Proceedings of the Edinburgh Mathematical Society (2004) **47**, 297–303 ©
DOI:10.1017/S0013091503000373 Printed in the United Kingdom

FREE QUANTUM ANALOGUES OF THE FIRST FUNDAMENTAL THEOREMS OF INVARIANT THEORY

JULIEN BICHON

*Laboratoire de Mathématiques Appliquées,
Université de Pau et des Pays de l'Adour, IPRA,
Avenue de l'université, 64000 Pau, France*
(julien.bichon@univ-pau.fr)

(Received 5 May 2003)

Abstract We formulate and prove a free quantum analogue of the first fundamental theorems of invariant theory. More precisely, the polynomial function algebras on matrices are replaced by free algebras, while the universal cosovereign Hopf algebras play the role of the general linear group.

Keywords: Hopf algebras; invariant theory; free algebras

2000 *Mathematics subject classification:* Primary 16W30

1. Introduction

This note was inspired by reading the paper by Goodearl *et al.* [**5**]. We have followed part of their introduction very closely and we have tried to keep, as much as possible, their terminology and notation in this paper.

Consider a fixed field K and positive integers $m, n, t > 0$. For integers $u, v > 0$, we write $M_{u,v}$ for the set of $u \times v$ matrices with entries in K. The general linear group $GL_t = GL_t(K)$ acts on the variety $V = M_{m,t} \times M_{t,n}$, with action given by

$$GL_t \times V \to V,$$
$$(g, (A, B)) \mapsto (Ag^{-1}, gB).$$

Thus GL_t acts on the algebra $\mathcal{O}(V) \cong \mathcal{O}(M_{m,t}) \otimes \mathcal{O}(M_{t,n})$. The first fundamental theorems of invariant theory describe the subalgebra $\mathcal{O}(V)^{GL_t}$ of invariants for this action in the following way. Consider the multiplication map

$$\theta : M_{m,t} \times M_{t,n} \to M_{m,n},$$
$$(A, B) \mapsto AB,$$

and denote by $\theta^* : \mathcal{O}(M_{m,n}) \to \mathcal{O}(M_{m,t}) \otimes \mathcal{O}(M_{t,n})$ the induced algebra morphism.

Theorem 1.1. *The ring of invariants* $\mathcal{O}(V)^{GL_t}$ *equals* $\operatorname{Im} \theta^*$.

Theorem 1.2. *Let X_{ij} $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ be the usual coordinate functions on $M_{m,n}$. Let $\mathcal{T}_{t+1}$ be the ideal of $\mathcal{O}(M_{m,n})$ generated by the $(t+1)\times(t+1)$ minors of the matrix (X_{ij}) over $\mathcal{O}(M_{m,n})$ (this ideal is zero if $t \geqslant \min\{m,n\}$). Then the kernel of θ^* is $\mathcal{T}_{t+1}$.*

These two theorems are respectively known as the first and second fundamental theorems of invariant theory (for GL_t) (see [**4**]). They give a full description of the algebra $\mathcal{O}(V)^{GL_t}$.

Goodearl *et al.* [**5**] generalized these theorems for quantized coordinate algebras. In this paper we prove analogues of these theorems for free algebras.

Let $u, v > 0$ be positive integers and denote by $A(u,v)$ the free algebra on uv generators. The natural analogue of the algebra morphism θ^* above is the algebra morphism (denoted by θ for simplicity)

$$\theta : A(m,n) \to A(m,t) \otimes A(t,m),$$
$$x_{ij} \mapsto \sum_{k=1}^{t} y_{ik} \otimes z_{kj},$$

where x_{ij}, y_{ij}, z_{ij} stand for generators of $A(m,n)$, $A(m,t)$ and $A(t,n)$, respectively. It is easy to show that, contrary to the commutative case, the morphism θ is always injective. The following question arises naturally: does there exist a quantum group G acting on $A_{(m,t)}$ and $A_{(t,n)}$ and such that $\operatorname{Im}\theta$ equals $(A_{(m,t)} \otimes A_{(t,n)})^G$? We show that the universal cosovereign Hopf algebras introduced in [**2**], which are natural free analogues of the general linear groups in quantum group theory, answer positively to this question. The key for proving this result is a representation theoretic property of these Hopf algebras, proved in [**3**].

Our work is organized as follows. In § 2 we recall the set-up of [**5**] for non-commutative analogues of the first fundamental theorems of invariant theory. In § 3 we recall some basic facts concerning the universal cosovereign Hopf algebras and state the main theorem. The proof is given in § 4.

