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CENTRAL ELEMENTS OF A^{**} FOR CERTAIN BANACH ALGEBRAS A WITHOUT BOUNDED APPROXIMATE IDENTITIES

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Abstract. In this paper we give a characterization of the central elements of the algebra A^{**} for a class of weakly sequentially complete Banach algebras A and present a detailed study of several questions related to Arens regularity of the algebra A.

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0. Introduction. Throughout the paper the letter A will denote a commutative, semisimple, weakly sequentially complete and completely continuous Banach algebra. The term "completely continuous" means that, for each $a \in A$, the multiplication operator $L_a : A \longrightarrow A$, $L_a(x) = ax$, is compact. Now let G be an arbitrary discrete group and $A(G)$ be its Fourier algebra and $VN(G)$ be its von Neumann algebra, as defined by P. Eymard in [**6**]. The algebra $A(G)$ possesses all the properties imposed on A, see [**6**] and e.g. [**16**; Theorem 3.6]. The algebra $A(G)$ is our main example and throughout the paper special attention will be paid to it. As is well known, on the second dual A^{**} of A there are two algebra multiplications extending that of A, known as the first and second Arens multiplications, whose constructions are recalled below. Assume that A^{**} is equipped with the first of them. Unless the algebra A is Arens regular, the algebra A^{**} is not commutative and a characterization of the central elements of A^{**} presents a certain interest. In this paper we study about the algebras A and A^{**} the following four questions. (a) What are the central elements of A^{**}? (b) When is A Arens regular? (c) When does Arens regularity of A imply that A is finite dimensional?; and (d) When is there a weakly compact homomorphism from A into $C_0(\Sigma)$ with an infinite dimensional range? Here Σ is the Gelfand spectrum of the algebra A. For the Fourier algebra $A(G)$ of an amenable discrete group G, as will be explained below, all these questions are solved. On the other hand, if G is a nonamenable discrete group, none of the these questions seems to be solved for the algebra $A(G)$. In the case where G is amenable, Lau and Losert in [**11**; Theorem 6.5] proved that the center of the algebra $VN(G)^*$ is $A(G)$. Again for G amenable, in [**13**] Lau and Wong proved that the algebra $A(G)$ is Arens regular if and only if G is finite. In the case where G is nonamenable, concerning Arens regularity of $A(G)$, the best known result seems to be the one obtained by Forrest [**7**], which says that if $A(G)$ is Arens regular then each amenable subgroup of G is finite. In the case where G is amenable, as proved by Granirer [**8**] and the author [**16**], weakly compact homomorphisms on $A(G)$ have a finite dimensional range and, as remarked by Granirer [**8**], this result fails if G is not amenable. In these kinds of problems, existence of a bounded approximate identity and knowledge of the center

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of the auxiliary algebra $(\overline{AA^*})^*$ play an essential role, as clearly displayed in the paper [**12**]. For a nonamenable discrete group G, the algebra $A(G)$ does not have a bounded approximate identity and we do not know the center of the algebra $(\overline{A(G)VN(G)^*})^*$. So we have had to follow a different path. Concerning the first question, our main result says that an element $m \in A^{**}$ is in the center of this algebra if and only if $mA^{**} \subseteq A$ and $A^{**}m \subseteq A$. Concerning the second and third questions, we give several necessary and sufficient conditions for A to be Arens regular or finite dimensional. At this point we remark that there exist nonreflexive Arens regular Banach algebras satisfying the conditions imposed on A. Concerning the fourth question, we show that there exist weakly compact homomorphisms $h : A \longrightarrow C_0(\Sigma)$ with infinite dimensional ranges if, and only if, the functional zero is in the weak sequential closure of Σ in A^*. The main ingredients of the proofs are weak sequential completeness and the fact that any von Neumann algebra has the Grothendieck property [**14**].

