ON QUASI-METRIZABILITY

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1. Introduction. The notion of a quasi-metric was introduced by Wilson **(7)** and has been studied by himself, Albert **(1)**, and Ribeiro **(6)** among others. In this paper, we extend and unify some of their work, and connect it with results of Csaszar **(2)** and Pervin **(4, 5)** on quasi-uniformities.

A *quasi-metric* on a space X (called weak metric by Ribeiro) is a function ρ on $X \times X$ such that

$$\rho(x, x) = 0$$

and

$$0 \leqslant \rho(x, z) \leqslant \rho(x, y) + \rho(y, z) < \infty$$

for $x, y, z \in X$.

The quasi-metric ρ is called an A-quasi-metric (called quasi-metric by Albert) if it satisfies the condition

(M1) $$x \neq y \text{ implies either } \rho(x, y) \neq 0 \text{ or } \rho(y, x) \neq 0.$$

It is called a W-quasi-metric (called quasi-metric by Wilson) if it satisfies the condition

(M2) $$x \neq y \text{ implies } \rho(x, y) \neq 0.$$

It is called a *pseudo-metric* if it satisfies the condition

(M3) $$\rho(x, y) = \rho(y, x).$$

It is called a *metric* if it satisfies (M2) and (M3).

For a quasi-metric ρ, the sphere of centre x and radius r, $S_\rho(x, r)$, is $\{y: \rho(x, y) < r\}$. (In view of the lack of symmetry one could also consider $\{y: \rho(y, x) < r\}$. This has in fact been done by Wilson and Albert.)

Any family F of quasi-metrics generates a topology on X with the family of all spheres $\{S_\rho(x, r): \rho \in F, x \in X, r > 0\}$ as a sub-base. If F is such that for any $\rho_1, \rho_2 \in F$ there exists $\rho_3 \in F$ with $\rho_3 \geqslant \rho_1$ and $\rho_3 \geqslant \rho_2$, then the family of all spheres is actually a base for the topology.

A simple way of generating quasi-metrics, used by Wilson and of interest to us here, is the following. For $A \subset X$, let

$$\rho_A(x, y) = \begin{cases} 1 & \text{if } x \in A \text{ and } y \in X - A, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly ρ_A is a quasi-metric but is not a pseudo-metric unless $A = \emptyset$ or $A = X$.

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Now, given a topological space (X, T) and a sub-base S for T, one checks immediately that the family of quasi-metrics $\{\rho_A: A \in S\}$ generates the topology T. This family also generates in a natural way a quasi-uniformity which in turn induces the topology T. To see this, let H be the family of sets $\alpha \subset X \times X$ such that

$$\alpha = \{(x, y): \rho_A(x, y) < r\}$$

for some $A \in S$ and $r > 0$, and U be the quasi-uniformity having H as a sub-base, i.e. let

$$U = \{\beta: \alpha_1 \cap \ldots \cap \alpha_n \subset \beta \subset X \times X \text{ for some } n = 1, 2, \ldots, \text{ and } \alpha_1, \ldots, \alpha_n \in H\}.$$

Then U satisfies all the conditions for a uniformity except that of symmetry. Moreover U induces the topology T since $G \in T$ if and only if for every $x \in G$ there exists $\beta \in U$ with $\{y: (x, y) \in \beta\} \subset G$.

The above remark yields the theorem due to Csaszar (**2**, p. 171) that every topological space is quasi-uniformizable. In case $S = T$, then U is the quasi-uniformity introduced by Pervin (**5**) from a different point of view.

We now turn our attention to situations where a single quasi-metric generates the topology.

2. Quasi-metrization theorems. We consider first the following definitions:

2.1. *Definitions.* Let (X, T) be a topological space.

1. T has a *σ-locally finite base* if and only if for every positive integer m there is a family B_m such that every point in X has a neighbourhood intersecting only a finite number of elements of B_m and $B_1 \cup B_2 \cup \ldots$ is a base for T.
2. T has a *σ-point finite base* if and only if for every positive integer m there is a family B_m such that, for every $x \in X$, $\{A \in B_m: x \in A\}$ is finite and $B_1 \cup B_2 \cup \ldots$ is a base for T.

Clearly, if T has a σ-locally finite base, then T has a σ-point finite base. However, the converse does not hold in general (see Example 3.1 below).