Throughout the paper we work over an arbitrary base field K.

2. The set-up for non-commutative invariant theory

Let us first recall the set-up, due to Goodearl *et al.*, for stating non-commutative analogues of the first fundamental theorems of invariant theory. Similar considerations were done independently by Banica [**1**] in the context of Kac algebras actions.

Let H be a Hopf algebra, let (A,ρ) be a right H-comodule algebra and let (B,λ) be a left H-comodule algebra, where

$$\rho : A \to A \otimes H \quad \text{and} \quad \lambda : B \to H \otimes B$$

are the coactions of H. One can turn A into a left H-comodule with the coaction $\rho' = \tau \circ (\mathrm{id}_A \otimes S) \circ \rho$, where $\tau : A \otimes H \to H \otimes A$ is the standard flip. Thus one can consider the tensor product $A \otimes B$ of the left H-comodules A and B, which is a left H-comodule, but which, in the non-commutative situation, is not any more a comodule algebra in general. However, we have the following result from [**5**] (see also [**1**]).

Proposition 2.1. *The set of coinvariants* $(A \otimes B)^{\mathrm{co}H}$ *is a subalgebra of* $A \otimes B$.

Here, it should be recalled that if M is a left H-comodule with coaction $\alpha : M \to H \otimes M$, the set of coinvariants is $M^{\mathrm{co}H} = \{x \in M \mid \alpha(x) = 1 \otimes x\}$. Denoting by $\mathcal{C}$ the category of left H-comodules and by I the trivial one-dimensional comodule, then $M^{\mathrm{co}H}$ is canonically identified with $\mathrm{Hom}_{\mathcal{C}}(I, M)$.

Fix positive integers $m, n, t > 0$. We denote by $A(m,n)$, $A(m,t)$ and $A(t,n)$ the free algebras on mn, mt and tn generators, respectively, with canonical generators denoted, respectively, by x_{ij}, y_{ij} and z_{ij}. The free analogue of the comultiplication map θ^* of the introduction is the algebra morphism

$$\theta : A(m,n) \to A(m,t) \otimes A(t,m),$$
$$x_{ij} \mapsto \sum_{k=1}^{t} y_{ik} \otimes z_{kj}.$$

It may be shown easily by direct computations that θ is injective.

Now consider a Hopf algebra H having a multiplicative matrix $u = (u_{ij}) \in M_t(H)$. This means that, for $i, j \in \{1, \dots, t\}$, we have $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and $\varepsilon(u_{ij}) = \delta_{ij}$. Then $A(m,t)$ is a right H-comodule algebra with coaction

$$\rho : A(m,t) \to A(m,t) \otimes H,$$
$$y_{ij} \mapsto \sum_{k=1}^{t} y_{ik} \otimes u_{kj}.$$

In the same way, $A(t,n)$ is a left H-comodule algebra with coaction

$$\lambda : A(t,n) \to H \otimes A(t,n),$$
$$z_{ij} \mapsto \sum_{k=1}^{t} u_{ik} \otimes z_{kj}.$$

Thus we are in the preceding situation and we can consider the algebra $(A(m,t) \otimes A(t,n))^{\mathrm{co}H}$. Similarly to Proposition 2.3 in [**5**], we have the following result.

Proposition 2.2. *The algebra* $(A(m,t) \otimes A(t,n))^{\mathrm{co}H}$ *is a subalgebra of* $A(m,t) \otimes A(t,n)$ *containing* $\mathrm{Im}\,\theta$.

It is then natural to ask whether a free analogue of the first fundamental theorem of invariant theory holds, that is, to wonder when the corestriction of the algebra morphism θ to a map $A(m,n) \to (A(m,t) \otimes A(t,m))^{\mathrm{co}H}$ is surjective. We will see that this is true for the universal cosovereign Hopf algebras [**2**]. In fact, the key property in order that the corestriction map of θ be surjective is that the tensor powers of the comodule U associated to the multiplicative matrix u be simple non-equivalent comodules. At the ring-theoretic level, this corresponds to the fact that the elements u_{ij} generate a free subalgebra on t^2 generators.

https://doi.org/10.1017/S0013091503000373 Published online by Cambridge University Press

