SEMIGROUPS ACTING ON CONTINUA

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A *semigroup* is a nonvoid Hausdorff space together with a continuous associative multiplication. (The latter phrase will generally be abbreviated to CAM and the multiplication in a semigroup will be denoted by juxtaposition unless the contrary is made explicit.)

Any Hausdorff space may be supplied with a CAM, and, for example, one may define $xy = x$ for all x and y. The addition of algebraic conditions may change the situation greatly and a circle together with a diameter does not admit a CAM with unit. It was shown in [W 1] (see [KW 1] for another example) that the space consisting of the curve $y = \sin(1/x)$, $0 < x \leqq 1$, together with its limit continuum, does not admit a CAM with unit. (This result follows readily from a result of Robert Hunter's [H].)

An *act* is such a continuous function

$$T \times X \to X$$

that T is a semigroup and X is a nonvoid Hausdorff space and, denoting the value of the anonymous function at the place (t, x) by tx, the associativity condition

$$t_1(t_2x) = (t_1t_2)x$$

holds for all $t_1, t_2 \in T$ and all $x \in X$. We shall refer to this situation as an action of T on X and say that T *acts* on X, or use similar terminology.

Again, any semigroup may act upon any space, for example one may put $tx = x$ for all $t \in T$ and all $x \in X$. Moreover, the situation in which T is a *group* is so well known as not to require explication. However, when T is merely a semigroup, very little is known without additional conditions on T and X of an algebraic and metric nature, and it is our intention here to inaugurate such an investigation, of a modest character.

Put in its simplest form, we shall give conditions under which a compact connected semigroup may not act upon the sinuscurve described in an earlier paragraph. In more detail, suppose that the space X contains an open dense half-line whose complement is a set C, that there is some $q \in X$ such that $Tq = X$, that T *acts unitarily* on X ($x \in Tx$ for each $x \in X$), and that a

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certain natural hypothesis is made to exclude trivial action of the sort just indicated — then C is (topologically) homogeneous.

It may be remarked that, from a paper by J. Aczel and A. D. Wallace (to appear), it can be concluded that X must indeed have the structure of a semigroup provided that T is commutative. It is also proved there that if T and X are compact, T acts unitarily, and $\{Tx \mid x \in X\}$ is a tower, then there exists $q \in X$ with $Tq = X$.

For material concerning *discrete* semigroups reference may be made to the books of Clifford-Preston [CP] and Ljapin [L] and for the general case to the excellent expository dissertation of Paalman-de Miranda [P-de M] and the forthcoming research monograph of Mostert-Hofmann.

Insofar as topology is concerned, we assume familiarity with much standard material and refer to Hocking-Young [HY], Hu [Hu], Kelley [K] and Wilder [Wi]. It does not follow that we adhere to the language and notation of any of these, but generally we note any departure from the customary rubric. In particular, we prefer A^*, A^0, and $F(A) = A^*\backslash A^0$ for the closure, interior and boundary of the set A. Where there may be confusion of meaning, topological usage will take precedence of algebraic usage. Thus to say a set is *closed*, is to mean that it is closed in the topology and *not* that it is a subsemigroup.

"Space" will include the quantifier "Hausdorff". A *continuum* is a compact connected space and a *bing* (Middle English, Old Norse— heap, pile) is a compact connected semigroup.

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Henceforth it will be supposed that $T \times X \to X$ is an *act*, as defined earlier. For A contained in X and M contained in T we write

$$MA = \{tx \mid t \in M \text{ and } x \in A\},$$
$$M^{(-1)}A = \{x \mid Mx \cap A \neq \square\}$$

and

$$M^{[-1]}A = \{x \mid Mx \subset A\}.$$

It will be observed that no differentiation is made between x and $\{x\}$ if it is not convenient to do so, and does not readily lead to confusion. In this vein we write $A\backslash x$ rather than $A\backslash\{x\}$, and so on. Also, inclusive quantifiers will be omitted if there is likely to be no misunderstanding.

It may be observed that

(1.1) $$M^{[-1]}A = X\backslash M^{(-1)}(X\backslash A).$$

Proof of the following have been given in [W 1] and [W 6] in various forms and in varying degrees of generality and we content ourselves with a brief sketch.

(1.2) (i) *If M is compact and if A is closed then $M^{(-1)}A$ is closed.*
(ii) *If A is open then $M^{(-1)}A$ is open.*
(iii) *If M is compact and if A is open then $M^{[-1]}A$ is open.*
(iv) *If A is closed then $M^{[-1]}A$ is closed.*
(v) *If M is compact then $\{x \mid A \subset Mx\} = \cap \{M^{(-1)}a \mid a \in A\}$ is closed and hence if A is also closed then $\{x \mid Mx = A\}$ is closed.*

For the proof of (i) it may be observed that

$$M^{(-1)}A = q((M \times X) \cap \alpha^{-1}(A))$$

where α is the (continuous) action-map, $\alpha(t, x) = tx$, and q is the projection of $T \times X$ onto X. From this, (iii) follows via (1.1). The others are similar.

