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A GENERALISATION OF THE CLUNIE–SHEIL-SMALL THEOREM

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Abstract

Clunie and Sheil-Small [‘Harmonic univalent functions’, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **9** (1984), 3–25] gave a simple and useful univalence criterion for harmonic functions, usually called the shear construction. However, the application of this theorem is limited to planar harmonic mappings that are convex in the horizontal direction. In this paper, a natural generalisation of the shear construction is given. More precisely, our results are obtained under the hypothesis that the image of a harmonic function is a union of two sets that are convex in the horizontal direction.

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1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. A function $f : \mathbb{D} \to \mathbb{C}$ is said to be harmonic if its real and imaginary parts are real harmonic, that is, they satisfy the Laplace equation. Since $\mathbb{D}$ is simply connected, it is well known that every such f can be written in the form

$$f(z) = h(z) + \overline{g(z)}, \quad z \in \mathbb{D},$$

where h and g are analytic in $\mathbb{D}$. The Jacobian J_f of f in terms of h and g is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2, \quad z \in \mathbb{D}.$$

Among all the harmonic functions in $\mathbb{D}$ one can distinguish those with nonvanishing Jacobian. In fact, it is proved that such harmonic functions are locally one-to-one. If the Jacobian of a harmonic function in $\mathbb{D}$ is positive, this function is locally one-to-one and sense preserving. More information about harmonic functions can be found in [2].

Clunie and Sheil-Small in [1] gave the following theorem, known as the shear construction.

THEOREM 1.1. *A function $f = h + \overline{g}$, harmonic in $\mathbb{D}$ with positive Jacobian, is a one-to-one sense-preserving mapping of $\mathbb{D}$ onto a domain that is convex in the direction of the real axis if and only if $h - g$ is an analytic one-to-one mapping of $\mathbb{D}$ onto a domain that is convex in the direction of the real axis.*

This theorem turns out to be a useful tool both as a univalence criterion and as a method of constructing harmonic mappings. In particular, it plays an important role in the study of harmonic mappings onto polygonal domains [5, 10, 15], onto a horizontal strip [9], onto a plane with a slit [12] and onto a plane with several slits [3, 6, 8]. Further interesting examples of harmonic mappings obtained in this way can be also found in [4, 7, 16].

In this paper, we generalise the theorem of Clunie and Sheil-Small. In Section 2 we show some auxiliary results. In Section 3 we use results from Section 2 to give new conditions for the univalence of planar harmonic mappings.

2. Topological properties

The proof of Theorem 1.1 of Clunie and Sheil-Small relies on the following lemma, which will also be useful in our considerations.

LEMMA 2.1. *Let D be a domain that is convex in the direction of the real axis and let p be a continuous real-valued function in D. Then the mapping $D \ni w \mapsto w + p(w)$ is one-to-one in D if and only if it is locally one-to-one. In this case, the image of D is convex in the direction of the real axis.*

Using this lemma, we will prove more general results and apply them to obtain new univalence criteria for harmonic mappings. First, we need the following definitions. For a given set D in the complex plane $\mathbb{C}$, we define the projection on the imaginary axis as

$$P(D) := \{a \in \mathbb{R} : \exists_{z \in D} \operatorname{Im} z = a\}.$$

We also define

$$\Lambda(D) := \{a \in \mathbb{R} : (D \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}) \text{ is a nonempty and connected set}\}.$$

One can immediately observe that a set $D \subset \mathbb{C}$ is convex in the direction of the real axis if and only if $P(D) = \Lambda(D)$.

We start our investigations with properties of P and Λ acting on a set D, which is a union of two domains that are convex in the direction of the real axis: that is, every horizontal line that meets D, meets it either in an open interval or a disjoint union of two open intervals.

