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FINITE p-SOLVABLE GROUPS WITH THREE p-REGULAR CONJUGACY CLASS SIZES

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Abstract Let G be a finite p-solvable group. We describe the structure of the p-complements of G when the set of p-regular conjugacy classes has exactly three class sizes. For instance, when the set of p-regular class sizes of G is $\{1, p^a, p^a m\}$ or $\{1, m, p^a m\}$ with $(m, p) = 1$, then we show that $m = q^b$ for some prime q and the structure of the p-complements of G is determined.

Keywords: finite p-solvable groups; conjugacy class sizes; p-regular elements

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1. Introduction

It is known that the structure of a finite group is strongly related to its set of conjugacy class sizes. In particular, in some papers it has been proved that certain properties of the sizes of p-regular conjugacy classes also affect the p-structure of G. In [**1**], Alemany *et al.* proved that if the set of conjugacy class sizes of p'-elements of a finite group G is $\{1, m\}$, then p-complements of G are nilpotent. In [**3**], the structure of the p-complements of a p-solvable group G has been described for the case in which the set of p-regular conjugacy class sizes of G is $\{1, m, n\}$ for arbitrary coprime integers $m, n > 1$. In fact, it is shown that G is solvable and the p-complements of G are quasi-Frobenius groups in which the inverse image of the kernel and complement are abelian. Also, in [**7**], it is proved that, if the set of conjugacy class sizes of all p'-elements of a finite p-solvable group G is $\{1, m, p^a, mp^a\}$, where m is a positive number not divisible by p, then m is a prime power and, furthermore, the p-complements of G are nilpotent.

We note that studying the p-structure of a finite group G from its set of p-regular conjugacy class sizes may be more difficult, even if one considers the p-solvability of G, since if H is a p-complement of G and $x \in H$, then $|x^H|$ does not divide $|x^G|$ in general. Furthermore, we are handicapped by the fact that there is no information on the elements whose order is divisible by p.

In this paper, we study the structure of the p-complements of a p-solvable group G, with $\{1, p^a, mp^a\}$ or $\{1, m, mp^a\}$ as the set of conjugacy class sizes of p'-elements of G, where $(p, m) = 1$. In fact, we prove the following two theorems, which are extensions for p-solvable groups of Itô's Theorem on groups with two class sizes (see, for example, [**9**, 33.6]).

Theorem A. *Let G be a p-solvable group with $\{1, m, p^a m\}$ as the set of conjugacy class sizes of p'-elements, where $(p, m) = 1$. Then $m = q^b$ for some prime q, and any p-complement H of G satisfies $H = Q \times K$ with Q a Sylow q-subgroup and K abelian.*

Theorem B. *Let G be a p-solvable group. If the set of conjugacy class sizes of p'-elements of G is $\{1, p^a, p^a m\}$, with $(p, m) = 1$, then $m = q^b$ for some prime q and some integer $b \geqslant 0$, and every p-complement H of G is either*

(i) *$H = Q \times K$, with Q a Sylow q-subgroup and K abelian, or*

(ii) *$H = QK$, with Q a normal abelian Sylow q-subgroup, K abelian and $Q\boldsymbol{O}_p(G) \trianglelefteq G$.*

Notice that the solvability of G is an easy consequence of both Theorem A and Theorem B. We also note that the methods we employ for proving Theorems A and B are quite different. In the proof of Theorem A we use the classification of the finite $\mathfrak{M}$-groups due to Schmidt, that is, those non-abelian groups in which all centralizers of non-central elements are abelian. In the proof of Theorem B a more detailed analysis is required.

We remark that the information obtained on the p-structure of a group G from its set of p-regular class sizes has important applications when studying the conjugacy class sizes of G in the ordinary case (see, for example, [**5**] and [**6**], in which the information is used to obtain the solvability or nilpotency of certain groups) and, as a consequence, in determining the structure of G.

Throughout this paper all groups are finite. If x is any element of a group G, we denote by x^G the conjugacy class of x in G and $|x^G|$ is called the conjugacy class size of x and also the index of x in G. If p is a prime number and n is an integer, then we use the notation n_p for the p-part of n, i.e. $n_p = p^\alpha$, where p^α divides n and $p^{\alpha+1}$ does not divide n. We will denote the set of p'-elements of G by $G_{p'}$ and the set of conjugacy classes of p'-elements of G by $\mathrm{cs}_{p'}(G)$. All further unexplained notation is standard.

2. Preliminary results

We will need some results on conjugacy class sizes of p-regular elements and of π-elements for a suitable set of primes π.

