# A VARIANT OF THE PROBLEM OF THE THIRTEEN SPHERES

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*To Professor H. S. M. Coxeter on his sixtieth birthday*

**1. Introduction.** We use the term *balls* for congruent, closed spheres no two of which have interior points in common. In Euclidean $n$-space let $N_n$ be the maximal number of balls which can touch a ball. Obviously, $N_2 = 6$. R. Hoppe (see **(1)**) proved that $N_3 = 12$, settling thereby a famous point of controversy between Newton and David Gregory, known as the problem of the thirteen spheres (see **(3)**). Simpler proofs were given by Günter **(6)**, Schütte and van der Waerden **(10)**, and Leech **(7)**.

By special constructions Coxeter **(3)** showed that $N_4 \geqslant 24$, $N_5 \geqslant 40$, $N_6 \geqslant 72$, $N_7 \geqslant 126$, and $N_8 \geqslant 240$. In order to obtain upper bounds, he accepted the "intuitively obvious" conjecture that in spherical $(n-1)$-space the packing density of a set of balls cannot exceed the density of $n$ balls all touching one another with respect to the simplex spanned by the centres of the balls. Coxeter pointed out that this hypothesis implies that $N_4 \leqslant 26$, $N_5 \leqslant 48$, $N_6 \leqslant 85$, $N_7 \leqslant 146$, and $N_8 \leqslant 244$. Some years ago Böröczky **(2)** established the validity of Coxeter's conjecture for $n = 4$, thus proving that $N_4 \leqslant 26$. This seems to be all we know about the numbers $N_n$.

Consider now a bunch of balls all touching a ball. We enlarge this bunch by adding new balls touching at least one ball of the original bunch. The problem we want to deal with is to find the maximal number $T_n$ of the balls contained in an $n$-dimensional bunch arising in this way. We shall prove that $T_2 = 18$, $56 \leqslant T_3 \leqslant 63$, and $168 \leqslant T_4 \leqslant 232$. Of course, we can continue the process of successively enlarging a bunch of balls, but in the next step the problem becomes extremely intricate, even in the plane.

**2. Neighbours.** Let **s** be a set of balls, and $a$ and $b$ two balls in **s**. The balls $a$ and $b$ are said to be *neighbours* of degree $k$ or $k$th neighbours in **s** if there is in **s** a subset of $k + 1$ balls, but no subset of less than $k + 1$ balls, containing $a$ and $b$ and having a connected point-set union. First neighbours, i.e. balls touching each other, will also simply be called neighbours.

Let $T$ be the number of all neighbours of $a$ of degree not more than $k$. Let $T_n^k$ be the maximum of $T$ for all sets of balls containing $a$. Thus, using the above notations, $N_n = T_n^1$ and $T_n = T_n^2$. We can also ask about the maximal number $M_n^k$ of neighbours of a ball of degree exactly $k$. Obviously,

$$T_n^k \leqslant M_n^1 + \ldots + M_n^k,$$

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but it is interesting to observe that, apart from some small values of $k$ depending on $n$, equality will not be attained generally.

**3. Lower bounds for $T_n$.** The example of the densest lattice-packing of circles shows that $T_2 \geqslant 18$ (Fig. 1). However, observe that there are other

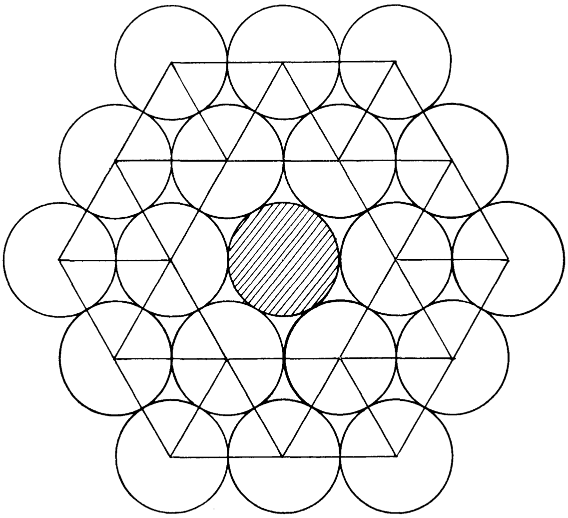

FIGURE 1

arrangements in which a circle has 18 neighbours of at most second degree. For, considering a circle having 6 first neighbours and a second neighbour chosen arbitrarily, we always can add 11 further second neighbours (Fig. 2).

