \exists z_{\langle\tau\rangle}(x(z) \wedge n(z)))$ $\rightarrow x = n))(x_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{S}(x_{\langle\langle\tau\rangle\rangle}, y_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\rightarrow$ I)
25	(25)	$E[n_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
26	(26)	$\mathbf{N}(n_{\langle\langle\tau\rangle\rangle}) \wedge \Diamond \exists z_{\langle\tau\rangle}(y_{\langle\langle\tau\rangle\rangle}(z) \wedge n_{\langle\langle\tau\rangle\rangle}(z))$	Asmp.($\rightarrow$ I)
24	(27)	$y_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[x_{\langle\langle\tau\rangle\rangle}]$	24; QL
	(28)	$\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}) \rightarrow \exists w_\tau(z_{\langle\tau\rangle}(w)$ $\wedge x_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle} - w))$	QL
29	(29)	$n_{\langle\langle\tau\rangle\rangle} = \mathbf{0}$	Asmp. ($\neg$I)
	(30)	$\Box\neg\exists z_{\langle\tau\rangle}(\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}](z) \wedge \mathbf{0}(z))$	KQL
24, 29	(31)	$\Box\neg\exists z_{\langle\tau\rangle}(y_{\langle\langle\tau\rangle\rangle}(z) \wedge n_{\langle\langle\tau\rangle\rangle}(z))$	27; 29; 30; QL
24, 26	(32)	$n_{\langle\langle\tau\rangle\rangle} \neq 0$	26; 29; 31; K
24, 25, 26	(33)	$\exists m_{\langle\langle\tau\rangle\rangle}(\mathbf{S}(m, n_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{N}(m))$	25; 26; 32; Lem. 3; QL
34	(34)	$\mathbf{S}(m_{\langle\langle\tau\rangle\rangle}, n_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{N}(m_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\exists$E)
35	(35)	$E[m_{\langle\langle\tau\rangle\rangle}]$	Asmp. ($\exists$E)
34	(36)	$n_{\langle\langle\tau\rangle\rangle} = \mathbf{S}(m_{\langle\langle\tau\rangle\rangle})$	34; QL
	(37)	$(\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}) \wedge \mathbf{S}[m_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}))$ $\rightarrow \exists w_\tau \exists u_\tau(z_{\langle\tau\rangle}(w) \wedge z_{\langle\tau\rangle}(u)$ $\wedge m_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle} - w) \wedge x_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle} - u))$	QL
	(38)	$\forall z_{\langle\tau\rangle}((\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}) \wedge \mathbf{S}[m_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}))$ $\rightarrow \exists w_\tau \exists u_\tau(z_{\langle\tau\rangle}(w) \wedge z_{\langle\tau\rangle}(u)$ $\wedge m_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle} - w) \wedge x_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle} - u)$ $\wedge E[z_{\langle\tau\rangle} - w] \wedge E[z_{\langle\tau\rangle} - u] \wedge z_{\langle\tau\rangle} - w$ $\approx z_{\langle\tau\rangle} - u)$	37; QL$_\mathrm{C}$
	(39)	$\forall z_{\langle\tau\rangle}((\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}) \wedge \mathbf{S}[m_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}))$ $\rightarrow \exists s_{\langle\tau\rangle} \exists t_{\langle\tau\rangle}(m_{\langle\langle\tau\rangle\rangle}(s)$ $\wedge x_{\langle\langle\tau\rangle\rangle}(t) \wedge s \approx t))$	38; QL
	(40)	$\exists z_{\langle\tau\rangle}((\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}) \wedge \mathbf{S}[m_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}))$ $\rightarrow \exists s_{\langle\tau\rangle} \exists t_{\langle\tau\rangle}(m_{\langle\langle\tau\rangle\rangle}(s)$ $\wedge x_{\langle\langle\tau\rangle\rangle}(t) \wedge s \approx t))$	39; QL
	(41)	$\Diamond \exists z_{\langle\tau\rangle}((\mathbf{S}[x_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}) \wedge \mathbf{S}[m_{\langle\langle\tau\rangle\rangle}](z_{\langle\tau\rangle}))$ $\rightarrow \Diamond \exists s_{\langle\tau\rangle} \exists t_{\langle\tau\rangle}(m_{\langle\langle\tau\rangle\rangle}(s)$ $\wedge x_{\langle\langle\tau\rangle\rangle}(t) \wedge s \approx t))$	40; K
24, 34	(42)	$\Diamond \exists z_{\langle\tau\rangle}(y_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle}) \wedge n_{\langle\langle\tau\rangle\rangle}(z_{\langle\tau\rangle}))$ $\rightarrow \Diamond \exists s_{\langle\tau\rangle} \exists t_{\langle\tau\rangle}(m_{\langle\langle\tau\rangle\rangle}(s)$ $\wedge x_{\langle\langle\tau\rangle\rangle}(t) \wedge s \approx t)$	27; 36; 41; QL
24, 26, 34	(43)	$\Diamond \exists s_{\langle\tau\rangle} \exists t_{\langle\tau\rangle}(m_{\langle\langle\tau\rangle\rangle}(s)$ $\wedge x_{\langle\langle\tau\rangle\rangle}(t) \wedge s \approx t)$	26; 42; PL
24	(44)	$\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$	24; QL

22, 24	(45)	$\Box \forall y_{\langle \tau \rangle}(x_{\langle \langle \tau \rangle \rangle}(y)$ $\to \forall z_{\langle \tau \rangle}(x_{\langle \langle \tau \rangle \rangle}(z) \leftrightarrow y \approx z))$	22; 44; Lem. 6; QL
22, 24, 34	(46)	$\Diamond \exists z_{\langle \tau \rangle}(x_{\langle \langle \tau \rangle \rangle}(z) \wedge m_{\langle \langle \tau \rangle \rangle}(z))$	43; 45; KQL
22, 24, 35	(47)	$(\mathbf{N}(m_{\langle \langle \tau \rangle \rangle}) \wedge \Diamond \exists z_{\langle \tau \rangle}(x_{\langle \langle \tau \rangle \rangle}(z)$ $\wedge m_{\langle \langle \tau \rangle \rangle}(z))) \to x_{\langle \langle \tau \rangle \rangle} = m_{\langle \langle \tau \rangle \rangle}$	22; 24; 35; QL
22, 24, 34, 35	(48)	$x_{\langle \langle \tau \rangle \rangle} = m_{\langle \langle \tau \rangle \rangle}$	34; 46; 47; PL
22, 24, 34, 35	(49)	$n_{\langle \langle \tau \rangle \rangle} = \mathbf{S}[x_{\langle \langle \tau \rangle \rangle}]$	36; 48; QL
22, 24, 34, 35	(50)	$y_{\langle \langle \tau \rangle \rangle} = n_{\langle \langle \tau \rangle \rangle}$	27; 49; QL
22, 24, 25, 26	(51)	$y_{\langle \langle \tau \rangle \rangle} = n_{\langle \langle \tau \rangle \rangle}$	33; 34; 35; 50; QL
22, 24	(52)	$\forall n_{\langle \langle \tau \rangle \rangle}((\mathbf{N}(n) \wedge \Diamond \exists z_{\langle \tau \rangle}(y_{\langle \langle \tau \rangle \rangle}(z)$ $\wedge n(z))) \to y_{\langle \langle \tau \rangle \rangle} = n)$	25; 26; 51; QL
22, 23, 24	(53)	$\mathbf{N}(y_{\langle \langle \tau \rangle \rangle})$	22; 23; 24; 44; Theo. 2; QL
22, 23, 24	(54)	$\mathbf{N}(y_{\langle \langle \tau \rangle \rangle}) \wedge \forall n_{\langle \langle \tau \rangle \rangle}((\mathbf{N}(n)$ $\wedge \Diamond \exists z_{\langle \tau \rangle}(y_{\langle \langle \tau \rangle \rangle}(z) \wedge n(z)))$ $\to y_{\langle \langle \tau \rangle \rangle} = n)$	52; 53; PL
22, 23, 24	(55)	$\lambda x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \wedge \forall n_{\langle \langle \tau \rangle \rangle}((\mathbf{N}(n)$ $\wedge \Diamond \exists z_{\langle \tau \rangle}(x(z) \wedge n(z)))$ $\to x = n))(y_{\langle \langle \tau \rangle \rangle})$	20; 21; 56; Theo. 5; QL
	(56)	$Her[\lambda x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \wedge \forall n_{\langle \langle \tau \rangle \rangle}((\mathbf{N}(n)$ $\wedge \Diamond \exists z_{\langle \tau \rangle}(x(z) \wedge n(z))) \to x = n))]$	22; 23; 24; QL
	(57)	$\forall x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \to \forall n_{\langle \langle \tau \rangle \rangle}((\mathbf{N}(n)$ $\wedge \Diamond \exists z_{\langle \tau \rangle}(x(z) \wedge n(z))) \to x = n))$	20; 21; 56; Theo. 6; QL

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

THEOREM 7 (Axiom $\mathsf{PA_2}$: $(\mathrm{SIC})_\tau^{\mathcal{R}}$). $\vdash \forall x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \to \Diamond \exists y_{\langle \tau \rangle}(x(y))) \to \forall x_{\langle \langle \tau \rangle \rangle} \forall y_{\langle \langle \tau \rangle \rangle}$ $\forall z_{\langle \langle \tau \rangle \rangle}((\mathbf{N}(x) \wedge \mathbf{N}(y) \wedge \mathbf{S}(x, z) \wedge \mathbf{S}(y, z)) \to x = y)$.

Proof of Theorem 7.

1	(1)	$\forall x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \to \Diamond \exists y_{\langle \tau \rangle}(x(y)))$	Asmp. $(\to \mathrm{I})$
2	(2)	$E[x_{\langle \langle \tau \rangle \rangle}]$	Asmp. (QGen)
3	(3)	$E[y_{\langle \langle \tau \rangle \rangle}]$	Asmp. (QGen)
4	(4)	$E[z_{\langle \langle \tau \rangle \rangle}]$	Asmp. (QGen)
5	(5)	$\mathbf{N}(x_{\langle \langle \tau \rangle \rangle}) \wedge \mathbf{N}(y_{\langle \langle \tau \rangle \rangle})$ $\wedge \mathbf{S}(x_{\langle \langle \tau \rangle \rangle}, z_{\langle \langle \tau \rangle \rangle}) \wedge \mathbf{S}(y_{\langle \langle \tau \rangle \rangle}, z_{\langle \langle \tau \rangle \rangle})$	Asmp. $(\to \mathrm{I})$
2, 4, 5	(6)	$\mathbf{N}(z_{\langle \langle \tau \rangle \rangle})$	2; 4; 5; Theo. 2; QL
1, 2, 4, 5	(7)	$\Diamond \exists w_{\langle \tau \rangle}(z_{\langle \langle \tau \rangle \rangle}(w))$	1; 2; 6; QL
	(8)	$(\mathbf{S}[x_{\langle \langle \tau \rangle \rangle}](w_{\langle \tau \rangle}) \wedge \mathbf{S}[y_{\langle \langle \tau \rangle \rangle}](w_{\langle \tau \rangle}))$ $\to \exists s_\tau \exists t_\tau(w(s) \wedge w(t) \wedge x_{\langle \langle \tau \rangle \rangle}(w - s)$ $\wedge y_{\langle \langle \tau \rangle \rangle}(w - t))$	QL

	(9)	$\exists w_{\langle \tau \rangle}(\mathbf{S}[x_{\langle\langle \tau \rangle\rangle}](w_{\langle \tau \rangle}) \wedge \mathbf{S}[y_{\langle\langle \tau \rangle\rangle}](w_{\langle \tau \rangle}))$	8; QL
		$\quad \rightarrow \exists w_{\langle \tau \rangle}\exists s_\tau \exists t_\tau(w(s) \wedge w(t)$	
		$\quad \wedge x_{\langle\langle \tau \rangle\rangle}(w-s) \wedge y_{\langle\langle \tau \rangle\rangle}(w-t))$	
	(10)	$\Diamond \exists w_{\langle \tau \rangle}(\mathbf{S}[x_{\langle\langle \tau \rangle\rangle}](w_{\langle \tau \rangle}) \wedge \mathbf{S}[y_{\langle\langle \tau \rangle\rangle}](w_{\langle \tau \rangle}))$	9; K
		$\quad \rightarrow \Diamond \exists w_{\langle \tau \rangle}\exists s_\tau \exists t_\tau(w(s) \wedge w(t)$	
		$\quad \wedge\, x_{\langle\langle \tau \rangle\rangle}(w-s) \wedge y_{\langle\langle \tau \rangle\rangle}(w-t))$	
5	(11)	$\mathbf{S}[x_{\langle\langle \tau \rangle\rangle}] = z_{\langle\langle \tau \rangle\rangle} = \mathbf{S}[y_{\langle\langle \tau \rangle\rangle}]$	5; QL
$1,2,4,5$	(12)	$\Diamond \exists w_{\langle \tau \rangle}(\mathbf{S}[x_{\langle\langle \tau \rangle\rangle}](w) \wedge \mathbf{S}[x_{\langle\langle \tau \rangle\rangle}](w))$	7; 11; KQL
$1,2,4,5$	(13)	$\Diamond \exists w_{\langle \tau \rangle}\exists s_\tau \exists t_\tau(w(s) \wedge w(t)$	10; 12; PL
		$\quad \wedge x_{\langle\langle \tau \rangle\rangle}(w-s) \wedge y_{\langle\langle \tau \rangle\rangle}(w-t))$	
$1,2,4,5$	(14)	$\Diamond \exists w_{\langle \tau \rangle}\exists u_{\langle \tau \rangle}(x_{\langle\langle \tau \rangle\rangle}(w) \wedge y_{\langle\langle \tau \rangle\rangle}(u) \wedge w \approx u)$	13; KQL_{C}
$2,5$	(15)	$\Box \forall y_{\langle \tau \rangle}(x_{\langle\langle \tau \rangle\rangle}(y) \rightarrow \forall z_{\langle \tau \rangle}(x_{\langle\langle \tau \rangle\rangle}(z) \leftrightarrow y \approx z))$	2; 5; Lem. 6; QL
$1,2,4,5$	(16)	$\Diamond \exists w_{\langle \tau \rangle}(x_{\langle\langle \tau \rangle\rangle}(w) \wedge y_{\langle\langle \tau \rangle\rangle}(w))$	14; 15; KQL
$1,2,3,4,5$	(17)	$\Diamond \exists w_{\langle \tau \rangle}(x_{\langle\langle \tau \rangle\rangle}(w) \wedge y_{\langle\langle \tau \rangle\rangle}(w))$	2; 3; 5; Lem. 7; QL
		$\quad \rightarrow x_{\langle\langle \tau \rangle\rangle} = y_{\langle\langle \tau \rangle\rangle}$	
$1,2,3,4,5$	(18)	$x_{\langle\langle \tau \rangle\rangle} = y_{\langle\langle \tau \rangle\rangle}$	16; 17; PL
	(19)	$\forall x_{\langle\langle \tau \rangle\rangle}(\mathbf{N}(x) \rightarrow \Diamond \exists y_{\langle \tau \rangle}(x(y))) \rightarrow$	1; 2; 3; 4; 5; 18; QL
		$\quad \forall x_{\langle\langle \tau \rangle\rangle}\forall y_{\langle\langle \tau \rangle\rangle}\forall z_{\langle \tau \rangle}((\mathbf{N}(x) \wedge \mathbf{N}(y)$	
		$\quad \wedge\, \mathbf{S}(x,z) \wedge \mathbf{S}(y,z)) \rightarrow x = y)$	

$\square$

Next we prove a couple of lemmas from which it will follow that, for every type τ, FP_τ is derivable in KQL_{C} from FP_{e}. The first of these results will make use of the following definitions:

DEFINITION 15 (Representative). *For all positive integers* n *and* $i \leq n$, *types* $\tau^1, ..., \tau^n$, *and every* τ^i-*singular term* α, *and where there is some* $j \leq n$ *such that* $\tau^j = e$ *only if* $\tau^i = e$, *let*

$$\uparrow [\alpha]_{\langle \tau^1, ..., \tau^n \rangle} := \lambda v_{\tau^1}^1 ... v_{\tau^n}^n (\bigwedge_{\substack{\tau^l = \tau^i, \\ 1 \leq l \leq n}} (v^i = \alpha) \wedge \bigwedge_{\substack{1 \leq j \leq n, \\ \tau^j \neq \tau^i}} (v^j = \Lambda_{\tau^j})),$$

where $v^1, ..., v^n$ *are the first distinct variables of, respectively, types* $\tau^1, ..., \tau^n$ *that occur free nowhere in* α. *We say that* $\uparrow [\alpha]_{\langle \tau^1, ..., \tau^n \rangle}$ *is the* $\langle \tau^1, ..., \tau^n \rangle$-*representative of* α.

DEFINITION 16 (Lift). *For all positive integers* n *and* $i \leq n$, *types* $\tau^1, ..., \tau^n$, *and every* $\langle \tau^i \rangle$-*singular term* β, *and where there is some* $j \leq n$ *such that* $\tau^j = e$ *only if* $\tau^i = e$, *let*

$$\Uparrow [\beta]_{\langle\langle \tau^1, ..., \tau^n \rangle\rangle} := \lambda v(\exists u(\beta(u) \wedge v = \uparrow [u]_{\langle \tau^1, ..., \tau^n \rangle})),$$

where v *and* u *are, respectively, the first distinct singular variables of, respectively, type* $\langle \tau^1, ..., \tau^n \rangle$ *and type* τ^i *that occur nowhere free in* β. *We say that* $\Uparrow [\beta]_{\langle\langle \tau^1, ..., \tau^n \rangle\rangle}$ *is the* $\langle\langle \tau^1, ..., \tau^n \rangle\rangle$-*lift of* β.

To give an example, let α be 'x_e', $n = 3$, $\tau_1 = e$, $\tau_2 = \langle e \rangle$ and $\tau_3 = e$. Then, α's $\langle e, \langle e \rangle, e \rangle$-representative is $\uparrow [x_e]_{\langle e, \langle e \rangle, e \rangle} := \lambda y_e z_{\langle e \rangle} w_e (y = x_e = w \land z = \Lambda)$. Further, let β be the variable '$u_{\langle e \rangle}$'. Then, $\Uparrow [u_{\langle e \rangle}]_{\langle e, \langle e \rangle, e \rangle} := \lambda t_{\langle e, \langle e \rangle, e \rangle}(\exists x_e(u(x) \land t = \uparrow [x]_{\langle e, \langle e \rangle, e \rangle}))$. Finally, $x_{\langle e \rangle}$ has no type $\langle e, \langle e \rangle \rangle$ representative, since $\langle e \rangle \neq e$.

Intuitively, $x_{\langle \tau^i \rangle}$ and its $\langle \langle \tau^1, \dots, \tau^n \rangle \rangle$-lift are equinumerous, for every $i \leq n$. We begin by establishing that this is indeed so.

LEMMA 8. $\vdash \Box \forall x_{\langle \tau^i \rangle}(x \approx \Uparrow [x]_{\langle \langle \tau^1, \dots, \tau^n \rangle \rangle} \land E[\Uparrow [x]_{\langle \langle \tau^1, \dots, \tau^n \rangle \rangle}])$, where $1 \leq i \leq n$ and, for every $j \leq n$, $\tau^j = e$ only if $\tau^i = e$.

Proof of Lemma 8.

1	(1)	$E[x_{\langle \sigma \rangle}]$	Asmp. (QGen)
1	(2)	$E[\Uparrow [x_{\langle \sigma \rangle}]_{\langle \langle \tau^1, \dots, \tau^n \rangle \rangle}]$	1; QL$_{\mathsf{C}}$
1	(3)	$x_{\langle \sigma \rangle} \approx \Uparrow [x_{\langle \sigma \rangle}]_{\langle \langle \tau^1, \dots, \tau^n \rangle \rangle}$	2; QL$_{\mathsf{C}}$
	(4)	$\Box \forall x_{\langle \sigma \rangle}(x \approx \Uparrow [x]_{\langle \langle \tau^1, \dots, \tau^n \rangle \rangle} \land E[\Uparrow [x]_{\langle \langle \tau^1, \dots, \tau^n \rangle \rangle}])$	1; 2; 3; KQL

$\square$

LEMMA 9. $\forall x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \to \exists y_{\langle \langle \sigma \rangle \rangle}(\mathbf{N}(y) \land \Box \forall z_{\langle \sigma \rangle} \forall u_{\langle \tau \rangle}((y(z) \land z \approx u) \to x(u))))$.

Proof of Lemma 9. The proof is by induction.