The set $A \subset X$ is an *M-ideal* if A is non-void and if $MA \subset A$. If T and X are compact and if X properly contains a T-ideal then it is known (e.g., [KW 1] and [W 1]) that there is a maximal proper ideal and that each such is *open*.

We make repeated use of the fact that

$$M^*A^* \subset (MA)^*$$

(which follows immediately from the continuity of the action) and, in particular,

$$TA^* \subset (TA)^*.$$

If t is an element of a semigroup then

$$\Gamma(t) = \{t, t^2, t^3, \cdots\}^*$$

and for useful properties reference is made to [P-de M], in particular, p. 22 *et seq*. (These results are due mainly to Hewitt, Koch and Numakura, *loc. cit.*)

An illustrative and basic example of an act is given as follows. Suppose that X is locally compact Hausdorff and that $M(X)$ is the set of all continuous functions taking X into X, so that $M(X)$ is a semigroup under *composition*, using the compact-open topology. Then $M(X)$ acts on X by evaluation, $(f, x) \to f(x)$. As a matter of orientation (and we do not use this fact) it may be observed that, recalling that T acts on X, the function

$$\Theta : T \to M(X)$$

defined by

$$\Theta(t)(x) = tx$$

is a continuous homomorphism which will be an *iseomorphism* (homeomorphic isomorphism) into provided that T is compact and that T is effective, which is to say that if $t \neq t'$, then $tx \neq t'x$ for some $x \in X$.

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Additional insight into acts may be given. A *left congruence* on a semigroup S is such an equivalence $\boldsymbol{C} \subset S \times S$ that $\triangle \boldsymbol{C} \subset \boldsymbol{C}$, $\triangle$ being the diagonal, If g is the natural map from S to $S/\boldsymbol{C}$ then, S being compact and $\boldsymbol{C}$ closed, there is a unique manner in which S may act upon $S/\boldsymbol{C}$ such that $sg(s') = g(ss')$. Thus any compact semigroup acts in a canonical manner upon any of its left quotients.

Now conversely, assume that S acts on X (both being compact) and suppose that $Sq = X$ for some $q \in X$. If $\boldsymbol{C}$ is defined as the set of all (s, s') such that $sq = s'q$ then $\boldsymbol{C}$ is a closed left congruence and there is a homeomorphism of X upon $S/\boldsymbol{C}$, and the canonical action of S on $S/\boldsymbol{C}$ mimics in all essential respects the original action of S on X.

A *congruence* on a semigroup S is a subset of $S \times S$ which is simultaneously an *equivalence* (reflexive, symmetric and transitive) and a *subsemigroup* of $S \times S$, using coordinatewise operations in the latter. If S is compact and if $\boldsymbol{C}$ is a *closed* (in the standard topology of $S \times S$) congruence on S, then $S/\boldsymbol{C}$ is a semigroup and the canonical function

$$g : S \to S/\boldsymbol{C}$$

is a continuous onto homomorphism. The topological parts of this construction are contained, among many other places, in Kelley [K] and the whole matter is essentially in the folklore of semigroups (but cf. [W 1] and [W 7]).

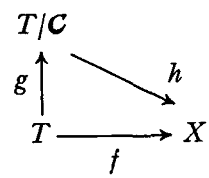
Removal of the harsh hypothesis that S be compact is very much an open question. If S were a group then the function g would be open, which would settle matters. But when S is just a semigroup there are examples to show that $S/\boldsymbol{C}$ need not be Hausdorff, even if it is assumed that S is the real line (but of course the operation is not addition). In this connection, reference is made to the papers of E. J. McShane [M] and B. J. Pettis [P], among others.

Reverting to the present instance, with T acting on X, we define $\boldsymbol{C} \subset T \times T$ by

$$\boldsymbol{C} = \{(t_1, t_2) \mid t_1 x = t_2 x \text{ for all } x \in X\}$$

and readily verify that $\boldsymbol{C}$ is a closed congruence on T. When T is compact, $T/\boldsymbol{C}$ is a semigroup and acts on X, and in fact, $T/\boldsymbol{C}$ is iseomorphic to a subset of $M(X)$ since $T/\boldsymbol{C}$ is compact and separates the points of X.

(1.3) PROPOSITION. *Using in part the notation above, suppose that, in the diagram*



the continuous function f is consistent with the action of T on X, in the sense that

$$(*) \qquad g(t) = g(t') \text{ implies } f(t) = f(t'), \text{ for all } t, t' \text{ in } T,$$

and suppose that T is compact. Then there is such a continuous function h that the diagram is analytic, $f = hg$. If f is bisconsistent, in the sense that (*) is an equivalence rather than merely an implication, then h is a homeomorphism into, $f(T)$ is a semigroup under the multiplication

$$x \circ x' = h(h^{-1}(x)h^{-1}(x')),$$

h is an iseomorphism onto this semigroup and f is similarly a homomorphism.