3. Universal cosovereign Hopf algebras and the theorem

Let $F \in GL_t$. Recall from [**2**] that the algebra $H(F)$ is defined to be the universal algebra with generators $(u_{ij})_{1\leqslant i,\, j\leqslant t}$, $(v_{ij})_{1\leqslant i,\, j\leqslant t}$ and relations

$$u\,{}^{\mathrm{t}}v = {}^{\mathrm{t}}vu = I_t = vF\,{}^{\mathrm{t}}uF^{-1} = F\,{}^{\mathrm{t}}uF^{-1}v,$$

where $u = (u_{ij})$, $v = (v_{ij})$ and I_t is the identity $t \times t$ matrix. It turns out (see [**2**]) that $H(F)$ is a Hopf algebra with comultiplication defined by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$, with counit defined by $\varepsilon(u_{ij}) = \varepsilon(v_{ij}) = \delta_{ij}$ and with antipode defined by $S(u) = {}^{\mathrm{t}}v$ and $S(v) = F\,{}^{\mathrm{t}}uF^{-1}$. Furthermore, $H(F)$ is a cosovereign Hopf algebra [**2**]: there exists an algebra morphism $\Phi : H(F) \to k$ such that $S^2 = \Phi * \mathrm{id} * \Phi^{-1}$. The Hopf algebras $H(F)$ have the following universal property (see [**2**, Theorem 3.2]).

Property 3.1. *Let H be a Hopf algebra and let V be a finite-dimensional H-comodule isomorphic with its bidual comodule V^{**}. Then there exists a matrix $F \in GL_t$ ($t = \dim V$) such that V is an $H(F)$-comodule and such that there exists a Hopf algebra morphism $\pi : H(F) \to H$ with $(1_V \otimes \pi) \circ \beta_V = \alpha_V$, where $\alpha_V : V \to V \otimes H$ and $\beta_V : V \to V \otimes H(F)$ denote the coactions of H and $H(F)$ on V, respectively. In particular, every finite type cosovereign Hopf algebra is a homomorphic quotient of a Hopf algebra $H(F)$.*

In view of this universal property, it is natural to say that the Hopf algebras $H(F)$ are the universal cosovereign Hopf algebras, or the free cosovereign Hopf algebras, and to see these Hopf algebras as natural analogues of the general linear groups in quantum group theory.

Coming back to the situation of § 1, we have the following result, which is a free quantum analogue of the first fundamental theorems of invariant theory.

Theorem 3.2. *Let $m, n, t > 0$ be positive integers and let $F \in GL_t$. Then the algebra morphism*

$$\theta : A(m,n) \to (A(m,t) \otimes A(t,n))^{\mathrm{co}H(F)}$$

is an isomorphism.

4. Proof of the theorem

4.1.

We begin with some general considerations. Let $\mathcal{C}$ be a K-linear strict tensor category, that is $\mathcal{C}$ is an abelian K-linear category and $\mathcal{C} = (\mathcal{C}, \otimes, I)$ is a strict tensor category (see [**6**]) such that the tensor product is K-linear in each variable. Let X, Y be some objects of $\mathcal{C}$. We define a K-algebra $\mathcal{C}(X,Y)$ in the following way. As a vector space,

$$\mathcal{C}(X,Y) = \bigoplus_{k \in \mathbb{N}} \mathrm{Hom}_{\mathcal{C}}(X^{\otimes k}, Y^{\otimes k}).$$

The product of $\mathcal{C}(X,Y)$ is defined on homogeneous elements

$$f_1 \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes k_1}, Y^{\otimes k_1}) \quad \text{and} \quad f_2 \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes k_2}, Y^{\otimes k_2})$$

https://doi.org/10.1017/S0013091503000373 Published online by Cambridge University Press

by

$$f_1 f_2 := f_1 \otimes f_2 \in \mathrm{Hom}_{\mathcal{C}}(X^{\otimes k_1+k_2}, Y^{\otimes k_1+k_2}).$$

It is obvious that $\mathcal{C}(X,Y)$ is an associative K-algebra.

Let $m,n \in \mathbb{N}^*$ and let U be an object of $\mathcal{C}$. Let $X = U^m$ be the direct sum of m copies of U. Let $p_i : U^m \to U$ and $v_i : U \to U^m$, $1 \leqslant i \leqslant m$, be the canonical morphisms such that $\sum_{i=1}^m v_i \circ p_i = 1_X$ and $p_i \circ v_j = \delta_{ij} 1_U$. Similarly, let $Y = U^n$ be the direct sum of n copies of U. Let $q_i : U^n \to U$ and $u_i : U \to U^n$, $1 \leqslant i \leqslant n$, be the canonical morphisms such that $\sum_{i=1}^m u_i \circ q_i = 1_Y$ and $q_i \circ u_j = \delta_{ij} 1_U$.

