Bull. Aust. Math. Soc. **97** (2018), 185–193
doi:10.1017/S0004972717000867

ON THE COMPLEMENT OF THE ZERO-DIVISOR GRAPH OF A PARTIALLY ORDERED SET

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(Received 15 July 2017; accepted 3 August 2017; first published online 2 November 2017)

Abstract

In this paper, it is proved that the complement of the zero-divisor graph of a partially ordered set is weakly perfect if it has finite clique number, completely answering the question raised by Joshi and Khiste ['Complement of the zero divisor graph of a lattice', *Bull. Aust. Math. Soc.* **89** (2014), 177–190]. As a consequence, the intersection graph of an intersection-closed family of nonempty subsets of a set is weakly perfect if it has finite clique number. These results are applied to annihilating-ideal graphs and intersection graphs of submodules.

2010 *Mathematics subject classification*: primary 05C25; secondary 06A07, 05C17.

Keywords and phrases: zero-divisor graph, intersection graph, annihilating-ideal graph, weakly perfect graph.

1. Introduction

The notion of a zero-divisor graph was introduced by Beck in 1988 to study colourings of commutative rings [4]. It has received considerable attention since 1999 when Anderson and Livingston highlighted its potential to illuminate algebraic structure [3]. The zero-divisor graph concept has since been extended to noncommutative rings [24], semigroups [12], and partially ordered sets [15]. By affording a graph-theoretic approach to the exploration of ideas in algebra, zero-divisor graphs expand the tools to explain algebraic phenomena. Similarly, the complement of a zero-divisor graph provides information that can be considered dual to that given by the zero-divisor graph. The complement of the zero-divisor graph is an appropriate tool to study intersection graphs of algebraic structures. The present investigation follows this approach by considering complements of zero-divisor graphs of partially ordered sets.

Given a commutative (multiplicative) semigroup S with 0, let $Z(S)$ be the set of zero-divisors of S. As in [12], the *zero-divisor graph* of S is the graph $\Gamma(S)$ whose vertices are the elements of $Z(S)\setminus\{0\}$, and distinct vertices x and y are adjacent if and

The first author is financially supported by the University Grants Commission, New Delhi, via Senior Research Fellowship Award Letter No. F.17-37/2008(SA-I).

only if $xy = 0$. If S is a commutative ring then this definition of $\Gamma(S)$ coincides with the definition in [3], and if S is the (multiplicative) monoid of ideals of a commutative ring R then $\Gamma(S)$ is the *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, which was introduced in [5]. The complement of $\mathbb{AG}(R)$ will be denoted by $\mathbb{AG}^c(R)$ and, more generally, the complement of $\Gamma(S)$ will be denoted by $\Gamma^c(S)$.

Let P be a partially ordered set (sometimes referred to as a *poset*) with the least element 0. Given any set $X \subseteq P$ with $X \neq \emptyset$, let $X^\vee = \{y \in P \mid y \geq x$ for every $x \in X\}$ and $X^\wedge = \{y \in P \mid y \leq x$ for every $x \in X\}$. If $x \in P$, the sets $\{x\}^\vee$ and $\{x\}^\wedge$ will be denoted by $x^\vee$ and $x^\wedge$, respectively.

A *zero-divisor* of P is any element of the set

$$Z(P) = \{x \in P \mid \text{there exists } y \in P\backslash\{0\} \text{ such that } \{x, y\}^\wedge = \{0\}\}.$$

As in [21], the *zero-divisor graph* of P is the graph $G(P)$ whose vertices are the elements of $Z(P)\backslash\{0\}$, such that two vertices x and y are adjacent if and only if $\{x, y\}^\wedge = \{0\}$. If $Z(P) \neq \{0\}$ then clearly $G(P)$ has at least two vertices and $G(P)$ is connected with diameter at most three [21, Proposition 2.1]. Throughout, the complement of $G(P)$ is denoted by $G^c(P)$.