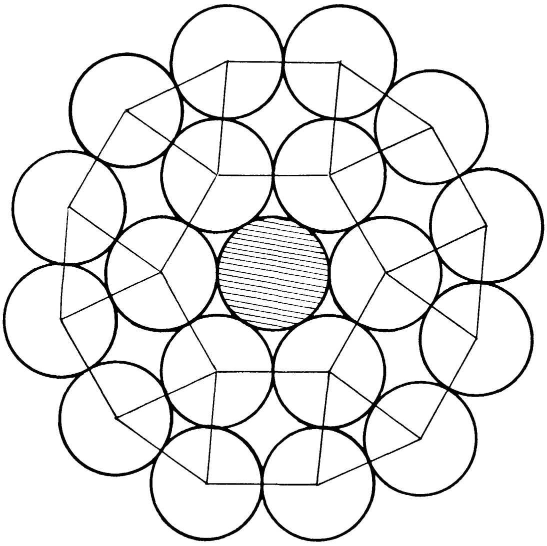

FIGURE 2

Turning to the case when $n = 3$, it is natural to start with the densest lattice-packing of balls. Consider a densest lattice-packing of circles in a horizontal plane, as well as the layer of balls having these circles as equators. In this layer a ball $a$ has 6 first and 12 second neighbours. Complete the layer by further layers so as to obtain a densest lattice-packing. In the layer above the original layer $a$ has 3 first and 9 second neighbours, while in the next layer it has no first neighbours but 6 second neighbours (Fig. 3). Together with the neighbours below $a$, this amounts to $6 + 12 + 2(3 + 9 + 6) = 54$ neighbours of first and second degree.

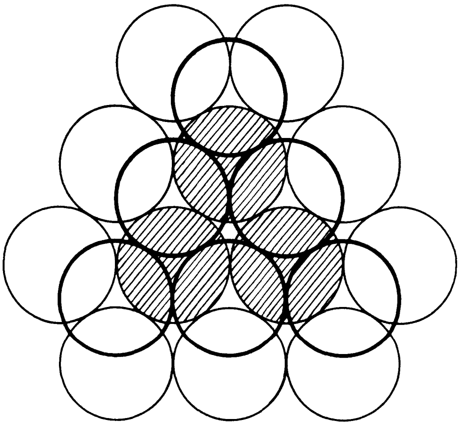

FIGURE 3

But there is an equally dense regular packing other than the lattice-packing described in 1883 by W. Barlow. To construct this so-called hexagonal close-packing, we can start with the above horizontal layer containing $a$ and the layer above it. The third layer arises by reflecting the layer of $a$ in the central plane of the second layer. This layer will contain 7 second neighbours of $a$ and not 6 as did the corresponding layer in the lattice-packing (Fig. 4). Recapitu-

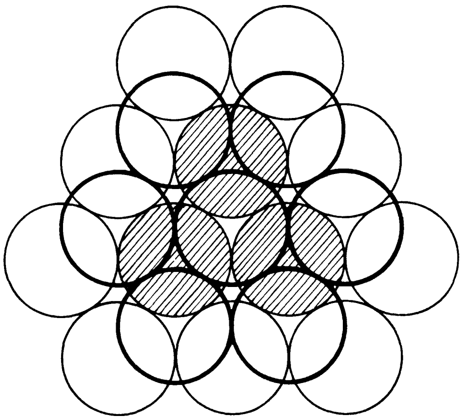

FIGURE 4

lating, in the hexagonal close-packing each ball has 12 first and 46 second neighbours. Thus $T_3 \geqslant 56$.