Lemma 4.1. *Assume that* $\mathrm{End}_{\mathcal{C}}(U) = K$. *Then the algebra morphism*

$$\psi : A(m,n) \to \mathcal{C}(X,Y),$$
$$x_{ij} \mapsto u_j \circ p_i$$

is an isomorphism

Proof. Let $i_1,\dots,i_k \in \{1,\dots,m\}$, $j_1,\dots,j_k \in \{1,\dots,m\}$. Then

$$\psi(x_{i_1 j_1} \cdots x_{i_k j_k}) = (u_{j_1} \otimes \cdots \otimes u_{j_k}) \circ (p_{i_1} \otimes \cdots \otimes p_{i_k}).$$

Thus ψ transforms a basis of $A(m,n)$ into a basis of $\mathcal{C}(X,Y)$ and is an isomorphism. □

4.2.

Consider again the K-linear strict tensor category $\mathcal{C}$. Let X be an object of $\mathcal{C}$. Recall (see [**6**]) that a right dual for X is a triplet (X^*, e, d), where X^* is an object of $\mathcal{C}$, while $e : X \otimes X^* \to I$ and $d : I \to X^* \otimes X$ are morphisms such that

$$(e \otimes 1_X) \circ (1_X \otimes d) = 1_X \quad \text{and} \quad (1_{X^*} \otimes e) \circ (d \otimes 1_{X^*}) = 1_{X^*}.$$

We then have, for all objects Y, Z of $\mathcal{C}$, isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(Y, X^* \otimes Z) \cong \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z),$$
$$f \mapsto (e \otimes 1_Z) \circ (1_X \otimes f).$$

Also recall that if X has a right dual (X^*, e, d), then $X^{\otimes n}$ has a right dual $(X^{*\otimes n}, e_n, d_n)$ for all $n \in \mathbb{N}^*$, where

$$e_n = e \circ \cdots \circ (1_{X^{\otimes n-1}} \otimes e \otimes 1_{X^{*\otimes n-1}}) \quad \text{and} \quad d_n = (1_{X^{*\otimes n-1}} \otimes d \otimes 1_{X^{\otimes n-1}}) \circ \cdots \circ d.$$

Recall finally that, in the tensor category of finite-dimensional left comodules over a Hopf algebra, every object has a right dual.

4.3.

Let $m, n, t > 0$ be positive integers and let $F \in GL_t$. Consider the multiplicative matrix $u = (u_{ij}) \in M_t(H(F))$. One associates a right $H(F)$-comodule U_{r} to u: as a vector space $U_{\mathrm{r}} = K^t$, with its standard basis $e_1, \dots, e_t$, and the coaction $\alpha : U_{\mathrm{r}} \to U_{\mathrm{r}} \otimes H(F)$ is given by $\alpha(e_i) = \sum_j e_j \otimes u_{ji}$. Similarly, one associates a left $H(F)$-comodule U_{l} to u: as a vector space $U_{\mathrm{l}} = K^t$, and the coaction $\beta : U_{\mathrm{l}} \to H(F) \otimes U_{\mathrm{l}}$ is given by $\beta(e_i) = \sum_j u_{ij} \otimes e_j$.

We have the following key result from [**3**, Corollary 2.6].

Proposition 4.2. *The left (respectively, right) $H(F)$-comodules $U_{\mathrm{l}}^{\otimes k}$, $k \in \mathbb{N}^*$ (respectively, $U_{\mathrm{r}}^{\otimes k}$, $k \in \mathbb{N}^*$), are simple non-equivalent left (respectively, right) $H(F)$-comodules, and have endomorphism algebras isomorphic with K.*

As a right $H(F)$-comodule algebra, $A(m, t)$ is naturally identified with the tensor algebra

$$T(U_{\mathrm{r}}^m) = \bigoplus_{i \in \mathbb{N}} (U_{\mathrm{r}}^m)^{\otimes i}.$$