In [4], it was conjectured that zero-divisor graphs of commutative rings R with unity (where every element of R was permitted to be a vertex) are weakly perfect, that is, the chromatic number and clique number are equal. This conjecture was shown to be false in [2, Theorem 2.1]. Nevertheless, the conjecture (frequently referred to as *Beck's conjecture*) has been confirmed in other contexts. For example, it was shown in [21, Corollary 2.4] that the zero-divisor graph $G(P)$ of a poset P is weakly perfect if its clique number $\omega(G(P))$ is finite (see also [15, Theorem 2.9]), and in [17, Theorem 3.3] it was proved that $G(P)$ is weakly perfect if P is a 0-distributive lattice such that $\omega(G^c(P)) < \infty$.

Let F be a collection of nonempty subsets of a set S. The *intersection graph* of F is the graph $\mathbb{I}(F)$ whose vertices are the elements of F, and distinct vertices x and y are adjacent if and only if $x \cap y \neq \emptyset$. The concept of an intersection graph was introduced by Bosak in 1964 [8]. Later, Csákány and Pollák studied the intersection graphs of proper nontrivial subgroups of finite groups in [10] and Zelinka continued the work on intersection graphs of proper nontrivial subgroups of finite abelian groups in [26].

Recently, 'intersection graphs of ideals' of rings were considered in [9]. Given a (not necessarily commutative) ring R, the *intersection graph of ideals* $IG(R)$ of R is the graph whose vertices are the proper nonzero left ideals of R, and distinct vertices I and J are adjacent if and only if $I \cap J \neq \{0\}$. More generally, the *intersection graph of submodules* $IG(M)$ of an R-module M is the graph whose vertices are the proper nonzero submodules of M and distinct vertices I and J are adjacent if and only if $I \cap J \neq \{0\}$. Observe that if $F = \{I\backslash\{0\} \mid I$ is a proper nonzero submodule of $M\}$, then $IG(M) \cong \mathbb{I}(F)$. The complement of $IG(M)$ (respectively, $\mathbb{I}(F)$) will be denoted by $IG^c(M)$ (respectively, $\mathbb{I}^c(F)$).

There are a number of contributions to the problem of determining whether annihilating-ideal graphs and intersection graphs are weakly perfect. It was proved in [6, Corollary 2.12] that if R is a *reduced* commutative ring (that is, if R is a commutative ring without nonzero nilpotent elements) such that $\omega(\mathbb{AG}(R)) < \infty$ then the annihilating-ideal graph $\mathbb{AG}(R)$ is weakly perfect. Moreover, [25, Proposition 3.7] gives some special cases under which the complement $\mathbb{AG}^c(R)$ of the annihilating-ideal graph is weakly perfect, and conditions were given in [1, Theorem 2.13] that guarantee that the complement $IG^c(M)$ of the intersection graph is weakly perfect for an R-module M.

In this paper we completely settle this problem in terms of complements of zero-divisor graphs of partially ordered sets. In fact, we prove the following result.

Theorem 1.1. *Let P be a (not necessarily finite) partially ordered set with 0 such that $Z(P) \neq \{0\}$ and $\omega(G^c(P)) < \infty$. Then $G(P)$ and $G^c(P)$ are weakly perfect.*

This extends [15, Theorem 2.9], generalises [17, Theorem 3.3], and completely answers the question raised in [17, Section 3]. As an application, it is shown that if F is a collection of nonempty subsets of a set S such that $F \cup \{\emptyset\}$ is closed under intersection and $\omega(\mathbb{I}(F)) < \infty$ then $\mathbb{I}(F)$ and $\mathbb{I}^c(F)$ are weakly perfect (Corollary 4.3). Moreover, $IG(M)$ and $IG^c(M)$ are weakly perfect for every R-module M such that $\omega(IG(M)) < \infty$, which generalises [1, Theorem 2.13]. Also, if R is a reduced commutative ring such that $Z(R) \neq \{0\}$ and $\omega(\mathbb{AG}^c(R)) < \infty$ then $\mathbb{AG}(R)$ and $\mathbb{AG}^c(R)$ are weakly perfect (Corollary 4.2). This extends [6, Corollary 2.12] and generalises [25, Proposition 3.7].