Lemma 2.1. *Let G be a finite group. All the conjugacy class sizes in $G_{p'}$ are then p-numbers if and only if G has abelian p-complements.*

Proof. See Lemma 2 of [**4**]. □

Lemma 2.2. *Suppose that G is a finite group and that p is not a divisor of the sizes of p-regular conjugacy classes. Then $G = P \times H$, where P is a Sylow p-subgroup and H is a p-complement of G.*

Proof. This is exactly Lemma 1 of [**8**]. □

Lemma 2.3. *Let x and y be a q-element and a q'-element, respectively, of a group G, such that $\boldsymbol{C}_G(x) \subseteq \boldsymbol{C}_G(y)$. Then $\boldsymbol{O}_q(G) \subseteq \boldsymbol{C}_G(y)$.*

Proof. It is enough to apply Thompson's $P \times Q$-Lemma [**11**, 8.2.8] to the action of $\langle x \rangle \times \langle y \rangle$ on $\boldsymbol{O}_q(G)$. □

Lemma 2.4. *Let G be a π-separable group. If $x \in G$ with $|x^G|$ a π-number, then $x \in \boldsymbol{O}_{\pi\pi'}(G)$.*

Proof. See Theorem C of [**2**]. □

The following result is an extension for p-regular elements of Itô's Theorem on groups having two class sizes.

Theorem 2.5. *Let G be a finite group. If the set of p-regular conjugacy class sizes of G is exactly $\{1, m\}$, then $m = p^a q^b$, with q a prime distinct from p and $a, b \geqslant 0$. If $b = 0$, then G has an abelian p-complement. If $b \neq 0$, then $G = PQ \times A$, with $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $A \subseteq \boldsymbol{Z}(G)$. Furthermore, if $a = 0$, then $G = P \times Q \times A$.*

Proof. This is Theorem A of [**2**]. □

Theorem 2.6. *Let G be a finite p-solvable group and let $\pi = \{p, q\}$ with q and p two distinct primes. Suppose that the sizes of the conjugacy classes of $G_{p'}$ are π-numbers. Then G is solvable, it has abelian π-complements and every p-complement of G has a normal Sylow q-subgroup.*

Proof. This is Theorem 5 of [**7**]. □

The following result extends Theorem 6 of [**7**] with an easier proof.

Theorem 2.7. *Let G be a finite p-solvable group and let $\pi = \{p, q\}$ with q and p two distinct primes. Suppose that the sizes of p-regular classes in G are π-numbers. Let q^b be the highest power of the prime q that divides the sizes of classes of p-regular elements in G. Suppose that there exists some q-element $x \in G$ such that $|x^G| = p^a q^b$, where $a, b \geqslant 0$. Then G has nilpotent p-complements and they have abelian Sylow subgroups for all primes distinct from q.*

Proof. We know that G is solvable by Theorem 2.6. Let K be a π-complement of G such that $K \subseteq \boldsymbol{C}_G(x)$. Notice that K is abelian by Theorem 2.6, and furthermore it can be assumed to be non-central in G, otherwise the result is trivial. Let y be any element in K and observe that $K \subseteq \boldsymbol{C}_G(xy) = \boldsymbol{C}_G(x) \cap \boldsymbol{C}_G(y) \subseteq \boldsymbol{C}_G(x)$. Notice that the index of xy in $\boldsymbol{C}_G(x)$ is a q'-number, and consequently a p-number. If we choose KQ_0 to be a p-complement of $\boldsymbol{C}_G(xy)$, then it is also a p-complement of $\boldsymbol{C}_G(x)$. Note that $x \in KQ_0$. Now, let H be a p-complement of G such that $KQ_0 \subseteq H$. We can then write $H = KQ$ with Q a Sylow q-subgroup of G that is normal in H by Theorem 2.6. Therefore, $Q_0 \subseteq Q$ and

$$\boldsymbol{C}_Q(xy) = Q \cap \boldsymbol{C}_G(xy) = Q_0 \quad \text{and} \quad \boldsymbol{C}_Q(x) = Q \cap \boldsymbol{C}_G(x) = Q_0.$$

Then $\boldsymbol{C}_Q(x) = \boldsymbol{C}_Q(xy) \subseteq \boldsymbol{C}_Q(y)$, so we can apply Thompson's Lemma to get $Q \subseteq \boldsymbol{C}_G(y)$ for all $y \in K$. Consequently, $H = Q \times K$ as desired. □

Theorem 2.8. *Let G be a finite group and let π be a set of primes. Suppose that the conjugacy class size of every π-element of G is a power of p for some fixed prime $p \notin \pi$. Then G has an abelian Hall π-subgroup H and $H\boldsymbol{O}_p(G) \trianglelefteq G$.*

Proof. This is part (a) of Theorem A of [**4**]. □

We use the above theorem to give a simplified proof of Theorem 2.9, which is moreover an extension of Theorem 7 of [**7**].

Theorem 2.9. *Let G be a p-solvable group whose conjugacy class sizes of p'-elements are $\{1, p^{a_1}, \dots, p^{a_r}, p^{c_1}q^b, \dots, p^{c_s}q^b\}$, where q is a prime distinct from p and $c_i \geqslant 0$, $b, a_i \geqslant 0$ for all i. Then any p-complement H of G is either*

(i) *$H = Q \times K$, with Q a Sylow q-subgroup and K abelian, or*

(ii) *$H = QK$, with Q a Sylow q-subgroup, Q and K both abelian, $Q \trianglelefteq H$ and $Q\boldsymbol{O}_p(G) \trianglelefteq G$.*