1. Notation and Preliminaries. Let A be a commutative Banach algebra, a, b two elements of A, f an element of A^* and m, n be two elements of A^{**}. We define the elements $f.a,\ a.f,\ m.f,\ f * n$ of A^* and nm and n^*m of A^{**} as follows.

$$
\begin{array}{ll}
< f.a, b > = < f, ab > & < a.f, b > = < f, ab > \\
< m.f, a > = < m, fa > & < f * m, a > = < m, af > \\
< nm, f > = < n, m.f > & < n * m, f > = < m, f * n >
\end{array}
$$

The operations $(n, m) \longmapsto n.m$ and $(n, m) \longmapsto n * m$ define two Banach algebra multiplications on A^{**}, known, respectively, as the first and second Arens multiplication of A^{**}. The basic properties of these operations are the following: $a.m = m.a = a * m = m * a$; for m fixed, the mapping $n \longmapsto nm$ is weak*-weak* continuous on A^{**}. However the mapping $m \longmapsto nm$ need not be weak*-weak* continuous, unless n is in the center of A^{**}. The properties of the second multiplication is symmetric to those of the first multiplication. For either multiplication, A is a subalgebra of A^{**}. If, for all n, m in A^{**}, the equality $nm = n * m$ holds, then algebra A is said to be Arens regular. From now on, we shall denote by A^{**} the algebra A^{**} equipped with the first Arens multiplication and consider A as a subalgebra of A^{**}. All the results we need and use about Arens multiplications can be found in the survey paper [**4**]. An element of A is said to be (weakly) compact if the operator $L_a : A \longrightarrow A$, defined by $L_a(b) = ab$, is (weakly) compact. The algebra A is an ideal in its second dual equipped with either of the Arens multiplications if and only if every element of A is weakly compact [**4**]. The algebra A is said to be c.c. (=completely continuous) if each element of A is compact. Now assume that A is an ideal in its second dual, and let m be an element of A^{**}. Define the mapping $L_m : A \longrightarrow A$ by $L_m(a) = ma$. Then the first and second adjoints of L_m are given by $L_m^*(f) = m.f = f * m$ and $L_m^{**}(n) = nm$. The algebra A being commutative, as one can see it readily, $nm = m * n$. Thus, an element m of A^{**} is in the center of the algebra A^{**} if and only if for all n in A^{**}, $mn = m * n$. By $A^{**}A^*$ and AA^* we denote, respectively, the subspaces $\{m.\varphi : m \in A^{**}$ and $\varphi \in A^*\}$ and $\{a.\varphi : a \in A$ and $\varphi \in A^*\}$ of A^*. Finally we remark that, algebra A being commutative, the centers of the algebras A^{**} and $(A^{**}, *)$ are the same.

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2. Central Elements of $A(G)^{**}$. *Throughout the paper A will be a commutative, semisimple, c.c. and weakly sequentially complete Banach algebra.* By Σ we shall denote the Gelfand space of A (the set of multiplicative functionals on A equipped with the weak* topology induced by $\sigma(A^*, A)$). For m in A^{**} (so, for a in A) by $\hat{m}$ we shall denote the function $\hat{m} : \Sigma \longrightarrow C$ defined by $< \hat{m}, f >= m(f)$. It is clear that if $\Gamma : A \longrightarrow C_0(\Sigma)$ is the Gelfand transform, Γ^{**} applies A^{**} into $\ell^\infty(\Sigma)$ and $\Gamma(m) = \hat{m}$. We now give some examples of Banach algebras satisfying the above conditions.

Examples 2.1. The following Banach algebras satisfy all the conditions we have imposed on A.

(a) The Fourier algebra $A(G)$ of any discrete group G. See [**6**] and e.g. [**16**; Theorem 3.6]. In particular, the group algebra $L^1(G)$ of a compact abelian group G.

(b) The space ℓ^1, considered as a Banach algebra with coordinatewise multiplication.

(c) The semigroup algebra $\ell^1(N)$, where the set of the positive integers N is equipped with the multiplication $pq = \min\{p, q\}$ [**9**; Example 11.1.5].

(d) The space

$$bv = \{(x_n)_{n\in N} \in C^N :\| (x_n) \| = | x_0 | + \Sigma_{n\in N} | x_{n+1} - x_n | < \infty\}$$

equipped with coordinatewise multiplication.