We continue to show that in the densest lattice-packing of 4-dimensional balls each ball has 24 neighbours and 144 second neighbours. Minkowski **(8)** proved that in 4-space (just as in 3- and 5-space) the balls with radius $1/\sqrt{2}$ centred at those points which have integral Cartesian coordinates with an even sum constitute a densest lattice-packing. In this lattice-packing let $(x_1, x_2, x_3, x_4)$ be the centre of a first or second neighbour of the ball centred at the origin. Obviously, $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leqslant 8$. Since $x_1 + x_2 + x_3 + x_4$ is even, $(x_1, x_2, x_3, x_4)$ must be one of the points given by the permutations of $(\pm 1, \pm 1, 0, 0,)$ $(\pm 1, \pm 1, \pm 1, \pm 1)$, $(\pm 2, 0, 0, 0)$, $(\pm 2, \pm 1, \pm 1, 0)$, and $(\pm 2, \pm 2, 0, 0)$. These five sets of points contain 24, 16, 8, 96, and 24 points, respectively. The points of the first set have a distance from $(0, 0, 0, 0)$ equal to $\sqrt{2}$. Thus each of these points is a centre of a first neighbour. The rest of the points have a distance from $(0, 0, 0, 0)$ greater than $\sqrt{2}$, but a distance equal to $\sqrt{2}$ from a suitable point of the first set. Therefore these points are all centres of a second neighbour. Thus $T_4 \geqslant 168$.

**4. Upper bounds for $T_n$.** Obviously, the angle spanned by the centres of two first neighbours of a ball at its centre is not less than 60°. A similar fact is expressed in the following theorem.

THEOREM. *The angle spanned by the centres of two second neighbours of a ball at its centre is not less than* 30°.

We suppose the balls to be of unit radius. A ball and its centre will be denoted by the same symbol.

First we consider the case when the second neighbours $C_1$ and $C_2$ of the ball $O$ are neighbours of the same neighbour $B$ of $O$. Since $BC_1 = BC_2 = BO = 2$, the triangle $OC_1C_2$ is contained in a circle of radius $\leqslant 2$ (Fig. 5). Since $C_1C_2 \geqslant 2$, in this circle the central angle spanned by $C_1$ and $C_2$ is $\geqslant 60°$. Hence

$$\angle C_1OC_2 \geqslant 30°.$$

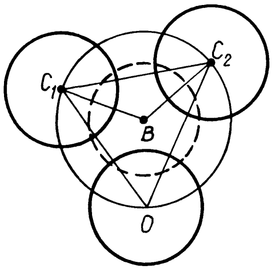


FIGURE 5

Now we suppose that there are two first neighbours $B'_1$ and $B'_2$ of $O$ such that $C_1$ is a neighbour of $B'_1$ and $C_2$ is a neighbour of $B'_2$. Obviously, we may assume that $\angle C_1OC_2 < 60°$. Rotate the triangle $OB'_iC_i$ about the axis $OC_i$ so as to obtain a triangle $OB_i C_i$ coplanar with the triangle $OC_1 C_2$ but not overlapping it ($i = 1, 2$). We claim that $B_1 B_2 \geqslant 2$, $B_1 C_2 \geqslant 2$, and $B_2 C_1 \geqslant 2$.

Let $b_1$, $b_2$, $b'_1$, $b'_2$, $c_1$, and $c_2$ be the central projections of the points $B_1$, $B_2$, $B'_1$, $B'_2$, $C_1$, and $C_2$, respectively, onto the surface of the ball $O$. Since

$$OC_i > OB_i = B_i C_i = 2,$$

we have $\angle B_i OC_i < 60°$ ($i = 1, 2$). Combining this with the assumption that $\angle C_1 OC_2 < 60°$, we see that the points $b_1$, $c_1$, $c_2$, and $b_2$ lie, in this order, on an open semicircle.

Let $p$ be a point on the arc $c_1 c_2$. Then

$$b_i p = b_i c_i + c_i p,$$

where $xy$ refers to the spherical distance between the points $x$ and $y$. Thus, in view of $b_i c_i = b'_i c_i$ and the triangle inequality

$$b'_i p \leqslant b'_i c_i + c_i p,$$

we have

$$\text{(1)} \qquad b_i p \geqslant b'_i p,$$

whence

$$\text{(2)} \qquad b_1 b_2 = b_1 p + pb_2 \geqslant b'_1 p + pb'_2 \geqslant b'_1 b'_2.$$

On the other hand, applying (1) in the case when $p$ is the extremity of the arc $c_1 c_2$ other than $c_i$, we obtain

$$\text{(3)} \qquad b_1 c_2 \geqslant b'_1 c_2, \qquad b_2 c_1 \geqslant b'_2 c_1.$$

Since in a triangle having two sides of given length the length of the third side is an increasing function of the opposite angle, the inequalities (2) and (3) imply that

$$B_1 B_2 \geqslant B'_1 B'_2 \geqslant 2, \qquad B_1 C_2 \geqslant B'_1 C_2 \geqslant 2, \qquad B_2 C_1 \geqslant B'_2 C_1 \geqslant 2,$$

as stated (Fig. 6).