Base case:

1	(1)	$E[z_{\langle \sigma \rangle}]$	Asmp. (QGen)
2	(2)	$E[u_{\langle \tau \rangle}]$	Asmp. (QGen)
3	(3)	$\mathbf{0}_{\langle \langle \sigma \rangle \rangle}(z_{\langle \sigma \rangle}) \land z_{\langle \sigma \rangle} \approx u_{\langle \tau \rangle}$	Asmp. ($\to$ I)
3	(4)	$\neg \exists w_\sigma(z_{\langle \sigma \rangle}(w))$	3; QL
3	(5)	$\neg \exists w_\tau(u_{\langle \sigma \rangle}(w))$	3; 4; QL
	(6)	$(\neg \exists w_\tau(u_{\langle \sigma \rangle}(w)) \land E[\mathbf{0}_{\langle \langle \tau \rangle \rangle}] \land E[u_{\langle \sigma \rangle}]) \to \mathbf{0}_{\langle \langle \tau \rangle \rangle}(u_{\langle \tau \rangle})$	QL
2, 3	(7)	$\mathbf{0}_{\langle \langle \tau \rangle \rangle}(u_{\langle \tau \rangle})$	2; 5;6; Lem. 1; PL
	(8)	$\Box \forall z_{\langle \sigma \rangle} \forall u_{\langle \tau \rangle}((\mathbf{0}_{\langle \langle \sigma \rangle \rangle}(z) \land z \approx u) \to \mathbf{0}_{\langle \langle \tau \rangle \rangle}(u))$	1; 2; 3; 7; KQL
	(9)	$\mathbf{N}(\mathbf{0}_{\langle \langle \sigma \rangle \rangle}) \land \mathbf{N}(\mathbf{0}_{\langle \langle \tau \rangle \rangle})$	Theo. 1; PL
	(10)	$\mathbf{N}(\mathbf{0}_{\langle \langle \tau \rangle \rangle}) \land \exists y_{\langle \langle \sigma \rangle \rangle}(\mathbf{N}(y)$ $\land \Box \forall z_{\langle \sigma \rangle} \forall u_{\langle \tau \rangle}((y(z) \land z \approx u) \to \mathbf{0}_{\langle \langle \tau \rangle \rangle}(u)))$	8; 9; Lem. 1; QL
	(11)	$E[\lambda x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \land \exists y_{\langle \langle \sigma \rangle \rangle}(\mathbf{N}(y)$ $\land \Box \forall z_{\langle \sigma \rangle} \forall u_{\langle \tau \rangle}((y(z) \land z \approx u) \to x(u))))]$	QL$_{\mathsf{C}}$
	(12)	$\lambda x_{\langle \langle \tau \rangle \rangle}(\mathbf{N}(x) \land \exists y_{\langle \langle \sigma \rangle \rangle}(\mathbf{N}(y)$ $\land \Box \forall z_{\langle \sigma \rangle} \forall u_{\langle \tau \rangle}((y(z) \land z \approx u) \to x(u))))(\mathbf{0}_{\langle \langle \tau \rangle \rangle})$	10; 11; Lem. 1; PL

Inductive case:

13	(13)	$E[n_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
14	(14)	$E[m_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
15	(15)	$\lambda x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \wedge \exists y_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(y)$ $\wedge \Box\forall z_{\langle\sigma\rangle}\forall u_{\langle\tau\rangle}((y(z) \wedge z \approx u)$ $\to x(u))))(n_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{S}(n_{\langle\langle\tau\rangle\rangle}, m_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$ I)
15	(16)	$\mathbf{N}(n_{\langle\langle\tau\rangle\rangle}) \wedge \exists y_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(y)$ $\wedge \Box\forall z_{\langle\sigma\rangle}\forall u_{\langle\tau\rangle}((y(z) \wedge z \approx u)$ $\to n_{\langle\langle\tau\rangle\rangle}(u)))$	15; QL
15	(17)	$m_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[n_{\langle\langle\tau\rangle\rangle}]$	15; QL
18	(18)	$\mathbf{N}(y_{\langle\langle\sigma\rangle\rangle}) \wedge \Box\forall z_{\langle\sigma\rangle}\forall u_{\langle\tau\rangle}((y_{\langle\langle\sigma\rangle\rangle}(z)$ $\wedge z \approx u) \to n_{\langle\langle\tau\rangle\rangle}(u))$	Asmp.($\exists$E)
19	(19)	$E[y_{\langle\langle\sigma\rangle\rangle}]$	Asmp. ($\exists$E)
18, 19	(20)	$\exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{S}(y_{\langle\langle\sigma\rangle\rangle}, t))$	18 ;19; Theo. 5; QL
21	(21)	$\mathbf{S}(y_{\langle\langle\sigma\rangle\rangle}, t_{\langle\langle\sigma\rangle\rangle})$	Asmp. ($\exists$E)
22	(22)	$E[t_{\langle\langle\sigma\rangle\rangle}]$	Asmp. ($\exists$E)
18, 19, 21, 22	(23)	$\mathbf{N}(t_{\langle\langle\sigma\rangle\rangle})$	18; 19; 21; 22; Theo. 2; QL
24	(24)	$E[Z_{\langle\sigma\rangle}]$	Asmp. (QGen)
25	(25)	$E[U_{\langle\tau\rangle}]$	Asmp. (QGen)
26	(26)	$\mathbf{S}[y_{\langle\langle\sigma\rangle\rangle}](Z_{\langle\sigma\rangle}) \wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}$	Asmp. ($\to$ I)
24, 25, 26	(27)	$\exists w_\sigma \exists r_\tau(Z_{\langle\sigma\rangle}(w) \wedge y(Z_{\langle\sigma\rangle} - w)$ $\wedge U_{\langle\tau\rangle}(r) \wedge Z_{\langle\sigma\rangle} - w \approx U_{\langle\tau\rangle} - r)$	24; 25; 26; QL$_\mathrm{C}$
	(28)	$\Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((\mathbf{S}[y_{\langle\langle\sigma\rangle\rangle}](Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to \exists w_\sigma \exists r_\tau(Z_{\langle\sigma\rangle}(w)$ $\wedge y(Z_{\langle\sigma\rangle} - w) \wedge U_{\langle\tau\rangle}(r)$ $\wedge Z_{\langle\sigma\rangle} - w \approx U_{\langle\tau\rangle} - r))$	24; 25; 26; 27; KQL
21	(29)	$\mathbf{S}[y_{\langle\langle\sigma\rangle\rangle}] = t_{\langle\langle\sigma\rangle\rangle}$	21; QL
21	(30)	$\Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t_{\langle\langle\sigma\rangle\rangle}(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to \exists w_\sigma \exists r_\tau(Z_{\langle\sigma\rangle}(w)$ $\wedge y(Z_{\langle\sigma\rangle} - w) \wedge U_{\langle\tau\rangle}(r)$ $\wedge Z_{\langle\sigma\rangle} - w \approx U_{\langle\tau\rangle} - r))$	28; 29; QL
18, 21	(31)	$\Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t_{\langle\langle\sigma\rangle\rangle}(Z_{\langle\sigma\rangle}) \wedge Z_{\langle\sigma\rangle}$ $\approx U_{\langle\tau\rangle}) \to \exists w_\sigma \exists r_\tau(Z_{\langle\sigma\rangle}(w)$ $\wedge y(Z_{\langle\sigma\rangle} - w) \wedge U_{\langle\tau\rangle}(r) \wedge Z_{\langle\sigma\rangle} - w$ $\approx U_{\langle\tau\rangle} - r \wedge n_{\langle\langle\tau\rangle\rangle}(U_{\langle\tau\rangle} - r)))$	18; 30; KQL$_\mathrm{C}$
	(32)	$E[\mathbf{S}[n_{\langle\langle\tau\rangle\rangle}]] \to \forall U_{\langle\tau\rangle}(\exists r_\tau(U(r)$ $\wedge n_{\langle\langle\tau\rangle\rangle}(U - r)) \to \mathbf{S}[n_{\langle\langle\tau\rangle\rangle}](U))$	QL
	(33)	$\Box E[\mathbf{S}[n_{\langle\langle\tau\rangle\rangle}]] \to \Box\forall U_{\langle\tau\rangle}(\exists r_\tau(U(r)$ $\wedge n_{\langle\langle\tau\rangle\rangle}(U - r)) \to \mathbf{S}[n_{\langle\langle\tau\rangle\rangle}](U))$	32; K

18, 21	(34)	$\Box E[\mathbf{S}[n_{\langle\langle\tau\rangle\rangle}]]$ $\to \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t_{\langle\langle\sigma\rangle\rangle}(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to \mathbf{S}[n_{\langle\langle\tau\rangle\rangle}](U))$	31; 33; K
13, 14	(35)	$\mathbf{N}(m_{\langle\langle\tau\rangle\rangle})$	13; 14; 15; 16; Theo. 2; QL
13, 14	(36)	$\Box E[m_{\langle\langle\tau\rangle\rangle}]$	14; 35; Lem. 5; QL
13, 14, 15	(37)	$\Box E[\mathbf{S}[n_{\langle\langle\tau\rangle\rangle}]]$	17; 36; QL
13, 14, 15, 18, 21	(38)	$\Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t_{\langle\langle\sigma\rangle\rangle}(Z_{\langle\sigma\rangle}) \wedge Z_{\langle\sigma\rangle}$ $\approx U_{\langle\tau\rangle}) \to \mathbf{S}[n_{\langle\tau\rangle}](U))$	34; 37; PL
13, 14, 15, 18, 21	(39)	$\Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t_{\langle\langle\sigma\rangle\rangle}(Z_{\langle\sigma\rangle}) \wedge Z_{\langle\sigma\rangle}$ $\approx U_{\langle\tau\rangle}) \to m_{\langle\langle\tau\rangle\rangle}(U))$	17; 38; QL
13, 14, 15, 18, 19, 21, 22	(40)	$\mathbf{N}(t_{\langle\langle\sigma\rangle\rangle}) \wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t_{\langle\langle\sigma\rangle\rangle}(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to m_{\langle\langle\tau\rangle\rangle}(U))$	23; 39; PL
13, 14, 15, 18, 19	(41)	$\exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(t) \wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to m_{\langle\langle\tau\rangle\rangle}(U)))$	20; 21; 22; 40 QL
13, 14, 15	(42)	$\exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(t) \wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to m_{\langle\langle\tau\rangle\rangle}(U)))$	16; 18; 19; 41 QL
13, 14, 15	(43)	$\exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(t) \wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to m_{\langle\langle\tau\rangle\rangle}(U)))$	16; 18; 19; 41 QL
13, 14, 15	(44)	$\mathbf{N}(m_{\langle\langle\tau\rangle\rangle}) \wedge \exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(t)$ $\wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to m_{\langle\langle\tau\rangle\rangle}(U)))$	35; 43; PL
13, 14, 15	(45)	$\lambda x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \wedge \exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(t)$ $\wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t(Z_{\langle\sigma\rangle}) \wedge Z_{\langle\sigma\rangle}$ $\approx U_{\langle\tau\rangle}) \to x(U))))(m_{\langle\langle\tau\rangle\rangle})$	11; 14; 44; QL
	(46)	$Her[\lambda x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \wedge \exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(t)$ $\wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t(Z_{\langle\sigma\rangle})$ $\wedge Z_{\langle\sigma\rangle} \approx U_{\langle\tau\rangle}) \to x(U))))]$	13; 15; 45; QL
	(47)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \exists t_{\langle\langle\sigma\rangle\rangle}(\mathbf{N}(t)$ $\wedge \Box\forall Z_{\langle\tau\rangle}\forall U_{\langle\tau\rangle}((t(Z_{\langle\sigma\rangle}) \wedge Z_{\langle\sigma\rangle}$ $\approx U_{\langle\tau\rangle}) \to x(U))))$	11; 12; 46; Theo. 6; QL

$\Box$

LEMMA 10. $\vdash \mathrm{FP}_e \to \mathrm{FP}_\tau$.

Proof of Lemma 10. The lemma's proof is by induction on the set of types. The base case is obvious, since $\vdash \mathrm{FP}_e \to \mathrm{FP}_e$, by PL. We show that $\vdash \mathrm{FP}_e \to \mathrm{FP}_{\langle\tau^1,\dots,\tau^n\rangle}$. So, as the inductive hypothesis, assume that $\vdash \mathrm{FP}_e \to \mathrm{FP}_{\tau^i}$, for all positive integers n and i such that $1 \le i \le n$. Now, let: $\tau : \langle\tau^1, \dots, \tau^n\rangle$, and

$$\xi := \begin{cases} e, & \text{if there is some } i \le n \text{ such that } \tau^i \text{ is } e, \\ \tau^1, & \text{otherwise.} \end{cases}$$

To prove that $\vdash \mathrm{FP}_e \to \mathrm{FP}_{\langle\tau^1,\dots,\tau^n\rangle}$, it suffices, given the inductive hypothesis, to show that $\vdash \mathrm{FP}_\xi \to \mathrm{FP}_{\langle\tau^1,\dots,\tau^n\rangle}$. This is what we'll now show:

1	(1)	$E[x_{\langle\langle\langle\tau^1,...,\tau^n\rangle\rangle\rangle\rangle}]$	Asmp. (QGen)
2	(2)	$\mathbf{N}(x_{\langle\langle\langle\tau^1,...,\tau^n\rangle\rangle\rangle\rangle})$	Asmp. ($\rightarrow$ I)
1,2	(3)	$\exists y_{\langle\langle\xi\rangle\rangle}(\mathbf{N}(y) \wedge \Box\forall z_{\langle\langle\tau^1,...,\tau^n\rangle\rangle}\forall u_{\langle\xi\rangle}((y(z) \wedge z \approx u)$ $\rightarrow x_{\langle\langle\langle\tau^1,...,\tau^n\rangle\rangle\rangle}(u)))$	1; 2; Lem. 9; QL
4	(4)	$\mathbf{N}(y_{\langle\langle\xi\rangle\rangle}) \wedge \Box\forall z_{\langle\langle\tau^1,...,\tau^n\rangle\rangle}\forall u_{\langle\xi\rangle}((y_{\langle\langle\xi\rangle\rangle}(z) \wedge z \approx u)$ $\rightarrow x_{\langle\langle\langle\tau^1,...,\tau^n\rangle\rangle\rangle}(u))$	Asmp. ($\exists$E)
5	(5)	$E[y_{\langle\langle\xi\rangle\rangle}]$	Asmp. ($\exists$E)
6	(6)	$\forall y_{\langle\langle\xi\rangle\rangle}(\mathbf{N}(y) \rightarrow \Diamond\exists w_{\langle\xi\rangle}(x(w)))$	Asmp. (FP$_\xi$)
4,5,6	(7)	$\Diamond\exists w_{\langle\xi\rangle}(y_{\langle\langle\xi\rangle\rangle}(w))$	4; 5; 6; QL
4,5,6	(8)	$\Diamond\exists w_{\langle\xi\rangle}\exists t_{\langle\langle\tau^1,...,\tau^n\rangle\rangle}(y_{\langle\langle\xi\rangle\rangle}(w) \wedge w \approx t)$	7; Lem. 8; KQL
4,5,6	(9)	$\Diamond\exists t_{\langle\langle\tau^1,...,\tau^n\rangle\rangle}(x_{\langle\langle\langle\tau^1,...,\tau^n\rangle\rangle\rangle}(t))$	4; 8; KQL
1,2,6	(10)	$\Diamond\exists t_{\langle\langle\tau^1,...,\tau^n\rangle\rangle}(x_{\langle\langle\langle\tau^1,...,\tau^n\rangle\rangle\rangle}(t))$	3; 4; 5; 9; QL
	(11)	$\forall y_{\langle\langle\xi\rangle\rangle}(\mathbf{N}(y) \rightarrow \Diamond\exists w_{\langle\xi\rangle}(x(w))) \rightarrow \forall x_{\langle\langle\langle\tau^1,...,\tau^n\rangle\rangle\rangle}(\mathbf{N}(x)$ $\rightarrow \Diamond\exists t_{\langle\langle\tau^1,...,\tau^n\rangle\rangle}(x(t)))$	1; 2; 6; 10; QL

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

LEMMA 11. *For every theorem φ of* PA$_2$, $\vdash$ FP $\rightarrow \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, *for every type τ, where* $\overline{\forall}(\varphi)_\tau^{\mathcal{R}}$ *is the result of prefixing $(\varphi)_\tau^{\mathcal{R}}$ with any sequence of universal quantifiers binding all variables occurring in $(\varphi)_\tau^{\mathcal{R}}$.*

Proof of Lemma 11. The proof is by induction on the length of a PA$_2$-derivation. The base case has the following subcases: (i) φ is an arithmetical axiom of PA$_2$; (ii) φ is a PL theorem; (iii) φ is an instance of $\forall1$, $\forall2$ or Ind; (iv) φ is an instance of =I; (v) φ is an instance of C$\forall$E; (vi) φ is an instance of Comp:

Subcase (i): φ is an arithmetical axiom of PA$_2$. Then $\vdash$ FP $\rightarrow (\varphi)_\tau^{\mathcal{R}}$, by Theorems 1–7. Hence, $\vdash \overline{\forall}(\text{FP} \rightarrow (\varphi)_\tau^{\mathcal{R}})$, by QL-reasoning. So, $\vdash \overline{\forall}\text{FP} \rightarrow \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by QL-reasoning. So, $\vdash$ FP $\rightarrow \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by QL-reasoning.

Subcase (ii): φ is a PL-theorem. Then $\vdash (\varphi)_\tau^{\mathcal{R}}$, since $(\cdot)_\tau^{\mathcal{R}}$ preserves Boolean connectives, and every PL-theorem is an axiom of KQL$_\mathtt{C}$. So, $\vdash \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by QL-reasoning. Hence, $\vdash$ FP $\rightarrow \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by PL-reasoning.

Subcase (iii): φ is an instance of $\forall1$, $\forall2$, or Ind. Then, $\vdash (\varphi)_\tau^{\mathcal{R}}$. So, $\vdash \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by QL-reasoning. Hence, $\vdash$ FP $\rightarrow \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by PL-reasoning.

Subcase (iv): φ is an instance of =I. Then $\varphi : v = v$. Let $\overline{\forall}$ be a sequence of the form $\forall^\star\forall(v)_\tau^{\mathcal{R}}\forall^\dagger$, where $\forall^\star$ and $\forall^\dagger$ are themselves sequences of universal quantifiers and $\forall(v)_\tau^{\mathcal{R}}$ does not occur in $\forall^\dagger$. Then, $\vdash (v = v)_\tau^{\mathcal{R}} \rightarrow \forall^\dagger((v = v)_\tau^{\mathcal{R}})$, by repeated applications of $\forall1$, $\forall2$ and Gen. By a further application of Gen we get that $\vdash \forall(v)_\tau^{\mathcal{R}}((v = v)_\tau^{\mathcal{R}} \rightarrow \forall^\dagger((v = v)_\tau^{\mathcal{R}}))$, and so $\vdash \forall(v)_\tau^{\mathcal{R}}((v = v)_\tau^{\mathcal{R}}) \rightarrow \forall(v)_\tau^{\mathcal{R}}\forall^\dagger((v = v)_\tau^{\mathcal{R}})$, by $\forall1$. Since $\vdash \forall(v)_\tau^{\mathcal{R}}((v = v)_\tau^{\mathcal{R}})$, by $\forall=$ and QL-reasoning, we thus obtain the result that $\vdash \forall(v)_\tau^{\mathcal{R}}\forall^\dagger((v = v)_\tau^{\mathcal{R}})$. So, $\vdash \forall^\star\forall(v)_\tau^{\mathcal{R}}\forall^\dagger((v = v)_\tau^{\mathcal{R}})$ by repeated applications of Gen. Therefore, $\vdash$ FP $\rightarrow \overline{\forall}((v = v)_\tau^{\mathcal{R}})$, by PL-reasoning.

Subcase (v): φ is an instance of C$\forall$E. Then, $\varphi := \forall v\psi \rightarrow \psi_\alpha^v$. We have that $\vdash E[(\alpha)_\tau^{\mathcal{R}}] \rightarrow (\forall(v)_\tau^{\mathcal{R}}(\psi)_\tau^{\mathcal{R}} \rightarrow (\psi_\alpha^v)_\tau^{\mathcal{R}})$. Now, α is either the constant '0', or else an individual variable u. Suppose first that it is the constant '0'. Then, $\vdash E[(\alpha)_\tau^{\mathcal{R}}]$, by Lemma 1. So, $\vdash \forall(v)_\tau^{\mathcal{R}}(\psi)_\tau^{\mathcal{R}} \rightarrow (\psi_\alpha^v)_\tau^{\mathcal{R}}$, by PL. That is, $\vdash (\varphi)_\tau^{\mathcal{R}}$. Therefore, $\vdash \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by QL-reasoning. Hence, $\vdash$ FP $\rightarrow \overline{\forall}(\varphi)_\tau^{\mathcal{R}}$, by PL-reasoning.