If f is also a translation relative to the given act, in the sense that

$$f(tt') = tf(t') \text{ for all } t, t' \text{ in } T,$$

and if C is a minimal T-ideal which intersects $f(T)$, then $C \subset f(T)$, C is a minimal left ideal of $(f(T), \circ)$, and if there is some $u \in T$ with $f(u) \in C$ and $uC = C$, then $(C, \circ)$ is a group.

Proof. The first paragraph is easily verified. To prove that $C \subset f(T)$, first notice that $f(T)$ is a T-ideal: for $Tf(T) = f(T^2)$ since f is a translation, and $T^2 \subset T$, so that $Tf(T) \subset f(T)$. This, together with the hypothesis that C is a minimal T-ideal intersecting $f(T)$, implies that $C \subset f(T)$.

Now let $x = f(t)$ be an arbitrary but fixed element of C. From the definition of $\circ$, one sees that $f(t_1) \circ f(t_2) = f(t_1 t_2)$, and f is a translation, so that $f(T) \circ f(t) = f(Tt) = Tf(t)$. Since $f(t) \in C$ and C is a minimal T-ideal, $Tf(t) = C$. That is, $f(T) \circ x = C$ for arbitrary $x \in C$, so that C is a minimal left ideal of $(f(T), \circ)$.

From the above, $(C, \circ)$ is a semigroup and $C \circ x = C$ for each $x \in C$ since C is a minimal left ideal. If there is some $x \in C$ such that $x \circ C = C$ also, then as is well known, C is a group. Thus we observe that $f(u) \in C$ by hypothesis, and prove that $f(u) \circ C = C$: from the previous paragraph, we have that

$$C = Tf(u) = f(Tu) \text{ and } f(u) \circ f(Tu) = f(uTu),$$

which when combined give

$$f(u) \circ C = f(uTu).$$

Since f is a translation,

$$f(uTu) = uf(Tu),$$

and $uf(Tu)$ equals uC since $f(Tu) = C$ by the above. Therefore

$$f(u) \circ C = uC,$$

and uC equals C by hypothesis, so we have the desired result,

$$f(u) \circ C = C.$$

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In the following corollary and in our later use of (1.3), we shall have, for some $a \in X$, $f(t) = ta$ (and thus $f(T) = Ta$) and the condition

(†) $ta = t'a$ implies $tx = t'x$ for all $t, t' \in T$ and all $x \in X$.

It is easy to see that f is a biconsistent translation.

NOTATION. $Q = \{x \in X \mid Tx = X\}$.

(1.31) COROLLARY. *If $X = Q$ and if there is some $a \in X$ which satisfies (†), then X is a left simple semigroup, hence X is iseomorphic to $E \times H$, where E is the set of idempotents of X and H is a maximal subgroup of X. If also, there is some $u \in X$ with $uX = X$, then X is a group.*

PROOF. Define $f : T \to X$ by $f(t) = ta$; then f is a biconsistent translation and $f(T) = Ta = X$ (since $a \in Q$), so that X is a semigroup by (1.3). It is clear that X is the only T-ideal since $X = Q$, so also by (1.3), X has no proper left ideals — i.e., X is left simple. Then by a result in [W 3], X is iseomorphic to $E \times H$.

If there is $u \in T$ such that $uX = X$, then by the above and (1.3), X is a group.

(1.4) PROPOSITION.

(i) *If T is compact and X is a continuum, then each maximal proper T-ideal is open and dense.*

(ii) *Suppose that $X \neq Q \neq \square$. Then $X \backslash Q$ is the unique maximal proper T-ideal, and if also T and X are continua, then $X \backslash Q$ is open, dense and connected.*

PROOF. (i) Let J be a maximal proper T-ideal and let $x \in X \backslash J$; $X \backslash J$ is closed since either $X \backslash J = \{x\}$ or $X \backslash J = \{y \in X \mid Ty = Tx\}$, which is closed because T is compact. Therefore J is open.

Since J is a proper open set and X is connected, $J \subsetneq J^*$; J^* is also a T-ideal and J is a maximal proper one, hence $J^* = X$, and thus J is dense.

(ii) $Q \cap Tx \neq \square$ if and only if $x \in Q$ (if there is some $q \in Q \cap Tx$, then $X = Tq \subset T^2x \subset Tx$, hence $X = Tx$ so that $x \in Q$; if $x \in Q$, obviously $x \in Q \cap Tx$). Therefore $Tx \subset X \backslash Q$ if and only if $x \in X \backslash Q$, i.e., $X \backslash Q = T^{[-1]}(X \backslash Q)$. This implies that $T(X \backslash Q) \subset X \backslash Q$; it is nonempty and proper, hence is a proper T-ideal, since $X \neq Q \neq \square$ by hypothesis; and if J is a set properly containing $X \backslash Q$, then J intersects Q, hence $TJ = X$, so that no proper T-ideal properly contains $X \backslash Q$.