(e) Any closed subalgebra of a Banach algebra that satisfies the properties imposed on A.

Now let a be an element in A. For f in Σ, we have $L_a^*(f) = f.a = < f, a > f$ so that $< f, a >$ is an eigenvalue of the compact operator L_a^*. This fact and well known spectral properties of the compact operators show that the Gelfand space Σ of A is discrete and the set

$$S_a = \left\{ f \in \Sigma : < f, a > \neq 0\} = \cup_{n \geqslant 1} \{f \in \Sigma : | < f, a > | \geqslant \frac{1}{n} \right\}$$

is countable. The weak* topology of A^* being weaker than its weak topology, the space (Σ, weak)—the set Σ endowed with the topology induced by the weak topology of A^*—is also discrete. The first main result of the paper is the following theorem.

Theorem 2.2. *Let m be an element of the algebra A^{**}. Then m is in the center of A^{**} if and only if $mA^{**} \subseteq A$ and $A^{**}m \subseteq A$.*

Proof. Assume first that m is in the center of A^{**}. Let us prove that the operator $L_m : A \longrightarrow A$, $L_m(a) = ma$, is weakly compact. To this end let $(a_n)_{n\in N}$ be a sequence in the unit ball of A. Put $S_n = \{f \in \Sigma : < f, a_n > \neq 0\}$ and $S = \cup_{n\in N} S_n$. The set S is countable so that the subspace $Span(S)$ of A^* generated by S is separable. Hence, by Cantor's diagonal process, from the sequence $(a_n)_{n\in N}$ we can extract a subsequence, denoted again $(a_n)_{n\in N}$, such that, for each φ in $Span(S)$, the sequence $(< \varphi, a_n >)_{n\in N}$ converges. Since, for each f in $\Sigma \backslash S$ and all n in N, $< f, a_n >= 0$ and $Span(\Sigma) = Span(S) + Span(\Sigma \backslash S)$, we see that, for each φ in $Span(\Sigma)$, the sequence $(< \varphi, a_n >)_{n\in N}$ converges too. The sequence $(a_n)_{\in N}$ being bounded, we conclude that for each φ in $\overline{Span(\Sigma)}$, the sequence $(<\varphi, a_n>)_{n\in N}$ converges. On the other hand, since m is in the center of A^{**}, for any φ in A^*, $\varphi * m = m.\varphi$ and, the functional $\varphi * m$, which is in A^*, actually is in $\overline{A^*A} = \overline{\{\varphi.a : \varphi \in A^* \textit{ and } a \in A\}}$. Indeed, if $(b_\alpha)_{\alpha\in I}$ is a net in A converging to m in the weak*−topology of A^{**}, then

$$<\varphi.b_\alpha, p>=<\varphi, b_\alpha p> \longrightarrow <\varphi, mp>=<\varphi, m*p>=<\varphi*m, p>$$

for any p in A^{**} so that $\varphi.b_\alpha \longrightarrow \varphi * m$ in the weak topology of A^* and $\varphi * m$ is in $\overline{A^*A}$. Now let us see that $\overline{A^*A} = \overline{Span(\Sigma)}$. The inclusion $\overline{Span(\Sigma)} \subseteq \overline{A^*A}$ is clear since, for f in Σ and a in A, $f.a =<f.a> f$. To prove the reverse inclusion, assume that, for some φ in A^* and a in A, we have $\varphi.a \notin \overline{Span(\Sigma)}$. Then, by The Hahn-Banach Theorem, there is some p in A^{**}, such that $<\varphi.a, p>\neq 0$ but $<f, p>=0$, for each f in Σ. As ap is in A, $<ap, f>=<a, f><p, f>=0$, and as A is semisimple, we conclude that $ap = 0$. However this contradicts the inequality $<\varphi, ap>\neq 0$. Hence, for all $\varphi \in A^*$, $m.\varphi \in \overline{Span(\Sigma)}$ and consequently the sequence $(<\varphi, ma_n>)_{n\in N}$ converges. This shows that the sequence $(ma_n)_{n\in N}$ is weakly Cauchy in A. As the algebra A is weakly sequentially complete, the sequence $(ma_n)_{n\in N}$ converges weakly to some point in A. This proves that the operator $L_m(a) = ma$ is weakly compact on A so that $L_m^{**}(A^{**}) = A^{**}m \subseteq A$. As m is in the center of A^{**}, $mA^{**} = A^{**}m \subseteq A$ too.