Suppose instead that α is the variable u. Then, $\vdash E[(u)^{\mathcal{R}}_\tau] \to (\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau \to (\psi)^{\mathcal{R}\,(v)^{\mathcal{R}}_\tau}_{\tau\,(u)^{\mathcal{R}}_\tau})$, and $\overline{\forall}$ is a sequence of the form $\forall^\star\forall(u)^{\mathcal{R}}_\tau\forall^\dagger$, where $\forall^\star$ and $\forall^\dagger$ are themselves sequences of universal quantifiers, and $\forall(u)^{\mathcal{R}}_\tau$ does not occur in $\forall^\dagger$. By repeated applications of Gen, $\forall1$ and $\forall2$ we get that $\vdash E[(u)^{\mathcal{R}}_\tau] \to \forall^\dagger(\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau \to (\psi)^{\mathcal{R}\,(v)^{\mathcal{R}}_\tau}_{\tau\,(u)^{\mathcal{R}}_\tau})$. A further application of Gen shows that $\vdash \forall(u)^{\mathcal{R}}_\tau(E[(u)^{\mathcal{R}}_\tau] \to \forall^\dagger(\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau \to (\psi)^{\mathcal{R}\,(v)^{\mathcal{R}}_\tau}_{\tau\,(u)^{\mathcal{R}}_\tau}))$, and we get that $\vdash \forall(u)^{\mathcal{R}}_\tau(E[(u)^{\mathcal{R}}_\tau]) \to \forall(u)^{\mathcal{R}}_\tau\forall^\dagger(\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau \to (\psi)^{\mathcal{R}\,(v)^{\mathcal{R}}_\tau}_{\tau\,(u)^{\mathcal{R}}_\tau})$, by a further application of $\forall1$. So, $\vdash \forall(u)^{\mathcal{R}}_\tau\forall^\dagger(\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau \to (\psi)^{\mathcal{R}\,(v)^{\mathcal{R}}_\tau}_{\tau\,(u)^{\mathcal{R}}_\tau})$, by $\forall E2$. Finally, repeated applications of Gen yield that $\vdash \forall^\star\forall(u)^{\mathcal{R}}_\tau\forall^\dagger(\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau \to (\psi)^{\mathcal{R}\,(v)^{\mathcal{R}}_\tau}_{\tau\,(u)^{\mathcal{R}}_\tau})$. That is, $\vdash \overline{\forall}(\varphi)^{\mathcal{R}}_\tau$. So, $\vdash \mathrm{FP} \to \overline{\forall}(\varphi)^{\mathcal{R}}_\tau$, by PL-reasoning.

Subcase (vi): φ is an instance of Comp. Then, $\varphi := \exists U\forall v^1 \ldots \forall v^n(U(v^1,\ldots,v^n) \leftrightarrow \psi)$. By $\mathrm{Comp}_{\mathrm{C}}$ we get $\vdash E_{\lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau} \to E[\lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau]$. Now, $\vdash E[\lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau] \to (E[(v^n)^{\mathcal{R}}_\tau] \to (\ldots \to (E[(v^1)^{\mathcal{R}}_\tau] \to ((\psi)^{\mathcal{R}}_\tau \leftrightarrow \lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau)((v^1)^{\mathcal{R}}_\tau,\ldots,(v^n)^{\mathcal{R}}_\tau))\ldots))$. So, repeated applications of Gen, $\forall1$, $\forall2$ and $\forall E2$ yield $\vdash E[\lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau] \to \forall(v^1)^{\mathcal{R}}_\tau\ldots\forall(v^n)^{\mathcal{R}}_\tau((\psi)^{\mathcal{R}}_\tau \leftrightarrow \lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau)((v^1)^{\mathcal{R}}_\tau,\ldots,(v^n)^{\mathcal{R}}_\tau)$. Hence, $\vdash E[\lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau] \to \exists(U)^{\mathcal{R}}_\tau\forall(v^1)^{\mathcal{R}}_\tau\ldots\forall(v^n)^{\mathcal{R}}_\tau((\psi)^{\mathcal{R}}_\tau \leftrightarrow (U)^{\mathcal{R}}_\tau((v^1)^{\mathcal{R}}_\tau,\ldots,(v^n)^{\mathcal{R}}_\tau))$, by QL-reasoning. So, $\vdash E_{\lambda(v^1)^{\mathcal{R}}_\tau\ldots(v^n)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau} \to \exists(U)^{\mathcal{R}}_\tau\forall(v^1)^{\mathcal{R}}_\tau\ldots\forall(v^n)^{\mathcal{R}}_\tau((\psi)^{\mathcal{R}}_\tau \leftrightarrow (U)^{\mathcal{R}}_\tau((v^1)^{\mathcal{R}}_\tau,\ldots,(v^n)^{\mathcal{R}}_\tau))$. But then, proceeding as in the proof of subcase (iv), we can derive, by repeated applications of Gen, $\forall1$, $\forall2$ and $\forall E2$, that $\vdash \overline{\forall}\exists(U)^{\mathcal{R}}_\tau\forall(v^1)^{\mathcal{R}}_\tau\ldots\forall(v^n)^{\mathcal{R}}_\tau((\psi)^{\mathcal{R}}_\tau \leftrightarrow (U)^{\mathcal{R}}_\tau((v^1)^{\mathcal{R}}_\tau,\ldots,(v^n)^{\mathcal{R}}_\tau))$. That is, $\vdash \overline{\forall}(\varphi)^{\mathcal{R}}_\tau$. So, $\vdash \mathrm{FP} \to \overline{\forall}(\varphi)^{\mathcal{R}}_\tau$. This concludes the proof of the base case.

For the inductive case, there are two subcases two consider: (i) φ is derived by an application of Gen; (ii) φ is derived by an application of MP.

Subcase (i): φ is derived by an application of Gen. Then, $\varphi := \forall v\psi$. By the induction hypothesis, we have that $\vdash \mathrm{FP} \to \overline{\forall}(\psi)^{\mathcal{R}}_\tau$, for any universal closure $\overline{\forall}(\psi)^{\mathcal{R}}_\tau$ of $(\psi)^{\mathcal{R}}_\tau$. But then, $\vdash \mathrm{FP} \to \overline{\forall}\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau$, for any universal closure $\overline{\forall}\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau$ of $\forall(v)^{\mathcal{R}}_\tau(\psi)^{\mathcal{R}}_\tau$. I.e., we have that $\vdash \mathrm{FP} \to \overline{\forall}(\varphi)^{\mathcal{R}}_\tau$.

Subcase (ii): φ is derived by an application of MP. Then, by the induction hypothesis, we have that $\vdash \mathrm{FP} \to \overline{\forall}(\psi)^{\mathcal{R}}_\tau$ and $\vdash \mathrm{FP} \to \overline{\forall}(\psi \to \varphi)^{\mathcal{R}}_\tau$. Let $\overline{\forall}$ be the sequence $\forall v^1\ldots\forall v^n$. Then, by repeated applications of $\forall E$, we get $\vdash \mathrm{FP} \to (E[v^n] \to (\ldots \to (E[v^1] \to (\psi)^{\mathcal{R}}_\tau)\ldots))$ and $\vdash \mathrm{FP} \to (E[v^n] \to (\ldots \to (E[v^1] \to (\psi \to \varphi)^{\mathcal{R}}_\tau)\ldots))$. By PL-reasoning we get $\vdash \mathrm{FP} \to (E[v^n] \to (\ldots \to (E[v^1] \to (\varphi)^{\mathcal{R}}_\tau)\ldots))$. But then, proceeding as in the proof of subcase (v) of the base case, we can derive, by repeated applications of Gen, $\forall1$, $\forall2$ and $\forall E2$, $\vdash \mathrm{FP} \to \overline{\forall}(\varphi)^{\mathcal{R}}_\tau$. This concludes the proof. $\square$

We thus obtain Theorem 8 as a straightforward corollary of Lemma 11:

THEOREM 8 (Interpretability). $\mathrm{KQL}^{\mathrm{FP}}_{\mathrm{C}}$ *interprets* PA_2 *via the* Russellian translation $(\cdot)^{\mathcal{R}}_\tau$, *for each type* τ. *More precisely, for every closed theorem* φ *of* PA_2, $(\varphi)^{\mathcal{R}}_\tau$ *is a theorem of* $\mathrm{KQL}^{\mathrm{FP}}_{\mathrm{C}}$, *for every type* τ.

§B. Proof that FP **implies its necessity.** We begin by observing that the following result, stronger than Theorem 6, is also derivable by using nothing but QL-reasoning:

LEMMA 12. $\vdash_{\mathrm{QL}} \forall y_{\langle\langle\langle\tau\rangle\rangle\rangle}((y(\boldsymbol{0}) \wedge Her[y]) \to (\boldsymbol{N}(x_{\langle\langle\tau\rangle\rangle}) \to y(x_{\langle\langle\tau\rangle\rangle})))$.

Owing to Lemma 12, it is easy to see that Lemma 5's proof can easily be adapted to establish the following lemmas:

LEMMA 13. $\vdash N(x) \to E[x]$.

LEMMA 14. $\vdash N(x) \to (\Box N(x) \land \Box E[x])$.

The following result can then be established:

LEMMA 15. $\vdash_{\text{S5QL}_C} \text{FP}_\tau \to \Box\text{FP}_\tau$.

Proof of Lemma 15.

(1)	$\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y))) \to (E[x_{\langle\langle\tau\rangle\rangle}]$ $\to (N(x_{\langle\langle\tau\rangle\rangle}) \to \Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))))$	QL
(2)	$N(x_{\langle\langle\tau\rangle\rangle}) \to E[x_{\langle\langle\tau\rangle\rangle}]$	Lem. 13
(3)	$\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y))) \to$ $(N(x_{\langle\langle\tau\rangle\rangle}) \to \Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	1; 2; PL
(4)	$\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$ $\to \Box(N(x_{\langle\langle\tau\rangle\rangle}) \to \Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	3; Lem. 14; S5
(5)	$\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y))) \to \Diamond\Box(N(x_{\langle\langle\tau\rangle\rangle})$ $\to \Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	4; K
(6)	$\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y))) \to (N(x_{\langle\langle\tau\rangle\rangle})$ $\to \Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	5; S5
(7)	$\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$ $\to \forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$	6; QL
(8)	$\Box\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$ $\to \Box\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$	7; K
(9)	$\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$ $\to \Box\forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$	8; S5

$\Box$

§C. Necessity of arithmetic. The main aim of this section is to prove that the result $(\varphi)_\tau^{\mathcal{A}}$ of translating each closed formula φ of PA_2 to PA_2 is, if true, necessarily so. That is, it is proven that $(\varphi)_\tau^{\mathcal{A}} \to \Box(\varphi)_\tau^{\mathcal{A}}$, for every closed formula φ of PA_2.

Say that a relation is *arithmetical* just in case it necessarily exists and is essentially and rigidly about natural numbers. That is:

DEFINITION 17 (Arithmetical relations). *Where α is a type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-singular term, and, for every* i *such that $1 \leq i \leq n$, v^i is a type $\langle\langle\tau\rangle\rangle$-singular variable which occurs nowhere in α:*

$Ar[\alpha] :=$

$$\Box\forall v^1_{\langle\langle\tau\rangle\rangle} \ldots \Box\forall v^n_{\langle\langle\tau\rangle\rangle} \Box(\alpha(v^1, \ldots, v^n) \to (\bigwedge_{1 \leq i \leq n}(N(v^i)) \land \Box(\alpha(v^1, \ldots, v^n)))) \land \Box E[\alpha].$$

Then, the proof that $(\varphi)_\tau^{\mathcal{A}} \to \Box(\varphi)_\tau^{\mathcal{A}}$, for every closed formula φ of PA_2, relies on the following lemma: if all the parameters occurring in $(\varphi)_\tau^{\mathcal{A}}$ are natural numbers or arithmetical relations, then $(\varphi)_\tau^{\mathcal{A}} \to \Box(\varphi)_\tau^{\mathcal{A}}$. It furthermore relies on the result that, for

every formula ψ of PA_2, if all the parameters occurring in $(\forall V\psi)^{\mathcal{A}}_\tau$ are either natural numbers or arithmetical relations, then if every *arithmetical* relation V is such that $(\psi)^{\mathcal{A}}_\tau$, then *every* relation V is such that $(\psi)^{\mathcal{A}}_\tau$.

This last result relies on showing that whenever some relation V is such that $(\psi)^{\mathcal{A}}_\tau$, there is a relation of a particular kind, defined in terms of V, which is arithmetical and such that $(\psi)^{\mathcal{A}}_\tau$. I'll now turn to the definition of this particular kind of arithmetical relation, which will be characterized partly in terms of plural resources.

I'll start by defining a relation's 'pluralization' and a plurality's 'relationization'. I'll then show that if U is the relationization of the pluralization of some relation R, then U is an arithmetical relation.

DEFINITION 18 (Pluralization). *For each type τ, type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-plural variable vv, type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-singular variable V, and where u is a type $\langle\langle\langle\tau\rangle\rangle\rangle_n$ singular variable distinct from V:*

$$\text{Pl}[vv, V] := \forall u(u \prec vv \leftrightarrow \exists z^1_{\langle\langle\tau\rangle\rangle} \dots \exists z^n_{\langle\langle\tau\rangle\rangle}(V(z^1, \dots, z^n) \wedge \bigwedge_{1 \leq i \leq n}(\boldsymbol{N}(z^i)) \wedge u$$
$$= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i)))).$$

So, the $yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$ are a *pluralization* of relation $x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$—that is, $\text{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ —just in case the $yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$ are relations each one of which holds of $w^1, \dots, w^n$ just in case each w^i is identical to some z^i such that z^i is a natural number and $z^1, \dots, z^n$ stand in relation $x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$.

DEFINITION 19 (Relationization). *For each type τ and type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-plural variable vv:*

$$\text{Rel}[vv] := \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(x \prec vv \wedge x(w^1, \dots, w^n))).$$

So, the *relationization* of the $yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$, $\text{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$, is the relation of being $w^1_{\langle\langle\tau\rangle\rangle}$, $\dots, w^n_{\langle\langle\tau\rangle\rangle}$ such that $w^1_{\langle\langle\tau\rangle\rangle}, \dots, w^n_{\langle\langle\tau\rangle\rangle}$ stand in some relation among the $yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$.

Proving the result that a relation is arithmetical if it is the relationization of the pluralization of some relation requires establishing a number of auxiliary lemmas about pluralities of 'ordered n-tuples' of natural numbers. I'll first introduce a type-theoretic characterization of ordered n-tuples' of natural numbers, and of pluralities of these. I'll then proceed to the proof of the auxiliary lemmas.

DEFINITION 20 (Ordered n-tuples of natural numbers). *For each type τ, and where α is a type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-singular term, for each i such that $1 \leq i \leq n$, v^i and u^i are type $\langle\langle\tau\rangle\rangle$-singular variables none of which occur in α, and such that v^i and u^j are distinct, for every i and j such that $1 \leq i, j \leq n$:*

$$\text{OrdN}[\alpha] := \exists v^1 \dots \exists v^n(\bigwedge_{1 \leq i \leq n}(\boldsymbol{N}(v^i)) \wedge \alpha = \lambda u^1 \dots u^n(\bigwedge_{1 \leq i \leq n}(u^i = v^i))).$$

Then, pluralities of ordered n-tuples are defined as follows:

DEFINITION 21 (Plurality of ordered n-tuples of natural numbers). *Where vv is a type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-plural variable, and u is a type $\langle\langle\langle\tau\rangle\rangle\rangle_n$-singular variable, for each type τ:*

$$\text{POrdN}[vv] := \forall u(u \prec vv \rightarrow \text{OrdN}[u]).$$

LEMMA 16. $\vdash \mathrm{OrdN}[\alpha] \to (\Box\mathrm{OrdN}[\alpha] \wedge \Box E[\alpha])$.

Proof of Lemma 16.

1	(1)	$\mathrm{OrdN}[\alpha]$	Asmp. ($\to$ I)
2	(2)	$\bigwedge_{1\leq i\leq n}(\mathbf{N}(v^i)) \wedge \alpha = \lambda u^1 \ldots u^n(\bigwedge_{1\leq i\leq n}(u^i = v^i))$	Asmp. ($\exists$E)
3	(3)	$\bigwedge_{1\leq i\leq n}(E(v^i))$	Asmp. ($\exists$E)
2, 3	(4)	$\bigwedge_{1\leq i\leq n}(\Box E(v^i)) \wedge \bigwedge_{1\leq i\leq n}(\Box\mathbf{N}(v^i))$	2; 3; Lem. 5; QL
2, 3	(5)	$\Box E[\lambda u^1 \ldots u^n(\bigwedge_{1\leq i\leq n}(u^i = v^i))]$	4; $\mathtt{KQL_C}$
	(6)	$\Box E[\lambda u^1 \ldots u^n(\bigwedge_{1\leq i\leq n}(u^i = v^i))]$	KQL
		$\quad\to (\alpha = \lambda u^1 \ldots u^n(\bigwedge_{1\leq i\leq n}(u^i = v^i))$	
		$\quad\to \Box\alpha = \lambda u^1 \ldots u^n(\bigwedge_{1\leq i\leq n}(u^i = v^i)))$	
2, 3	(7)	$\Box\alpha = \lambda u^1 \ldots u^n(\bigwedge_{1\leq i\leq n}(u^i = v^i))$	2; 5; 6; PL
2, 3	(8)	$\bigwedge_{1\leq i\leq n}(\Box E(v^i)) \wedge \bigwedge_{1\leq i\leq n}(\Box\mathbf{N}(v^i))$	4; 7; PL
		$\quad\wedge \Box\alpha = \lambda u^1 \ldots u^n(\bigwedge_{1\leq i\leq n}(u^i = v^i))$	
2, 3	(9)	$\Box\mathrm{OrdN}[\alpha]$	8; KQL
2, 3	(10)	$\Box E[\alpha]$	2; 5; QL
2, 3	(11)	$\Box\mathrm{OrdN}[\alpha] \wedge \Box E[\alpha]$	9; 10; PL
1	(12)	$\Box\mathrm{OrdN}[\alpha] \wedge \Box E[\alpha]$	1; 2; 3; 11; QL
	(13)	$\mathrm{OrdN}[\alpha] \to (\Box\mathrm{OrdN}[\alpha] \wedge \Box E[\alpha])$	1; 12; PL

$\Box$

LEMMA 17. $\vdash_{\mathtt{S5PQL_C}} (\mathrm{POrdN}[vv] \wedge E[vv]) \to (\Box\mathrm{POrdN}[vv] \wedge \Box E[vv])$.

Proof of Lemma 17.