Proof. If there exists a q-element of index $q^b p^{c_i}$ for some i, then case (i) follows by Theorem 2.7. Otherwise, the index of every q-element is a p-number. So, by Theorem 2.8, G has an abelian Sylow q-subgroup Q and $Q\boldsymbol{O}_p(G) \trianglelefteq G$ and thus, if H is a p-complement of G, containing Q, then $Q \trianglelefteq H$. Also by Theorem 2.6 we get that G is solvable and it has an abelian $\{p, q\}$-complement. So we have case (ii). □

Corollary 2.10. *Let G be a p-solvable group.*

(a) *If $\mathrm{cs}_{p'}(G) = \{1, q^b, p^a q^b\}$, where q is a prime distinct from p, then any p-complement H of G satisfies $H = Q \times K$ with Q a Sylow q-subgroup and K abelian.*

(b) *If $\mathrm{cs}_{p'}(G) = \{1, p^a, p^a q^b\}$, where q is a prime distinct from p, then every p-complement H of G, is either*

  (i) *$H = Q \times K$, with Q a Sylow q-subgroup and K abelian, or*

  (ii) *$H = QK$, with Q a normal abelian Sylow q-subgroup, K abelian and $Q\boldsymbol{O}_p(G) \trianglelefteq G$.*

Proof. Case (a) is an immediate consequence of Theorem 2.7 and case (b) is a consequence of Theorem 2.9. □

3. Groups with three p-regular class sizes

In order to prove Theorems A and B, it is sufficient to show that when the set of p-regular class sizes of a group G is $\{1, p^a, p^a m\}$ or $\{1, m, p^a m\}$, with $(m, p) = 1$, then m is a prime power, q^b. We shall prove this in Theorems 3.1 and 3.2. The main results then follow by Corollary 2.10. Note that if $a = 0$, then G has two p-regular class sizes, so Theorems A and B are immediate consequences of Theorem 2.5. Also note that if $b = 0$, then G has abelian p-complements by Lemma 2.1.

Theorem 3.1. *If G is a p-solvable group such that $\mathrm{cs}_{p'}(G) = \{1, m, p^a m\}$ with $(p, m) = 1$, then $m = q^b$ for some prime q.*

Proof. Take H to be a p-complement of G. We prove the theorem in the following steps.

Step 1. For every non-central p-regular element x of G, we may assume that there exist at least two primes q and r, distinct from p, such that r and q divide $|\boldsymbol{C}_G(x)|/|\boldsymbol{Z}(G)|$.

Suppose that there exists a non-central p-regular element $x \in G$ such that, for some prime $q \neq p$, $|\boldsymbol{C}_G(x)|_{p'}/|\boldsymbol{Z}(G)|_{p'}$ is a q-number. On the other hand, we may assume that there exists a non-central r-element y in G, with r distinct from p and q, since otherwise G is the direct product of a $\{p, q\}$-group and a central factor, and so the result follows. Therefore, r is a divisor of $|\boldsymbol{C}_G(y)|_{p'}/|\boldsymbol{Z}(G)|_{p'}$. Note that $|\boldsymbol{C}_G(x)|_{p'} = |G|_{p'}/m = |\boldsymbol{C}_G(y)|_{p'}$, and so we get a contradiction.

Step 2. If x is a non-central element of H such that $|x^G| = m$, then $|x^H| = m$. Moreover, $\boldsymbol{C}_H(x) = T_x Q_x$, with Q_x a Sylow q-subgroup of $\boldsymbol{C}_H(x)$, where q is a prime divisor of the order of x, and T_x a normal abelian q'-subgroup of $\boldsymbol{C}_H(x)$. Furthermore, $\boldsymbol{C}_H(x)$ is a p-complement of $\boldsymbol{C}_G(x)$.

Let x be a non-central element of H with $|x^G| = m$. So $G = H\boldsymbol{C}_G(x)$, and consequently $|H : \boldsymbol{C}_H(x)| = |G : \boldsymbol{C}_G(x)| = m$, as desired. Also, $|\boldsymbol{C}_G(x) : \boldsymbol{C}_H(x)| = |G : H|$ implies that $\boldsymbol{C}_H(x)$ is a p-complement of $\boldsymbol{C}_G(x)$. By the minimality of the class size of x, we can certainly assume that x is a q-element for some prime q distinct from p. Let y be a non-central $\{p, q\}'$-element in $\boldsymbol{C}_G(x)$ (note that by Step 1 such elements exist). Then $\boldsymbol{C}_G(xy) = \boldsymbol{C}_G(x) \cap \boldsymbol{C}_G(y) \subseteq \boldsymbol{C}_G(x)$, and so by the hypotheses, y has index 1 or p^a in $\boldsymbol{C}_G(x)$. Now, we apply Theorem 2.9, so $\boldsymbol{C}_G(x)$ has abelian $\{p, q\}$-complements, and $T_x \boldsymbol{O}_p(\boldsymbol{C}_G(x)) \trianglelefteq \boldsymbol{C}_G(x)$, for every $\{p, q\}$-complement T_x of $\boldsymbol{C}_G(x)$. Since $\boldsymbol{C}_H(x)$ is a p-complement of $\boldsymbol{C}_G(x)$, we may assume that $T_x \subseteq \boldsymbol{C}_H(x)$, and, as a consequence, $T_x \trianglelefteq \boldsymbol{C}_H(x)$. So we get the result.

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Step 3. H is an $\mathfrak{M}$-group.