2. Preliminary concepts and definitions

A *clique* in G is any complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$, is the maximum order of any clique in G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum cardinality of colours required to colour every vertex of G so that no two adjacent vertices are assigned the same colour. If $\omega(G) = \chi(G)$ then G is called *weakly perfect*. Clearly $\omega(G) \leq \chi(G)$. Thus, given a clique K in G of order n such that $\omega(G) = n$, it follows that G is weakly perfect if and only if there exists a function f from the vertices of G into the vertices of K such that vertices x and y of G are not adjacent in G whenever $f(x) = f(y)$.

Let P be a partially ordered set with zero. An element $a \in P$ is an *atom* if $a > 0$ and, for every $x \in P$, the inequalities $0 \leq x \leq a$ imply that either $0 = x$ or $x = a$. Also, P is called *atomic* if for every x with $0 \neq x \in P$, there exists an atom $a \in P$ such that $a \leq x$.

Throughout, $\mathbb{N}$, $K_{\mathcal{D}}$ (where $\mathcal{D}$ is a set), K_n and $\mathbb{Z}_n$ ($n \in \mathbb{N}$) denote the set of positive integers, the complete graph with vertex-set $\mathcal{D}$, the complete graph of order n and the ring of integers modulo n, respectively. All graphs G are simple and undirected with vertex-set denoted by $V(G)$. If $x \in V(G)$ then $N(x)$ denotes the set of vertices of G that are adjacent to x. More generally, if $\emptyset \neq X \subseteq V(G)$ then $N(X) = \bigcup_{x \in X} N(x)$.

For references on rings and modules, partially ordered sets, and graph theory, see [19], [11] and [7], respectively.

3. Weakly perfect zero-divisor graphs of partially ordered sets

In this section it is assumed that P is a partially ordered set with zero such that $Z(P) \neq \{0\}$ and $\omega(G^c(P)) < \infty$. The following lemma, which shows that such partially ordered sets are atomic, will be used freely throughout this section.

Lemma 3.1. *Let P be a partially ordered set with zero such that $Z(P) \neq \{0\}$ and $\omega(G^c(P)) < \infty$. Then $|x^\vee \cap V(G^c(P))| < \infty$ for every $x \in P\backslash\{0\}$. Moreover, P is atomic.*

Proof. Observe that $x^\vee \cap V(G^c(P))$ is a clique in $G^c(P)$ for every $x \in Z(P)\backslash\{0\}$, which verifies the first statement (it is clear that $x^\vee \cap V(G^c(P)) = \emptyset$ if $x \in P\backslash Z(P)$). If $x \in Z(P)\backslash\{0\}$ is not bounded below by an atom then $x^\wedge$ contains an infinite chain, which contradicts $\omega(G^c(P)) < \infty$. Furthermore, for every $x \in P$, if $a \in P$ is an atom then either $a \leq x$ or $x \wedge a = 0$, and hence every element of $P\backslash Z(P)$ is bounded below by an atom (in fact, by every atom) of P. Therefore, P is atomic. □

For an atomic partially ordered set P with a finite numbers of atoms, it was proved in [16, Corollary 2.11] that $G(P)$ is weakly perfect by showing that the clique and the chromatic numbers of $G(P)$ are equal to the number of atoms in P. The following theorem extends this result by relaxing the 'finite' condition on the set of atoms. Recall that if P is atomic and $Z(P) \neq \{0\}$ then the set of atoms of P induces a maximal clique K of $G(P)$ such that any two vertices x and y of $G(P)$ are adjacent if and only if $V(K) \subseteq N(x) \cup N(y)$ [18, Theorem 4.4]. This observation is easily checked by noting that an atom a is not adjacent to a vertex x in $G(P)$ if and only if $a \leq x$, and two vertices x and y of $G(P)$ are not adjacent in $G(P)$ if and only if the set $\{x, y\}^\wedge$ contains an atom.