To prove the reverse implication, assume that $A^{**}m \subseteq A$ and $mA^{**} \subseteq A$. Then, for each n in A^{**}, the products nm and mn are in A and since $<nm, f>=<n, f><m, f>=<mn, f>$ for each f in Σ, A being semisimple, we get that $nm = mn$. This being true for all n in A^{**}, we conclude that m is in the center of A^{**}.

Remark 2.3. Let m be an element of A^{**}. If m is in the center of A^{**} then, as seen above, the mapping $L_m : A \longrightarrow A$, $L_m(a) = ma$, is weakly compact. But, if for some $m \in A^{**}$, L_m is weakly compact, we can not say that m is in the center of A^{**}. Indeed, for any m in the annihilator of Σ in A^{**}, $L_m(A) = mA = \{0\}$ so that L_m is weakly compact but m need not to be in the center of A^{**}, see Theorem 3.2. below. We also remark that from the inclusions $mA^{**} \subseteq A$ and $A^{**}m \subseteq A$, we can not deduce that $m \in A$, see Remark 3.4. below.

Corollary 2.4. *If the algebra A has a bounded approximate identity, then the center of A^{**} is A.*

Proof. Indeed, in this case A^{**} has a right unit E so that for m in A^{**}, $mE = m$. This fact and the inclusion $mA^{**} \subseteq A$ show that the center of A^{**} is A.

Thus, if G is a compact abelian group and $A = L^1(G)$ is its group algebra, then the center of A^{**} is A. For a completely different proof of this result (for a not necessarily commutative G) see the paper [**10**].

3. Arens Regularity of A(G) and Related Questions. In this section we study the following three questions: a) When is the algebra A Arens regular? b) When does Arens regularity of A imply that A is finite dimensional?; and c) When is there a weakly compact homomorphism $h : A \longrightarrow C_0(\Sigma)$ with an infinite dimensional range? *The letter A will have the same signification as in the previous section. Moreover in this*

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section we assume that A^* is a von Neumann algebra. The reader will observe that most of the results below remain valid without this hypothesis but for the unity of the statements we put this condition as a blanket assumption.

For the proof of the next theorem we recall that: a) A Banach space X is said to be "weakly compactly generated" if, for some weakly compact subset E of X, X is the closed linear span of E. A weakly compactly generated dual Banach space has the RNP [**1**; p.76, Corollary 4.1.10]. b) As proved by Pfitzner in [**14**], any von Neumann algebra B has the so called Grothendieck property. i.e. In the space B^*, weak* convergent sequences are weakly convergent. Any continuous linear operator from a space having the Grothendieck property into a weakly compactly generated Banach space is weakly compact [**5**; p.179].

By $\Sigma^{\perp}$ we denote the annihilator of Σ in A^{**} and by Z the center of A^{**}. For the proof of the next theorem we need the following lemma. Before this we remark that, for m in $\Sigma^{\perp}$, $mA = \{0\}$ since A is semisimple and A is an ideal in A^{**}. This implies that $A^{**}m = \{0\}$. (But $mA^{**} \neq \{0\}$, unless $m \in Z$).