Corresponding to the characterization of a metric space as a regular T_1-space having a σ-locally finite base (see Kelley (**3**, p. 127)), we have the following:

2.2. THEOREM. *Let (X, T) be a topological space. If T has a σ-point finite base, then there exists a quasi-metric ρ which generates T.*

Proof. Let $\{B_m: m = 1, 2, \ldots\}$ be a family given by definition 2.1.2 and set

$$\bar{\rho}_m(x, y) = \sum_{A \in B_m} \rho_A(x, y),$$

$$S_m(x, y) = \frac{\bar{\rho}_m(x, y)}{1 + \bar{\rho}_m(x, y)},$$

$$\rho(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} S_m(x, y).$$

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Since, for each $x \in X$ and $m = 1, 2, \ldots$, only a finite number of elements $A \in B_m$ have $x \in A$ and only for these can $\rho_A(x, y) \neq 0$ we see that $\bar{\rho}_m(x, y) < \infty$ and hence $\bar{\rho}_m$ is a quasi-metric. Therefore, S_m and ρ are also quasi-metrics. We now check that ρ generates T.

Let U be open and $x \in U$. Then there exists k and $A \in B_k$ such that $x \in A \subset U$. Let $r = 1/2^{k+1}$. If $\rho(x, y) < r$, then $S_k(x, y) < \frac{1}{2}$; hence $\bar{\rho}_k(x, y) < 1$ and $\rho_A(x, y) = 0$ and $y \in A$. Thus, $S_\rho(x, r) \subset A \subset U$.

On the other hand, given $x \in X$ and $r > 0$, choose n so that $1/2^n < r$ and $x \in A$ for some $A \in B_1 \cup B_2 \cup \ldots$ and set

$$U = \bigcap \{A : A \in B_m \text{ for some } m = 1, 2, \ldots, n \text{ and } x \in A\}.$$

Then U is open and $x \in U$. If $y \in U$, then $S_m(x, y) = 0$ for $m = 1, 2, \ldots, n$ and hence, since $S_m \leqslant 1$ for all m,

$$\rho(x, y) \leqslant \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n} < r.$$

Thus, $U \subset S_\rho(x, r)$.

2.3. Corollary. *Let (X, T) be a topological space. If T has a countable base, then there exists a quasi-metric ρ which generates T.*

Unlike the metric situation, the converse of Theorem 2.2 does not hold in general even when the space is normal (see Example 3.2 below).

Next, we consider conditions under which the nature of the quasi-metric can be improved.

2.4. Lemma. *Let (X, T) be a topological space and ρ be a quasi-metric which generates T. Then*

1. *ρ is an A-quasi-metric if and only if the space is T_0.*
2. *ρ is a W-quasi-metric if and only if the space is T_1.*

Proof. Immediate from the definitions.

2.5. Theorem. *A regular, compact, quasi-metrizable topological space (X, T) is pseudo-metrizable.*

Proof. Let ρ be a quasi-metric which generates the topology T and for any subsets S, T of X let

$$\rho(S, T) = \inf\{\rho(s, t) : s \in S \text{ and } t \in T\}.$$

We first show:

2.5.1. *T has a σ-locally finite base.*

Well order the elements of X by some relation $<$ and let i, j, k be positive integers. For each $x \in X$, choose open sets $A(x, n)$ for $n = 1, 2, \ldots$ such that

$$x \in A(x, n) \subset \bar{A}(x, n) \subset A(x, n + 1) \subset S_\rho(x, 1/i)$$

and set

$$B(x) = \bigcup_{n=1}^{\infty} A(x, n).$$

Next let

$$C(x, n) = \{t: \rho(\{t\}, A(x, n)) \leqslant 1/j\}$$

and $D(x, n)$ be an open set, if one exists, such that

$$C(x, n) \subset D(x, n) \subset B(x)$$

and

$$\rho(X - B(x), D(x, n)) \geqslant 1/k.$$

Let

$$N(x) = \{n: D(x, n) \text{ exists}\},$$
$$U(x) = \cup_{n \in N(x)} A(x, n),$$
$$U'(x) = \cup_{n \in N(x)} D(x, n),$$
$$V(x) = U(x) - \overline{\cup_{y<x} U'(y)},$$
$$P(i, j, k) = \{V(x): x \in X\}.$$

We first check that any $s \in X$ has a neighbourhood α which intersects at most one set in $P(i, j, k)$. Indeed, let y be the first element in X such that $\rho(\{s\}, V(y)) < 1/j$. Then $s \in U'(y)$. Let $\alpha = U'(y) \cap S_\rho(s, 1/j)$. Then for $x \neq y$ we have $\alpha \cap V(x) = \emptyset$.