Theorem 3.2. *Let P be a partially ordered set with zero such that $Z(P) \neq \{0\}$. If P is atomic and $\mathscr{A}$ is the set of all atoms of P then $\omega(G(P)) = \chi(G(P)) = |\mathscr{A}|$.*

Proof. As noted above, $\mathscr{A}$ induces a maximal clique K of $G(P)$ such that any two vertices x and y of $G(P)$ are adjacent if and only if $V(K) \subseteq N(x) \cup N(y)$. Consider a function $f : V(G(P)) \to V(K)$ that satisfies $f(x) \in V(K)\backslash N(x)$ for every $x \in V(G(P))$ (note that $V(K)\backslash N(x) \neq \emptyset$ by the maximality of K, and so the existence of such a function f is guaranteed by the Axiom of Choice). If $x, y \in V(G(P))$ such that $f(x) = f(y)$ then $(V(K)\backslash N(x)) \cap (V(K)\backslash N(y)) \neq \emptyset$ (because it contains the element $f(x) = f(y)$), that is, $V(K)\backslash(N(x) \cup N(y)) \neq \emptyset$. Hence, $x \notin N(y)$ by the choice of the clique K. Thus, f is a colouring of $G(P)$. □

It follows from [15, Theorem 2.9] that if $\omega(G(P)) < \infty$ then $G(P)$ is weakly perfect. In [17, Theorem 3.3] it was shown that, for a 0-distributive lattice P, the graph $G(P)$ is weakly perfect if $\omega(G^c(P)) < \infty$. This result is generalised in the next corollary, which follows immediately from Lemma 3.1 and Theorem 3.2.

Corollary 3.3. *Let P be a partially ordered set with zero such that $Z(P) \neq \{0\}$. If $\omega(G^c(P)) < \infty$ then $G(P)$ is weakly perfect.*

https://doi.org/10.1017/S0004972717000867 Published online by Cambridge University Press

Let P be a partially ordered set. The *comparability graph* of P is the graph whose vertices are the elements of P, and distinct vertices x and y are adjacent if and only if either $x \leq y$ or $y \leq x$. Hence, for P with zero, $G^c(P)$ can be regarded as a generalisation of the comparability graph of P (more precisely, the subgraph of the comparability graph of P induced by the nonzero zero-divisors of P is a subgraph of $G^c(P)$). Similarly, there is a sense in which the complement of the comparability graph of P generalises the zero-divisor graph of P.

Recall that a graph G is *perfect* if every induced subgraph of G is weakly perfect (in particular, perfect graphs are weakly perfect). Recently, perfect zero-divisor graphs of 0-distributive lattices and reduced rings have been characterised [23, Theorem 1.4]. The next two results (whose original statements were given in the purely partially ordered set-theoretic terms of chains and antichains) are well known. It was shown later in [20, perfect graph theorem] that a finite graph G is perfect if and only if G^c is perfect.

Theorem 3.4 [22, Theorem 2]. *Let P be a finite partially ordered set. Then the comparability graph of P is perfect.*

Theorem 3.5 [13, Theorem 1.1]. *Let P be a finite partially ordered set. Then the complement of the comparability graph of P is perfect.*

Let P be a partially ordered set with zero. Then $G(P)$ need not be perfect (and hence, by [20, perfect graph theorem], $G^c(P)$ need not be perfect) even if P is a Boolean algebra. In fact, the subgraph of $G(\mathbb{Z}_2^6)$ induced by the set $\{(0,0,1,1,1,0),$ $(1,1,0,0,0,0), (0,0,0,1,1,1), (0,1,1,0,0,0), (1,0,0,0,0,1)\}$ is a cycle of length five. Moreover, in contrast to the perfect graph theorem, a weakly perfect graph need not have a weakly perfect complement. For example, the graph constructed by introducing a new vertex to a cycle of length five that is adjacent to precisely two nonadjacent vertices of the cycle is not weakly perfect, but its complement is weakly perfect. On the other hand, the goal in this section is to establish Theorem 1.1, which augments Theorems 3.4 and 3.5.