Now assume further that T and X are continua. $X \backslash Q$ is open and dense by the preceding proof. Suppose that $X \backslash Q$ is not connected, so that $X \backslash Q = U \cup V$, where U and V are disjoint nonempty open sets. Observe that $X \backslash Q = T^{[-1]}U \cup T^{[-1]}V$ since $x \in Q$ implies $Tx = X$ so that $Tx \not\subset U$ and $Tx \not\subset V$, hence $x \notin T^{[-1]}U \cup T^{[-1]}V$; conversely, $x \in X \backslash Q$ implies

$Tx \subset X \backslash Q$ since $X \backslash Q$ is a T-ideal, and Tx is a continuum, so either $Tx \subset U$ or $Tx \subset V$. Therefore at least one of $T^{[-1]}U$ and $T^{[-1]}V$ is nonempty, say $T^{[-1]}U \neq \square$. Both sets are proper and they are open by (1.2) (iii). Thus X being connected implies that there is some $x \in (T^{[-1]}U)^* \backslash T^{[-1]}U$. Because $T^{[-1]}V$ is an open set disjoint from $T^{[-1]}U$, $x \notin T^{[-1]}V$, so that $x \in (T^{[-1]}U)^* \cap Q$. Therefore $X = Tx \subset T(T^{[-1]}U)^*$; by continuity, $T(T^{[-1]}U)^* \subset (TT^{[-1]}U)^*$; by definition, $TT^{[-1]}U \subset U$, and thus $X \subset U^*$, which contradicts our assumption that V is a nonempty open set disjoint from U.

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The following lemma differs from similar statements in [F], [W 1], [W 4], and [W 6] only in that we do not require the semigroup to be connected. The fact that connectedness is unnecessary proves very useful in (2.3), where we apply it to a semigroup of the form $\Gamma(t)$.

(2.1) LEMMA. *Let X be a continuum, let $H \subset X$, and let S be a compact semigroup acting on X. If $F(H) \neq \square$ and there exists an S-ideal in H, then there is some $p \in F(H)$ such that $Sp \subset H^*$.*

PROOF. Let G be a component of $H \cap S^{[-1]}H$. ($H \cap S^{[-1]}H$ is nonempty since there is an S-ideal L contained in H by hypothesis, and $\square \neq SL \subset L \subset H$ implies $\square \neq L \subset H \cap S^{[-1]}H$). Then $G \cup SG \subset H$, hence $G^* \cup SG^* \subset H^*$ by continuity of the action. Suppose that $(G^* \cup SG^*) \cap F(H) = \square$; then $G^* \cup SG^* \subset H^0$, which is to say, $G^* \subset H^0 \cap S^{[-1]}H^0$. Since S is compact, $S^{[-1]}H^0$ is open by 1.2 (iii), so $H^0 \cap S^{[-1]}H^0$ is open; it is also a proper subset of X since it is contained in H and $F(H) \neq \square$. Of course $H^0 \cap S^{[-1]}H^0 \subset H \cap S^{[-1]}H$, so we have that $G = G^*$ and that G must be a component of $H^0 \cap S^{[-1]}H^0$. But then G is a component of a proper open subset of a continuum, whose closure does not intersect the boundary of the open set, and this is impossible (see [HY], for example). Therefore there must be some $p \in (G^* \cup SG^*) \cap F(H)$, and then $Sp \subset S(G^* \cup SG^*) = SG^* \cup S^2G^* \subset SG^* \subset H^*$.

A subset N of a continuum is a *nodal set* iff N is a nondegenerate continuum and $F(N)$ is exactly one point. When a set B is the intersection of all the nodal sets containing it, we will say that $B = D(B)$. Faucett has proved that if A is the complement of a maximal proper ideal of a bing and if $A = D(A)$, then cardinal $A = 1$ [F]. We have proved a more general result (2.3), that if T is a bing acting on a continuum X, and if A is a *subset* of the complement of a maximal proper T-ideal, then $A = D(A)$ implies that cardinal $A = 1$. We shall use without proof the following facts:

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(α) $D(A)$ is a continuum [R].
(β) If $A = D(A)$ and F is a continuum intersecting A, then $A \cap F$ is a continuum [R].
(γ) If $A = D(A)$ is contained in an open set U and $y \notin U$, then there is a nodal set N containing A such that $y \notin N$ and $F(N) \in U$ [R].
(δ) If $\boldsymbol{N}$ is a collection of nodal sets, if $A_0 = \cap \boldsymbol{N} \neq \square$, and if $Z = \{F(N) \mid N \in \boldsymbol{N}\}$, then $A_0 \cap Z^* \neq \square$ (easily proved using (γ)).

The following lemma contains the heart of the proof of (2.3), our generalization of Faucett's theorem. The reason we define $\boldsymbol{N}$, which may be a proper subset of $\boldsymbol{M}$, and $A_0 = \cap \boldsymbol{N}$, is that (2.2) (ii) need not be true if one states it for A and $\{F(M) \mid M \in \boldsymbol{M}\}$, rather than for A_0 and $Z = \{F(N) \mid N \in \boldsymbol{N}\}$.