1	(1)	$\mathrm{POrdN}[vv] \wedge E[vv]$	Asmp. ($\to$ I)
1	(2)	$\forall u(u \prec vv \to \mathrm{OrdN}[u])$	1; PL
3	(3)	$E[u]$	Asmp. (QGen)
4	(4)	$u \prec vv$	Asmp. ($\to$ I)
1, 3, 4	(5)	$\mathrm{OrdN}[u]$	2; 3; 4; QL
1, 3, 4	(6)	$\Box E[u]$	5; Lem. 16; PL
1, 3	(7)	$u \prec vv \to \Box E[uu]$	4; 6; PL
1	(8)	$\forall u(u \prec vv \to \Box E[u])$	3; 7; QL
1	(9)	$E[vv]$	1; PL
1	(10)	$\Box E[vv]$	8; 9; PQL ($\Box E_P$)
11	(11)	$\Diamond u \prec vv$	Asmp. ($\to$ I)
	(12)	$u \prec vv \to \Box(E[vv] \to (u \prec vv \wedge E[u]))$	PQL ($\prec$ I)
	(13)	$\Box E[vv] \to (u \prec vv \to \Box(u \prec vv \wedge E[u]))$	12; K
	(14)	$\Box\Box E[vv] \to \Box(u \prec vv \to \Box(u \prec vv \wedge E[u]))$	13; K
	(15)	$\Box\Box E[vv] \to (\Diamond u \prec vv \to \Diamond\Box(u \prec vv \wedge E[u]))$	14; K
	(16)	$\Box E[vv] \to (\Diamond u \prec vv \to (u \prec vv \wedge E[u]))$	15; S5

1, 11	(17)	$u \prec vv \land E[u]$	10; 11; 16; PL
1, 11	(18)	$\mathrm{OrdN}[u]$	2; 17; QL
1, 11	(19)	$\Box\mathrm{OrdN}[u]$	18; Lem. 16; PL
1	(20)	$\Diamond u \prec vv \to \Box\mathrm{OrdN}[u]$	11; 19; PL
1	(21)	$\Box(u \prec vv \to \mathrm{OrdN}[u])$	20; K
	(22)	$(\mathrm{POrdN}[vv] \land E[vv]) \to \Box(u \prec vv \to \mathrm{OrdN}[u])$	1; 21; PL
	(23)	$\Box((\mathrm{POrdN}[vv] \land E[vv]) \to \Box(u \prec vv \to \mathrm{OrdN}[u]))$	22; K
	(24)	$\Diamond(\mathrm{POrdN}[vv] \land E[vv]) \to \Diamond\Box(u \prec vv \to \mathrm{OrdN}[u])$	23; K
	(25)	$\Diamond(\mathrm{POrdN}[vv] \land E[vv]) \to (u \prec vv \to \mathrm{OrdN}[u])$	24; S5
	(26)	$\Diamond(\mathrm{POrdN}[vv] \land E[vv]) \to \forall u(u \prec vv \to \mathrm{OrdN}[u])$	25; QL
	(27)	$\Box\Diamond(\mathrm{POrdN}[vv] \land E[vv]) \to \Box\forall u(u \prec vv \to \mathrm{OrdN}[u])$	26; K
	(28)	$(\mathrm{POrdN}[vv] \land E[vv]) \to (\Box\mathrm{OrdN}[vv] \land \Box E[vv])$	1; 10; 27; Def. OrdN[·]; S5

$\Box$

LEMMA 18. $\vdash \mathrm{OrdN}[\alpha] \to \Box(\alpha(w^1, \dots, w^n) \to (\bigwedge_{1 \le i \le n}(\mathbf{N}(w^i)) \land \Box\alpha(w^1, \dots, w^n)))$.

Proof of Lemma 18.

1	(1)	$\mathrm{OrdN}[\alpha]$	Asmp. $(\to I)$
2	(2)	$\alpha(w^1, \dots, w^n)$	Asmp. $(\to I)$
3	(3)	$\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \land \alpha = \lambda u^1 \dots u^n(\bigwedge_{1 \le i \le n}(u^i = v^i))$	Asmp. $(\exists E)$
4	(4)	$\bigwedge_{1 \le i \le n}(E[v^i])$	Asmp. $(\exists E)$
2, 3	(5)	$\lambda u^1 \dots u^n(\bigwedge_{1 \le i \le n}(u^i = v^i))(w^1, \dots, w^n)$	2; 3; QL
2, 3	(6)	$(\bigwedge_{1 \le i \le n}(w^i = v^i))$	5; KQL
3, 4	(7)	$\bigwedge_{1 \le i \le n}(\Box E[v^i])$	3; 4; Lem. 5; KQL
2, 3, 4	(8)	$\bigwedge_{1 \le i \le n}(\Box(w^i = v^i))$	6; 7; KQL
2, 3, 4	(9)	$\bigwedge_{1 \le i \le n}(\Box E[w^i_\sigma])$	6; 7; KQL
3, 4	(10)	$\Box E[\lambda u^1 \dots u^n(\bigwedge_{1 \le i \le n}(u^i = v^i))]$	7; KQL$_\mathsf{C}$
2, 3, 4	(11)	$\Box\lambda u^1 \dots u^n(\bigwedge_{1 \le i \le n}(u^i = v^i))(w^1, \dots, w^n)$	8; 9, 10; KQL
2, 3, 4	(12)	$\Box\alpha(w^1, \dots, w^n)$	3; 11; QL
2, 3, 4	(13)	$\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \land \Box\alpha(w^1, \dots, w^n)$	3; 12; PL
1, 2	(14)	$\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \land \Box\alpha(w^1, \dots, w^n)$	1; 3; 4; 13; QL
1	(15)	$\alpha(w^1, \dots, w^n) \to (\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \land \Box\alpha(w^1, \dots, w^n))$	2; 14; PL
	(16)	$\Box\mathrm{OrdN}[\alpha] \to \Box(\alpha(w^1, \dots, w^n) \to (\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \land \Box\alpha(w^1, \dots, w^n)))$	1; 15; K
	(17)	$\mathrm{OrdN}[\alpha] \to \Box(\alpha(w^1, \dots, w^n) \to (\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \land \Box\alpha(w^1, \dots, w^n)))$	16; Lem. 16; PL

$\Box$

LEMMA 19. $\vdash_{\mathsf{S5PQL_C}} (\mathrm{POrdN}[vv] \land E[vv]) \to \Box\forall w^1 \dots \Box\forall w^n \Box(\mathrm{Rel}[yy](w^1, \dots, w^n) \to (\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \land \Box\mathrm{Rel}[yy](w^1, \dots, w^n)))$.

Proof of Lemma 19.

1	(1)	$\text{POrdN}[vv] \wedge E[vv]$	Asmp. ($\to$ I)
2	(2)	$\text{Rel}[vv](w^1, \dots, w^n)$	Asmp. ($\to$ I)
2	(3)	$\exists x(x \prec vv \wedge x(w^1, \dots, w^n))$	2; QL
4	(4)	$x \prec vv \wedge x(w^1, \dots, w^n)$	Asmp. ($\exists$E)
5	(5)	$E[x]$	Asmp. ($\exists$E)
1, 4, 5	(6)	$\text{OrdN}[x]$	1; 4; 5; QL
1, 4, 5	(7)	$\bigwedge_{1 \le i \le n}(\mathbf{N}(w^i)) \wedge \Box x(w^1, \dots, w^n)$	6; Lem. 18; QL
1, 4, 5	(8)	$\Box E[x]$	6; Lem. 16; PL
1	(9)	$\Box E[vv]$	1; Lem. 17; PL
1	(10)	$\Box x \prec vv$	9; KPQL ($\prec$ I)
1, 4, 5	(11)	$\Box E[x] \wedge \Box x \prec vv \wedge \Box x(w^1, \dots, w^n)$	7; 8; 10; PL
1, 4, 5	(12)	$\Box \exists x(x \prec vv \wedge x(w^1, \dots, w^n))$	11; KQL
1, 4, 5	(13)	$\bigwedge_{1 \le i \le n}(\Box E[w^i])$	7; Lem. 14; PL
1, 4, 5	(14)	$\Box E[\text{Rel}[vv]]$	9; KQL$_\text{C}$
1, 4, 5	(15)	$\Box \text{Rel}[vv](w^1, \dots, w^n)$	12; 13; 14; KQL
1, 4, 5	(16)	$\bigwedge_{1 \le i \le n}(\mathbf{N}(w^i)) \wedge \Box \text{Rel}[vv](w^1, \dots, w^n)$	7; 16; PL
1, 2	(17)	$\bigwedge_{1 \le i \le n}(\mathbf{N}(w^i)) \wedge \Box \text{Rel}[vv](w^1, \dots, w^n)$	3; 4; 5; 16; QL
	(18)	$\Box(\text{POrdN}[vv] \wedge E[vv]) \to \Box(\text{Rel}[vv](w^1, \dots, w^n)$ $\to (\bigwedge_{1 \le i \le n}(\mathbf{N}(w^i)) \wedge \Box \text{Rel}[vv](w^1, \dots, w^n)))$	1; 2; 17; KPL
	(19)	$(\text{POrdN}[vv] \wedge E[vv]) \to$ $\Box \forall w^1 \dots \Box \forall w^n \Box(\text{Rel}[vv](w^1, \dots, w^n)$ $\to (\bigwedge_{1 \le i \le n}(\mathbf{N}(w^i)) \wedge \Box \text{Rel}[vv](w^1, \dots, w^n)))$	18; Lem. 17; S5

$\Box$

LEMMA 20. $\vdash_{\text{PQL}} \text{Pl}[vv, V] \to \text{POrdN}[vv]$.

Proof of Lemma 20.

1	(1)	$\text{Pl}[vv, V]$	Asmp. ($\to$ I)
2	(2)	$E[u]$	Asmp. (QGen)
3	(3)	$u \prec vv$	Asmp. ($\to$ I)
1, 2, 3	(4)	$\exists z^1 \dots z^n(\bigwedge_{1 \le i \le n}(\mathbf{N}(z^i)) \wedge u$ $= \lambda w^1 \dots w^n(\bigwedge_{1 \le i \le n}(w^i = z^i)))$	1; 2; 3; QL
1, 2	(5)	$u \prec vv \to \text{OrdN}[u]$	3; 4; Def. OrdN[·]; PL
1	(6)	$\text{POrdN}[vv]$	2; 5; QL
	(7)	$\text{Pl}[vv, V] \to \text{POrdN}[vv]$	1; 6; PL

$\Box$

LEMMA 21. $\vdash_{\text{S5PQL}_\text{C}} (\text{Pl}[vv, V] \wedge E[vv]) \to \Box \forall w^1 \dots \Box \forall w^n \Box(\text{Rel}[vv](w^1, \dots, w^n) \to (\bigwedge_{1 \le i \le n}(\mathbf{N}(v^i)) \wedge \Box \text{Rel}[vv](w^1, \dots, w^n)))$.

Proof of Lemma 21. A trivial consequence of Lemmas 19 and 20. $\Box$

LEMMA 22. $\vdash_{\text{S5PQL}_\text{C}} (\text{Pl}[vv, V] \wedge E[vv]) \to \Box E[\text{Rel}[vv]].$

Proof of Lemma 22.

1	(1)	$\text{Pl}[vv, V] \wedge E[vv]$	Asmp. ($\to$ I)
1	(2)	$\text{POrdN}[vv]$	1; Lem. 20; PL
1	(3)	$\Box E[vv]$	1; 2; Lem. 17; PL
1	(4)	$\Box E[\text{Rel}[vv]]$	3; KQL$_\text{C}$
	(5)	$(\text{Pl}[vv, V] \wedge E[vv]) \to \Box E[\text{Rel}[vv]]$	1; 4; PL

□

We are now in a position to offer a proof of Theorem 9, a crucial element in our case for the claim the truths of pure arithmetic are all metaphysically necessary (Section 5). The proof of this result makes use of some basic facts about arithmetical relations:

LEMMA 23. $\vdash_{\text{S5PQL}_\text{C}} (\text{Pl}[vv, V] \wedge E[vv]) \to Ar[\text{Rel}[vv]].$

Proof of Lemma 23. This lemma is a straightforward consequence of Lemmas 21 and 22. □

LEMMA 24. $\vdash_{\text{KQL}} Ar[\Lambda].$

Proof of Lemma 24. A straightforward corollary of Lemma 2. □

LEMMA 25. $\vdash_{\text{S5PQL}} Ar[v_{\langle\langle\langle\tau\rangle\rangle\rangle_n}] \to \Box Ar[v_{\langle\langle\langle\tau\rangle\rangle\rangle_n}].$

Proof of Lemma 25. A straightforward consequence, by S5-reasoning, of the definition of $Ar[\cdot]$. □

Now, consider the following notational conventions. For each type τ and formula φ of MT, and where $\alpha^1, ..., \alpha^n$ are all the type τ-singular parameters occurring in φ, let

$$\mathbf{N}[\varphi] := \bigwedge_{1 \leq i \leq n} (\mathbf{N}(\alpha^i)).$$

Similarly, where $\beta^1, ..., \beta^m$ are all the type $\langle\langle\tau\rangle\rangle_n$-singular parameters occurring in φ, for any n, let

$$Ar_\tau[\varphi] := \bigwedge_{1 \leq i \leq n} (Ar[\beta^i]).$$

Finally, for each type τ and formula φ of MT, let $\overline{Ar}_\tau[\varphi] := \mathbf{N}[\varphi] \wedge Ar_\tau[\varphi]$. Then:

LEMMA 26. *For every formula φ of* PA$_2$ *in which the* n*-ary predicate variable V occurs free*:

(A) $\vdash_{\text{PQL}_\text{C}} \mathbf{N}[(\varphi)_\tau^{\mathcal{A}}] \to \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\text{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}] \to ((\varphi)_\tau^{\mathcal{A}} \leftrightarrow ((\varphi)_\tau^{\mathcal{A}})_{\text{Rel}[yy]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})).$

(B) $\vdash_{\text{QL}_\text{C}} \mathbf{N}[(\varphi)_\tau^{\mathcal{A}}] \to (\neg \exists z_{\langle\langle\tau\rangle\rangle}^1 ... \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1 \leq i \leq n} (\mathbf{N}(z^i)) \wedge V(z^1, ..., z^n)) \to ((\varphi)_\tau^{\mathcal{A}} \leftrightarrow$
$((\varphi)_\tau^{\mathcal{A}})_{\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})).$

Proof of Lemma 26. The proof is by induction on the complexity of φ. We prove (A) and (B) in one go, considering the base and inductive cases for each one of them.

Base case: the only interesting subcase is when $\varphi := U(\alpha^1, ..., \alpha^n)$ and $V = U$. Case (A):

1	(1)	$\mathbf{N}[(\varphi)^{\mathcal{A}}_\tau]$	Asmp. ($\to$ I)
2	(2)	$E[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. (QGen)
3	(3)	$\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. ($\to$ I)
2	(4)	$E[\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(x \prec yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$ $\wedge x(w^1,\dots,w^n)))]$	2; $\mathtt{QL_C}$
5	(5)	$V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$ I)
1	(6)	$\bigwedge_{1\le i\le n}(\mathbf{N}(\alpha^i_{\langle\langle\langle\tau\rangle\rangle\rangle_n}))$	1; PL
1	(7)	$\bigwedge_{1\le i\le n}(E[\alpha^i_{\langle\langle\tau\rangle\rangle}])$	6; Lem. 13; PL
1	(8)	$E[\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = \alpha^i_{\langle\langle\tau\rangle\rangle}))]$	7; $\mathtt{QL_C}$
1	(9)	$\bigwedge_{1\le i\le n}(\alpha^i_{\langle\langle\tau\rangle\rangle} = \alpha^i_{\langle\langle\tau\rangle\rangle})$	7; QL
1	(10)	$\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = \alpha^i_{\langle\langle\tau\rangle\rangle}))$ $(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	7; 8; 9; QL
1, 5	(11)	$\exists z^1_{\langle\langle\tau\rangle\rangle} \dots \exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n)$ $\wedge \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = \alpha^i_{\langle\langle\tau\rangle\rangle}))$ $= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = z^i)))$	5; 6; 7; 8; QL
1, 3, 5	(12)	$\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = \alpha^i_{\langle\langle\tau\rangle\rangle}))$ $\prec yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$	3; 8; 11; QL
1, 3, 5	(13)	$\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = \alpha^i_{\langle\langle\tau\rangle\rangle}))$ $\prec yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \wedge \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}$ $(\bigwedge_{1\le i\le n}(w^i = \alpha^i_{\langle\langle\tau\rangle\rangle}))(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	10; 12; PL
1, 3, 5	(14)	$\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(x \prec yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \wedge x(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle}))$	8; 13; QL
1, 2, 3, 5	(15)	$\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(x \prec yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$ $\wedge x(w^1,\dots,w^n)))(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle}))$	4; 7; 14; QL
1, 2, 3	(16)	$V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$ $\to \mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}](\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	5; 15; Def. Rel[·]; PL
17	(17)	$\mathrm{Rel}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$ I)
17	(18)	$\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(x \prec yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \wedge x(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle}))$	17; Def. Rel[·]; QL
19	(19)	$x_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \prec yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \wedge x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\exists$E)
20	(20)	$E[x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. ($\exists$E)
3, 19, 20	(21)	$\exists z^1_{\langle\langle\tau\rangle\rangle} \dots \exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n) \wedge x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$ $= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = z^i)))$	3; 19; 20; QL
22	(22)	$\bigwedge_{1\le i\le n}(\mathbf{N}(z^i_{\langle\langle\tau\rangle\rangle}))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1_{\langle\langle\tau\rangle\rangle},\dots,z^n_{\langle\langle\tau\rangle\rangle}) \wedge x_{\langle\langle\langle\tau\rangle\rangle\rangle_n}$ $= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\le i\le n}(w^i = z^i_{\langle\langle\tau\rangle\rangle}))$	Asmp. ($\exists$E)
23	(23)	$\bigwedge_{1\le i\le n}(E[z^i_{\langle\langle\tau\rangle\rangle}])$	Asmp. ($\exists$E)

19, 22	(24)	$\lambda w^1_{\langle\langle\tau\rangle\rangle}\dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\leq i\leq n}(w^i=z^i_{\langle\langle\tau\rangle\rangle}))$ $(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	19; 22; QL
19, 22	(25)	$\bigwedge_{1\leq i\leq n}(\alpha^i_{\langle\langle\tau\rangle\rangle}=z^i_{\langle\langle\tau\rangle\rangle})$	24; QL
19, 22	(26)	$V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	22; 25; QL
3, 19, 20	(27)	$V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	21; 22; 23; 26; QL
3, 17	(28)	$V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	18; 19; 20; 27; QL
3	(29)	$\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}](\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$ $\rightarrow V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	17; 28; PL
1, 2, 3	(30)	$(\varphi)^{\mathcal{A}}_\tau \leftrightarrow ((\varphi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}$	16; 29; PL
1, 2	(31)	$Pl[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\rightarrow ((\varphi)^{\mathcal{A}}_\tau \leftrightarrow ((\varphi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]})$	3; 30; PL
1	(32)	$\forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Pl[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\rightarrow ((\varphi)^{\mathcal{A}}_\tau \leftrightarrow ((\varphi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}))$	2; 31; QL
	(33)	$\mathbf{N}[(\varphi)^{\mathcal{A}}_\tau] \rightarrow \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Pl[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\rightarrow ((\varphi)^{\mathcal{A}}_\tau \leftrightarrow ((\varphi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}))$	1; 32; PL

Case (B):

1	(1)	$\mathbf{N}[(\varphi)^{\mathcal{A}}_\tau]$	Asmp. ($\rightarrow$ I)
2	(2)	$\neg\exists z^1_{\langle\langle\tau\rangle\rangle}\dots\exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\leq i\leq n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n))$	Asmp. ($\rightarrow$ I)
3	(3)	$V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\neg$I)
1	(4)	$\bigwedge_{1\leq i\leq n}(\mathbf{N}(\alpha^i_{\langle\langle\tau\rangle\rangle}))$	1; PL
1	(5)	$\bigwedge_{1\leq i\leq n}(E[\alpha^i_{\langle\langle\tau\rangle\rangle}])$	4; Lem. 13; PL
1, 3	(6)	$\exists z^1_{\langle\langle\tau\rangle\rangle}\dots\exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\leq i\leq n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n))$	3; 4; 5; QL
1, 2	(7)	$\neg V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	2; 3; 6; PL
1	(8)	$\neg\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	Lem. 2
1, 2	(9)	$V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle}) \leftrightarrow \Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\alpha^1_{\langle\langle\tau\rangle\rangle},\dots,\alpha^n_{\langle\langle\tau\rangle\rangle})$	7; 8; PL
1	(10)	$\neg\exists z^1_{\langle\langle\tau\rangle\rangle}\dots\exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\leq i\leq n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n)) \rightarrow ((\varphi)^{\mathcal{A}}_\tau \leftrightarrow ((\varphi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	2; 9; PL
	(11)	$\mathbf{N}[(\varphi)^{\mathcal{A}}_\tau] \rightarrow (\neg\exists z^1_{\langle\langle\tau\rangle\rangle}\dots\exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1\leq i\leq n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n)) \rightarrow ((\varphi)^{\mathcal{A}}_\tau \leftrightarrow ((\varphi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	1; 10; PL

Inductive case (i): $\varphi := \neg\psi$. Cases (A) and (B) follow straightforwardly from the inductive hypothesis.