Let $x \in H$ be a non-central element. We distinguish two possibilities for the index of x in G.

First suppose that $|x^G| = m$. We may assume that x is a q-element for some prime q. By Step 2 we have $\boldsymbol{C}_H(x) = T_x Q_x$, where T_x is the normal abelian q-complement of $\boldsymbol{C}_H(x)$ and Q_x is a Sylow q-subgroup of $\boldsymbol{C}_H(x)$. Let $y \in T_x$ be a non-central r-element for some prime $r \neq q$.

Assume first that $|y^G| = m$. Then $|\boldsymbol{C}_H(x)| = |\boldsymbol{C}_H(y)|$ and $\boldsymbol{C}_H(y) = L_y R_y$, where L_y is the normal abelian r-complement of $\boldsymbol{C}_H(y)$ and R_y is a Sylow r-subgroup of $\boldsymbol{C}_H(y)$, by Step 2. So $Q_y \subseteq L_y$ is the normal abelian Sylow q-subgroup of $\boldsymbol{C}_H(y)$. Therefore, $x \in Q_y$ and hence $Q_y \subseteq \boldsymbol{C}_H(x) \cap \boldsymbol{C}_H(y)$. Also, since T_x is abelian, $T_x \subseteq \boldsymbol{C}_H(x) \cap \boldsymbol{C}_H(y)$, which implies that $\boldsymbol{C}_H(x) = \boldsymbol{C}_H(y) = Q_y \times T_x$, and we deduce that $\boldsymbol{C}_H(x)$ is abelian.

Now we assume that $|y^G| = p^a m$. Then by the minimality of $|\boldsymbol{C}_G(y)|$ we have $\boldsymbol{C}_G(y) = P_y R_y \times K_y$, where P_y and R_y are some Sylow p-subgroup and r-subgroup of $\boldsymbol{C}_G(y)$, respectively, and K_y is abelian. On the other hand, since $\boldsymbol{C}_G(xy) = \boldsymbol{C}_G(x) \cap \boldsymbol{C}_G(y) \subseteq \boldsymbol{C}_G(y)$, the minimality of $|\boldsymbol{C}_G(y)|$ implies that $\boldsymbol{C}_G(xy) = \boldsymbol{C}_G(y) \subseteq \boldsymbol{C}_G(x)$. So we may assume that $R_y \subseteq T_x$ and, consequently, R_y is abelian. Hence, $R_y \times K_y$ is an abelian p-complement of $\boldsymbol{C}_G(y)$. Also $\boldsymbol{C}_G(y) \subseteq \boldsymbol{C}_G(x)$, whence every p-complement of $\boldsymbol{C}_G(y)$ is a p-complement of $\boldsymbol{C}_G(x)$. Now, from the fact that $\boldsymbol{C}_H(x)$ is a p-complement of $\boldsymbol{C}_G(x)$ we get that $\boldsymbol{C}_H(x)$ is abelian.

Suppose that $|x^G| = p^a m$. First we assume that there exists some non-central p-regular element α such that $\boldsymbol{C}_G(x) \subsetneq \boldsymbol{C}_G(\alpha)$. So $|\alpha^G| = m$. Since $\boldsymbol{C}_H(x)$ is a p'-subgroup of $\boldsymbol{C}_G(\alpha)$, there exists $g \in G$ such that $\boldsymbol{C}_H(x) \subseteq \boldsymbol{C}_{H^g}(\alpha)$, where $\boldsymbol{C}_{H^g}(\alpha)$ is a p-complement of $\boldsymbol{C}_G(\alpha)$. By the above argument, $\boldsymbol{C}_{H^g}(\alpha)$ is abelian, whence $\boldsymbol{C}_H(x)$ is abelian too.

Suppose that there exists no non-central p'-element α such that $\boldsymbol{C}_G(x) \subsetneq \boldsymbol{C}_G(\alpha)$. Hence, we may certainly assume that x is an r-element for some prime $r \neq p$. So we can write $\boldsymbol{C}_G(x) = P_x R_x \times K_x$, where P_x and R_x are some Sylow p-subgroup and r-subgroup of $\boldsymbol{C}_G(x)$ and K_x is abelian. By Step 1 there exists a non-central q-element $w \in K_x$ for some prime $q \notin \{p, r\}$. Thus, $\boldsymbol{C}_G(wx) = \boldsymbol{C}_G(x) \cap \boldsymbol{C}_G(w) = \boldsymbol{C}_G(x) \subseteq \boldsymbol{C}_G(w)$, which implies that $\boldsymbol{C}_G(x) = \boldsymbol{C}_G(w)$, by the hypotheses. On the other hand, we have $\boldsymbol{C}_G(w) = P_w Q_w \times L_w$, where P_w and Q_w are some Sylow p-subgroup and q-subgroup of $\boldsymbol{C}_G(w)$ and L_w is abelian. So $R_w \subseteq L_w$ is the normal abelian Sylow r-subgroup of $\boldsymbol{C}_G(x)$. Hence, $\boldsymbol{C}_G(x)$ has abelian p-complements. Consequently, every p'-subgroup of $\boldsymbol{C}_G(x)$ is abelian and, in particular, $\boldsymbol{C}_H(x)$ is abelian too.