LEMMA 3.1. *If $\Sigma^{\perp}$ is contained in Z then Σ is relatively weakly compact in A^*.*

Proof. Assume that $\Sigma^{\perp} \subseteq Z$. Then for any n and m in A^{**}, $nm - mn$ is in $\Sigma^{\perp}$. So, for any p in A^{**}, $p(mn - nm) = (mn - nm)p$. Hence, by the above remark, $p(mn - nm) = 0 = (mn - nm)p$ so that, for all n, m, p in A^{**}, we have $pmn = nmp$. For $p = m$, we get that $m^2n = nm^2$ for all n in A^{**}. This shows that m^2 is in Z so that, by Theorem 2.1 above, m^3 is in A. It follows that, for any $\varepsilon > 0$, the set

$$K_{\varepsilon} = \{f \in \Sigma : |< f, m >| \geqslant \varepsilon\} = \{f \in \Sigma : |< f, m^3 >| \geqslant \varepsilon^3\}$$

is finite. This proves that the second adjoint of the Gelfand transform maps A^{**} into $C_0(\Sigma)$. From this we conclude that the Gelfand transform $\Gamma : A \longrightarrow C_0(\Sigma)$ is weakly compact. As Σ is contained in the image under Γ^* of the closed unit ball of $\ell^1(\Sigma)$, we deduce that Σ is relatively weakly compact in A^*.

As an immediate corollary of this lemma we have the following result.

COROLLARY 3.2. *If the algebra A is Arens regular then Σ is relatively compact in $(A^*, weak)$.*

At this point we recall that the algebra $A = \ell^1$ (example 2.1 (b) above) is Arens regular (so its spectrum is relatively weakly compact in ℓ^{∞}) but it is not finite dimensional.

THEOREM 3.3. *For the algebra A the following assertions are equivalent.*

(a) *A is Arens regular.*
(b) *$A^{**}A^{**} \subseteq A$.*
(c) *$\Sigma^{\perp} \subseteq Z$*
(d) *For any m in A^{**}, $A^{**}m = \{0\}$ implies that $mA^{**} = \{0\}$.*
(e) *$\overline{A^{**}A^*} = \overline{AA^*}$.*
(f) *For each m in A^{**}, the mapping $L_m : A \longrightarrow A$, $L_m(a) = ma$, is weakly compact.*
(g) *For each m in A^{**}, the mapping $S_m : A^{**} \longrightarrow A^{**}$, $S_m(n) = nm$, is weakly compact.*
(h) *A^{**} is Arens regular.*

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Proof. Since A is Arens regular if, and only if, $Z = A^{**}$, by Theorem 2.1 above, the equivalence of the assertions (a) and (b) is clear.

b)$\Longrightarrow$c) and c)$\Longrightarrow$d). Implication b)$\Longrightarrow$c) being obvious, we prove implication c)$\Longrightarrow$d). To prove this implication assume that c) holds. Let $m \in A^{**}$ be such that $A^{**}m = \{0\}$. The algebra A being semisimple, this implies that $m \in \Sigma^{\perp}$. Hence, by c), $m \in Z$. It follows that $mA^{**} = A^{**}m = \{0\}$.

d)$\Longrightarrow$e). Assume that d) holds. Let $m \in A^{**}$ and $\varphi \in A^*$. If we had $m.\varphi \notin \overline{AA^*}$, we would have an n in A^{**} such that $< m.\varphi, n >=< \varphi, nm > \neq 0$ but $< a\psi, n >= < \psi, na >= 0$ for all $a \in A$ and $\psi \in A^*$. This implies that $an = na = 0$ for all a in A. This in turn implies that $nm = 0$, which contradicts the inequality $< \varphi, nm > \neq 0$ and proves e).

e)$\Longrightarrow$f). Assume that e) holds. Let us first prove that then c) holds. To see this let m be an element in $\Sigma^{\perp}$. Then $A^{**}m = \{0\}$ and, by e), for any $\varphi \in A^*$ and $n \in A^{**}, n.\varphi \in \overline{AA^*}$. We have to show that, for all $n \in A^{**}, mn = 0$. Assume, for a contradiction, that for some n in $A^{**}, mn \neq 0$. Then, for some $\varphi \in A^*$, $< mn, \varphi >=< m, n.\varphi > \neq 0$. On the other hand, since $n.\varphi \in \overline{AA^*}, n.\varphi = a.\psi$ for some $a \in A$ and $\psi \in A^*$. Hence, $< m, n.\varphi >=< m, a.\psi >=< ma, \psi >= < am, \varphi >= 0$, contradicting the inequality $< mn, \varphi > \neq 0$. This contradiction proves that $\Sigma^{\perp} \subseteq Z$ so that, by Lemma 3.1, the space $\overline{Span(\Sigma)}$ is weakly compactly generated. Now fix an $m \in A^{**}$ and consider the mapping $\tau_m : A^* \longrightarrow A^*$, defined by $\tau_m(\varphi) = m.\varphi$. By e), τ_m applies A^* into $\overline{AA^*}$. As we have seen in the course of the proof of Theorem 2.1, $\overline{AA^*} = \overline{Span(\Sigma)}$. From this, since A^* has the Grothendieck property and the space $\overline{Span(\Sigma)}$ is weakly compactly generated, we conclude that the mapping τ_m is weakly compact. As $(L_m)^* = \tau_m$, we conclude that f) holds.