Next we check that $\cup_{i,j,k} P(i, j, k)$ is a base for T. Let α be open and $s \in \alpha$. Take $r > 0$ so that $S_\rho(s, 2r) \subset \alpha$ and let

$$E = \{y: y < s \text{ and } \rho(s, y) \geqslant r\}.$$

Since $\bar{E}$ is compact and $s \notin \bar{E}$, there is a positive integer i such that $1/i < r$ and $s \notin S_\rho(y, 1/i)$ for any $y \in E$, for otherwise there would be $y \in \bar{E}$ with $\rho(y, s) = 0$. For such an i, let x be the first element in X with $s \in B(x)$. Since $B(x) \subset S_\rho(x, 1/i)$, we see that $x \notin E$. But $x \leqslant s$; hence $\rho(s, x) < r$. Let n be such that $s \in A(x, n)$. Since $\rho(X - B(x), \bar{A}(x, n)) > 0$, there exist positive integers j, k for which $D(x, n)$ exists. For these integers, $s \in U(x)$ and, for any $y < x$, since $s \notin B(y)$ we have

$$\rho(\{s\}, U'(y)) \geqslant 1/k.$$

Hence

$$s \in U(x) - \overline{\cup_{y<x} U'(y)} = V(x).$$

Finally, since $V(x) \subset S_\rho(x, 1/i)$ and $\rho(s, x) < r$, we have

$$V(x) \subset S_\rho(s, r + 1/i) \subset S_\rho(s, 2r) \subset \alpha.$$

We next show:

2.5.2. *T is pseudo-metrizable.*

Given (i, j, k) and (i', j', k'), triples of positive integers and $F \subset P(i', j', k')$, we have, in view of the local discreteness of $P(i', j', k')$,

$$\overline{\cup_{\beta \in F} \beta} = \cup_{\beta \in F} \bar{\beta}.$$

Thus for $\alpha \in P(i, j, k)$, if

$$\alpha' = \bigcup\{\beta \in P(i', j', k') : \bar{\beta} \subset \alpha\},$$

we have $\bar{\alpha}' \subset \alpha$. By considering all such possible pairs and rearranging the $((i, j, k), (i', j', k'))$ into a sequence, let B_m, for $m = 1, 2, \ldots$, consist of pairs of open sets (α_0, α_1) such that

(i) $\bar{\alpha}_0 \subset \alpha_1$ and each point in X has a neighbourhood which intersects at most one α_1,

(ii) for any open U and $x \in U$ there exist $\alpha \in B_1 \cup B_2 \cup \ldots$ with $x \in \alpha_0 \subset \alpha_1 \subset U$.

By Urysohn's lemma, for each $\alpha \in B_m$, let f_α be a continuous function on X to $[0; 1]$ such that

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x \in \bar{\alpha}_0, \\ 1 & \text{if } x \in X - \alpha_1, \end{cases}$$

and set

$$d_m(x, y) = \sum_{\alpha \in B_m} |f_\alpha(x) - f_\alpha(y)|,$$

$$d(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} d_m(x, y).$$

For any x and y, at most two terms in the summation defining $d_m(x, y)$ can be different from zero. Thus $d_m \leqslant 2$ and d is a pseudo-metric. To check that it generates the topology, let U be open and $x \in U$. Choose m and $\alpha \in B_m$ so that $x \in \alpha_0 \subset \alpha_1 \subset U$ and let $r = 1/2^m$. If $d(x, y) < r$, then $d_m(x, y) < 1$ and $|f_\alpha(x) - f_\alpha(y)| < 1$ so that $y \in \alpha_1 \in U$. Thus, $S_d(x, r) \subset U$. On the other hand, given $x \in X$ and $r > 0$, let k be an integer with $4/2^k < r$ and U be a neighbourhood of x such that

$$|f_\alpha(x) - f_\alpha(y)| < \tfrac{1}{4}r \qquad \text{for } y \in U, \alpha \in \bigcup_{m=1}^{k} B_m.$$

Then $d_m(x, y) < \frac{1}{2}r$ for $m = 1, \ldots, k$ and hence

$$d(x, y) < \sum_{m=1}^{k} \frac{1}{2^m}\frac{r}{2} + \sum_{m=k+1}^{\infty} \frac{2}{2^m} < \frac{r}{2} + \frac{2}{2^k} < r$$

for $y \in U$. Thus, $U \subset S_d(x, r)$.

3. Examples. In this section, we give counter-examples to some natural conjectures which also show that the condition of quasi-metrizability is only slightly more restrictive than the first axiom of countability.