LEMMA 2.2. *Let D_1, D_2 be two domains that are convex in the direction of the real axis with a nonempty intersection. Then*

$$P(D_1 \cap D_2) = \Lambda(D_1 \cup D_2) \cap [P(D_1) \cap P(D_2)].$$

Proof. Let D_1, D_2 be two domains that are convex in the direction of the real axis with a nonempty intersection. If $a \in P(D_1 \cap D_2)$, then there exists $w \in D_1 \cap D_2$ such that $\operatorname{Im} w = a$. This means that

$$w \in (D_1 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}) \cap (D_2 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}).$$

Next, observe that the sets $D_1 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ and $D_2 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ are nonempty and connected, since both domains D_1 and D_2 are convex in the direction of the real axis. In addition $D_1 \cap D_2 \neq \emptyset$. Thus $(D_1 \cup D_2) \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ is nonempty and connected and, consequently, $a \in \Lambda(D_1 \cup D_2)$. Obviously, $a \in P(D_1) \cap P(D_2)$, so the inclusion $P(D_1 \cap D_2) \subset \Lambda(D_1 \cup D_2) \cap [P(D_1) \cap P(D_2)]$ holds.

If $a \in \Lambda(D_1 \cup D_2) \cap [P(D_1) \cap P(D_2)]$, then the set $(D_1 \cup D_2) \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ is nonempty and connected. Next, observe that the sets $D_1 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ and $D_2 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ are nonempty, open and connected intervals, since D_1 and D_2 are open and convex in the direction of the real axis. Hence there is $w \in D_1 \cap D_2$ with $\operatorname{Im} w = a$. Thus $a \in P(D_1 \cap D_2)$ and $\Lambda(D_1 \cup D_2) \cap [P(D_1) \cap P(D_2)] \subset P(D_1 \cap D_2)$, which completes the proof. □

Lemma 2.3. *Let D_1, D_2 be two domains that are convex in the direction of the real axis with a nonempty intersection, and let $q : D_1 \cup D_2 \to \mathbb{C}$ be a continuous function such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for all $z \in D_1 \cup D_2$. Then $\Lambda(D_1 \cup D_2) = \Lambda(q(D_1) \cup q(D_2))$ if and only if $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$.*

Proof. Let D_1, D_2 be two domains that are convex in the direction of the real axis with $D_1 \cap D_2 \neq \emptyset$. Then the sets $q(D_1)$, $q(D_2)$ are domains that are convex in the direction of the real axis and $q(D_1) \cap q(D_2) \neq \emptyset$. It is also clear that

$$P(q(D_1)) = P(D_1), \quad P(q(D_2)) = P(D_2), \tag{2.1}$$

and

$$\begin{aligned} P(D_1) \cup P(D_2) = P(D_1 \cup D_2) &= P(q(D_1 \cup D_2)) = P(q(D_1) \cup q(D_2)) \\ &= P(q(D_1)) \cup P(q(D_2)). \end{aligned}$$

Assume that $\Lambda(D_1 \cup D_2) = \Lambda(q(D_1) \cup q(D_2))$. By Lemma 2.2 and (2.1),

$$\begin{aligned} P(D_1 \cap D_2) &= \Lambda(D_1 \cup D_2) \cap [P(D_1) \cap P(D_2)] \\ &= \Lambda(q(D_1) \cup q(D_2)) \cap [P(q(D_1)) \cap P(q(D_2))] = P(q(D_1) \cap q(D_2)), \end{aligned}$$

which completes the first part of the proof.

Now assume $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$. Since $\Lambda(D_1 \cup D_2) \subset P(D_1 \cup D_2)$, we need consider only two cases.