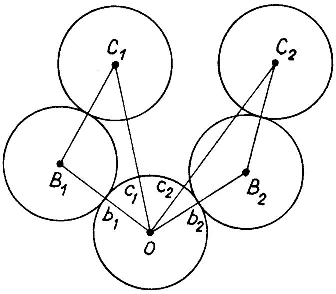


FIGURE 6

Recapitulating, we have constructed a convex pentagon $C_1 B_1 OB_2 C_2$ such that $C_1 B_1 = B_1 O = OB_2 = B_2 C_2 = 2$, while the side $C_1 C_2$ and the diagonals $B_1 C_2$, $B_2 C_1$, and $B_1 B_2$ are not less than 2 and the diagonals $OC_1$ and $OC_2$ are greater than 2. These properties of the pentagon will enable us to prove that $\angle C_1 OC_2 \geqslant 30°$.

We may suppose that either the side $C_1C_2$ or one of the diagonals $B_1C_2$, $B_2C_1$, and $B_1B_2$ is of length 2. Otherwise rotate the triangle $OB_1C_1$ about $O$ so as to decrease $\angle C_1OC_2$ until the required position ensues. The cases when $B_1C_2 = 2$ or $B_2C_1 = 2$ have been settled above. For now $C_1$ and $C_2$ are neighbours of the same neighbour of $O$. Thus we have to consider only the cases when either $B_1B_2 = 2$ or $C_1C_2 = 2$.

Suppose that $B_1B_2 = 2$ and $C_1C_2 > 2$. Rotate $C_1$ about $B_1$ and $C_2$ about $B_2$ so as to decrease $\angle C_1OC_2$ until we have $C_1C_2 = 2$. In this position the radius of the circle passing through $O$, $C_1$, and $C_2$ equals 2 (Fig. 7). Thus

$$\angle C_1OC_2 = 30°.$$

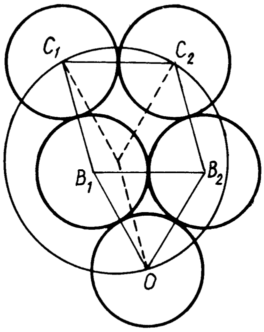


FIGURE 7

We still have to scrutinize the case when $C_1C_2 = 2$ but all diagonals of the equilateral pentagon $C_1B_1OB_2C_2$ are greater than 2 (Fig. 8). Consider the sides

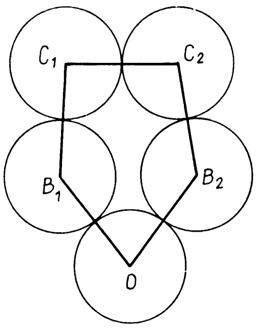


FIGURE 8

of the pentagon as rigid bars each of length 2 which are connected at the vertices by joints. Suppose that $OC_1 \geqslant OC_2$. Fix the position of the bars $OB_2$ and $B_2C_2$ and rotate the bar $C_1C_2$ about $C_2$ so as to decrease $\angle C_1OC_2$. This operation will come to an end only if $\angle OB_1C_1$ becomes equal to 180°. But in this position either $B_1B_2 < 2$ or $B_1C_2 < 2$ or $B_1B_2 = B_1C_2 = 2$ (Fig. 9). Therefore we can rotate the bar $C_1C_2$ until one of the diagonals becomes equal to 2, thus arriving at a situation discussed previously.

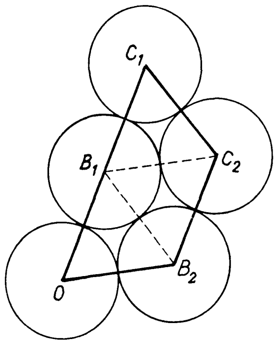


FIGURE 9

This completes the proof of the theorem.

In Euclidean $n$-space, let $O$ be a ball having $M_n^2$ second neighbours. Project the centres of the second neighbours of $O$ radially onto $O$. In the spherical $(n - 1)$-space consisting of the boundary of $O$ draw spheres with radius 15° about the projections. According to our theorem these spheres form a packing. An upper bound of the packing density yields an upper bound for $M_n^2$.