The hypotheses of Theorem 1.1 imply that $G(P)$ is weakly perfect by Corollary 3.3. The following three lemmas will complete most of the work in proving that these conditions also imply $G^c(P)$ is weakly perfect. It is sufficient (and will be more convenient) to verify that the result holds for a partially ordered set Q_P having an atom α with $|\alpha^\vee \cap V(G^c(Q_P))| = \omega(G^c(Q_P))$, provided that $G^c(P)$ is a subgraph of $G^c(Q_P)$ and $\omega(G^c(Q_P)) = \omega(G^c(P))$. The next lemma guarantees the existence of such a partially ordered set. Recall that every subset S of a partially ordered set P induces a subposet of P whose elements are the members of S with order inherited from P.

Lemma 3.6. *Let P be a partially ordered set with zero such that $Z(P) \neq \{0\}$ and $\omega(G^c(P)) < \infty$. There exists a partially ordered set Q_P such that $G^c(P)$ is a subgraph of $G^c(Q_P)$, and Q_P contains an atom α with $|\alpha^\vee \cap V(G^c(Q_P))| = \omega(G^c(Q_P)) = \omega(G^c(P))$.*

Proof. Let K be a clique in $G^c(P)$ of maximum cardinality. As $|V(K)| = \omega(G^c(P))$ is finite, the subposet of P induced by $V(K)$ contains a minimal element v. Let α be

https://doi.org/10.1017/S0004972717000867 Published online by Cambridge University Press

an element (not in P) and set $Q_P = P \cup \{\alpha\}$ (as sets). Extend the partial order on P to Q_P by $\alpha^\wedge = \{0, \alpha\}$ and $\alpha^\vee = \{\alpha\} \cup (\bigcup\{x^\vee \mid x \in V(K)\backslash\{v\}\}) \cup (P\backslash Z(P))$. As any two elements of $V(K)\backslash\{v\}$ have a nonzero lower bound in common, it is easy to check that any two distinct elements of $\alpha^\vee \cap V(G^c(P))$ are adjacent in $G^c(P)$. In fact, the equality $\alpha^\vee \cap V(G^c(P)) = V(K)\backslash\{v\}$ holds by the maximality of $|V(K)|$. Moreover, since α is an atom of Q_P, if $y \in V(G^c(P))\backslash\alpha^\vee$ then $\{\alpha, y\}^\wedge = \{0\}$ (in Q_P). It follows that $G^c(Q_P)$ is the graph obtained from $G^c(P)$ by introducing a new vertex α such that $N(\alpha) = V(K)\backslash\{v\}$. Now it is straightforward to check that Q_P satisfies the statements of the lemma. □

Henceforth, let $\mathscr{A}$ be the set of atoms of P. For every A with $\emptyset \neq A \subseteq \mathscr{A}$, there will be no harm in abusing notation by letting $G^c(A)$ denote the subgraph of $G^c(P)$ such that $V(G^c(A)) = (\bigcup\{a^\vee \mid a \in A\}) \cap V(G^c(P))$ and distinct vertices $x, y \in V(G^c(A))$ are adjacent in $G^c(A)$ if and only if the set $\{x, y\}$ has a nonzero lower bound in the subposet of P induced by $\{0\} \cup (\bigcup\{a^\vee \mid a \in A\})$ ($G^c(A)$ may not be the complement of the zero-divisor graph of the induced subposet since its vertices need not be the set of nonzero zero-divisors of the subposet). In particular, $G^c(P) = G^c(\mathscr{A})$ by Lemma 3.1, and Lemma 3.6 shows that generality is not lost by proving the '$G^c(P)$ is weakly perfect' portion of Theorem 1.1 only in the special case when there exists an $\alpha \in \mathscr{A}$ such that the clique $K = G^c(\{\alpha\})$ in $G^c(P)$ satisfies $|V(K)| = \omega(G^c(P))$.