Step 4. For every non-central element $x \in H$, m is a divisor of $|x^H|$. In particular, if H is a normal subgroup of G, then the theorem follows.

Let $x \in H$ be a non-central element. Since $\boldsymbol{C}_H(x)$ is a p'-subgroup of $\boldsymbol{C}_G(x)$, there exists $g \in G$ such that $\boldsymbol{C}_H(x) \subseteq \boldsymbol{C}_{H^g}(x)$, where $\boldsymbol{C}_{H^g}(x)$ is a p-complement of $\boldsymbol{C}_G(x)$. On the other hand, $m = |G : \boldsymbol{C}_G(x)|_{p'} = |H^g : \boldsymbol{C}_{H^g}(x)|$. Therefore, m divides $|x^H|$, as desired. If H is a normal subgroup of G, then $|x^H|$ divides $|x^G|$ for every non-central element $x \in H$, and by the fact that m is a divisor of $|x^H|$ we get that $\mathrm{cs}(H) = \{1, m\}$; thus, by applying Ito's Theorem on groups with two class sizes (see [**9**, 33.6]), we obtain that m is a prime power.

Step 5. Conclusion.

As we proved in Step 3, H is an $\mathfrak{M}$-group. So by applying the classification of finite $\mathfrak{M}$-groups (see [**13**, Theorem 9.3.12]), we have the following possibilities, each of which leads either to the fact that m is a prime power or to a contradiction.

- Assume that $H = A \times Q$, where A is abelian and Q is a q-group, for some prime q. Let x be an element in $Q \setminus \boldsymbol{Z}(H)$. So $|x^G|$ is a $\{p, q\}$-number, whence m is a q-power, as desired.

- Assume that H is non-abelian and has a normal abelian subgroup N of index q for some prime q distinct from p. Notice that $N \not\subseteq \boldsymbol{Z}(H)$, and hence, if $x \in N \setminus \boldsymbol{Z}(H)$, then $|x^G|$ is a $\{p, q\}$-number, whence m is a q-power.

- Suppose that $H/\boldsymbol{Z}(H)$ is a Frobenius group, with Frobenius kernel $K/\boldsymbol{Z}(H)$ and Frobenius complement $L/\boldsymbol{Z}(H)$, where K and L are abelian. It follows that $\mathrm{cs}(H) = \{1, |K/\boldsymbol{Z}(H)|, |L/\boldsymbol{Z}(H)|\}$, and so, by Step 4, $m = 1$, the theorem is trivially true.

- Suppose that $H/\boldsymbol{Z}(H)$ is a Frobenius group, with Frobenius kernel $K/\boldsymbol{Z}(H)$ and Frobenius complement $L/\boldsymbol{Z}(H)$, where K is abelian and $L/\boldsymbol{Z}(H)$ is a q-group for some prime q. It is easy to see that $\mathrm{cs}(H) = \{1, |L/\boldsymbol{Z}(H)|, |K/\boldsymbol{Z}(H)|\,|x^L| \colon x \in L \setminus \boldsymbol{Z}(H)\}$, and so, by Step 4, m is a q-power.

- Assume that $H/\boldsymbol{Z}(H) \cong S_4$, and $V/\boldsymbol{Z}(H)$ is the Klein 4-group, where V is non-abelian. Then for every $x \in H \setminus \boldsymbol{Z}(H)$ we have that $\boldsymbol{C}_H(x)/\boldsymbol{Z}(H)$ is a maximal cyclic subgroup of $H/\boldsymbol{Z}(H)$ (see [**13**, p. 521]). One can then easily obtain $\mathrm{cs}(H) = \{1, 6, 8, 12\}$, whence $m = 2$ by Step 4.

- Let $H/\boldsymbol{Z}(H) \cong \mathrm{PSL}(2, q^h)$ for some prime q. Note that $\boldsymbol{Z}(H) = \boldsymbol{Z}(G)_{p'}$. By Lemma 2.4, if x is a p-regular element in G such that $|x^G| = m$, then $x \in \boldsymbol{O}_{p'}(G)$. Therefore, $\boldsymbol{O}_{p'}(G)/\boldsymbol{Z}(H)$ is a non-trivial normal subgroup of $\mathrm{PSL}(2, q^h)$, so $H = \boldsymbol{O}_{p'}(G)$ is a normal subgroup of G, and the result follows by Step 4.