In the first case, if $a \in P(D_1) \cap P(D_2)$, then $a \in \Lambda(D_1 \cup D_2)$ if and only if $a \in \Lambda(q(D_1) \cup q(D_2))$, since Lemma 2.2 and equalities (2.1) give

$$\begin{aligned} \Lambda(D_1 \cup D_2) \cap [P(D_1) \cap P(D_2)] &= \Lambda(q(D_1) \cup q(D_2)) \cap [P(q(D_1)) \cap P(q(D_2))] \\ &= \Lambda(q(D_1) \cup q(D_2)) \cap [P(D_1) \cap P(D_2)]. \end{aligned}$$

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In the second case, if $a \in P(D_1 \cup D_2) \setminus (P(D_1) \cap P(D_2))$, then $a \in \Lambda(D_1 \cup D_2)$ if and only if $a \in \Lambda(q(D_1) \cup q(D_2))$. Indeed,

$$(P(D_1) \setminus P(D_2)) \cup (P(D_2) \setminus P(D_1)) \subset \Lambda(D_1 \cup D_2),$$

and, by (2.1),

$$(P(D_1) \setminus P(D_2)) \cup (P(D_2) \setminus P(D_1)) \subset \Lambda(q(D_1) \cup q(D_2)).$$

Consequently,

$$\begin{aligned}
&\Lambda(D_1 \cup D_2) \cap [P(D_1 \cup D_2) \setminus (P(D_1) \cap P(D_2))] \\
&\quad = \Lambda(D_1 \cup D_2) \cap [(P(D_1) \setminus P(D_2)) \cup (P(D_2) \setminus P(D_1))] \\
&\quad = (P(D_1) \setminus P(D_2)) \cup (P(D_2) \setminus P(D_1)) \\
&\quad = \Lambda(q(D_1) \cup q(D_2)) \cap [(P(D_1) \setminus P(D_2)) \cup (P(D_2) \setminus P(D_1))] \\
&\quad = \Lambda(q(D_1) \cup q(D_2)) \cap [P(D_1 \cup D_2) \setminus (P(D_1) \cap P(D_2))].
\end{aligned}$$

This completes the second part of the proof. □

Lemma 2.4. *Let D_1 and D_2 be two domains with a nonempty intersection and such that $D_1 \cup D_2$ is simply connected. Then $P(D_1)$, $P(D_2)$ and $P(D_1 \cap D_2)$ are open intervals.*

Proof. It is evident that $P(D)$ is open and connected since D is open and connected. The connectedness of the set $D_1 \cap D_2$ follows from the Janiszewski theorem [11, page 268, Theorem 2]. □

We use these lemmas to prove the following theorems.

Theorem 2.5. *Let D_1, D_2 be two domains that are convex in the direction of the real axis and let $q : D_1 \cup D_2 \to \mathbb{C}$ be a continuous, locally one-to-one function such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for all $z \in D_1 \cup D_2$. Then the following conditions are equivalent:*

(1) $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$;
(2) $\Lambda(D_1 \cup D_2) = \Lambda(q(D_1) \cup q(D_2))$;
(3) *q is one-to-one.*

Proof. The proof of (1) ⇔ (2) follows immediately from Lemma 2.3. We need only to prove (1) ⇔ (3). If D_1, D_2 are two disjoint domains that are convex in the direction of the real axis, then our claim follows immediately from Lemma 2.1. Hence, we only consider the case $D_1 \cap D_2 \neq \emptyset$.

Assume that q is one-to-one in $D_1 \cup D_2$. Then it is clear that q is locally one-to-one. Since q is continuous and one-to-one in $D_1 \cup D_2$, it is a homeomorphism on $D_1 \cup D_2$. Hence $q(D_1 \cap D_2) = q(D_1) \cap q(D_2)$, which implies that $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$.