Denote the restriction of a function $f : X \to Y$ to a subset $U \subseteq X$ by $f|_U$. For any $\alpha \in \mathscr{A}$ such that the clique $K = G^c(\{\alpha\})$ in $G^c(P)$ satisfies $|V(K)| = \omega(G^c(P))$, define an order $\preceq$ on the set

$$W_\alpha = \{(A, f) \mid \alpha \in A \subseteq \mathscr{A} \text{ and } f : V(G^c(A)) \to V(K) \text{ is a colouring of } G^c(A)\}$$

by $(A, f) \preceq (B, g)$ if and only if $A \subseteq B$ and $g|_{V(G^c(A))} = f$. Since $G^c(P) = G^c(\mathscr{A})$, the proof of Theorem 1.1 will be complete once it is shown that there exists a function $f : V(G^c(\mathscr{A})) \to V(K)$ such that $(\mathscr{A}, f) \in W_\alpha$.

Lemma 3.7. *Let P be a partially ordered set with zero and $Z(P) \neq \{0\}$. Suppose $\alpha \in \mathscr{A}$ is such that the clique $K = G^c(\{\alpha\})$ in $G^c(P)$ satisfies $|V(K)| = \omega(G^c(P)) < \infty$. Then W_α contains a maximal element.*

Proof. Note that $W_\alpha \neq \emptyset$ since, for example, $(\{\alpha\}, \iota) \in W_\alpha$ (where ι is the identity function). Suppose that $\mathscr{C} = \{(A_i, f_{A_i}) \mid i \in I\}$ (I an indexing set) is a chain in W_α. Then $(\bigcup_{i\in I} A_i, f) \in W_\alpha$, where the function $f : V(G^c(\bigcup_{i\in I} A_i)) \to V(K)$ is given by $f(v) = f_{A_i}(v)$ for every $i \in I$ with $v \in V(G^c(A_i))$. The element $(\bigcup_{i\in I} A_i, f)$ is clearly an upper bound of $\mathscr{C}$, so the result holds by Zorn's lemma. □

The next lemma is the final result prior to the proof of Theorem 1.1. The inequalities in the remainder of this section are between finite cardinalities by the assumption $\omega(G^c(P)) < \infty$ together with Lemma 3.1.

Lemma 3.8. *Let P be a partially ordered set with zero and $Z(P) \neq \{0\}$. Suppose $\alpha \in \mathscr{A}$ is such that the clique $K = G^c(\{\alpha\})$ in $G^c(P)$ satisfies $|V(K)| = \omega(G^c(P)) < \infty$. Let $(A, f) \in W_\alpha$ and $a \in \mathscr{A}\backslash A$. If $|(a^\vee\backslash V(G^c(A))) \cap V(G^c(P))| \leq |V(K)\backslash f(a^\vee \cap V(G^c(A)))|$ then there is a function $F : V(G^c(A \cup \{a\})) \to V(K)$ such that $(A \cup \{a\}, F) \in W_\alpha$ and $(A, f) \prec (A \cup \{a\}, F)$.*