- Finally, assume that $H/\boldsymbol{Z}(H) \cong \mathrm{PGL}(2, q^h)$ for some prime q. Note that $\boldsymbol{Z}(H) = \boldsymbol{Z}(G)_{p'}$. Since $\boldsymbol{O}_{p'}(G)$ can be assumed to be a proper non-central subgroup of H, we deduce that $\boldsymbol{O}_{p'}(G)/\boldsymbol{Z}(H) \cong \mathrm{PSL}(2, q^h)$. Therefore, any class size of $\mathrm{PSL}(2, q^h)$ divides a p-regular class size of G and, consequently, their least common multiple, which is $|\mathrm{PSL}(2, q^h)|$, divides m. Let $x \in H \setminus \boldsymbol{Z}(G)$, such that $|x^G| = m$; we then have $|x^H| = m$. Since $|\mathrm{PSL}(2, q^h)| = |\boldsymbol{O}_{p'}(G)|/|\boldsymbol{Z}(H)|$ divides m, there exists an integer t such that $|\boldsymbol{O}_{p'}(G) : \boldsymbol{Z}(H)|t = m = |H : \boldsymbol{C}_H(x)|$. This implies that $|\boldsymbol{C}_H(x) : \boldsymbol{Z}(H)|t = |H : \boldsymbol{O}_{p'}(G)| = 2$, and so $t = 1$. Therefore, $|H/\boldsymbol{Z}(H)| = |H/\boldsymbol{O}_{p'}(G)|\,|\boldsymbol{O}_{p'}(G)/\boldsymbol{Z}(H)| = 2m$. Let y be a non-central element in H. There exists $g \in G$ such that $\boldsymbol{C}_H(y) \subseteq \boldsymbol{C}_{H^g}(y)$, where $\boldsymbol{C}_{H^g}(y)$ is a p-complement of $\boldsymbol{C}_G(y)$, so $m = |G : \boldsymbol{C}_G(y)|_{p'} = |H^g : \boldsymbol{C}_{H^g}(y)|$. Taking into account that $\boldsymbol{Z}(H) \subset \boldsymbol{C}_H(y) \subseteq \boldsymbol{C}_{H^g}(y) \subseteq H^g$ and $2m = |H^g/\boldsymbol{Z}(H)|$, it follows that $\boldsymbol{C}_H(y) = \boldsymbol{C}_{H^g}(y)$, so $m = |H^g : \boldsymbol{C}_H(y)| = |H : \boldsymbol{C}_H(y)|$ for every $y \in H \setminus \boldsymbol{Z}(H)$. By Ito's Theorem on groups with two class sizes, m is a prime power, which contradicts the fact that $|\mathrm{PSL}(2, q^h)|$ divides m. □

Theorem 3.2. *Let G be a p-solvable group and suppose that $\mathrm{cs}_{p'}(G) = \{1, p^a, p^a m\}$ with $(p, m) = 1$. Then $m = q^b$ for some prime q.*

Proof. We will proceed by minimal counterexample to prove that m is a prime power. Let G be a group of minimal order satisfying the hypotheses and such that m is not a prime power. Notice that if w is a p'-element of index p^a, then by minimality of its index, w certainly can be assumed to be a q-element for some prime $q \neq p$. For the rest of the proof, we will fix the prime q and a q-element w of index p^a. Let H be a p-complement of G such that $H \subseteq \boldsymbol{C}_G(w)$.

Step 1. If $y \in H$ is a q'-element, then $|y^H| = 1$ or m. As a consequence, $H = QR \times A$, where Q and R are Sylow q- and r-subgroups of H, respectively, and A is abelian, and $m = q^b r^c$ with $b, c > 0$ for some prime $r \neq p, q$.

Let y be any q'-element of H. Then $\boldsymbol{C}_G(wy) = \boldsymbol{C}_G(w) \cap \boldsymbol{C}_G(y) \subseteq \boldsymbol{C}_G(w)$, so by the hypotheses y may have index 1 or m in $\boldsymbol{C}_G(w)$. Now, since $\boldsymbol{C}_G(w) = H\boldsymbol{C}_G(wy)$ and $\boldsymbol{C}_H(wy) = \boldsymbol{C}_H(y)$, it follows that $|H : \boldsymbol{C}_H(y)| = |\boldsymbol{C}_G(w) : \boldsymbol{C}_G(wy)| = 1$ or m. If every q'-element of H has index 1 in H, then H has a central q-complement. Therefore, every element of H is centralized by a $\{p, q\}$-complement of G, so its index is a $\{p, q\}$-number and m would be a power of q, a contradiction. Therefore, both numbers, 1 and m, appear as indexes of q'-elements in H, so we can apply Theorem 2.5. Since we have assumed that m is not a prime power, this completes the step.

Step 2. If x is an s-element for any prime $s \neq q, p$ and y is a q-element such that both x and y have index p^a, then $\boldsymbol{C}_G(x) = \boldsymbol{C}_G(y)^g$ for some $g \in G$.

Let H_1 be a p-complement of G contained in $\boldsymbol{C}_G(y)$. It is clear that there exists some $g \in G$ such that $H_1^g \subseteq \boldsymbol{C}_G(x)$. Then $y^g \in \boldsymbol{C}_G(x)$ and, clearly, $|\boldsymbol{C}_G(x) : \boldsymbol{C}_G(y^g x)|$ must be equal to 1 or m. As m is a p' number, we can take P_x to be a Sylow p-subgroup of $\boldsymbol{C}_G(x)$ such that $P_x \subseteq \boldsymbol{C}_G(y^g x)$. In particular, we have $P_x \subseteq \boldsymbol{C}_G(y^g)$. By considering the orders, P_x is a Sylow p-subgroup of $\boldsymbol{C}_G(y^g)$ and thus $\boldsymbol{C}_G(y^g) = H_1^g P_x = \boldsymbol{C}_G(x)$, as desired.