Inductive case (ii): $\varphi := \psi \wedge \chi$. Case (A):

	(1)	$\mathbf{N}[(\psi)_\tau^{\mathcal{A}}] \to \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\psi)_{\langle\langle\tau\rangle\rangle}^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	Ind.Hyp.
	(2)	$\mathbf{N}[(\chi)_\tau^{\mathcal{A}}] \to \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\chi)_\tau^{\mathcal{A}} \leftrightarrow ((\chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	Ind. Hyp.
3	(3)	$\mathbf{N}[(\psi \wedge \chi)_\tau^{\mathcal{A}}]$	Asmp. $(\to \mathrm{I})$
3	(4)	$\mathbf{N}[(\psi)_\tau^{\mathcal{A}}]$	3; PL
3	(5)	$\mathbf{N}[(\chi)_\tau^{\mathcal{A}}]$	3; PL
3	(6)	$\forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\psi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	1; 4; PL
3	(7)	$\forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\chi)_\tau^{\mathcal{A}} \leftrightarrow ((\chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	2; 5; PL
8	(8)	$E[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. (QGen)
9	(9)	$\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. $(\to \mathrm{I})$
3, 8, 9	(10)	$(\psi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	6; 8; 9; QL
3, 8, 9	(11)	$(\chi)_\tau^{\mathcal{A}} \leftrightarrow ((\chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	7; 8; 9; QL
	(12)	$(\psi \wedge \chi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}} \wedge (\chi)_\tau^{\mathcal{A}})$	Def. $(\cdot)_\tau^{\mathcal{A}}$
	(13)	$((\psi \wedge \chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}} \leftrightarrow (((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}$ $\wedge ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	Def. $(\cdot)_\tau^{\mathcal{A}}$
3, 8, 9	(14)	$(\psi \wedge \chi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi \wedge \chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}$	10; 11; 12; 13; PL
3, 8	(15)	$\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}] \to ((\psi \wedge \chi)_\tau^{\mathcal{A}}$ $\leftrightarrow ((\psi \wedge \chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	9; 14; PL
3	(16)	$\forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}] \to ((\psi \wedge \chi)_\tau^{\mathcal{A}}$ $\leftrightarrow ((\psi \wedge \chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	8; 15; QL
	(17)	$\mathbf{N}[(\psi \wedge \chi)_\tau^{\mathcal{A}}] \to \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\psi \wedge \chi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi \wedge \chi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	3; 16; PL

The proof of case (B) (inductive case (ii)) proceeds in a similar fashion, and so is left to the reader.

Inductive case (iii): $\varphi := \forall v \psi$. Case (A):

	(1)	$\mathbf{N}[(\psi)_\tau^{\mathcal{A}}] \to \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\psi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	Ind. Hyp.
2	(2)	$\mathbf{N}[(\forall v \psi)_\tau^{\mathcal{A}}]$	Asmp. $(\to \mathrm{I})$
3	(3)	$\mathbf{N}((v)_\tau^{\mathcal{A}})$	Asmp. $(\to \mathrm{I})$
2, 3	(4)	$\mathbf{N}[(\psi)_\tau^{\mathcal{A}}]$	2; 3; PL

2, 3	(5)	$\forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\psi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	1; 4; PL
6	(6)	$E[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. (QGen)
2, 3, 6	(7)	$\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\psi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	5; 6; QL
8	(8)	$\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. ($\to$ I)
2, 3, 6, 8	(9)	$(\psi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	7; 8; PL
2, 6, 8	(10)	$(\mathbf{N}((v)_\tau^{\mathcal{A}}) \to (\psi)_\tau^{\mathcal{A}}) \leftrightarrow (\mathbf{N}((v)_\tau^{\mathcal{A}})$ $\to ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	3; 9; PL
2, 6, 8	(11)	$\forall(v)_\tau^{\mathcal{A}}(\mathbf{N}((v)_\tau^{\mathcal{A}}) \to (\psi)_\tau^{\mathcal{A}}) \leftrightarrow \forall(v)_\tau^{\mathcal{A}}(\mathbf{N}((v)_\tau^{\mathcal{A}})$ $\to ((\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	10; QL
2, 6	(12)	$\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\forall v\psi)_\tau^{\mathcal{A}} \leftrightarrow (\forall v\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}})$	8; 11; PL
2	(13)	$\forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\forall v\psi)_\tau^{\mathcal{A}} \leftrightarrow (\forall v\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	6; 12; QL
	(14)	$\mathbf{N}[(\forall v\psi)_\tau^{\mathcal{A}}] \to \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$ $\to ((\forall v\psi)_\tau^{\mathcal{A}} \leftrightarrow (\forall v\psi)_\tau^{\mathcal{A}})_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	2; 13; QL

The proof of (B) (inductive case (iii)) proceeds in a similar fashion. The proof of inductive case (iv), $\varphi := \forall V\psi$ (cases (A) and (B)), is similar to that of inductive case (iii), and so is also left out. This concludes the proof. □

LEMMA 27. $\vdash_{\mathrm{S5PQL_C}} \overline{Ar}[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar(V) \to (\psi)_\tau^{\mathcal{A}}) \to \forall V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}((\psi))_\tau^{\mathcal{A}}))$, for every formula ψ of $\mathrm{PA_2}$.

Proof of Lemma 27.

1	(1)	$\overline{Ar}[(\forall V\psi)_\tau^{\mathcal{A}}]$	Asmp. ($\to$ I)
2	(2)	$\forall V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar(V) \to (\psi)_\tau^{\mathcal{A}})$	Asmp. ($\to$ I)
3	(3)	$\exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\neg(\psi)_\tau^{\mathcal{A}})$	Asmp. ($\neg$I)
4	(4)	$E[V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. ($\exists$E)
5	(5)	$\neg(\psi)_\tau^{\mathcal{A}}$	Asmp. ($\exists$E)
	(6)	$\exists z_{\langle\langle\tau\rangle\rangle}^1 \dots \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1\le i\le n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1, \dots, z^n))$ $\vee \neg \exists z_{\langle\langle\tau\rangle\rangle}^1 \dots \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1\le i\le n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1, \dots, z^n))$	PL
7	(7)	$\exists z_{\langle\langle\tau\rangle\rangle}^1 \dots \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1\le i\le n}(\mathbf{N}(z^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1, \dots, z^n))$	Asmp. ($\to$ I)
8	(8)	$\bigwedge_{1\le i\le n}(\mathbf{N}(z_{\langle\langle\tau\rangle\rangle}^i))$ $\wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z_{\langle\langle\tau\rangle\rangle}^1, \dots, z_{\langle\langle\tau\rangle\rangle}^n)$	Asmp. ($\exists$E)

9	(9)	$\bigwedge_{1 \leq i \leq n}(E[z^i_{\langle\langle\tau\rangle\rangle}])$	Asmp. ($\exists$E)
9	(10)	$E[\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle}))]$	9; $\mathtt{QL_C}$
9	(11)	$\lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle}))$	10; $\mathtt{QL}$
		$= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle}))$	
8, 9	(12)	$\bigwedge_{1 \leq i \leq n}(\mathbf{N}(z^i_{\langle\langle\tau\rangle\rangle}))$	8; 11; $\mathtt{PL}$
		$\wedge\ V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1_{\langle\langle\tau\rangle\rangle}, \dots, z^n_{\langle\langle\tau\rangle\rangle})$	
		$\wedge\ \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle}))$	
		$= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle}))$	
8, 9	(13)	$\exists z^1_{\langle\langle\tau\rangle\rangle} \dots \exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(\mathbf{N}(z^i_{\langle\langle\tau\rangle\rangle}))$	9; 12; $\mathtt{QL}$
		$\wedge\ V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1_{\langle\langle\tau\rangle\rangle}, \dots, z^n_{\langle\langle\tau\rangle\rangle})$	
		$\wedge\ \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle}))$	
		$= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle})))$	
8, 9	(14)	$\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \exists z^1_{\langle\langle\tau\rangle\rangle} \dots \exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(\mathbf{N}(z^i_{\langle\langle\tau\rangle\rangle}))$	10; 13; $\mathtt{QL}$
		$\wedge\ V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1_{\langle\langle\tau\rangle\rangle}, \dots, z^n_{\langle\langle\tau\rangle\rangle}) \wedge x$	
		$= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle})))$	
7	(15)	$\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \exists z^1_{\langle\langle\tau\rangle\rangle} \dots \exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(\mathbf{N}(z^i_{\langle\langle\langle\tau\rangle\rangle\rangle}))$	7; 8; 9; 14; $\mathtt{QL}$
		$\wedge\ V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1_\sigma, \dots, z^n_{\langle\langle\tau\rangle\rangle}) \wedge x$	
		$= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle})))$	
	(16)	$\exists x_{\langle\langle\langle\tau\rangle\rangle\rangle_n} \exists z^1_{\langle\langle\tau\rangle\rangle} \dots \exists z^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(\mathbf{N}(z^i_{\langle\langle\tau\rangle\rangle}))$	PQL (PluComp)
		$\wedge\ V_{\sigma_n}(z^1_{\langle\langle\tau\rangle\rangle}, \dots, z^n_{\langle\langle\tau\rangle\rangle}) \wedge x$	
		$= \lambda w^1_{\langle\langle\tau\rangle\rangle} \dots w^n_{\langle\langle\tau\rangle\rangle}(\bigwedge_{1 \leq i \leq n}(w^i = z^i_{\langle\langle\tau\rangle\rangle})))$	
		$\rightarrow \exists yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}])$	
7	(17)	$\exists yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}])$	15; 16; $\mathtt{PL}$
	(18)	$\mathbf{N}[(\psi)^{\mathcal{A}}_\tau] \rightarrow \forall yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Lem. 26(A)
		$\rightarrow ((\psi)^{\mathcal{A}}_\tau \leftrightarrow ((\psi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]})$	
1	(19)	$\mathbf{N}[(\psi)^{\mathcal{A}}_\tau]$	1; $\mathtt{PL}$
20	(20)	$E[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. ($\exists$E)
21	(21)	$\mathrm{Pl}[yy, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Asmp. ($\exists$E)
1, 20, 21	(22)	$(\psi)^{\mathcal{A}}_\tau \leftrightarrow ((\psi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}$	18; 19; 20; 21; $\mathtt{QL}$
1, 5, 20, 21	(23)	$\neg((\psi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}$	5; 22; $\mathtt{PL}$
20, 21	(24)	$Ar[\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]]$	20; 21; Lem. 23; $\mathtt{PL}$
1, 5, 20, 21	(25)	$Ar[\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]] \wedge \neg((\psi)^{\mathcal{A}}_\tau)^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}_{\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]}$	23; 24; $\mathtt{PL}$
1, 5, 20, 21	(26)	$E[\mathrm{Rel}[yy_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]]$	20; $\mathtt{QL_C}$
1, 5, 20, 21	(27)	$\exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)^{\mathcal{A}}_\tau))$	25; 26; $\mathtt{QL}$
1, 5, 7	(28)	$\exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)^{\mathcal{A}}_\tau))$	17; 20; 21; 27$\mathtt{QL}$

1, 3, 7	(29)	$\exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)_\tau^{\mathcal{A}}))$	3; 4; 5; 28QL
1, 3	(30)	$\exists z_{\langle\langle\tau\rangle\rangle}^1 \dots \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1\leq i\leq n}(\mathbf{N}(z^i))$	7; 29; PL
		$\wedge\, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n))$	
		$\to \exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)_\tau^{\mathcal{A}}))$	
31	(31)	$\neg\exists z_{\langle\langle\tau\rangle\rangle}^1 \dots \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1\leq i\leq n}(\mathbf{N}(z^i))$	Asmp. $(\to I)$
		$\wedge\, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n))$	
	(32)	$\mathbf{N}[(\psi)_\tau^{\mathcal{A}}] \to (\neg\exists z_{\langle\langle\tau\rangle\rangle}^1 \dots \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1\leq i\leq n}$	Lem. 26(B)
		$(\mathbf{N}(z^i)) \wedge V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n)) \to ((\psi)_\tau^{\mathcal{A}}$	
		$\leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}))$	
1, 31	(33)	$(\psi)_\tau^{\mathcal{A}} \leftrightarrow ((\psi)_\tau^{\mathcal{A}})_{\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}$	19; 31; 32; PL
1, 5, 31	(34)	$\neg((\psi)_\tau^{\mathcal{A}})_{\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}$	5; 33; PL
	(35)	$Ar[\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Lem. 24
1, 5, 31	(36)	$Ar[\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}] \wedge \neg((\psi)_\tau^{\mathcal{A}})_{\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}^{V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}}$	34; 35; PL
	(37)	$E[\Lambda_{\langle\langle\langle\tau\rangle\rangle\rangle_n}]$	Lem. 2
1, 5, 31	(38)	$\exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)_\tau^{\mathcal{A}}))$	36; 37; QL
1, 3, 31	(39)	$\exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)_\tau^{\mathcal{A}}))$	3; 4; 5; 38; QL
1, 3	(40)	$\neg\exists z_{\langle\langle\tau\rangle\rangle}^1 \dots \exists z_{\langle\langle\tau\rangle\rangle}^n (\bigwedge_{1\leq i\leq n}(\mathbf{N}(z^i))$	31; 39PL
		$\wedge\, V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(z^1,\dots,z^n))$	
		$\to \exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)_\tau^{\mathcal{A}}))$	
1, 3	(41)	$\exists V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar[V] \wedge \neg((\psi)_\tau^{\mathcal{A}}))$	6; 30; 40; PL
1, 2	(42)	$\forall V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}((\psi)_\tau^{\mathcal{A}})$	2; 3; 41; QL
	(43)	$\overline{Ar}[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}(Ar(V)$	1; 2; 42; PL
		$\to (\psi)_\tau^{\mathcal{A}}) \to \forall V_{\langle\langle\langle\tau\rangle\rangle\rangle_n}((\psi)_\tau^{\mathcal{A}}))$	

$\square$

The next lemma is the crux of the proof here given that pure arithmetic is necessary:

LEMMA 28. *For each type τ, $\overline{Ar}[(\varphi)_\tau^{\mathcal{A}}] \vdash_{\mathsf{S5PQL_C}} (\varphi)_\tau^{\mathcal{A}} \to \Box(\varphi)_\tau^{\mathcal{A}}$, for every formula φ of* PA$_2$ *and every type τ.*

Proof of Lemma 28. The proof is by induction on the complexity of φ.
Base case (i): $\varphi := N(\alpha)$. Then, $(\varphi)_\tau^{\mathcal{A}} = \mathbf{N}(\alpha_{\langle\langle\tau\rangle\rangle})$. So:

1	(1)	$\mathbf{N}(\alpha_{\langle\langle\tau\rangle\rangle})$	Asmp. $(\to I)$
1	(2)	$\Box\mathbf{N}(\alpha_{\langle\langle\tau\rangle\rangle})$	1; Lem. 14; PL
	(3)	$\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}] \to (\mathbf{N}(\alpha_{\langle\langle\tau\rangle\rangle}) \to \Box\mathbf{N}(\alpha_{\langle\langle\tau\rangle\rangle}))$	1; 2; PL

This establishes base case (i). Base case (ii): $\varphi := S(\alpha,\beta)$. Then, $(\varphi)_\tau^{\mathcal{A}} = \mathbf{S}(\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle})$. $\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}] = \bigwedge_{\substack{\delta\in\{\alpha_{\langle\langle\tau\rangle\rangle},\beta_{\langle\langle\tau\rangle\rangle}\}, \\ \delta\neq\ulcorner 0_{\langle\langle\tau\rangle\rangle}\urcorner}}(\mathbf{N}(\delta))$. So:

	(1)	$\mathbf{N}(\mathbf{0}_{\langle\langle\tau\rangle\rangle})$	Theo. 1
2	(2)	$\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}]$	Asmp. ($\to$ I)
2	(3)	$\mathbf{N}(\mathbf{0}_{\langle\langle\tau\rangle\rangle}) \wedge \overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}]$	1; 2; PL
4	(4)	$\mathbf{S}(\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$ I)
4	(5)	$\beta_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[\alpha_{\langle\langle\tau\rangle\rangle}]$	4; QL
2	(6)	$\Box E[\beta_{\langle\langle\tau\rangle\rangle}]$	3; Lem. 14; PL
2	(7)	$\Box\beta_{\langle\langle\tau\rangle\rangle} = \beta_{\langle\langle\tau\rangle\rangle}$	6; KQL
2, 4	(8)	$\Box\mathbf{S}[\alpha_{\langle\langle\tau\rangle\rangle}] = \beta_{\langle\langle\tau\rangle\rangle}$	5; 7; QL
2	(9)	$\Box E[\alpha_{\langle\langle\tau\rangle\rangle}]$	3; Lem. 14; PL
	(10)	$\Box E[\mathbf{S}]$	KQL_C
2, 4	(11)	$\Box\mathbf{S}(\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle})$	6, 8, 9; 10; KQL
2	(12)	$\mathbf{S}(\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle}) \to \Box\mathbf{S}(\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle})$	4; 11; PL
	(13)	$\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}] \to (\mathbf{S}(\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle}) \to \Box\mathbf{S}(\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle}))$	2; 12; PL

This establishes base case (ii). Base case (iii): $\varphi := V(\alpha^1, \dots, \alpha^n)$. Then, $(\varphi)_\tau^{\mathcal{A}} = V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}(\alpha_{\langle\langle\tau\rangle\rangle}^1, \dots, \alpha_{\langle\langle\tau\rangle\rangle}^n)$, and $\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}] = Ar(V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}) \wedge \bigwedge_{\substack{\delta \in \{\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle}\}, \\ \delta \neq \ulcorner 0_{\langle\langle\tau\rangle\rangle}\urcorner}} (\mathbf{N}(\delta))$. So we have that:

	(1)	$\mathbf{N}(\mathbf{0}_{\langle\langle\tau\rangle\rangle})$	Theo. 1
2	(2)	$\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}]$	Asmp. ($\to$ I)
2	(3)	$\mathbf{N}(\mathbf{0}_{\langle\langle\tau\rangle\rangle}) \wedge \overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}]$	1; 2; PL
2	(4)	$\Box\forall v_{\langle\langle\tau\rangle\rangle}^1 \dots \Box\forall v_{\langle\langle\tau\rangle\rangle}^n \Box(V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}(v^1, \dots, v^n)$ $\to (\bigwedge_{1 \leq i \leq n}(\mathbf{N}(v^i)) \wedge \Box V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}(v^1, \dots, v^n)))$ $\wedge \Box E[\overline{V}_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}]$	2; QL
2	(5)	$\bigwedge_{1 \leq i \leq n}(\mathbf{N}(\alpha_{\langle\langle\tau\rangle\rangle}^i))$	3; PL
2	(6)	$\bigwedge_{1 \leq i \leq n}(E[\alpha_{\langle\langle\tau\rangle\rangle}^i])$	5; Lem. 14; PL
2	(7)	$V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}(\alpha_{\langle\langle\tau\rangle\rangle}^1, \dots, \alpha_{\langle\langle\tau\rangle\rangle}^n)$ $\to \Box V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}(\alpha_{\langle\langle\tau\rangle\rangle}^1, \dots, \alpha_{\langle\langle\tau\rangle\rangle}^n)$	4; 6; TQL
	(8)	$\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}] \to (V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}(\alpha_{\langle\langle\tau\rangle\rangle}^1, \dots, \alpha_{\langle\langle\tau\rangle\rangle}^n)$ $\to \Box V_{\langle\langle\langle\tau\rangle\rangle, \dots, \langle\langle\tau\rangle\rangle\rangle}(\alpha_{\langle\langle\tau\rangle\rangle}^1, \dots, \alpha_{\langle\langle\tau\rangle\rangle}^n))$	2; 7; PL

This establishes base case (iii). Base case (iv): $\varphi := \alpha = \beta$. Then, $(\varphi)_\tau^{\mathcal{A}} = \bigwedge_{\substack{\delta \in \{\alpha_{\langle\langle\tau\rangle\rangle}, \beta_{\langle\langle\tau\rangle\rangle}\}, \\ \delta \neq \ulcorner 0_{\langle\langle\tau\rangle\rangle}\urcorner}} (\mathbf{N}(\delta))$. So:

	(1)	$\mathbf{N}(\mathbf{0}_{\langle\langle\tau\rangle\rangle})$	Theo. 1
2	(2)	$\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}]$	Asmp. ($\to$ I)
2	(3)	$\mathbf{N}(\alpha_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{N}(\beta_{\langle\langle\tau\rangle\rangle})$	1; 2; PL
4	(4)	$\alpha_{\langle\langle\tau\rangle\rangle} = \beta_{\langle\langle\tau\rangle\rangle}$	Asmp. ($\to$ I)

2	(5)	$\Box(E[\alpha_{\langle\langle\tau\rangle\rangle}])$	3; Lem. 14; PL
2	(6)	$\Box\alpha_{\langle\langle\tau\rangle\rangle} = \alpha_{\langle\langle\tau\rangle\rangle}$	3; KQL
2, 4	(7)	$\Box\alpha_{\langle\langle\tau\rangle\rangle} = \beta_{\langle\langle\tau\rangle\rangle}$	4; 6; QL
2	(8)	$\alpha_{\langle\langle\tau\rangle\rangle} = \beta_{\langle\langle\tau\rangle\rangle} \to \Box\alpha_{\langle\langle\tau\rangle\rangle} = \beta_{\langle\langle\tau\rangle\rangle}$	4; 7; PL
	(9)	$\overline{Ar}_\tau[(\varphi)_\tau^{\mathcal{A}}] \to (\alpha_{\langle\langle\tau\rangle\rangle} = \beta_{\langle\langle\tau\rangle\rangle} \to \Box\alpha_{\langle\langle\tau\rangle\rangle} = \beta_{\langle\langle\tau\rangle\rangle})$	2; 8; PL