For $n = 2$ we use the trivial upper bound 1 for the packing density, obtaining $M_2^2 \leqslant 360/30 = 12$. Since $M_2^1 = 6$, we have $T_2 \leqslant 6 + 12 = 18$.

For $n = 3$ the truth of Coxeter's "intuitively obvious" conjecture has been proved by Fejes Tóth **(4)**. This yields

$$M_3^2 \leqslant \frac{12 \operatorname{arccot} \sqrt{(1 + \sqrt{3})}}{6 \operatorname{arccot} \sqrt{(1 + \sqrt{3})} - \pi} \approx 53.11,$$

in consequence of which $M_3^2 \leqslant 53$. But using Robinson's **(9)** sharper density bound, we obtain by some computation which we omit here $M_3^2 \leqslant 51$. Hence

$$T_3 \leqslant M_3^1 + M_3^2 \leqslant 12 + 51 = 63.$$

To give an upper bound for $M_4^2$, we use Böröczky's theorem mentioned above which claims the truth of the "intuitively obvious" conjecture for $n = 4$. But since the volume of a regular spherical tetrahedron is a non-elementary function of its dihedral angle, it is not quite easy to find the numerical value of the corresponding bound. The computation may be performed by an elegant general method suggested by Coxeter. Unfortunately, the numerical values of the function involved in Coxeter's formula were not available to us. Therefore we used another function previously tabulated by G. Krámmer for another purpose.

In spherical 3-space consider a packing of $s$ balls each of radius $R$ and volume $t$. Let $ABCD$ be a regular tetrahedron with an edge-length equal to $2R$. Let $v$ be the volume of this tetrahedron, $2\delta$ its dihedral angle, $r$ its in-radius, and

$\sigma = 4(6\delta - \pi)$ the sum of its vertex angles. Then Böröczky's theorem implies that

$$\frac{st}{2\pi^2} \leqslant \frac{\sigma}{4\pi} t/v,$$

i.e.

$$s \leqslant \frac{2\pi(6\delta - \pi)}{v}.$$

The function we referred to is $v = v(r)$, which may be expressed by the explicit formula **(5)**

$$v = 12\sqrt{2} \int_0^r \frac{\sin x}{\sqrt{(1 + 2\sin^2 x)}} \arctan \frac{\sqrt{6}\sin x}{\sqrt{(1 + 2\sin^2 x)}}\, dx.$$

In order to express $\delta$ and $r$ in terms of $R$, we use the following relations:

$$\begin{array}{lll}
\sin EF = \tan R \cot 60°, & \angle BFE = 60°, & \\
\cos ED = \sec R \cos 2\mathrm{R}, & CD = 2EC = 2R, & \\
\cos 2\delta = \cot ED \tan EF, & ED = EA, & \angle AEF = 2\delta, \\
\tan r = \sin EF \tan \delta, & GF = r, & \angle GEF = \delta,
\end{array}$$

where $E$ is the mid-point of the edge $BC$, $F$ is the centre of the face $BCD$, and $G$ is the centre of the tetrahedron (Fig. 10).

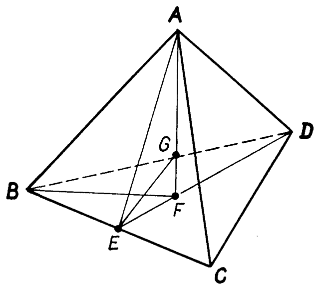


FIGURE 10

In the case when $R = 15°$ we obtain $2\delta \approx 71.5192°$ and $r \approx 6.3567°$. We reproduce the corresponding data of Krámmer's table:

| $r$ | $10^6 v$ | | |
|---|---|---|---|
| 6.3° | 17,866 | | |
| | | 846 | |
| 6.4° | 18,712 | | 26 |
| | | 872 | |
| 6.5° | 19,584 | | 26 |
| | | 898 | |
| 6.6° | 20,482 | | |

By quadratic interpolation we obtain

$$10^6 v(6.3567°) \approx 17{,}866 + 0.567 \times 846 + \tfrac{1}{2} \times 0.567(0.567 - 1)26 \approx 18{,}342,$$

whence

$$\frac{2\pi(6\delta - \pi)}{v} \approx \frac{34.5576 \times \pi^2}{90 \times 0.018342} \approx 206.6,$$

showing that $M_4^2 \leqslant 206$. Since $M_4^1 \leqslant 26$, we have $T_4 \leqslant 232$.

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