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Proof. Observe that if $v \in (a^\vee \backslash V(G^c(A))) \cap V(G^c(P))$ then v is not adjacent to any element of $V(G^c(A))\backslash a^\vee$ (in the graph $G^c(A \cup \{a\})$) since, otherwise, $v \in b^\vee$ for some $b \in A$, which contradicts $v \notin V(G^c(A))$. Hence, intuitively, after colouring the vertices of $G^c(A)$ by f, any vertex $v \in a^\vee \backslash V(G^c(A))$ of $G^c(A \cup \{a\})$ can be coloured by any element of $V(K)$ that has not already been used to colour an element of $a^\vee \cap V(G^c(A))$. That is, more precisely, since $|(a^\vee \backslash V(G^c(A))) \cap V(G^c(P))| \leq |V(K)\backslash f(a^\vee \cap V(G^c(A)))|$, the inequality $(A, f) \leq (A \cup \{a\}, F)$ holds. Here, $(A \cup \{a\}, F)$ is the element of W_α with $F : V(G^c(A \cup \{a\})) \to V(K)$ defined such that $F|_{V(G^c(A))} = f$ and $F|_{(a^\vee \backslash V(G^c(A))) \cap V(G^c(P))}$ is any injection into $V(K)\backslash f(a^\vee \cap V(G^c(A)))$. $\square$

Proof of Theorem 1.1. As noted in the discussion following the proof of Lemma 3.6, it can be assumed that there exists $\alpha \in \mathscr{A}$ such that the clique $K = G^c(\{\alpha\})$ in $G^c(P)$ satisfies $|V(K)| = \omega(G^c(P)) < \infty$. By Lemma 3.7, there exists a maximal element (A, f) of W_α. If $A \neq \mathscr{A}$ then pick $a \in \mathscr{A}\backslash A$. By Lemma 3.8, it remains to prove that $|(a^\vee \backslash V(G^c(A))) \cap V(G^c(P))| \leq |V(K)\backslash f(a^\vee \cap V(G^c(A)))|$. This will contradict the maximality of (A, f) so that $A = \mathscr{A}$, that is, so that $f : V(G^c(\mathscr{A})) = V(G^c(P)) \to V(K)$ is a colouring of $G^c(P)$. But

$$|V(K)\backslash f(a^\vee \cap V(G^c(A)))| = |V(K)| - |f(a^\vee \cap V(G^c(A)))|$$

since $f(a^\vee \cap V(G^c(A)) \subseteq V(K)$, and $|f(a^\vee \cap V(G^c(A)))| \leq |a^\vee \cap V(G^c(A))|$ holds in general, so if $|(a^\vee \backslash V(G^c(A))) \cap V(G^c(P))| > |V(K)\backslash f(a^\vee \cap V(G^c(A)))|$ then

$$\begin{aligned}|a^\vee \cap V(G^c(P))| &= |(a^\vee \backslash V(G^c(A))) \cap V(G^c(P))| + |a^\vee \cap V(G^c(A))| \\ &> |V(K)\backslash f(a^\vee \cap V(G^c(A)))| + |a^\vee \cap V(G^c(A))| \\ &= |V(K)| - |f(a^\vee \cap V(G^c(A)))| + |a^\vee \cap V(G^c(A))| \\ &\geq |V(K)|.\end{aligned}$$

Since $a^\vee \cap V(G^c(P))$ induces a clique in $G^c(P)$, this contradicts the maximality of K. Therefore, $|(a^\vee \backslash V(G^c(A))) \cap V(G^c(P))| \leq |V(K)\backslash f(a^\vee \cap V(G^c(A)))|$.

4. Applications

Let R be a reduced commutative ring such that $Z(R) \neq \{0\}$. In this section, Theorem 1.1 is applied to $\Gamma(R)$, $\Gamma^c(R)$, $\mathbb{AG}(R)$ and $\mathbb{AG}^c(R)$ and to the intersection graphs of the intersection-closed families of nonempty subsets. As a consequence, it is proved that if R is a (not necessarily reduced or commutative) ring and M is an R-module such that $\omega(IG^c(M)) < \infty$ then $IG(M)$ and $IG^c(M)$ are weakly perfect. In particular, this result applies to the intersection graphs of ideals of rings.

Let R be a reduced commutative ring. It was observed in [18, Remark 3.4] that there exists a partially ordered set P with zero such that $\Gamma(R) \cong G(P)$. Therefore, the following corollary holds by Theorem 1.1.