Step 3. Every s-element of G has index 1 or $p^a m$ in G for any prime $s \neq p, q$. Also, for every s-element x, we have $\boldsymbol{C}_G(x) = P_x S_x \times T_x$, where P_x and S_x are a Sylow p-subgroup and a Sylow s-subgroup of $\boldsymbol{C}_G(x)$, respectively, and T_x is abelian.

Suppose that ρ is a non-central s-element such that $|\rho^G| = p^a$. Then by the last step we have $\boldsymbol{C}_G(w) = \boldsymbol{C}_G(\rho)^g$ for some $g \in G$.

Let $z \in H$ be an element of prime power order. If $(o(z), q) = 1$, then, by Step 1, we conclude that the index of z in H is 1 or m. Let z be a q-element. Since $z \in \boldsymbol{C}_G(w) = \boldsymbol{C}_G(\rho)^g$, we conclude that $\boldsymbol{C}_G(z\rho^g) = \boldsymbol{C}_G(z) \cap \boldsymbol{C}_G(\rho^g) \subseteq \boldsymbol{C}_G(\rho^g)$. Therefore, $|\boldsymbol{C}_G(\rho^g) : \boldsymbol{C}_G(z\rho^g)| = 1$ or m, and as a consequence $\boldsymbol{C}_G(\rho^g) = H\boldsymbol{C}_G(z\rho^g)$. Now it is easy to see that $|z^H| = |H : \boldsymbol{C}_H(z)| = |\boldsymbol{C}_G(\rho^g) : \boldsymbol{C}_G(z\rho^g)| = 1$ or m.

Now we shall prove that m is a prime power, which is a contradiction. By Step 1 we have that H is solvable, which means that there must exist some prime q such that $\boldsymbol{Z}(H)_q < \boldsymbol{O}_q(H)$. If every r-element of H is central in H for every prime r dividing $|H|$ distinct from q, then m is certainly a q-power. So, let x be a non-central r-element of

H for such a prime r. We take Q to be a Sylow q-subgroup of $\boldsymbol{C}_H(x)$. Let us consider the action of $Q \times \langle x \rangle$ on $Q_0 = \boldsymbol{O}_q(H)$. We claim that $\boldsymbol{C}_{Q_0}(Q) \subseteq \boldsymbol{C}_{Q_0}(x)$. In fact, if $z \in \boldsymbol{C}_{Q_0}(Q)$ is non-central in H, then $\langle Q, z \rangle \leqslant \boldsymbol{C}_H(z) < H$. However, by the above paragraph, $|\boldsymbol{C}_H(z)|_q = |\boldsymbol{C}_H(x)|_q = |Q|$, so, in particular, $z \in Q \cap Q_0 \subseteq \boldsymbol{C}_{Q_0}(x)$ as claimed. We apply Thompson's $P \times Q$-Lemma to get $x \in \boldsymbol{C}_H(\boldsymbol{O}_q(H))$, and thus show that every Sylow r-subgroup of H lies in $\boldsymbol{C}_H(\boldsymbol{O}_q(H))$ for every $r \neq q$. This means that $|H : \boldsymbol{C}_H(\boldsymbol{O}_q(H))|$ is a q-number. However, if we take $w \in \boldsymbol{O}_q(H) \setminus \boldsymbol{Z}(H)$, then $\boldsymbol{C}_H(\boldsymbol{O}_q(H)) \subseteq \boldsymbol{C}_H(w)$ and, consequently, m is a q-number too, as desired.

Now let x be a non-central s-element, in which case we have $|x^G| = p^a m$. If y is an $\{s, p\}'$-element in $\boldsymbol{C}_G(x)$, then $\boldsymbol{C}_G(yx) = \boldsymbol{C}_G(y) \cap \boldsymbol{C}_G(x) = \boldsymbol{C}_G(x) \subseteq \boldsymbol{C}_G(y)$, which implies that $\boldsymbol{C}_G(x) = P_x S_x \times T_x$, where P_x and S_x are some Sylow p-subgroup and s-subgroup of $\boldsymbol{C}_G(x)$, respectively, and T_x is abelian.

Step 4. Every non-central $\{r, p\}'$-element has class size p^a. As a consequence, G is a $\{p, q, r\}$-group.

First we claim that every q-element has class size 1 or p^a.

Suppose that α is a q-element of index $p^a m$. Take a p-complement H_1 of G such that $\boldsymbol{C}_{H_1}(\alpha)$ is a p-complement of $\boldsymbol{C}_G(\alpha)$. Note that $\alpha \in H_1$. By using Step 3, there exists a non-central r-element $\beta \in G$ such that $|\beta^G| = p^a m$. Hence, $|\boldsymbol{C}_G(\alpha)| = |\boldsymbol{C}_G(\beta)|$ and $|\boldsymbol{C}_{H_1}(\alpha)/(\boldsymbol{Z}(G) \cap H_1)|_r = |\boldsymbol{C}_G(\beta)/\boldsymbol{Z}(G)|_r > 1$. So we conclude that there exists a non-central r-element $\gamma \in \boldsymbol{C}_{H_1}(\alpha)$, whence $|\gamma^G| = p^a m$. Moreover, $\boldsymbol{C}_G(\alpha\gamma) = \boldsymbol{C}_G(\alpha) \cap \boldsymbol{C}_G(\gamma)$, and by the maximality of the index of α and γ, we conclude that $\boldsymbol{C}_G(\alpha) = \boldsymbol{C}_G(\gamma) = \boldsymbol{C}_G(\alpha\gamma)$.