For the converse, we assume that q is locally one-to-one and $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$. The fact that $\operatorname{Im} q(z) = \operatorname{Im} z$ for all $z \in D_1 \cup D_2$, together with

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Lemma 2.1, ensures that q is one-to-one in $(D_1 \cup D_2) \cap \{z \in \mathbb{C} : \operatorname{Im} z \in P(D_1 \cap D_2)\}$. Assume that q is not one-to-one in

$$\widetilde{D} := (D_1 \cup D_2) \cap \{z \in \mathbb{C} : \operatorname{Im} z \notin P(D_1 \cap D_2)\}.$$

Then there exist $a \in P(\widetilde{D})$ and $z_1, z_2 \in D_1 \cup D_2$ with $a = \operatorname{Im} z_1 = \operatorname{Im} z_2$, $\operatorname{Re} z_1 \neq \operatorname{Re} z_2$ and $q(z_1) = q(z_2)$. This is only possible if $z_1 \in D_1$ and $z_2 \in D_2$ or $z_1 \in D_2$ and $z_2 \in D_1$, which means that $a \in P(q(D_1) \cap q(D_2))$. But, by the definition of $\widetilde{D}$, we have $a \notin P(D_1 \cap D_2)$, which gives a contradiction to the assumption that $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$. Thus q is one-to-one in $\widetilde{D}$. Now the fact that $\operatorname{Im} q(z) = \operatorname{Im} z$ for all $z \in D_1 \cup D_2$ implies that q is one-to-one in $D_1 \cup D_2$, and this completes the proof. □

THEOREM 2.6. *Let D_1, D_2 be two domains that are convex in the direction of the real axis with a nonempty intersection such that $D_1 \cup D_2$ is simply connected, and let $q : D_1 \cup D_2 \to \mathbb{C}$ be a continuous, locally one-to-one function such that* $\operatorname{Im} q(z) = \operatorname{Im} z$ *for all $z \in D_1 \cup D_2$ and $q(D_1) \cup q(D_2)$ is simply connected. Then the following conditions hold:*

(1) $P(D_1 \cap D_2) = P(q(D_1) \cap q(D_2))$;
(2) $\Lambda(D_1 \cup D_2) = \Lambda(q(D_1) \cup q(D_2))$;
(3) *q is one-to-one.*

PROOF. By Lemma 2.1, q is one-to-one on D_1 and q is one-to-one on D_2. Consequently, q on $D_1 \cup D_2$ takes on every value in $q(D_1) \cup q(D_2)$ once or twice and every value in $q(D_1 \cap D_2)$ exactly once. Hence q is one-to-one on $D_1 \cup D_2$, by a theorem of Ortel and Smith [14, Theorem 1] (see also [13] for more general results). This proves (3). Now the conditions (1) and (2) hold true by Theorem 2.5. □

3. Harmonic mappings

In this section we apply the results obtained in the previous section to the theory of harmonic mappings.

THEOREM 3.1. *Let $f = h + \overline{g}$ be a harmonic function in $\mathbb{D}$ such that $J_f > 0$ in $\mathbb{D}$. If $\Lambda((h - g)(\mathbb{D})) = \Lambda(f(\mathbb{D}))$, then the following statements are equivalent:*

(1) *f is a one-to-one mapping and $f(\mathbb{D})$ is a union of two nondisjoint domains that are convex in the direction of the real axis;*
(2) *$h - g$ is a one-to-one analytic mapping and $(h - g)(\mathbb{D})$ is a union of two nondisjoint domains that are convex in the direction of the real axis.*

PROOF. (1) ⇒ (2). Assume that $f(\mathbb{D}) = D_1 \cup D_2$, where $D_1, D_2 \subset \mathbb{C}$ are domains that are convex in the direction of the real axis with a nonempty intersection. Since f is one-to-one in the unit disk, there exists $f^{-1} : D_1 \cup D_2 \to \mathbb{D}$ and the composition $q := (h - g) \circ f^{-1}$ is a well-defined continuous function on $D_1 \cup D_2$. Moreover,

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$q(w) = (h-g)(f^{-1}(w)) = w - 2\,\mathrm{Re}\, g(f^{-1}(w))$ for all $w \in D_1 \cup D_2$. Thus q satisfies the assumptions of Theorem 2.5. Additionally, by $\Lambda((h-g)(\mathbb{D})) = \Lambda(f(\mathbb{D}))$,

$$\Lambda(D_1 \cup D_2) = \Lambda(q(D_1) \cup q(D_2)) \tag{3.1}$$

and, in consequence, q is one-to-one in $\mathbb{D}$, by Theorem 2.5. Hence $h - g$ is one-to-one in $\mathbb{D}$, since f is. Additionally, both sets $q(D_1)$ and $q(D_2)$ are domains that are convex in the direction of the real axis, by Lemma 2.1, and their intersection is nonempty by (3.1) and Theorem 2.5.