f)$\Longrightarrow$g). Assume that f) holds. Then, since for any $m \in A^{**}, L_m^{**} = S_m$, g) holds.

g)$\Longrightarrow$a). Assume that g) holds. Then f) also holds and, for any $m \in A^{**}$, $A^{**}m \subseteq A$. It follows that, $A^{**}A^{**} \subseteq A$, and by Theorem 2.1, A is Arens regular.

Implication h)$\Longrightarrow$a) being obvious, it remains to show that a)$\Longrightarrow$ h). To prove this implication, assume that A is Arens regular. For $F \in A^{***}$ and $m \in A^{**}, F.m$ is the functional defined on A^{**} by $< F.m, n >=< F, mn >$. Since $A^{**}A^{**} \subseteq A$ and $A^{***} = A^* \oplus A^{\perp}, F$ is of the form $F = f + \mu$ and $F.m = f.m$. It follows that

$$\{F.m : m \in A^{**}, || m || \leqslant 1\} = \{f.m : m \in A^{**}, || m || \leqslant 1\}.$$

Since A is Arens regular, the set $\{f.a : a \in A, || a || \leqslant 1\}$ is relatively weakly compact in A^*, and consequently

$$\{f.m : m \in A^{**}, || m || \leqslant 1\} \subseteq \overline{\{f.a : a \in A, || a || \leqslant 1\}}.$$

From this we conclude that the algebra A^{**} is Arens regular.

Remark 3.4. The inclusion $A^{**}A^{**} \subseteq A$ does not imply that the algebra A is reflexive. Indeed, let $A = \ell^1$ (Example 2.1(b) above). Then, since $A^{**} = \ell^1 \oplus c_o^{\perp}, A^{**}A^{**} \subseteq A$ but A is not reflexive. Observe also that although $A = \ell^1$ is Arens regular, has an (unbounded) approximate identity, and $\overline{A^*A} = c_o \neq \ell^{\infty} = A^*$.

For the sake of completeness we include the statement of the following known result. One part of this proposition is proved in [**16**; Proposition 2.8] and the other part in [**2**; Corollary 2.8]. We also recall that Σ is closed in (A^*, *weak*) whenever A has a bounded approximate identity [**16**; p. 361].

Proposition 3.5. *Let G be any locally compact group and* $A(G)$ *be its Fourier algebra. Then the spectrum of the algebra* $A(G)$ *is closed in* $(VN(G), weak)$ *if and only if the group G is amenable.*

Theorem 3.6. *For the algebra A the following assertions are equivalent.*

(a) *The dimension of A is finite.*
(b) *The algebra A is Arens regular and* Σ *is closed in* $(A^*, weak)$.
(c) *For m in* A^{**} *and* φ in A^*, $m.\varphi = 0$ *implies that* $< m, \varphi >= 0$.
(d) $A^* = \overline{Span\Sigma}$.

Proof. Implication (a)$\Longrightarrow$(b) is clear. To prove the reverse implication, assume that (b) holds. Then, by Corollary 3.2. above, (Σ, *weak*) is compact. Since (Σ, *weak*) is also discrete, we conclude that Σ is finite. The algebra A being semisimple, the dimension of A is finite.

As a finite dimensional semisimple Banach algebra is necessarily unital, implication (a)$\Longrightarrow$(c) is clear.