3.1. *A regular, Hausdorff, quasi-metric space which is not normal.*

Let S be the family of uncountable subsets of $(0; 1]$ and for each $A \in S$, let $A_1, A_2, \ldots$ be a disjoint, countable family of non-empty sets with $A = A_1 \cup A_2 \cup \ldots$. Let

$$X = (0; 1] \cup \{(t, A, n) : 0 \leqslant t \leqslant 1, A \in S, n = 1, 2, \ldots\}$$

and ρ be the quasi-metric on X such that for $x \neq y$:

$$\rho(x, y) = \begin{cases} 1/j & \text{if } x = (0, A, n), y = (t, A, n), t \in A_j, \\ 1/n & \text{if } x \in (0; 1], y = (x, A, n), \\ 1 & \text{otherwise.} \end{cases}$$

Let $C_1 = \{(0, A, n): A \in S, n = 1, 2, \ldots\}$ and $C_2 = (0; 1]$. Then C_1 and C_2 are closed and disjoint; in fact $\rho(C_1, C_2) = \rho(C_2, C_1) = 1$. If U and V are open, $C_1 \subset U$, $C_2 \subset V$, then $U \cap V \neq \emptyset$, for let

$$B_n = \{t \in C_2: S_\rho(t, 1/n) \subset V\}.$$

Then for some positive integer n, B_n is uncountable, i.e. $B_n \in S$. Let $x = (0, B_n, n)$. Since $x \in U$, there exists $t \in B_n$ with $(t, B_n, n) \in U$ and, on the other hand,

$$(t, B_n, n) \in S_\rho(t, 1/n) \subset V.$$

Thus, the topology generated by ρ is not normal. It is Hausdorff and regular since spheres are open and closed. It clearly has a σ-point finite base but not a σ-locally finite base since it is regular but not normal.

3.2. *A Hausdorff, normal, quasi-metric space which is not metrizable and has no σ-point finite base.*

Let $X = [0; 1]$ and

$$\rho(x, y) = \begin{cases} y - x & \text{if } x \leqslant y, \\ 1 & \text{if } y < x. \end{cases}$$

Then spheres are half-open intervals. If A, B are closed, disjoint sets, for $x \in A$ let $t_x > x$ and $[x; t_x) \cap B = \emptyset$ and set

$$U = \cup_{x \in A} [x; t_x).$$

Then U is open and $A \subset U$. If $y \in \bar{U} \cap B$, for each positive integer n, let $s_n \in U$ and $\rho(y, s_n) < 1/n$, i.e. $s_n - y < 1/n$. Then there exists $x_n \in A$ with $s_n \in [x_n, t_{x_n}) \subset U$. Thus, $y < x_n \leqslant s_n$ and $\rho(y, x_n) < 1/n$ so that $y \in A$, which is impossible. Therefore, the topology generated by ρ is normal, but is not metrizable since it is separable but has no countable base. In fact, it has no σ-point finite base, for if $B_1 \cup B_2 \cup \ldots$ is a base, then, for some m, B_m has an uncountable number of sets each containing an interval of positive diameter. Hence some point belongs to an infinite number of them.

3.3. *A compact, Hausdorff space in which every open set is $\mathfrak{F}_\sigma$ but which is not quasi-metrizable.*

Let

$$X = (\{0\} \times [0; 1)) \cup (\{1\} \times (0; 1]).$$

For $r > 0$ and $x = (0, s)$ let

$$\alpha_r(x) = \{x\} \cup \{(i, t): i = 0, 1 \text{ and } s < t < s + r\}$$

and for $x = (1, s)$ let

$$\alpha_r(x) = \{x\} \cup \{(i, t) : i = 0, 1 \text{ and } s - r < t < s\}.$$

Let T be the topology having $\{\alpha_r(x) : x \in X \text{ and } r > 0\}$ as a base. Then any set of the form

$$(\{0\} \times [a; b)) \cup (\{1\} \times (a; b])$$

is compact and any open set is the union of a countable number of such sets. T is not metrizable since it is separable and has no countable base. In view of Theorem 2.5 it is not even quasi-metrizable.

4. Remarks. Our Corollary 2.3 in conjunction with Lemma 2.4 was proved by Wilson **(7)** for a T_1-space, by Albert **(1)** for a T_0-space, and by Ribeiro **(6)** in the general situation. Ribeiro's proof was much more indirect than ours.

The proof of 2.5.2 actually shows that a normal space with a σ-locally finite base is pseudo-metrizable. This extends slightly the well-known fact that a normal T_1-space with a σ-locally finite base is metrizable (see Kelley **(3,** p. 128**)**). Our proof, although related to that of Kelley, is somewhat different in spirit.

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