The proof of (2) $\Rightarrow$ (1) is essentially the same as that of (1) $\Rightarrow$ (2). □

As a consequence of Theorem 3.1, we obtain a generalisation of Theorem 1.1 of Clunie and Sheil-Small.

THEOREM 3.2. *Let $f = h + \overline{g}$ be a harmonic function in $\mathbb{D}$ such that $J_f > 0$ in $\mathbb{D}$. If $(h-g)(\mathbb{D})$ and $f(\mathbb{D})$ are nonempty simply connected domains, then the following statements are equivalent:*

(1) *f is a one-to-one mapping and $f(\mathbb{D})$ is a union of two nondisjoint domains that are convex in the direction of the real axis;*
(2) *$h - g$ is a one-to-one analytic mapping and $(h-g)(\mathbb{D})$ is a union of two nondisjoint domains that are convex in the direction of the real axis.*

PROOF. (1) $\Rightarrow$ (2). Assume that $f(\mathbb{D}) = D_1 \cup D_2$, where $D_1, D_2 \subset \mathbb{C}$ are domains that are convex in the direction of the real axis with a nonempty intersection. Then the function

$$D_1 \cup D_2 \ni w \mapsto q(w) := (h-g)(f^{-1}(w)) = w - 2\,\mathrm{Re}\, g(f^{-1}(w))$$

is well defined and continuous in $D_1 \cup D_2$ since f is one-to-one in $\mathbb{D}$. Since $(h-g)(\mathbb{D})$ and $f(\mathbb{D})$ are nonempty simply connected domains, the desired theorem follows from Theorems 2.6 and 3.1.

The proof of (2) $\Rightarrow$ (1) is essentially the same as that of (1) $\Rightarrow$ (2). □

REMARK 3.3. Suppose that $S_H^0(C)$ denotes the class of all those normalised sense-preserving harmonic functions $f = h + \overline{g}$ that are defined on the unit disk $\mathbb{D}$, which can be proved to be univalent in $\mathbb{D}$ using Theorem 3.2. Now consider the subclass $S_H^0(S)$ defined in [17] by

$$S_H^0(S) := \{h + \overline{g} \in S_H^0 : h + e^{i\theta} g \in S \text{ for some } \theta \in \mathbb{R}\},$$

where S denotes the class of normalised univalent analytic functions defined on $\mathbb{D}$ and S_H^0 denotes the class of all normalised sense-preserving harmonic mappings on $\mathbb{D}$, introduced in [1]. A simple observation shows that $S_H^0(C) \subset S_H^0(S)$. Hence the coefficient conjecture of Clunie and Sheil-Small holds true for the class $S_H^0(C)$. Moreover, the growth theorem, covering theorem, lower bounds and upper bounds of $J_f(z)$, $|h'(z)|$, $|g'(z)|$, that were proved in [17] for the class $S_H^0(S)$, remain true for the functions in the class $S_H^0(C)$. Very recently, criteria for functions belonging to the class $S_H^0(S)$ have also been established in [18].

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If, in Theorem 3.2, one omits the assumption that both $f(\mathbb{D})$ and $(h-g)(\mathbb{D})$ are simply connected, then the theorem is no longer true, as shown in the following example.