To prove implication (c)$\Longrightarrow$(d), assume that assertion (c) holds. Let m be in $\Sigma^\perp$. Then, for any $\varphi \in A^*$, $m.\varphi = 0$. Hence by hypothesis, $< \varphi, m >= 0$. This being true for any $\varphi \in A^*$ and $m \in \Sigma^\perp$, we conclude that $\Sigma^\perp = \{0\}$ so that $\overline{Span\Sigma} = A^*$.

Finally, to prove implication (d)$\Longrightarrow$(a), assume that (d) holds. This implies that the algebra A is Arens regular so that Σ is relatively weakly compact and A^* is weakly compactly generated. As a weakly compactly generated dual space has the RNP [**1**; p.76], A^* has the RNP. However, since A is weakly sequentially complete, it contains an isomorphic copy of ℓ^1 by Rosenthal's ℓ^1-Theorem [**15**] unless it is reflexive. Since the dual of a Banach space containing an isomorphic copy of ℓ^1 cannot have the RNP [**1**; p.75, Corollary 4.1.7], A must be reflexive. Since a reflexive von Neumann algebra is finite dimensional, we conclude that the dimension of A is finite.

Now let G be a discrete group. If G is amenable then, as proved in [**13**] by Lau and Wong, the algebra $A(G)$ is Arens regular if, and only if, the group G is finite. The question whether this result also holds for every nonamenable G seems to be still open. Concerning this question, the best known result, as far as we know, is the following one obtained by B. Forrest [**7**]: If $A(G)$ is Arens regular then every amenable subgroup of G is finite. Next we present a completely different proof of this result.

Corollary 3.7. *Let G be a discrete group. If the algebra* $A(G)$ *is Arens regular then every amenable subgroup of G is finite.*

Proof. Assume that $A(G)$ is Arens regular, and let H be an amenable subgroup of G. The Fourier algebra $A = A(H)$ of H, being isometrically isomorphic to a quotient algebra of $A(G)$ [**6**], is also Arens regular. Hence by Proposition 3.5 and Theorem 3.6, $A(H)$ is finite dimensional, so H is finite.

At this point we remark that, if $A(G)$ is Arens regular then its spectrum (i.e. G) is relatively weakly compact in $VN(G)$ so that every subset of G which is closed in $(VN(G), weak)$ is finite. This reproves the above corollary (Consider the quotient homomorphism $h : A(G) \longrightarrow A(H)$).

Next we consider weakly compact homomorphisms from A into $C_0(\Sigma)$. If the algebra A is Arens regular then, by Corollary 3.2, every homomorphism $h : A \longrightarrow C_0(\Sigma)$ is weakly compact but need not have a finite dimensional range (take e.g. $A = \ell^1$). On the other hand, if Σ is closed in $(A^*, weak)$ then every weakly compact homomorphism $h : A \longrightarrow C_0(\Sigma)$ has a finite dimensional range [**16**; Theorem 2.14]. Denote by $\overline{\Sigma}^{sw}$ the sequential closure of Σ in $(A^*, weak)$. Thus an element f of A^* is in $\overline{\Sigma}^{sw}$ if, and only if, there is a sequence $(f_n)_{n\in N}$ in Σ that converges weakly to f. Then we have the following result.

Theorem 3.8. *There exists a weakly compact homomorphism* $h : A \longrightarrow C_0(\Sigma)$ *with infinite dimensional range if and only if* $0 \in \overline{\Sigma}^{sw}$.

Proof. We first recall that a continuous linear operator T from a Banach space X into c_0 is weakly compact if, and only if, there exists a sequence $(f_n)_{n\in N}$ in X^*, converging weakly to zero, such that $T(x) = (< f_n, x >)_{n\in N}$ [**4**; p.108]. Now assume that $0 \in \overline{\Sigma}^{sw}$. Then there exists an infinite sequence $(f_n)_{n\in N}$ in Σ that converges weakly to zero. Let $h : A \longrightarrow C_0(\Sigma)$ be the mapping defined by $h(a) = (< f_n, a >)_{n\in N}$. Then h is a weakly compact homomorphism whose range is infinite dimensional. Conversely, if $h : A \longrightarrow C_0(\Sigma)$ is a weakly compact homomorphism with infinite dimensional range, then, as one can easily see it, h is necessarily of the above form for some infinite sequence $(f_n)_{n\in N}$ in Σ that converges weakly to zero. It follows that $0 \in \overline{\Sigma}^{sw}$.