Now transform $A(m, t)$ and $T(U_{\mathrm{r}}^m)$ into left $H(F)$-comodules in the manner of § 2. Then it is immediate that, as a left $H(F)$-comodule, $T(U_{\mathrm{r}}^m)$ is identified with $T(U_{\mathrm{l}}^{m*})$. We have an isomorphism

$$\begin{aligned} \phi_1 : A(m, t) &\to T(U_{\mathrm{l}}^{m*})^{\mathrm{op}}, \\ y_{ij} &\mapsto v_i(e_j)^*, \end{aligned}$$

which is both an algebra isomorphism and an $H(F)$-comodule isomorphism (we use the notations of § 4.1). Similarly, $A(t, n)$ is identified with $T(U_{\mathrm{l}}^n)$ by the following left $H(F)$-comodule algebra isomorphism:

$$\begin{aligned} \phi_2 : A(t, n) &\to T(U_{\mathrm{l}}^n), \\ z_{ij} &\mapsto u_j(e_i). \end{aligned}$$

We have now all the ingredients necessary to prove Theorem 3.2. We work in the K-linear tensor category $\mathcal{C}$ of left $H(F)$-comodules. Since $\mathcal{C}$ is a concrete tensor category of vector spaces, we can proceed as if it was strict and the considerations of §§ 4.1 and 4.2 are valid. We put $U = U_{\mathrm{l}}$. We have

$$\begin{aligned} (T(U^{m*}) \otimes T(U^n))^{\mathrm{co}H(F)} &\cong \mathrm{Hom}_{\mathcal{C}}(I, T(U^{m*}) \otimes T(U^n)) \\ &\cong \mathrm{Hom}_{\mathcal{C}}\Big(I, \bigoplus_{i,j \in \mathbb{N}} (U^{m*})^{\otimes i} \otimes (U^n)^{\otimes j}\Big) \\ &\cong \bigoplus_{i,j \in \mathbb{N}} \mathrm{Hom}_{\mathcal{C}}(I, (U^{m*})^{\otimes i} \otimes (U^n)^{\otimes j}) \\ &\cong \bigoplus_{i,j \in \mathbb{N}} \mathrm{Hom}_{\mathcal{C}}((U^m)^{\otimes i}, (U^n)^{\otimes j}) \quad \text{(by § 4.2)} \\ &\cong \bigoplus_{i \in \mathbb{N}} \mathrm{Hom}_{\mathcal{C}}((U^m)^{\otimes i}, (U^n)^{\otimes i}) \qquad \text{(by Proposition 4.2)} \\ &= \mathcal{C}(U^m, U^n). \end{aligned}$$

https://doi.org/10.1017/S0013091503000373 Published online by Cambridge University Press

It is a straightforward matter to check that, for $i_1, \dots, i_p \in \{1, \dots, m\}$ and $j_1, \dots, j_p \in \{1, \dots, n\}$, the isomorphism just considered transforms the element

$$\sum_{k_1,\dots,k_p} v_{i_p}(e_{k_p})^* \otimes \cdots \otimes v_{i_1}(e_{k_1})^* \otimes u_{j_1}(e_{k_1}) \otimes \cdots \otimes u_{j_p}(e_{k_p})$$

into the element

$$(u_{j_1} \otimes \cdots \otimes u_{j_k}) \circ (p_{i_1} \otimes \cdots \otimes p_{i_k})$$

of $\mathcal{C}(U^m, U^n)$. Hence this isomorphism, composed with $(\phi_1 \otimes \phi_2) \circ \theta$, yields the isomorphism ψ of Lemma 4.1, and θ is itself an isomorphism. This concludes the proof of Theorem 3.2.

Remark 4.3. Theorem 3.2 is still valid if one replaces the Hopf algebra $H(F)$ by $H(t)$, the free Hopf algebra generated by the matrix coalgebra M_t^* (see [**7**]). This is clear from the proof and from the fact that as $H(F)$ is a quotient of $H(t)$, Proposition 4.2 remains valid for $H(t)$.

References

1. T. Banica, Subfactors associated to compact Kac algebras, *Integ. Eqns Operat. Theory* **39**(1) (2000), 1–14.
2. J. Bichon, Cosovereign Hopf algebras, *J. Pure Appl. Alg.* **157**(2–3) (2001), 121–133.
3. J. Bichon, Corepresentation theory of universal cosovereign Hopf algebras, preprint math.QA/0211059 (2002).
4. C. De Concini and C. Procesi, A characteristic free approach to invariant theory, *Adv. Math.* **21** (1976), 330–354.
5. K. R. Goodearl, T. H. Lenagan and L. Rigal, The first fundamental theorem of coinvariant theory for the quantum general linear group, *Publ. Res. Inst. Math. Sci. Kyoto* **36**(2) (2000), 269–296.
6. C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155 (Springer, 1995).
7. M. Takeuchi, Free Hopf algebras generated by coalgebras, *J. Math. Soc. Jpn* **23**(4) (1971), 581–562.