This establishes base case (iv). Inductive case (i): $\varphi := \neg\psi$. Since $\overline{Ar}_\tau[(\psi)_\tau^{\mathcal{A}}] = \overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}]$, we have that:

(1)	$\overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}] \to ((\psi)_\tau^{\mathcal{A}} \to \Box(\psi)_\tau^{\mathcal{A}})$	Ind. hyp.
(2)	$\Box\overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}] \to \Box((\psi)_\tau^{\mathcal{A}} \to \Box(\psi)_\tau^{\mathcal{A}})$	1; K
(3)	$\Box\overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}] \to (\Box\Diamond\neg(\psi)_\tau^{\mathcal{A}} \to \Box\neg(\psi)_\tau^{\mathcal{A}})$	2; K
(4)	$\Box\overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}] \to (\neg(\psi)_\tau^{\mathcal{A}} \to \Box\neg(\psi)_\tau^{\mathcal{A}})$	3; S5
(5)	$\overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}] \to \Box\overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}]$	Theo. 1; Lems. 14 & 25; K
(6)	$\overline{Ar}_\tau[(\neg\psi)_\tau^{\mathcal{A}}] \to ((\neg\psi)_\tau^{\mathcal{A}} \to \Box(\neg\psi)_\tau^{\mathcal{A}})$	4; 5; PL

This establishes inductive case (i). Inductive case (ii): $\varphi := \psi \wedge \chi$. We have that:

(1)	$(\overline{Ar}_\tau[(\psi)_\tau^{\mathcal{A}}] \to ((\psi)_\tau^{\mathcal{A}} \to \Box(\psi)_\tau^{\mathcal{A}})) \wedge (\overline{Ar}_\tau[(\chi)_\tau^{\mathcal{A}}]$ $\to ((\chi)_\tau^{\mathcal{A}} \to \Box(\chi)_\tau^{\mathcal{A}}))$	Ind. hyp.
(2)	$(\overline{Ar}_\tau[(\psi)_\tau^{\mathcal{A}}] \wedge \overline{Ar}_\tau[(\chi)_\tau^{\mathcal{A}}]) \to (((\psi)_\tau^{\mathcal{A}} \wedge (\chi)_\tau^{\mathcal{A}})$ $\to (\Box(\psi)_\tau^{\mathcal{A}} \wedge \Box(\chi)_\tau^{\mathcal{A}}))$	1; PL
(3)	$(\overline{Ar}_\tau[(\psi)_\tau^{\mathcal{A}}] \wedge \overline{Ar}_\tau[(\chi)_\tau^{\mathcal{A}}]) \to (((\psi)_\tau^{\mathcal{A}} \wedge (\chi)_\tau^{\mathcal{A}})$ $\to \Box((\psi)_\tau^{\mathcal{A}} \wedge (\chi)_\tau^{\mathcal{A}}))$	2; K
(4)	$\overline{Ar}_\tau[(\psi \wedge \chi)_\tau^{\mathcal{A}}] \to ((\psi \wedge \chi)_\tau^{\mathcal{A}} \to \Box(\psi \wedge \chi)_\tau^{\mathcal{A}})$	3; Def.$(\cdot)_\tau^{\mathcal{A}}$; PL

This establishes inductive case (ii). Inductive case (iii): $\varphi := \forall x\psi$. We have that:

(1)	$\overline{Ar}_\tau[(\psi)_\tau^{\mathcal{A}}] \to ((\psi)_\tau^{\mathcal{A}} \to \Box(\psi)_\tau^{\mathcal{A}})$	Ind. hyp.
(2)	$\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to ((\psi)_\tau^{\mathcal{A}}$ $\to \Box(\psi)_\tau^{\mathcal{A}}))$	1; Def.$\overline{Ar}_\tau[\cdot]$; PL
(3)	$\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to ((\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to (\psi)_\tau^{\mathcal{A}})$ $\to (\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to \Box(\psi)_\tau^{\mathcal{A}}))$	2; PL
(4)	$\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ $\to \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to \Box(\psi)_\tau^{\mathcal{A}}))$	3; QL
(5)	$\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ $\to (E[x_{\langle\langle\tau\rangle\rangle}] \to (\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to \Box(\psi)_\tau^{\mathcal{A}})))$	4; QL
(6)	$\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ $\to (\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to \Box(\psi)_\tau^{\mathcal{A}}))$	5; Lem. 13; PL
(7)	$\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ $\to \Box(\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to (\psi)_\tau^{\mathcal{A}}))$	6; Lem. 14; S5

(8) $\quad \Box\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to \Box(\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ 7; K
$\qquad \to \Box(\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to (\psi)_\tau^{\mathcal{A}}))$

(9) $\quad \Box\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ 8; K
$\qquad \to \Diamond\Box(\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to (\psi)_\tau^{\mathcal{A}}))$

(10) $\quad \Box\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ 9; S5
$\qquad \to (\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \to (\psi)_\tau^{\mathcal{A}}))$

(11) $\quad \Box\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ 10; QL
$\qquad \to \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}}))$

(12) $\quad \Box\Box\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\Box\Diamond\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x)$ 11; K
$\qquad \to (\psi)_\tau^{\mathcal{A}}) \to \Box\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}}))$

(13) $\quad \Box\overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ 12; S5
$\qquad \to \Box\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}}))$

(14) $\quad \overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to (\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}})$ 13; Theo. 1; Lems. 14 & 25; K
$\qquad \to \Box\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to (\psi)_\tau^{\mathcal{A}}))$

(15) $\quad \overline{Ar}_\tau[(\forall x\psi)_\tau^{\mathcal{A}}] \to ((\forall x\psi)_\tau^{\mathcal{A}} \to \Box(\forall x\psi)_\tau^{\mathcal{A}})$ 14; Def.$(\cdot)_\tau^{\mathcal{A}}$

This establishes inductive case (iii). Inductive case (iv): $\varphi := \forall V\psi$. We have that:

(1) $\quad \overline{Ar}_\tau[(\psi)_\tau^{\mathcal{A}}] \to ((\psi)_\tau^{\mathcal{A}} \to \Box(\psi)_\tau^{\mathcal{A}})$ Ind. hyp.

(2) $\quad \overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to (Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}]$ 1; Def.$\overline{Ar}_\tau[\cdot]$; PL
$\qquad \to ((\psi)_\tau^{\mathcal{A}} \to \Box(\psi)_\tau^{\mathcal{A}}))$

(3) $\quad \overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to ((Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}] \to (\psi)_\tau^{\mathcal{A}})$ 2; PL
$\qquad \to (Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}] \to \Box(\psi)_\tau^{\mathcal{A}}))$

(4) $\quad \overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to ((\psi)_\tau^{\mathcal{A}} \to (Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}]$ 3; PL
$\qquad \to \Box(\psi)_\tau^{\mathcal{A}}))$

(5) $\quad \overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$ 4; QL
$\qquad \to \forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}(Ar_\tau[V] \to \Box(\psi)_\tau^{\mathcal{A}}))$

(6) $\quad \overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$ 5; QL
$\qquad \to (E[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}] \to (Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}]$
$\qquad \to \Box(\psi)_\tau^{\mathcal{A}})))$

(7) $\quad \overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$ 6; T
$\qquad \to (Ar_\tau[V]_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle} \to \Box(\psi)_\tau^{\mathcal{A}}))$

(8) $\quad \overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$ 7; Lem. 25; S5
$\qquad \to \Box(Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}] \to (\psi)_\tau^{\mathcal{A}}))$

(9) $\quad \Box\overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to \Box(\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$ 8; K
$\qquad \to \Box(Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}] \to (\psi)_\tau^{\mathcal{A}}))$

(10) $\quad \Box\overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\Diamond\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$ 9; K
$\qquad \to \Diamond\Box(Ar_\tau[V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}] \to (\psi)_\tau^{\mathcal{A}}))$

(11) $\quad \Box\overline{Ar}_\tau[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\Diamond\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$ 10; S5
$\qquad \to (Ar_\tau[V] \to (\psi)_\tau^{\mathcal{A}}))$

(12)	$\Box\overline{Ar_\tau}[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\Diamond\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$	11; QL
	$\quad \to \forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}(Ar_\tau[V] \to (\psi)_\tau^{\mathcal{A}}))$	
(13)	$\Box\overline{Ar_\tau}[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\Diamond\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$	12; Lem. 27; T
	$\quad \to \forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}}))$	
(14)	$\Box\Box\overline{Ar_\tau}[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\Box\Diamond\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$	13; K
	$\quad \to \Box\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}}))$	
(15)	$\Box\overline{Ar_\tau}[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$	14; S5
	$\quad \to \Box\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}}))$	
(16)	$\overline{Ar_\tau}[(\forall V\psi)_\tau^{\mathcal{A}}] \to \Box\overline{Ar_\tau}[(\forall V\psi)_\tau^{\mathcal{A}}]$	Theo. 1; Lems. 14 & 25; S5
(17)	$\overline{Ar_\tau}[(\forall V\psi)_\tau^{\mathcal{A}}] \to (\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}})$	15; 16; PL
	$\quad \to \Box\forall V_{\langle\langle\langle\tau\rangle\rangle,...,\langle\langle\tau\rangle\rangle\rangle}((\psi)_\tau^{\mathcal{A}}))$	

This establishes inductive case (iv), and the lemma.   □

Theorem 9 is then an immediate corollary of Lemma 28:

THEOREM 9. $\vdash_{\mathsf{S5PQL_C}} (\varphi)_\tau^{\mathcal{A}} \to \Box(\varphi)_\tau^{\mathcal{A}}$, *for every closed formula* φ *of* PA$_2$ *and every type* τ.

§D. Relationship between S5PQL$_{\mathsf{SA,C}}$ **and** S5PQL$_{\mathsf{SA,C}}$. In this section is shown that the theorems of S5PQL$_{\mathsf{SA,C}}$ are the same as those of the following system:

DEFINITION 22 (System S5PQL$_{\mathsf{SA,C}}$). S5PQL$_{\mathsf{SA,C}}$ *is that axiomatic system whose axioms consist of the result of prefixing any sequence (including the empty sequence) of metaphysical necessity operators and universal quantifiers, in any order, to* S5PQL$_{\mathsf{SA,C}}$'s *axioms, and whose only inference rule is* MP.

The following lemma will be useful in what follows.

LEMMA 29. *If* $\vdash_{\mathsf{S5PQL_{SA,C}}} \varphi$, *then* $\vdash_{\mathsf{S5PQL_{SA,C}}} \Box\varphi$ *and* $\vdash_{\mathsf{S5PQL_{SA,C}}} \forall v\varphi$.

Proof of Lemma 29. The proof is by induction on the length of a $\vdash_{\mathsf{S5PQL_{SA,C}}}$-derivation.
Base case: φ is an axiom. Clearly, $\vdash_{\mathsf{S5PQL_{SA,C}}} \Box\varphi$ and $\vdash_{\mathsf{S5PQL_{SA,C}}} \forall v\varphi$, as the result of prefixing a necessity operator or a universal quantifier to any axiom of S5PQL$_{\mathsf{SA,C}}$ is itself an axiom of S5PQL$_{\mathsf{SA,C}}$.
Inductive case: suppose that $\vdash_{\mathsf{S5PQL_{SA,C}}} \varphi$. Then, by the inductive hypothesis: (i) $\vdash_{\mathsf{S5PQL_{SA,C}}} \Box\psi$; (ii) $\vdash_{\mathsf{S5PQL_{SA,C}}} \forall v\psi$; (iii) $\vdash_{\mathsf{S5PQL_{SA,C}}} \Box(\psi \to \varphi)$; and (iv) $\vdash_{\mathsf{S5PQL_{SA,C}}} \forall v(\psi \to \varphi)$. We have that $\Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$ is an axiom of S5PQL$_{\mathsf{SA,C}}$. So, by (i) and (iii), via two applications of MP, we get that $\vdash_{\mathsf{S5PQL_{SA,C}}} \Box\varphi$. Similarly, we have that $\forall v(\psi \to \varphi) \to (\forall v\psi \to \forall v\varphi)$ is an axiom of S5PQL$_{\mathsf{SA,C}}$. So, by (ii) and (iv), via two applications of MP, we get that $\vdash_{\mathsf{S5PQL_{SA,C}}} \forall v\varphi$. This concludes the proof.   □

The main result of this section is then the following:

LEMMA 30. $\vdash_{\mathsf{S5PQL_{SA,C}}} \varphi$ *if and only if* $\vdash_{\mathsf{S5PQL_{SA,C}}} \varphi$.

Proof of Lemma 30. I'll start by showing that if $\vdash_{\mathsf{S5PQL_{SA,C}}} \varphi$, then $\vdash_{\mathsf{S5PQL_{SA,C}}} \varphi$. The proof is by induction on the length of a $\vdash_{\mathsf{S5PQL_{SA,C}}}$-derivation. The base case is trivial, as any axiom of S5PQL$_{\mathsf{SA,C}}$ is an axiom of S5PQL$_{\mathsf{SA,C}}$. For the inductive case, there are

three subcases to consider: (i) φ is proved via an application of Gen; (ii) φ is proved via an application of Nec; and (iii) φ is proved via an application of MP.

Subcase (i): φ is proved via an application of Gen. Then $\varphi := \forall v \psi$. By the inductive hypothesis, $\vdash_{\text{S5PQL}_{\text{SA,C}}} \psi$. But then, $\vdash_{\text{S5PQL}_{\text{SA,C}}} \forall v \psi$, by Lemma 29.

Subcase (ii): φ is proved via an application of Nec. Then $\varphi := \Box \psi$. By the inductive hypothesis, $\vdash_{\text{S5QL}_{\text{SA,C}}} \psi$. But then, $\vdash_{\text{S5QL}_{\text{SA,C}}} \Box \psi$, by Lemma 29.

Subcase (iii): φ is proved via an application of MP. By the inductive hypothesis, $\vdash_{\text{S5QL}_{\text{SA,C}}} \psi$ and $\vdash_{\text{S5QL}_{\text{SA,C}}} \psi \to \varphi$. But then, $\vdash_{\text{S5QL}_{\text{SA,C}}} \varphi$, by MP.

So, $\vdash_{\text{S5PQL}_{\text{SA,C}}} \varphi$, then $\vdash_{\text{S5PQL}_{\text{SA,C}}} \varphi$. To see that if $\vdash_{\text{S5PQL}_{\text{SA,C}}} \varphi$, then $\vdash_{\text{S5PQL}_{\text{SA,C}}} \varphi$, note that all axioms of S5PQL$_{\text{SA,C}}$ are theorems of S5PQL$_{\text{SA,C}}$, owing to S5PQL$_{\text{SA,C}}$'s rules Nec and Gen. So, whatever is derivable from S5PQL$_{\text{SA,C}}$'s axioms via MP is derivable in S5PQL$_{\text{SA,C}}$ by MP, Nec and Gen. This concludes the proof. □

§E. Relations between FP, GENERATION and INFAX. In this section I will begin by showing that GENERATION implies FINITARY PLENITUDE. In order to do so, it will be useful to appeal to some facts concerning the smaller than relation between natural numbers. This relation is defined as follows:

DEFINITION 23 (Smaller than). *For all types τ, let*

$$<_{\langle\langle\langle\tau\rangle\rangle,\langle\langle\tau\rangle\rangle\rangle} := \lambda x_{\langle\langle\tau\rangle\rangle} y_{\langle\langle\tau\rangle\rangle} (\forall z_{\langle\langle\langle\tau\rangle\rangle\rangle}((Her(z) \land \forall u_{\langle\langle\tau\rangle\rangle}(S(x,u) \to z(u))) \to z(y))).$$

Then, the relevant results about the *smaller than* relation are the following:

LEMMA 31. $\vdash_{\text{QL}_\text{C}} \forall x_{\langle\langle\tau\rangle\rangle} y_{\langle\langle\tau\rangle\rangle}(S(x,y) \to x < y)$.

LEMMA 32. $\vdash_{\text{QL}_\text{C}} \forall x_{\langle\langle\tau\rangle\rangle} \forall y \forall z((x < y \land y < z) \to x < z)$.

Proof of Lemmas 31 and 32. The two lemmas follow straightforwardly by QL$_\text{C}$-reasoning. □

LEMMA 33. $\vdash_{\text{QL}_\text{C}} \forall x_{\langle\langle\tau\rangle\rangle} \forall y_{\langle\langle\tau\rangle\rangle}((N(x) \land N(y)) \to x < y \lor x = y \lor y < x)$.

Proof of Lemma 33. Provable from Lemmas 1, 12, 31, and 32, and Theorem 6, by QL$_\text{C}$-reasoning. □

Now, consider the following definition:

DEFINITION 24 (GEN). *For every type τ, type $\langle\langle\tau\rangle\rangle$-singular term α and $\langle\tau\rangle$-singular term β,*

$$\text{GEN}[\alpha, \beta] := (N(\alpha) \land \alpha(\beta)) \land \forall y_{\langle\langle\tau\rangle\rangle} \forall z_{\langle\langle\tau\rangle\rangle}((N(y) \land S(y,z)) \to \Box(y(\beta) \to \Diamond z(\beta))).$$

We will prove a couple of auxiliary lemmas involving GEN[·, ·].

LEMMA 34. $\vdash_{\text{S5QL}_\text{C}} \forall x_{\langle\langle\tau\rangle\rangle}(N(x) \to \forall u_{\langle\langle\tau\rangle\rangle} \forall w_{\langle\tau\rangle}((\text{GEN}[u,w] \land u \leq x) \to \Diamond x(w)))$, *for every type τ.*

Proof of Lemma 34.