Concerning the problems tackled in this paper, a certain number of questions remain to be clarified. Below we have enumerated them as remarks and questions.

4. Remarks and questions. Let A be as in Section 3.

1) Corollary 3.2. shows that if the algebra A is Arens regular then its spectrum Σ is relatively weakly compact in A^*. We do not know if the converse of this result is true.

2) If Σ is not closed in $(A^*, weak)$, then $\overline{\Sigma}^{w} = \Sigma \cup \{0\}$ [**16**; p. 361]. We do not know if, even for $A = A(G)$, in this case, we have $0 \in \overline{\Sigma}^{sw}$.

3) If $A^{**} = A \oplus \Sigma^{\perp}$ then A is Arens regular. We do not know if the converse result is true. We do not know either, when A is separable, whether the quotient space $A^{**}/\Sigma^{\perp}$, which is a commutative semisimple Banach algebra, is separable. In which case the functional zero would be in $\overline{\Sigma}^{sw}$.

4) Assume that Σ is closed in $(A^*, weak)$. Is then $Z = A$? At this point we remark that Σ may be closed in $(A^*, weak)$ even if A has no bounded approximate identity, see example 2.9 in [**16**].

5) It is still not known if there exits an infinite group G for which $A(G)$ is Arens regular.

REFERENCES

1. R. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodym property*, Lecture Notes in Mathematics No. 993, (Springer-Verlag, 1983).

2. C. Chou and G. Xu, Weak convergence in von Neumann algebras of non-amenable groups, *Proc. Amer. Math. Soc.*, to appear.

3. J. Dazord, Factoring operators through c_0, *Math. Ann.* **220** (1976), 105–122.

4. J. Duncan and S. A. R. Husseiniun, The second dual of a Banach algebra. *Proc. Roy. Soc. Edinburgh, Ser. A* **84** (1979), 309–325.

5. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, No. 5 (Amer. Math. Soc., 1977).

6. P. Eymard, L'algebre de Fourier d'un groupe locallement compact, *Bull. Soc. Math. France* **92** (1964), 181–236.

7. B. Forrest, Arens regularity and discrete groups, *Pacific J. Math.* **15** (1991), 217–227.

8. E. Granirer, A survey on some functional analytic properties of the Fourier algebra $A(G)$ of a locally compact group, *Southeast Asian Bull. Math.* **20** (1996), 1–12.

9. E. Hewitt and H. Zuckermann, The ℓ^1- algebra of a commutative semi-group. *Trans. Amer. Math. Soc.* **83** (1956), 70–97.

10. N. Isik, J. Pym and A. Ülger, The second dual of the group algebra of a compact group, *J. London Math. Soc. (2)* **35** (1987), 135–148.

11. A. T. Lau and V. Losert, The C^*-algebra generated by operators with compact support on a locally compact group, *J. Funct. Anal.* **112** (1993), 1–30.

12. A. T. Lau and A. Ülger, Topological centers of certain dual algebras, *Trans. Amer. Math. Soc.* **348** (1996), 1191–1212.

13. A. T. Lau and J. C. S. Wong, Weakly almost periodic elements in $L^\infty(G)$ of a locally compact group, *Proc. Amer. Math. Soc.* **107** (1989), 1031–1036.

14. H. Pfitzner, Weak compactness in the dual of a C^*- algebra is determined commutatively, *Math. Ann.* **298** (1994), 349–371.

15. H. Rosenthal, A characterization of Banach spaces containing ℓ^1, *Proc. Nat. Acad. Sci. U.S.A.* **71** (1974), 2411–2413.

16. A. Ülger, Some results about the spectrum of commutative Banach algebras under the weak topology and applications, *Monatsh. Math.* **121** (1996), 353–379.