EXAMPLE 3.4. Consider a horizontal shear of the rotated Koebe function with the dilatation equal to iz. From the equations

$$h(z) - g(z) = \frac{z}{(1-iz)^2},$$
$$g'(z) = izh'(z),$$

we get

$$h(z) = \frac{-6iz - 3z^2 + iz^3}{6(i+z)^3}, \quad g(z) = \frac{3z^2 + iz^3}{6(i+z)^3},$$

and

$$f(z) = h(z) + \overline{g(z)} = \frac{-6iz - 3z^2 + iz^3}{6(i+z)^3} + \overline{\left(\frac{3z^2 + iz^3}{6(i+z)^3}\right)}.$$

Now, using the transformation

$$w = u + iv := \frac{1+iz}{1-iz},$$

which maps the unit disk onto the right half-plane, $\{w \in \mathbb{C} : \operatorname{Re} w > 0\}$, we get

$$h(z) - g(z) = \frac{1}{4i}(w^2 - 1), \quad h(z) + g(z) = \frac{1}{6i}(w^3 - 1).$$

Consequently,

$$f(z) = \operatorname{Re}(h(z) + g(z)) + i \operatorname{Im}(h(z) - g(z)) = \frac{1}{6} \operatorname{Im}(w^3 - 1) - \frac{i}{4} \operatorname{Re}(w^2 - 1).$$

After some calculations,

$$f(z) = \frac{1}{6} v(3u^2 - v^2) - \frac{i}{4}(u^2 - v^2 - 1), \tag{3.2}$$

where $u > 0$ and $v \in \mathbb{R}$.

Clearly, the function $h - g$ maps the unit disk onto the plane with the slit along the imaginary axis: more precisely, onto $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z \geq \frac{1}{4} \text{ and } \operatorname{Re} z = 0\}$, which is a simply connected domain. On the other hand, the formula (3.2), allows us to find the image of the unit disk via the function f, by studying which parts of the vertical lines of the complex plane belong to $f(\mathbb{D})$. First, observe that $\operatorname{Re} f(z) = 0$ if and only if $v = 0$ or $v^2 = 3u^2$. Thus if $\operatorname{Re} f(z) = 0$, either $\operatorname{Im} f(z) = (1 - u^2)/4$ with $u > 0$ (if $v = 0$), or $\operatorname{Im} f(z) = (1 + 2u^2)/4$ with $u > 0$ (if $v^2 = 3u^2$). Consequently, the point $i/4$ does not belong to $f(\mathbb{D})$.

Now assume that $\operatorname{Re} f(z) = c$ with $c \neq 0$. Since $v \neq 0$, $u^2 = (v^3 + 6c)/3v$ and

$$\operatorname{Im} f(z) = \frac{2v^3 + 3v - 6c}{12v} \quad \text{where } v \in (-\infty, 0) \cup (0, +\infty).$$

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If $c < 0$ and $v \in (-\infty, 0)$, then

$$\lim_{v\to-\infty} \frac{2v^3 + 3v - 6c}{12v} = +\infty, \quad \lim_{v\to 0^-} \frac{2v^3 + 3v - 6c}{12v} = -\infty,$$

and the whole vertical line $w = c$ belongs to $f(\mathbb{D})$. If $c > 0$ and $v \in (0, +\infty)$, then

$$\lim_{v\to+\infty} \frac{2v^3 + 3v - 6c}{12v} = +\infty, \quad \lim_{v\to 0^+} \frac{2v^3 + 3v - 6c}{12v} = -\infty,$$

and, again, the whole vertical line $w = c$ belongs to $f(\mathbb{D})$. Hence $f(\mathbb{D}) = \mathbb{C} \setminus \{i/4\}$, which is not a simply connected domain. The function f fails to satisfy the assumptions of Theorem 3.1 and straightforward calculations show that $f(1/\sqrt{3}) = f(-1/\sqrt{3}) = (3i/8)$. Thus f is not univalent in $\mathbb{D}$.

Remark 3.5. Recall that Theorem 1.1 can be reformulated and it remains valid for a function that is convex in any fixed direction. Our results can also be rewritten in this fashion.

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