1	(1)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
2	(2)	$N(x_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$I)
3	(3)	$E[u_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
4	(4)	$E[w_{\langle\tau\rangle}]$	Asmp. (QGen)

5	(36)	$\text{GEN}[u_{\langle\langle\tau\rangle\rangle}, w_{\langle\tau\rangle}] \wedge u_{\langle\langle\tau\rangle\rangle} \leq x_{\langle\langle\tau\rangle\rangle}$	Asmp. $(\rightarrow\text{I})$
6	(37)	$E[a_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
7	(38)	$E[b_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
8	(39)	$\lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y) \wedge \Diamond y(w))(a_{\langle\langle\tau\rangle\rangle})$ $\wedge \mathbf{S}(a_{\langle\langle\tau\rangle\rangle}, b_{\langle\langle\tau\rangle\rangle})$	Asmp. $(\rightarrow\text{I})$
8	(40)	$\mathbf{N}(a_{\langle\langle\tau\rangle\rangle}) \wedge \Diamond a_{\langle\langle\tau\rangle\rangle}(w)$	8; QL
6, 7, 8	(41)	$\mathbf{N}(b_{\langle\langle\tau\rangle\rangle})$	6; 7; 8; 9; Theo. 2; QL
5, 6, 7, 8	(42)	$\square(a_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}) \rightarrow b_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}))$	5; 6; 7; 8; 9; QL
5, 6, 7, 8	(43)	$\Diamond a_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}) \rightarrow \Diamond\Diamond b_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	11; T
5, 6, 7, 8	(44)	$\Diamond b_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	9; 12; S5
4	(45)	$E[\lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y) \wedge \Diamond y(w))]$	4; QL_C
5, 6, 7, 8	(46)	$\lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y) \wedge \Diamond y(w))(b_{\langle\langle\tau\rangle\rangle})$	7; 10; 13; 14; QL
5	(47)	$Her[\lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y) \wedge \Diamond y(w))]$	6; 7; 8; 15; QL
5	(48)	$u_{\langle\langle\tau\rangle\rangle} < x_{\langle\langle\tau\rangle\rangle} \rightarrow ((Her[\lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y)$ $\wedge \Diamond y(w_{\langle\tau\rangle}))] \wedge \forall y_{\langle\langle\tau\rangle\rangle}(\mathbf{S}(u_{\langle\langle\tau\rangle\rangle}, y)$ $\rightarrow \lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y) \wedge \Diamond y(w_{\langle\tau\rangle}))(y_{\langle\langle\tau\rangle\rangle})))$ $\rightarrow \lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y) \wedge \Diamond y(w_{\langle\tau\rangle}))(x_{\langle\langle\tau\rangle\rangle}))$	14; QL
5	(49)	$\mathbf{N}(u_{\langle\langle\tau\rangle\rangle}) \wedge u_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	5; PL
19	(50)	$E[y_{\langle\langle\tau\rangle\rangle}]$	Asmp.
20	(51)	$\mathbf{S}(u_{\langle\langle\tau\rangle\rangle}, y)$	Asmp.
3, 5, 19, 20	(52)	$\square(u_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}) \rightarrow \Diamond y_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}))$	3; 5; 18; 19; 20; QL
5	(53)	$\Diamond u_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	5; T
3, 5, 19, 20	(54)	$\Diamond y_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	21; 22; S5
3, 5, 19, 20	(55)	$\mathbf{N}(y_{\langle\langle\tau\rangle\rangle})$	3; 18; 19; 20; Theo. 2; QL
3, 4, 5, 19, 20	(56)	$\lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y) \wedge \Diamond y(w_{\langle\tau\rangle}))(y_{\langle\langle\tau\rangle\rangle})$	14; 19; 23; 24; QL
3, 4, 5	(57)	$\forall y_{\langle\langle\tau\rangle\rangle}(\mathbf{S}(u_{\langle\langle\tau\rangle\rangle}, y) \rightarrow \lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y)$ $\wedge \Diamond y(w_{\langle\tau\rangle}))(y_{\langle\langle\tau\rangle\rangle}))$	19; 20; 25; QL
3, 4, 5	(58)	$u_{\langle\langle\tau\rangle\rangle} < x_{\langle\langle\tau\rangle\rangle} \rightarrow \lambda y_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(y)$ $\wedge \Diamond y(w_{\langle\tau\rangle}))(x_{\langle\langle\tau\rangle\rangle}))$	16; 17; 26; PL
3, 4, 5	(59)	$u_{\langle\langle\tau\rangle\rangle} < x_{\langle\langle\tau\rangle\rangle} \rightarrow \Diamond x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	27; QL
5	(60)	$u_{\langle\langle\tau\rangle\rangle} = x_{\langle\langle\tau\rangle\rangle} \rightarrow x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	18; QL
5	(61)	$u_{\langle\langle\tau\rangle\rangle} = x_{\langle\langle\tau\rangle\rangle} \rightarrow \Diamond x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	29; T
3, 4, 5	(62)	$u_{\langle\langle\tau\rangle\rangle} \leq x_{\langle\langle\tau\rangle\rangle} \rightarrow \Diamond x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	28; 30; PL
3, 4	(63)	$(\text{GEN}[u_{\langle\langle\tau\rangle\rangle}, w_{\langle\tau\rangle}] \wedge u_{\langle\langle\tau\rangle\rangle} \leq x_{\langle\langle\tau\rangle\rangle})$ $\rightarrow \Diamond x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	5; 31; PL
	(64)	$\forall u_{\langle\langle\tau\rangle\rangle} \forall w_{\langle\tau\rangle}((\text{GEN}[u_{\langle\langle\tau\rangle\rangle}, w_{\langle\tau\rangle}] \wedge u_{\langle\langle\tau\rangle\rangle}$	3; 4; 32; QL
	(65)	$\mathbf{N}(x_{\langle\langle\tau\rangle\rangle}) \rightarrow \forall u_{\langle\langle\tau\rangle\rangle} \forall w_{\langle\tau\rangle}((\text{GEN}[u_{\langle\langle\tau\rangle\rangle},$ $w_{\langle\tau\rangle}] \wedge u_{\langle\langle\tau\rangle\rangle} \leq x_{\langle\langle\tau\rangle\rangle}) \rightarrow \Diamond x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}))$	2; 33; PL
	(66)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \forall u_{\langle\langle\tau\rangle\rangle} \forall w_{\langle\tau\rangle}((\text{GEN}[u_{\langle\langle\tau\rangle\rangle},$ $w_{\langle\tau\rangle}] \wedge u_{\langle\langle\tau\rangle\rangle} \leq x) \rightarrow \Diamond x(w_{\langle\tau\rangle})))$	1; 34; QL

$\square$

LEMMA 35. $\vdash_{\text{KQL}_\text{C}} \forall x_{\langle\langle\tau\rangle\rangle} \forall u_{\langle\langle\tau\rangle\rangle}(x < u \rightarrow (\Diamond \exists w_{\langle\tau\rangle}(u(w)) \rightarrow \Diamond \exists w_{\langle\tau\rangle}(x(w))))$.

Proof of Lemma 35.

1	(1)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
2	(2)	$E[u_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
3	(3)	$x_{\langle\langle\tau\rangle\rangle} < u_{\langle\langle\tau\rangle\rangle}$	Asmp. ($\rightarrow$I)
	(4)	$E[\lambda y_{\langle\langle\tau\rangle\rangle}(\neg\Diamond\exists w_{\langle\tau\rangle}(y(w)))]$	QL$_{\text{C}}$
5	(5)	$E[a_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
6	(6)	$E[b_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
7	(7)	$\lambda y_{\langle\langle\tau\rangle\rangle}(\neg\Diamond\exists w_{\langle\tau\rangle}(y(w)))(a_{\langle\langle\tau\rangle\rangle}) \wedge \mathbf{S}(a_{\langle\langle\tau\rangle\rangle}, b_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\rightarrow$I)
7	(8)	$\neg\Diamond\exists w_{\langle\tau\rangle}(a_{\langle\langle\tau\rangle\rangle}(w))$	7; QL
7	(9)	$b_{\langle\langle\tau\rangle\rangle} = \mathbf{S}[a_{\langle\langle\tau\rangle\rangle}]$	7; QL
10	(10)	$E[w_{\langle\tau\rangle}]$	Asmp. (QGen)
10	(11)	$\mathbf{S}[a_{\langle\langle\tau\rangle\rangle}](w_{\langle\tau\rangle}) \rightarrow \exists z_{\tau}(w_{\langle\tau\rangle}(z) \wedge a_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle} - z)$ $\wedge\, E[w_{\langle\tau\rangle} - z])$	10; QL$_{\text{C}}$
	(12)	$\exists w_{\langle\tau\rangle}(\mathbf{S}[a_{\langle\langle\tau\rangle\rangle}](w)) \rightarrow \exists w_{\langle\tau\rangle}(a_{\langle\langle\tau\rangle\rangle}(w))$	10; 11; QL
	(13)	$\Diamond\exists w_{\langle\tau\rangle}(\mathbf{S}[a_{\langle\langle\tau\rangle\rangle}](w)) \rightarrow \Diamond\exists w_{\langle\tau\rangle}(a_{\langle\langle\tau\rangle\rangle}(w))$	12; K
7	(14)	$\Diamond\exists w_{\langle\tau\rangle}(b_{\langle\langle\tau\rangle\rangle}(w)) \rightarrow \Diamond\exists w_{\langle\tau\rangle}(a_{\langle\langle\tau\rangle\rangle}(w))$	9; 13; QL
7	(15)	$\neg\Diamond\exists w_{\langle\tau\rangle}(b_{\langle\langle\tau\rangle\rangle}(w))$	8; 14; PL
6, 7	(16)	$\lambda y_{\langle\langle\tau\rangle\rangle}(\neg\Diamond\exists w_{\langle\tau\rangle}(y(w)))(b_{\langle\langle\tau\rangle\rangle})$	4; 6; 15; QL
	(17)	$Her[\lambda y_{\langle\langle\tau\rangle\rangle}(\neg\Diamond\exists w_{\langle\tau\rangle}(y(w)))]$	5; 6; 7; 16; QL
3	(18)	$\forall z_{\langle\langle\tau\rangle\rangle}(\mathbf{S}(x_{\langle\langle\tau\rangle\rangle}, z) \rightarrow \lambda y_{\langle\langle\tau\rangle\rangle}(\neg\Diamond\exists w_{\langle\tau\rangle}(y(w)))(z))$ $\rightarrow \neg\Diamond\exists w_{\langle\tau\rangle}(u_{\langle\langle\tau\rangle\rangle}(w))$	3; 4; 17; QL
19	(19)	$\Diamond\exists w_{\langle\tau\rangle}(u_{\langle\langle\tau\rangle\rangle}(w))$	Asmp. ($\rightarrow$I)
20	(20)	$\neg\Diamond\exists w_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(w))$	Asmp. ($\neg$I)
1, 20	(21)	$\lambda y_{\langle\langle\tau\rangle\rangle}(\neg\Diamond\exists w_{\langle\tau\rangle}(y_{\langle\langle\tau\rangle\rangle}(w)))(x_{\langle\langle\tau\rangle\rangle})$	1; 4; 20; QL
1, 20	(22)	$\forall z_{\langle\langle\tau\rangle\rangle}(\mathbf{S}(x_{\langle\langle\tau\rangle\rangle}, z)$ $\rightarrow \lambda y_{\langle\langle\tau\rangle\rangle}(\neg\Diamond\exists w_{\langle\tau\rangle}(y_{\langle\langle\tau\rangle\rangle}(w)))(z_{\langle\langle\tau\rangle\rangle}))$	1; 17; 21; QL
1, 3, 19	(23)	$\Diamond\exists w_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(w))$	18; 19; 20; 22; PL
	(24)	$\forall x_{\langle\langle\tau\rangle\rangle}\forall u_{\langle\langle\tau\rangle\rangle}(x < u \rightarrow (\Diamond\exists w_{\langle\tau\rangle}(u(w))$ $\rightarrow \Diamond\exists w_{\langle\tau\rangle}(x(w))))$	1; 2; 3; 19; 23; QL

$\square$

Finally comes the proof that GENERATION implies FP:

LEMMA 36. $\vdash_{\text{S5QL}_{\text{SA,C}}}$ GENERATION $\rightarrow$ FP.

Proof of Lemma 36. It is shown that $\vdash_{\text{S5QL}_{\text{SA,C}}} \exists u_{\langle\langle\tau\rangle\rangle}\exists w_{\langle\tau\rangle}(\text{GEN}[u, w]) \rightarrow \text{FP}_{\tau}$, as GENERATION—i.e., $\exists x_{\langle e\rangle}\exists y_{\langle\langle e\rangle\rangle}((\mathbf{N}(y) \wedge y(x)) \wedge \forall w_{\langle\langle e\rangle\rangle}\forall z_{\langle\langle e\rangle\rangle}((\mathbf{N}(w) \wedge \mathbf{S}(w, z)) \rightarrow \square(w(x) \rightarrow \Diamond z(x))))$—is just $\exists u_{\langle\langle e\rangle\rangle}\exists w_{\langle e\rangle}(\text{GEN}[u, w])$.

1	(1)	$\exists u_{\langle\langle\tau\rangle\rangle}\exists w_{\langle\tau\rangle}(\text{GEN}[u, w])$	Asmp. ($\rightarrow$I)
2	(2)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
3	(3)	$\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\rightarrow$I)

4	(4)	$\mathrm{GEN}[u, w]$	Asmp. ($\exists$E)
5	(5)	$E[u_{\langle\langle\tau\rangle\rangle}]$	Asmp. ($\exists$E)
6	(6)	$E[w_{\langle\tau\rangle}]$	Asmp. ($\exists$E)
4	(7)	$\mathbf{N}(u_{\langle\langle\tau\rangle\rangle})$	4; PL
2, 3, 4, 5	(8)	$u_{\langle\langle\tau\rangle\rangle} \leq x_{\langle\langle\tau\rangle\rangle} \vee x_{\langle\langle\tau\rangle\rangle} < u_{\langle\langle\tau\rangle\rangle}$	2; 3; 5; 7; Lem. 33; QL
2, 3, 4, 5, 6	(9)	$u_{\langle\langle\tau\rangle\rangle} \leq x_{\langle\langle\tau\rangle\rangle} \to \Diamond x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle})$	2; 3; 4; 5; 6; Lem. 34; QL
	(10)	$x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}) \to \exists w_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(w))$	QL$_{\mathrm{SA}}$
	(11)	$\Diamond x_{\langle\langle\tau\rangle\rangle}(w_{\langle\tau\rangle}) \to \Diamond \exists w_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(w))$	10; K
2, 3, 4, 5, 6	(12)	$u_{\langle\langle\tau\rangle\rangle} \leq x_{\langle\langle\tau\rangle\rangle} \to \Diamond \exists w_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(w))$	9; 11; PL
4	(13)	$\Diamond \exists w_{\langle\tau\rangle}(u_{\langle\langle\tau\rangle\rangle}(w))$	4; TQL$_{\mathrm{SA}}$
2, 4, 5	(14)	$x_{\langle\langle\tau\rangle\rangle} < u_{\langle\langle\tau\rangle\rangle} \to \Diamond \exists w_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(w))$	2; 5; 13; QL
2, 3, 4, 5, 6	(15)	$\Diamond \exists w_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(w))$	8; 12; 14; PL
4, 5, 6	(16)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Diamond \exists w_{\langle\tau\rangle}(x(w)))$	2; 3; 15; QL
	(17)	$\exists u_{\langle\langle\tau\rangle\rangle}\exists w_{\langle\tau\rangle}(\mathrm{GEN}[u, w])$ $\to \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Diamond \exists w_{\langle\tau\rangle}(x(w)))$	1; 4; 5; 6; 16; QL

$\square$

I'll now turn to mapping the relationships between FP and INFAX. Consider the following definition:

DEFINITION 25 (Infinity). *Let*:

(a) $Z[x_{\langle\langle\tau\rangle\rangle}] := \forall y_{\langle\tau\rangle}(\neg\exists z_{\tau}(y(z)) \to x(y))$; (b) $A[x_{\langle\langle\tau\rangle\rangle}] := \forall y_{\langle\tau\rangle}\forall w_{\langle\tau\rangle}\forall z_{\tau}((x(y) \wedge \forall u_{\tau}(w(u) \leftrightarrow (y(u) \vee u = z))) \to x(w))$; (c) $Inf[z_{\langle\tau\rangle}, x_{\langle\langle\tau\rangle\rangle}] := Z[x_{\langle\langle\tau\rangle\rangle}] \wedge A[x_{\langle\langle\tau\rangle\rangle}] \wedge \neg x(z)$; (d) $Inf[z_{\langle\tau\rangle}] := \exists x_{\langle\langle\tau\rangle\rangle}(Z[x_{\langle\langle\tau\rangle\rangle}] \wedge A[x_{\langle\langle\tau\rangle\rangle}] \wedge \neg x(z))$.

$Inf[z_{\langle\tau\rangle}]$ states that there are infinitely many instances of $z_{\langle\tau\rangle}$. In what follows are proven some results involving the following theses, and FP:

Axiom of Infinity (**InfAx$_\tau$:**) There are infinitely many type τ-entities;

$$Inf[\lambda z_{\tau}(z = z)].$$

Numerical Actuality$_\tau$ (**NA$_\tau$**): Every natural number has some instance;

$$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \exists y_{\langle\tau\rangle}(x(y))).$$

Essential Instances$_{\langle\tau^1,\dots,\tau^n\rangle}$ (**EI$_{\langle\tau^1,\dots,\tau^n\rangle}$**): Whatever possibly stands in a type $\langle\tau^1, \dots, \tau^n\rangle$-relation, stands in that relation whenever the relation exists

$$\Box\forall y_{\langle\tau^1,\dots,\tau^n\rangle}\Box\forall x_{\tau^1}^1 \dots \Box\forall x_{\tau^n}^n\Box(y(\overline{x}) \to \Box(E[y] \to y(\overline{x}))).$$

Identity as Coextensionality$_{\langle\tau^1,\dots,\tau^n\rangle}$ (**IC$_{\langle\tau^1,\dots,\tau^n\rangle}$**): Coextensive type $\langle\tau^1, \dots, \tau^n\rangle$-relations are identical;

$$\forall y_{\langle\tau^1,\dots,\tau^n\rangle}\forall z_{\langle\tau^1,\dots,\tau^n\rangle}(\forall x_{\tau^1}^1 \dots \forall x_{\tau^n}^n(y(\overline{x}) \leftrightarrow z(\overline{x})) \to y = z).$$

LEMMA 37. $\vdash \forall y_{\langle\langle\tau\rangle\rangle}\forall x_{\langle\langle\tau\rangle\rangle}\forall z_{\langle\tau\rangle}((Inf[\lambda z_{\tau}(z = z), x] \wedge \mathbf{N}(y) \wedge y(z)) \to x(z))$.

Proof of Lemma 37. Provable from Lemma 1 and Theorem 6, by QL$_{\mathrm{C}}$-reasoning. $\square$

LEMMA 38. $\vdash \forall x_{\langle\langle\tau\rangle\rangle}\forall y_{\langle\langle\tau\rangle\rangle}\forall z_{\langle\tau\rangle}\forall u_{\langle\tau\rangle}((\mathbf{N}(x) \wedge \mathbf{S}(x, y) \wedge x(z) \wedge y(u)) \to \neg\exists f_{\langle\langle\tau\rangle,\langle\tau\rangle\rangle}$ $(Inj[f, u, z]))$.

Proof of Lemma 38. Provable from Lemmas 1 and 4 and Theorems 1, 2, and 6. □

LEMMA 39. $\vdash \text{INFAx}_\tau \leftrightarrow \text{NA}_\tau$.

Proof of Lemma 39. Provable, by QL_C-reasoning, from Lemmas 1, 2, 4, 37, and 38 and Theorems 1, 2, and 6. □

LEMMA 40. $\vdash_{\text{S5QL}_{\text{SA,C}}} (\text{FP}_\tau \wedge \text{EI}_{\langle\langle\tau\rangle\rangle}) \rightarrow \text{INFAx}_\tau$.

Proof of Lemma 40.

1	(1)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \Diamond\exists y_{\langle\tau\rangle}(x(y)))$	Asmp. (FP_τ)
2	(2)	$\Box\forall x_{\langle\langle\tau\rangle\rangle}\Box\forall y_{\langle\tau\rangle}\Box(x(y) \rightarrow \Box(E[x] \rightarrow x(y)))$	Asmp. $(\text{EI}_{\langle\langle\tau\rangle\rangle})$
2	(3)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \Box\forall y_{\langle\tau\rangle}(x(y)$ $\rightarrow (\Box E[x] \rightarrow \Box x(y))))$	2; TQL
4	(4)	$\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$	Asmp. $(\rightarrow\text{I})$
5	(5)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
2, 4, 5	(6)	$\Box\forall y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y) \rightarrow (\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow \Box x_{\langle\langle\tau\rangle\rangle}(y)))$	3; 4; 5; QL
2, 4, 5	(7)	$\Box\forall y_{\langle\tau\rangle}(\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow (x_{\langle\langle\tau\rangle\rangle}(y) \rightarrow \Box x_{\langle\langle\tau\rangle\rangle}(y)))$	6; KQL
2, 4, 5	(8)	$\Box(E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow \forall y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y) \rightarrow \Box x_{\langle\langle\tau\rangle\rangle}(y)))$	7; KQL
2, 4, 5	(9)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow \Box\forall y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y) \rightarrow \Box x_{\langle\langle\tau\rangle\rangle}(y))$	8; KQL
	(10)	$x_{\langle\langle\tau\rangle\rangle}(y_{\langle\tau\rangle}) \rightarrow E[y_{\langle\tau\rangle}]$	SA
	(11)	$x_{\langle\langle\tau\rangle\rangle}(y_{\langle\tau\rangle}) \rightarrow \exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	10; QL
	(12)	$\Box(x_{\langle\langle\tau\rangle\rangle}(y_{\langle\tau\rangle})) \rightarrow \Box\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	11; K
2, 4, 5	(13)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow \Box\forall y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)$ $\rightarrow \Box\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	9; 12; KQL
2, 4, 5	(14)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow \Box(\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$ $\rightarrow \exists y_{\langle\tau\rangle}\Box\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	13; KQL
2, 4, 5	(15)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow \Box(\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$ $\rightarrow \Box\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	14; KQL
2, 4, 5	(16)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow (\Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$ $\rightarrow \Diamond\Box\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	15; KQL
2, 4, 5	(17)	$\Box E[x_{\langle\langle\tau\rangle\rangle}] \rightarrow (\Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$ $\rightarrow \exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y)))$	16; S5
4, 5	(18)	$\Box E[x_{\langle\langle\tau\rangle\rangle}]$	4; 5; Lem. 5; QL
1, 4, 5	(19)	$\Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	1; 4; 5; QL
1, 2, 4, 5	(20)	$\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	17; 18; 19; PL
1, 2	(21)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \exists y_{\langle\tau\rangle}(x(y)))$	4; 5; 20; QL
	(22)	$(\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \Diamond\exists y_{\langle\tau\rangle}(x(y)))\wedge$ $\Box\forall x_{\langle\langle\tau\rangle\rangle}\Box\forall y_{\langle\tau\rangle}\Box(x(y) \rightarrow \Box(E[x] \rightarrow x(y))))$ $\rightarrow \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \exists y_{\langle\tau\rangle}(x(y)))$	1; 2; 21; PL
	(23)	$(\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \rightarrow \Diamond\exists y_{\langle\tau\rangle}(x(y)))$ $\wedge \Box\forall x_{\langle\langle\tau\rangle\rangle}\Box\forall y_{\langle\tau\rangle}\Box(x(y) \rightarrow \Box(E[x] \rightarrow x(y))))$ $\rightarrow \exists x_{\langle\langle\tau\rangle\rangle}(Inf[\lambda z_\tau(z=z),x])$	22; Lem. 39; PL

□

LEMMA 41. $\vdash_{\text{S5QL}_\text{C}} (\text{FP}_\tau \wedge \text{IC}_{\langle\langle\tau\rangle\rangle}) \rightarrow \text{INFAx}$.