(2.2) Lemma. *Let A be a nonempty set in a continuum X such that $A = D(A)$, and let $\boldsymbol{M}$ be all the nodal sets containing A. Fix N_0 in $\boldsymbol{M}$, let $\boldsymbol{N} = \{N \in \boldsymbol{M} \mid N \subset N_0\}$, let $Z = \{F(N) \mid N \in \boldsymbol{N}\}$ and let $A_0 = \cap \boldsymbol{N}$. Then*

(i) $A \cap Z^* = A_0 \cap Z^*$.

(ii) *If B is a continuum intersecting both A_0 and $X \backslash A_0$, then $B \supset A_0 \cap Z^*$.*

(iii) *Suppose that T is a semigroup acting on X and that $t \in T$, $a' \in A_0 \cap Z^*$ such that $ta' \in A_0$. Then A_0 contains a $\Gamma(t)$-ideal.*

Proof. (i) Suppose $M \in \boldsymbol{M}$ and $Z^* \not\subset M$. Since M is closed this implies that $Z \not\subset M$, so there is some $N \in \boldsymbol{N}$ such that $F(N) \notin Z$. Now M is connected, $M \cap N \neq \square$ since $A \subset M \cap N$, and $F(N)$ is a cutpoint of X not contained in M, hence $M \subset N$. Therefore $M \subset N_0$, so that $M \in \boldsymbol{N}$. We have proved that if $M \in \boldsymbol{M} \backslash \boldsymbol{N}$, then $Z^* \subset M$. Now $A = [\cap \boldsymbol{N}] \cap [\cap (\boldsymbol{M} \backslash \boldsymbol{N})]$ by definition, hence $A \cap Z^* = [\cap \boldsymbol{N}] \cap Z^*$, which is precisely $A_0 \cap Z^*$.

(ii) First observe that $Z^* \cap N = (Z \cap N)^*$ for $N \in \boldsymbol{N}$: let $p \in Z^* \cap N$, so that either $p = F(N)$, in which case $p \in Z \cap N$, or else $p \in Z^* \cap N^0$, so that surely $p \in (Z \cap N)^*$. The other inclusion is obvious.

Suppose now that B is a continuum intersecting both A_0 and $X \backslash A_0$, so that B intersects $X \backslash N$ for some $N \in \boldsymbol{N}$. If we can show that B contains $N \cap Z$, then B is closed so that B will contain $(N \cap Z)^* = N \cap Z^*$, which contains $A_0 \cap Z^*$. Thus let $p \in N \cap Z$. In case $p = F(N)$, then $p \in B$ since B is a connected set intersecting both $X \backslash N$ and N (since $A_0 \subset N$). Otherwise $p \in N \backslash F(N)$; since $p \in Z$, there is some $N_1 \in \boldsymbol{N}$ such that $p = F(N_1)$. We will prove below that $N_1 \subset N$, from which it is clear that $F(N_1)$ separates $X \backslash N$ from $A_0 \subset N_1$, hence the connected set B must contain $F(N_1) = p$. To prove that $N, N_1 \in \boldsymbol{N}$ and $F(N_1) \in N$ imply $N_1 \subset N$, first note that $(X \backslash N)^*$ is connected since its boundary is a point and X is connected; therefore,

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since $p = F(N_1)$ is a cutpoint and $p \notin (X \backslash N)^*$, either (1) $(X \backslash N)^* \subset N_1$ or (2) $(X \backslash N)^* \subset X \backslash N_1$. It is not possible that (1) holds: for we know that $N \cup N_1 \subset N_0$ by definition of N, hence $(X \backslash N_0)^* \subset (X \backslash N)^*$ and, if $(X \backslash N)^*$ were contained in N_1, we would have $(X \backslash N_0)^* \subset N_1 \subset N_0$. But this implies that $X \backslash N_0 = \square$, which is false because N_0 is nodal (a closed set with non-empty boundary cannot have an empty complement). Therefore it must be true that (2) holds, which clearly implies that $N_1 \subset N$.

(iii) We are given a set A_0; for $n \geqq 1$, define A_n to be $tA_{n-1} \cap A_0$. Observe that $A_0 = D(A_0)$, so that A_0 is a continuum by (α); then by induction, using these facts, (β) and the continuity of t, each A_n is a continuum. One also easily shows by induction that $tA_n \subset tA_{n-1}$, so that if there were a nonempty A_n such that $tA_n \subset A_0$, then A_n would be a $\Gamma(t)$-ideal in A_0 and we would be done. Suppose therefore, in the remainder of the proof, that whenever $A_n \neq \square$, $tA_n \not\subset A_0$; we will first show, by induction, that this implies $A_n \neq \square$ for each n, and then we will use this fact to exhibit a $\Gamma(t)$-ideal in A_0. We will prove each A_n nonempty by showing that there is some $a_n \in A_n$ such that $ta_n = a'$. We lean heavily on the hypotheses that $a' \in Z^* \cap A_0$ and $ta' \in A_0$, and on the fact, stated as (2.2) (ii), that $Z^* \cap A_0$ has the property that a continuum intersecting both A_0 and its complement must contain $Z^* \cap A_0$. (Thus $Z^* \cap A_0$ behaves somewhat like a *C-set*; see § 3 for definition.) First observe that $A_0 \neq \square$ by hypothesis, hence tA_0 is a nonempty subcontinuum by continuity of the action. Also, tA_0 intersects both $X \backslash A_0$ (by supposition) and A_0 (since $ta' \in tA_0 \cap A_0$). Hence $tA_0 \supset Z^* \cap A_0$ by (2.2) (ii), so there must be some $a_0 \in A_0$ such that $ta_0 = a'$. Now suppose that $n \geqq 0$ and that we have $a_n \in A_n$ such that $ta_n = a'$; then $a' \in tA_n \cap A_0 = A_{n+1}$ so that $ta' \in tA_{n+1} \cap A_0$. Also, tA_{n+1} is a continuum and it intersects $X \backslash A_0$ by supposition, hence again by (2.2) (ii), we have $Z^* \cap A_0 \subset tA_{n+1}$. Therefore there is some $a_{n+1} \in A_{n+1}$ such that $ta_{n+1} = a'$. Therefore $A_n \neq \square$ for each n; also, $t : X \to X$ is a continuous function, hence there is a $\Gamma(t)$-ideal in A_0 by the following remark.