Corollary 4.1. *Let R be a reduced commutative ring such that $Z(R) \neq \{0\}$. If $\omega(\Gamma^c(R)) < \infty$ then $\Gamma(R)$ and $\Gamma^c(R)$ are weakly perfect.*

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Let I and J be ideals of a commutative ring R. If R is reduced then $IJ = \{0\}$ if and only if $I \cap J = \{0\}$. In this case, $\mathbb{AG}(R)$ is the zero-divisor graph of the lattice P (under inclusion) of ideals of R; that is, $\mathbb{AG}(R) = G(P)$. Hence, Corollary 4.2, which extends [6, Corollary 2.12] and generalises [25, Proposition 3.7], is an immediate consequence of Theorem 1.1.

Corollary 4.2. *Let R be a reduced commutative ring such that $Z(R) \neq \{0\}$. If $\omega(\mathbb{AG}^c(R)) < \infty$ then $\mathbb{AG}(R)$ and $\mathbb{AG}^c(R)$ are weakly perfect.*

If F is a collection of nonempty subsets of a set S such that $F \cup \{\emptyset\}$ is closed under intersection (in which case $F \cup \{\emptyset\}$ is a meet-semilattice under inclusion), then the subgraph of $\mathbb{I}(F)$ induced by the nonzero zero-divisors of $F \cup \{\emptyset\}$ is $G^c(F \cup \{\emptyset\})$. In fact, $\mathbb{I}(F)$ is the *join* $G^c(F \cup \{\emptyset\}) + K_{\mathcal{D}}$ of the graphs $G^c(F \cup \{\emptyset\})$ and $K_{\mathcal{D}}$ (that is, the graph obtained from the union $G^c(F \cup \{\emptyset\}) \cup K_{\mathcal{D}}$ by letting every vertex of $G^c(F \cup \{\emptyset\})$ be adjacent to every vertex of $K_{\mathcal{D}}$), where $\mathcal{D}$ is the set of nonzero-divisors of $F \cup \{\emptyset\}$. In particular, $\omega(\mathbb{I}(F)) = \omega(G^c(F \cup \{\emptyset\})) + |\mathcal{D}|$. Therefore, it is straightforward to check that if $\omega(\mathbb{I}(F)) < \infty$ (so that $\omega(G^c(F \cup \{\emptyset\})) < \infty$, and hence $G(F \cup \{\emptyset\})$ and $G^c(F \cup \{\emptyset\})$ are weakly perfect by Theorem 1.1) then $\mathbb{I}(F)$ and $\mathbb{I}^c(F)$ are weakly perfect.

Let R be a (not necessarily commutative) ring, and suppose that M is an R-module. Recall that $IG(M) \cong \mathbb{I}(F)$, where $F = \{I \backslash \{0\} \mid I$ is a proper nonzero submodule of $M\}$. In this case, $F \cup \{\emptyset\}$ is closed under intersection, and hence the above discussion shows that if $\omega(IG(M)) < \infty$ then $IG(M)$ and $IG^c(M)$ are weakly perfect. These remarks are summarised in the next corollary, which generalises [1, Theorem 2.13].

Corollary 4.3. *Let F be a collection of nonempty subsets of a set S. If $F \cup \{\emptyset\}$ is closed under intersection and $\omega(\mathbb{I}(F)) < \infty$, then $\mathbb{I}(F)$ and $\mathbb{I}^c(F)$ are weakly perfect. In particular, if R is a (not necessarily commutative) ring and M is an R-module such that $\omega(IG(M)) < \infty$, then $IG(M)$ and $IG^c(M)$ are weakly perfect.*

Remark 4.4. Note that the hypothesis '$F \cup \{\emptyset\}$ is closed under intersection' of Corollary 4.3 cannot be omitted. For example, it is well known that, for any graph G of order $n < \infty$, there exists a set S of cardinality at most $n^2/4$ containing subsets $S_1, \ldots, S_n$ such that $\mathbb{I}(\{S_1, \ldots, S_n\}) \cong G$ [14, Theorem 1].

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