Proof of Lemma 41.

1	(1)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$	Asmp. (FP_τ)
2	(2)	$\forall y_{\langle\langle\tau\rangle\rangle}\forall z_{\langle\langle\tau\rangle\rangle}(\forall x_{\langle\tau\rangle}(y(x) \leftrightarrow z(x)) \to y = z)$	Asmp. $(\mathrm{IC}_{\langle\langle\tau\rangle\rangle})$
3	(3)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
4	(4)	$\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$I)
5	(5)	$\neg\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	Asmp. ($\neg$I)
	(6)	$\neg\exists y_{\langle\tau\rangle}(\Lambda(y))$	Lem. 2
5	(7)	$\forall y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y) \leftrightarrow \Lambda(y))$	5; 6; QL
2, 3, 5	(8)	$x_{\langle\langle\tau\rangle\rangle} = \Lambda$	2; 3; 7; Lem. 2; QL
	(9)	$\Box\neg\exists y_{\langle\tau\rangle}(\Lambda(y))$	6; K
2, 3, 5	(10)	$\Box\neg\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	8; 9; QL
1, 3, 4	(11)	$\Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	1; 3; 4; QL
1, 2, 3, 4	(12)	$\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	5; 10; 11; KQL
1, 2	(13)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \exists y_{\langle\tau\rangle}(x(y))$	3; 4; 12; KQL
1, 2	(14)	$\exists x_{\langle\langle\tau\rangle\rangle}(Inf[\lambda z_\tau(z = z), x])$	13; Lem. 39; PL
	(15)	$(\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$ $\wedge\ \forall y_{\langle\langle\tau\rangle\rangle}\forall z_{\langle\langle\tau\rangle\rangle}(\forall x_{\langle\tau\rangle}(y(x) \leftrightarrow z(x)) \to y = z))$ $\to \exists x_{\langle\langle\tau\rangle\rangle}(Inf[\lambda z_\tau(z = z), x])$	1; 2; 14; PL

$\Box$

LEMMA 42. $\vdash_{\mathrm{TQL}} \mathrm{INFAx}_\tau \to \mathrm{FP}_\tau$.

Proof of Lemma 42.

1	(1)	$\exists x_{\langle\langle\tau\rangle\rangle}(Inf[\lambda z_\tau(z = z), x])$	Asmp. (INFAx_τ)
1	(2)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \exists y_{\langle\tau\rangle}(x(y)))$	1; Lem. 39; PL
3	(3)	$E[x_{\langle\langle\tau\rangle\rangle}]$	Asmp. (QGen)
4	(4)	$\mathbf{N}(x_{\langle\langle\tau\rangle\rangle})$	Asmp. ($\to$I)
1, 3, 4	(5)	$\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	2; 3; 4; QL
1, 3, 4	(6)	$\Diamond\exists y_{\langle\tau\rangle}(x_{\langle\langle\tau\rangle\rangle}(y))$	5; K
1	(7)	$\forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$	3; 4; 6; QL
	(8)	$\exists x_{\langle\langle\tau\rangle\rangle}(Inf[\lambda z_\tau(z = z), x])$ $\to \forall x_{\langle\langle\tau\rangle\rangle}(\mathbf{N}(x) \to \Diamond\exists y_{\langle\tau\rangle}(x(y)))$	1; 7; PL

$\Box$

§F. Model theory. A basic model theory for PMT, based on the work of Gallin [28], Stalnaker [71] and Fritz and Goodman [25], is here offered. I start by defining *frames* and *inhabited structures* (henceforth called 'structures').

DEFINITION 26 (Frame, Inhabited structure). *A frame is a pair* $\mathbf{F} = \langle W, U \rangle$ *where* $\exists x \in W$, $\mathbf{U}$ *is a function with domain* $\mathbf{W}$ *such that* $\bigcup_{w \in W}(U(w)) \neq \emptyset$, *and*: *(i)*

$\mathbf{F}_e := \bigcup_{w \in W}(\boldsymbol{U}(w))$; and (ii) $\mathbf{F}_{\langle \tau^1, ..., \tau^n \rangle} := (\wp(\bigtimes_{1 \leq i \leq n}(\boldsymbol{F}_{\tau^i})))^{\boldsymbol{W}}$, for all $\tau^1, ..., \tau^n \in \mathcal{T}$ and $n \in \mathbb{Z}^+$. An inhabited structure is a triple $\mathbf{G} = \langle \boldsymbol{W}, \boldsymbol{U}, \boldsymbol{D} \rangle$ where $\langle \boldsymbol{W}, \boldsymbol{U} \rangle$ is a frame and $\mathbf{D}$ a function with domain $\mathbf{W} \times \mathcal{T}$ such that: (i) $\mathbf{D}_{w,e} = \boldsymbol{U}(w)$; and (ii) $\mathbf{D}_{w,\langle \tau^1, ..., \tau^n \rangle} \subseteq \boldsymbol{F}_{\langle \tau^1, ..., \tau^n \rangle}$, for all $\tau^1, ..., \tau^n \in \mathcal{T}$.

The set $\mathbf{W}$ represents the collection of all possible worlds, and the type $\langle \tau^1, ..., \tau^n \rangle$-relations existing at a world w are represented by set-theoretic functions in $\mathbf{D}_{w,\langle \tau^1, ..., \tau^n \rangle}$. Since different functions in $\mathbf{D}_{w,\langle \tau^1, ..., \tau^n \rangle}$ may have the same image at some world, this representation makes room for some distinct relations possibly sharing their extension. Pluralities of type τ-entities existing at a world w are represented by nonempty members of $\wp(\mathbf{D}_{w,\tau})$.[64] I'll now turn to the standard definitions of *variable-assignment*, *value*, and *satisfaction*:

DEFINITION 27 (Variable-assignments). *A* variable-assignment σ of a structure $\mathbf{G}$ is a *function with domain the set of variables of* PMT *and which assigns to each singular type τ-variable* v *(plural type τ-variable vv) a value $\sigma(v) \in \bigcup_{w \in W}(\boldsymbol{D}_{w,\tau})$ (a value $\sigma(vv) \in \bigcup_{w \in W}(\wp(\boldsymbol{D}_{w,\tau}) \setminus \{\emptyset\}))$, for every $\tau \in \mathcal{T}$. If σ is a variable-assignment,* v *a singular type τ-variable (plural type τ-variable), and $o \in \bigcup_{w \in W}(\boldsymbol{D}_{w,\tau})$ ($o \in \bigcup_{w \in W}(\wp(\boldsymbol{D}_{w,\tau}) \setminus \{\emptyset\}))$, $\sigma[o/v]$ is a variable-assignment just like σ except (perhaps) that $\sigma[o/v](v) = o$. As$_{\mathbf{G}}$ is the collection of all variable-assignments of* $\mathbf{G}$.

DEFINITION 28 (Value). *The* value *of an expression ξ of* PMT *relative to $\sigma \in$ As$_{\mathbf{G}}$ and structure $\mathbf{G}$, $\mathbf{V}_{\mathbf{G}}^{\sigma}(\xi)$, is defined as follows (where, if ξ is an singular term or a formula, then $\mathbf{V}_{\mathbf{G}}^{\sigma}(\xi)$ is a function with domain $\mathbf{W}$):*

(a) $\mathbf{V}_{\mathbf{G}}^{\sigma}(v) = \sigma(v)$; *(b)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(vv) = \sigma(vv)$; *(c)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(=_{\langle \rho, \rho \rangle})(w) = \{\langle o, o \rangle : o \in \boldsymbol{D}_{w,\rho}\}$; *(d)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(vv = uu)(w) = \{\langle\rangle : \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(vv) = \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(uu) \in \wp(\boldsymbol{D}_{w,\delta}) \setminus \{\emptyset\}\}$; *(e)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(\beta(\overline{\alpha}))(w) = \{\langle\rangle : \langle \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\alpha^1), ..., \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\alpha^n) \rangle \in \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\beta)(w)\}$; *(f)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(\neg\varphi)(w) = \{\langle\rangle : \langle\rangle \notin \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\varphi)(w)\}$; *(g)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(\varphi \wedge \psi)(w) = \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\varphi)(w) \cap \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\psi)(w)$; *(h)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(\Box\varphi)(w) = \bigcap_{u \in W}(\boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\varphi)(u))$; *(i)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(\alpha \prec vv)(w) = \{\langle\rangle : \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\alpha) \in \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(vv) \ \& \ \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(vv) \in \wp(\boldsymbol{D}_{w,\delta}) \setminus \{\emptyset\}\}$; *(j)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(\forall v\varphi)(w) = \bigcap_{o \in \boldsymbol{D}_{w,\delta}}(\boldsymbol{V}_{\boldsymbol{G}}^{\sigma[o/v]}(\varphi)(w))$; *(k)* $\mathbf{V}_{\mathbf{G}}^{\sigma}(\forall vv\varphi)(w) = \bigcap_{o \in \wp(\boldsymbol{D}_{w,\delta}) \setminus \{\emptyset\}}(\boldsymbol{V}_{\boldsymbol{G}}^{\sigma[o/vv]}(\varphi)(w))$;

(l) $\mathbf{V}_{\mathbf{G}}^{\sigma}(\lambda v^1 ... v^n(\varphi))(w) = \{\langle o^1, ..., o^n \rangle \in \boldsymbol{D}_{w,\langle \tau^1, ..., \tau^n \rangle} : \langle\rangle \in \boldsymbol{V}_{\boldsymbol{G}}^{\sigma[o^1/v_{\tau}^1]...[o^n/v_{\tau}^n]}(\varphi)(w)\}$.

DEFINITION 29 (Satisfaction). *Structure $\mathbf{G}$, $w \in \boldsymbol{W}$ and $\sigma \in$ As$_{\mathbf{G}}$ jointly satisfy formula φ, $\mathbf{G}, w, \sigma \vDash \varphi$, if and only if $\langle\rangle \in \boldsymbol{V}_{\boldsymbol{G}}^{\sigma}(\varphi)(w)$. $\mathbf{G}$ satisfies φ, $\mathbf{G} \vDash \varphi$, if and only if $\mathbf{G}, w, \sigma \vDash \varphi$, for all $\sigma \in$ As$_{\mathbf{G}}$ and $w \in \boldsymbol{W}$. A class of structures satisfies a collection of formulae if and only if every member of the class satisfies every formula in the collection.*

The value of $\lambda v^1 ... v^n(\varphi)$ in a structure $\mathbf{G}$ does not shift from world to world. For instance, $\mathbf{G}, w, \sigma \vDash u = \lambda v^1 ... v^n\varphi \rightarrow \Box(E[u] \rightarrow u = \lambda v^1 ... v^n\varphi)$: complex predicates satisfy an object language formulation of the claim that they are rigid designators. Indeed, the following lemma establishes that complex predicates do not constitute counterexamples to Ind in the context of our modal type theory:[65]

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[64] Of course, this talk of 'representation' need not be seen as having anything but heuristic value.

[65] This lemma is a consequence of a more general result according to which if $\mathbf{V}_{\mathbf{G}}^{\sigma}(\alpha) = \mathbf{V}_{\mathbf{G}}^{\sigma}(\gamma)$, then $\mathbf{V}_{\mathbf{G}}^{\sigma}(\varphi) = \mathbf{V}_{\mathbf{G}}^{\sigma}(\varphi')$, where φ is any formula or term of PMT and φ' is a formula or term which results from φ by having γ occur at some places where α occurs, re-lettering bound variables to ensure that no variables free in $\alpha = \beta$ are bound in φ or φ'. We leave its proof to the reader.

LEMMA 43. *For every instance φ of* Ind *and every inhabited structure* **G**, $\mathbf{G} \vDash \varphi$.

From UPPER's proponents' standpoint, not all structures represent ways modal reality might be. One requirement is that they be FP- and SA(*seriously actualist*)-structures:

DEFINITION 30 (FP-structures). *A structure* $\mathbf{G} = \langle W, U, D \rangle$ *is an* FP-*structure if and only if, for every $n \in N$, some $w \in W$ is such that $n \leq \boldsymbol{D}_{w,e}$.*

DEFINITION 31 (SA-functions, SA-structures). $\mathbf{A}_{\mathbf{G}, \langle \tau^1, \dots, \tau^n \rangle} = \{f \in \boldsymbol{F}_{\langle \tau^1, \dots, \tau^n \rangle} : \forall w \in W(f(w) \subseteq \bigtimes_{1 \leq i \leq n}(\boldsymbol{D}_{w, \tau^i}))\}$ *is the set of* **G***'s type $\langle \tau^1, \dots, \tau^n \rangle$ SA-functions, for all $\tau^1, \dots, \tau^n \in \mathcal{T}$ and structures* **G**. *A structure* $\mathbf{G} = \langle W, U, D \rangle$ *is an* SA-*structure if and only if, for all $\tau^1, \dots, \tau^n \in \mathcal{T}$, and every $w \in W$, $\mathbf{D}_{w, \langle \tau^1, \dots, \tau^n \rangle} \subseteq \boldsymbol{A}_{\mathbf{G}, \langle \tau^1, \dots, \tau^n \rangle}$.*

FP&SA-structures satisfy all of S5PQL$_{\text{SA}}^{\text{FP}}$'s theorems. Further, –*closed* structures (in the sense of [25, definition 6]) constitute a natural class of structures which satisfy Comp$_C$ [25, sec. 5.1, proposition 8]. The notion can be glossed as follows. Say that an entity is distinguishable from another from a world's standpoint just in case, roughly, there is some relation in the domain of that world such that the first entity falls under that relation at some world, but the other doesn't, or vice versa. Then, a structure is closed just in case each world's domain contains all and only the entities which, from that world's standpoint, are distinguishable from all other entities. It is natural to think that, from the standpoint of UPPER's proponents, the inhabited structures which represent genuine ways modal reality might be are all UPPER-*structures*, in the following sense:

DEFINITION 32 (UPPER-structures). *To be an* UPPER-*structure just is to be an* FP-, SA- *and closed structure.*

Now, consider the following structures, where, for each inhabited structure **S** and world w of **S**, $Fix(w, \mathbf{S})$ is the set of *structure automorphisms* fixed to world w of structure **S** (this notion is precisely defined in [25, definition 15, p. 668], and so I'll omit the details), $\mathcal{N}$ is the set of finite von Neumann ordinals, $n^{\mathcal{N}}$ is the nth von Neumann ordinal for each $n \in \mathbb{N}$, and $1^* := \{1^{\mathcal{N}}\}$:

DEFINITION 33 (Structures $\mathbf{G}^\circ, \mathbf{G}^\bullet$). $\mathbf{G}^\circ = \langle W^\circ, U^\circ, D^\circ \rangle$, *where*:
(*i*) $\mathbf{W}^\circ = \mathcal{N} \cup \{1^*\}$; (*ii*) *for all $w \in W^\circ$, $\mathbf{D}_{w,e}^\circ = w$ and $\mathbf{D}_{w, \langle \tau^1, \dots, \tau^n \rangle}^\circ := (\{\emptyset\})^{W^\circ}$ for all $\tau^1, \dots, \tau^n \in \mathcal{T}$. $\mathbf{G}^\bullet := \langle W^\circ, U^\circ, D^\bullet \rangle$, where, for all $w \in W^\circ$, $\mathbf{D}_{w,e}^\bullet := \boldsymbol{D}_{w,e}^\circ$ and $\mathbf{D}_{w, \langle \tau^1, \dots, \tau^n \rangle}^\bullet = \{f \in \boldsymbol{F}_{\langle \tau^1, \dots, \tau^n \rangle} : \forall \pi(\pi \in Fix(w, \boldsymbol{G}^\circ) \Rightarrow \pi(f) = f) \text{ and } f \in \boldsymbol{A}_{\boldsymbol{G}^\bullet}, \langle \tau^1, \dots, \tau^n \rangle\}$, for all $\tau^1, \dots, \tau^n \in \mathcal{T}$.*

$\mathbf{G}^\bullet$'s existence is a consequence of ZF. For each $n \in \mathbb{N}$, $\mathbf{D}_{(n+1)^{\mathcal{N}}, e}^\bullet = \mathbf{D}_{n^{\mathcal{N}}, e}^\bullet \cup \{n^{\mathcal{N}}\}$, and $\mathbf{D}_{n^{\mathcal{N}}, e}^\bullet = n$. Just like $1^{\mathcal{N}}$, world 1^*'s type e-domain has exactly one element—respectively, $0^{\mathcal{N}}$ and $1^{\mathcal{N}}$. Clearly, $\mathbf{G}^\bullet \vDash$ CONTINGENTISM$_e$, and $\mathbf{G}^\bullet \vDash \square\neg$INFAX. Since $0^{\mathcal{N}}$ and $1^{\mathcal{N}}$ are indistinguishable (as individuals) from the standpoint of $0^{\mathcal{N}}$, some relations defined in terms of them do not exist at $0^{\mathcal{N}}$ (even though they will exist at all other worlds, owing to $1^{\mathcal{N}}$ and 1^* being distinguishable from the standpoint of those other worlds). So, $\mathbf{G}^\bullet \vDash$ CONTINGENTISM$_\tau$ for every $\tau \in \mathcal{T}$. Clearly, $\mathbf{G}^\bullet$ is an FP- and SA-structure. That $\mathbf{G}^\bullet$ is –*closed* is a consequence of Fritz and Goodman's Proposition

6 [25, sec. 4.3], since it is *generated*, in the sense of [25, sec. 4.2, definition 15], by structure $\mathbf{G}^\circ$. Hence, $\mathbf{G}^\bullet$ is an UPPER-structure, and so:[66]

LEMMA 44. $\mathbf{G}^\bullet$ *satisfies all theorems of* $\mathtt{S5PQL}^{\mathtt{FP}}_{\mathtt{SA,C}}$, CONTINGENTISM$_\tau$ *for every type* τ, *and* $\Box\neg$INFAX.

Finally, consider the following inhabited structure, whose existence is guaranteed by ZF:

DEFINITION 34 (Structure **N**). $\mathbf{N} := \langle \mathcal{N}, \mathbf{U}, \mathbf{D} \rangle$, *where for every* $w \in \mathbf{W}$, $\mathbf{D}_{w,e} = w$ *and* $\mathbf{D}_{w,\langle \tau^1,\ldots,\tau^n \rangle} = \mathbf{A}_{\mathbf{H},\langle \tau^1,\ldots,\tau^n \rangle}$.

To conclude we note the following result:

LEMMA 45. **N** *satisfies all theorems of* $\mathtt{S5PQL}^{\mathtt{FP}}_{\mathtt{SA,N}}$, CONTINGENTISM$_e$, *and* $\Box\neg$INFAX.

Granting ZF's consistency, Lemmas 44 and 45 establish the joint consistency of FP with nontrivial modal type theories, and so also with the weak system $\mathtt{KQL}^{\mathtt{FP}}_{\mathtt{C}}$. Given the track record, vis-à-vis their *consistency*, of systems purporting to contain arithmetic, these lemmas are not an insignificant achievement.

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[66] As an anonymous reviewer observes, in $\mathbf{G}^\bullet$ it is only at a world without individuals that some possibly existing relation does not exist. This could be easily dealt with by having the type e domain of world $0^\mathcal{N}$ consist instead of a nonempty set containing none of $0^\mathcal{N}$ and $1^\mathcal{N}$. Moreover, if to the model are added two worlds, each with exactly one individual in its type e domain, where these individuals are different and none is a finite von Neumann ordinal, then it will be the case that, from the standpoint of every world whatsoever, there could have been some type τ relation which does not exist at that world, for every type $\tau \neq e$. As suggested by the reviewer, another model which achieves the same effect has as its worlds the finite sets of finite von Neumann ordinals (with worlds type e domains being equated to the worlds themselves). Since the model offered in the main text suffices to establish Lemma 44, I'll here disregard these and other models along similar lines.

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