Remark. *Let $t : X \to X$ be a continuous function, let A_0 be a compact subset of X, and define, inductively,*

$$A_{n+1} = t(A_n) \cap A_0.$$

If each of the sets A_n is nonempty, then there is a nonempty closed set B contained in every A_n such that $t(B) \subset B$.

Proof. Immediately from the definition there is, for each n, an element $x_n \in A_0$ such that

$$\{x_n, t(x_n), \cdots, t^n(x_n)\} \subset A_0.$$

For $n \geqq 1$, let

$$B_n = \{x_n, x_{n+1}, \cdots, \}^*,$$

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so that these sets form a tower of closed subsets of the compact set A_0, and hence that

$$B_0 = \cap \{B_n \mid n \geqq 1\} \subset A_0$$

is nonempty. It follows that $t^k(B_0) \subset A_0$ for every $k \geqq 1$, and from this that

$$B = (\cup \{t^k(B_0) \mid k \geqq 1\})^*$$

is the desired set.

(2.3) PROPOSITION. *Suppose that T acts on X, X is a continuum, J is a maximal proper T-ideal and A is a nonempty subset of $X \backslash J$ such that $A = D(A)$. If either*

(a) *T is a continuum, or*

(b) *T is Γ-compact ($\Gamma(t)$ is compact for each $t \in T$) and there is a T-ideal contained in $X \backslash N_0$ for some nodal set N_0 containing A,*

then cardinal $A = 1$.

PROOF. If $a \in X \backslash J$, then $J \cup Ta$ is clearly a T-ideal; J is a maximal proper T-ideal, hence either $J \cup Ta = J$ or $J \cup Ta = X$. The former implies that $J \cup a = X$ and we are done. Therefore, suppose for the rest of this proof that $J \cup Ta = X$ for each $a \in X \backslash J$. Then in particular, since $J \subset X \backslash A$, $(X \backslash A) \cup Ta = X$ for each $a \in A$. We will find an $a' \in A$ such that $Ta' \subset (X \backslash A) \cup a'$, which clearly implies that $A = a'$, the desired conclusion.

Let us first prove that given the other hypotheses, (a) implies (b). Obviously, T compact implies T Γ-compact. Let $x \in J$ (which is nonempty by definition of T-ideal) and note that $Tx \subset J \subset X \backslash A$ and that Tx is a continuum since T is and since the action is continuous. Therefore $X \backslash Tx$ is open so that if $y \in Tx$, then by (γ), there is a nodal set N_0 containing A such that $y \notin N_0$ and $F(N_0) \in X \backslash Tx$. Tx is connected, in the complement of the cutpoint $F(N_0)$, and intersects $X \backslash N_0$, so we conclude that $Tx \subset X \backslash N_0$. Since Tx is a T-ideal, (b) is satisfied.

Assume (b) now, let $\mathbf{N} = \{N \mid A \subset N \subset N_0 \text{ and } N \text{ is nodal}\}$, let $A_0 = \cap \mathbf{N}$, and let $Z = \{F(N) \mid N \in \mathbf{N}\}$. Choose $a' \in A_0 \cap Z^*$, which is nonempty by (δ); note that $A_0 \cap Z^* = A \cap Z^*$ by (2.2) (i), so that $a' \in A$; and suppose that $ta' \in A$. We will show that $ta' = a'$, hence $Ta' \subset (X \backslash A) \cup a'$, which is the desired result. Since $A \subset A_0$, we have $a', ta' \in A_0$, so (2.2) (iii) asserts that there exists a $\Gamma(t)$-ideal in A_0. It is clear that each $N \in \mathbf{N}$ also contains this $\Gamma(t)$-ideal, and for each $N \in \mathbf{N}$, $X \backslash N$ contains a T-ideal, hence a $\Gamma(t)$-ideal (by (b), since $X \backslash N_0 \subset X \backslash N$). Finally, the action map $T \times X \to X$ restricted to $\Gamma(t) \times X$ is an action of the compact semigroup $\Gamma(t)$ on the continuum X, so by (2.1), since $F(N) = F(X \backslash N) =$ one point, we have $\Gamma(t)F(N) \subset N^*$ and $\Gamma(t)F(N) \subset (X \backslash N)^*$. That is, $\Gamma(t)F(N) = F(N)$. This

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is true for each $N \in \mathbf{N}$, which is to say $\Gamma(t)z = z$ for each $z \in Z$. Now $a' \in Z^*$ and the action is continuous, hence also $\Gamma(t)a' = a'$, so that, in particular, $ta' = a'$.

(2.3.1) COROLLARY. *Let S be a bing and J be a maximal proper (left, right or two-sided) ideal of S. If A is a nonempty subset of $S \backslash J$ and $A = D(A)$, then cardinal $A = 1$.*

PROOF. First suppose that J is a maximal proper left ideal of S. The multiplication of S is an action of S on itself (on the left) and, with respect to this action, J is a maximal proper S-ideal. S is a continuum so that cardinal $A = 1$ by (2.3). Left-right duality gives the same result when J is a maximal proper right ideal of S.

Suppose now that J is a maximal proper two-sided ideal of S. One can check that the space $T = S \times S$ with the multiplication

$$(x, y)(x', y') = (xx', y'y)$$

is a semigroup, and that

$$T \times S \to S$$

defined by $((x, y), s) = xsy$ is an action of T on S. T is a continuum and one can see without difficulty that J is a maximal proper T-ideal, so that we may again use (2.3) to conclude that cardinal $A = 1$.

3

We will use without proof the following facts.

(ε) *Let X be a continuum containing an open dense half-line, W, and let $C = X \backslash W$. Then C is a C-set, i.e., a continuum which intersects C and is not contained in C, must contain C.*

(ρ) *A locally connected subcontinuum which intersects a nondegenerate C-set is contained in it (follows from a result in [W 5]).*

(3.1) PROPOSITION. *Let X be a continuum containing an open dense half line, W, let $C = X \backslash W$, and suppose that cardinal $C > 1$. Let T be a bing acting unitarily on X such that $\square \neq Q \neq X$. Then Q is a single element, the endpoint q of X, $C \subset Tx$ for each $x \in X$, and either*

(i) *$TC \not\subset C$, so that TC is homeomorphic with X and the Q-set of the act $T \times TC \to TC$ is all of TC; or, disjunctively,*

(ii) *$TC \subset C$, C is the unique minimal T-ideal and C is a homogeneous space.*

If also there is some $a \in X$ such that

(†) $\quad ta = t'a$ *implies* $tx = t'x$ *for all* $x \in X$,

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then Ta has the structure of a semigroup, C is the minimal ideal of Ta and a group.

PROOF. Since T and X are continua, $X \backslash Q$ is connected and dense in X by (1.4), so Q must be a subset of $C \cup q$, where q is the endpoint of X. Q is the complement of a maximal proper T-ideal by (1.4), $C = D(C)$, and cardinal $C > 1$ by hypothesis, hence Q cannot contain C by (2.3); thus to prove that $Q = q$, we must show that $Q \cap C \neq \Box$ implies $Q \supset C$. Whether or not Q intersects C, $C = B_1 \cup B_2$, where

$$B_1 = \{x \in C \mid Tx \supset C\},\ B_2 = \{x \in C \mid Tx \subset C\}.$$

This follows from (ε) since, for each $x \in C$, Tx is a continuum and $x \in Tx \cap C$. If $Q \cap C \neq \Box$, then $Q \cap C = B_1$: for obviously $Q \cap C \subset B_1$, and if $x \in B_1$, $Q \cap Tx \neq \Box$, hence $x \in Q$ (see proof of (1.4) (ii)). Therefore if $Q \cap C \neq \Box$,

$$C = (Q \cap C) \cup \{x \in C \mid Tx \subset C\},$$

which are both closed sets by continuity and compactness. They are disjoint by continuity, and C is connected by either (α) or (ε), hence if $Q \cap C \neq \Box$ then $C = Q \cap C$, which is false.

We prove next that $C \subset Tx$ for each $x \in X$, which observation was made to the authors by K. Sigmon. Suppose first that there is some $x \in W$ such that $Tx \subset W$. Let $t \in T$ such that $tu \in C$ and let A be the arc in W joining x and u. Since tA is a locally connected continuum intersecting C and since C is a nondegenerate C-set, tA must be contained in C by (δ); but this contradicts $tx \in W \cap tA$. Therefore, for each $x \in W$, $Tx \cap C \neq \Box$. Also, $x \in Tx \cap W$ for each $x \in W$, Tx is a continuum, and C is a C-set, hence Tx must contain C for each $x \in W$. Because W is dense and the act is continuous, Tx must contain C for each $x \in C$ as well.

(i) Suppose $TC \not\subset C$ and let $x \in C$ such that $Tx \not\subset C$. Since $C \subset Tx$, Tx is homeomorphic with X, and since $T(Tx) \subset Tx$, T acts on Tx via a restriction of the original action. Let Q_x be the Q-set for this restricted action: i.e., $Q_x = \{y \in Tx \mid Ty = Tx\}$. Since $x \in C \cap Q_x$, $Q_x \neq \Box$ and Q_x is not just the endpoint of Tx, hence by the first assertion of this theorem, we conclude that $Q_x = Tx$. Therefore $Tx = TC$ and the proof of (i) is complete.

(ii) Suppose that $TC \subset C$. We proved above that $C \subset Tx$ for each $x \in X$, which is to say, C is a subset of every T-ideal; thus, when $TC \subset C$, $Tx = C$ for each $x \in C$ and C is the unique minimal T-ideal.

We prove that C is homogeneous by a series of assertions:

(1) *For each $x \in W$, Tx is the continuum irreducible between C and x.* For Tx is a continuum containing C and x, hence Tx is homeomorphic with X. T acts on Tx, $x \in Q_x = \{y \in Tx \mid Ty = Tx\}$ and $C \subset Tx \backslash Q_x$, hence Q_x

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contains only the endpoint of Tx by the first assertion proved above. Therefore, x is the endpoint of Tx.

(2) *T contains an idempotent e which acts as identity for X.* For there exists $t \in T$ such that $tq = q$, since $Tq = X$ by (i) above; then tX is a continuum containing q and intersecting C $(tC \subset tX \cap C)$, hence $tX = X$. Therefore $t^n X = X$ for each $n \geqq 1$, hence $yX = X$ for each $y \in \Gamma(t)$. $\Gamma(t)$ contains an idempotent e since $\Gamma(t)$ is compact [P-de M].

(3) *Let $J = \{t \in T \mid tW \subset W\}$; then $T \backslash J = \{t \in T \mid tX \subset C\}$, and J is open.* The bracketed set is closed by (1.2) (iv), and it is clear that $T\backslash J$ contains it. Conversely let $t \in T\backslash J$, so that $ty \in C$ for some $y \in W$. If A is an arc in W containing y, then tA is a locally connected continuum intersecting C, hence $tA \subset C$ by (ζ); W is the union of a family of such arcs, hence $tW \subset C$. Therefore, $t \in T\backslash J$ implies $tW \subset C$, hence $tX \subset C$.

(4) *If $t \in J^*$ then $tC = C$, hence t is a homeomorphism of C onto itself.* By continuity of the action, we have only to prove that $tC = C$ for each $t \in J$, so let $t \in J$. Then tX is a continuum not contained in but intersecting C, hence $C \subset tX$ by (ε). $tX = tW \cup tC$ and $tW \subset W$, hence $C \subset tC$. Then, since T is compact, t is a homeomorphism of C onto itself by the Swelling Lemma [W 1], [W 2].

(5) *Let J_0 be the component of J which contains e, and let $T_0 = J_0^*$; then $T_0 x = Tx$ for each $x \in X$.* If $J = T$, then $T_0 = T$ and we are done, so suppose $J \neq T$. Then J_0 is a component of a proper open subset of the continuum T, hence $J_0^* = T_0$ intersects the boundary of J [HY]. Let $t \in T_0\backslash J$; then $tX \subset C$ by (3), so that $T_0 x$ intersects C for any $x \in X$. Also, $x = ex \in T_0 x$, and it is clear that $T_0 x$ is a subcontinuum of Tx. Thus if $x \in W$, $T_0 x$ is a continuum containing C and x; hence $T_0 x \supset Tx$, by (1). It is clear that $T_0 x \subset Tx$ for any $x \in X$, hence $T_0 x = Tx$ for each $x \in W$. Continuity then gives $T_0 x = Tx$ for each $x \in X$.

(6) *C is homogeneous.* Since $T_0 x = C$ for each $x \in C$, by (5), and since each member of T_0 is a homeormorphism of C onto itself by (4), C must be homogeneous.

We suppose now that there is some $a \in X$ satisfying (†); then, as remarked in § 1, $t \to ta$ is a biconsistent translation of T onto Ta, so that Ta has a semigroup structure with C as minimal left ideal, by (1.3). Since C is the unique minimal T-ideal, it is the unique minimal left ideal of the semigroup Ta, hence is the minimal ideal of Ta [P-de M]. According to (1.3), to prove that C is also a group, we have only to produce some $u \in T$ with $ua \in C$ and $uC = C$; but there exists $u \in T_0$ with $ua \in C$ since $T_0 a = Ta \supset C$, and $uC = C$ by (4).

Case (i) of the theorem is not vacuous. The dual of a construction due to Koch and Wallace, p. 282, [KW 2], shows that any continuum X with an isolated arc A admits the structure of a semigroup with the endpoint

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of A as right unit and with $Xb = B$ for each $b \in B = (X \backslash A)^*$. Thus if we take X as in the theorem and let A be an arc in X containing q, then X with the semigroup mentioned is a bing acting on itself as described in case (i).

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