Now consider the action of $\langle \alpha \rangle \times \langle \gamma \rangle$ on $\boldsymbol{O}_q(H_1)$ and $\boldsymbol{O}_r(H_1)$ and, by Lemma 2.3, we deduce that $\boldsymbol{O}_q(H_1) \times \boldsymbol{O}_r(H_1) \subseteq \boldsymbol{C}_G(\alpha) = \boldsymbol{C}_G(\gamma)$. In particular, $\gamma \in \boldsymbol{C}_{H_1}(\boldsymbol{F}(H_1)) \subseteq \boldsymbol{F}(H_1)$, since, by Step 1, H_1 is a solvable group that can be described as $H_1 = Q_1 R_1 \times A_1$, where Q_1 and R_1 are some Sylow q- and r-subgroups of H_1, respectively, and A_1 is abelian. Therefore, $\gamma \in \boldsymbol{O}_r(H_1)$.

Now we shall show that $R_1 \subseteq \boldsymbol{C}_G(\alpha)$, which provides a contradiction, since α has index $p^a m$, which is divisible by r.

Let $\eta \in R_1$ be a non-central r-element. Then, by Step 3, $\boldsymbol{C}_G(\eta) = P_\eta R_\eta \times T_\eta$, where P_η and R_η are some Sylow p-subgroup and r-subgroup of $\boldsymbol{C}_G(\eta)$, respectively, and T_η is abelian. So $\boldsymbol{C}_{H_1}(\eta) \subseteq (R_\eta \times T_\eta)^x$ for some $x \in \boldsymbol{C}_G(\eta)$. Since $|H_1 : \boldsymbol{C}_{H_1}(\eta)| = m = |G : \boldsymbol{C}_G(\eta)|_{p'}$, by Step 1 we deduce that $|\boldsymbol{C}_{H_1}(\eta)| = |\boldsymbol{C}_G(\eta)|_{p'}$. Therefore, $\boldsymbol{C}_{H_1}(\eta) = (R_\eta \times T_\eta)^x$. By changing the notation we may assume that $\boldsymbol{C}_{H_1}(\eta) = R_\eta \times T_\eta$. Now we consider the action of $R_\eta \times T_\eta$ on $\boldsymbol{O}_r(H_1)$ by conjugation. We claim that $\boldsymbol{C}_{\boldsymbol{O}_r(H_1)}(R_\eta) \subseteq \boldsymbol{C}_{\boldsymbol{O}_r(H_1)}(T_\eta)$.

If z is a non-central element in $\boldsymbol{C}_{\boldsymbol{O}_r(H_1)}(R_\eta)$, then $\langle R_\eta, z \rangle \subseteq \boldsymbol{C}_G(z)$ and, since $|\boldsymbol{C}_G(z)|_r = |\boldsymbol{C}_G(\eta)|_r = |R_\eta|$, we deduce that $z \in R_\eta$ and hence $z \in \boldsymbol{C}_{\boldsymbol{O}_r(H_1)}(T_\eta)$. So it follows that $\boldsymbol{C}_{\boldsymbol{O}_r(H_1)}(R_\eta) \subseteq \boldsymbol{C}_{\boldsymbol{O}_r(H_1)}(T_\eta)$. Now, by using Thompson's $P \times Q$-Lemma, we have $T_\eta \subseteq \boldsymbol{C}_{H_1}(\boldsymbol{O}_r(H_1)) \subseteq \boldsymbol{C}_{H_1}(\gamma)$, which implies that $\alpha \in T_\gamma = T_\eta$, where T_γ is the $\{r, p\}$-complement of $\boldsymbol{C}_G(\gamma)$, and so $\alpha \in \boldsymbol{C}_G(\eta)$ and hence $R_1 \subseteq \boldsymbol{C}_G(\alpha)$, as we claimed.

Now let g be any $\{r, p\}'$-element of G, which can be assumed to belong to H. Then we have $g = g_q z$, where g_q is the q-part of g and z is an element in A. Since $z \in \boldsymbol{C}_G(g_q)$

and g_q has index p^a in G, we deduce that there exists $t \in G$ such that $z \in H^t \subseteq \boldsymbol{C}_G(g_q)$. So $z \in A^t$ and, by the fact that A^t is central in H^t, we have $H^t \subseteq \boldsymbol{C}_G(z)$. Then $H^t \subseteq \boldsymbol{C}_G(z) \cap \boldsymbol{C}_G(g_q) = \boldsymbol{C}_G(g)$, which implies that $|g^G| = 1$ or p^a.

Therefore, by Step 3 and the above argument, we get that every s-element of G is central for every $s \notin \{p, q, r\}$. Hence, the $\{p, q, r\}$-complement of G is central, and so by minimal counterexample we conclude that G is a $\{p, q, r\}$-group.

Step 5. Let P_w be a Sylow p-subgroup of $\boldsymbol{C}_G(w)$. Then any p'-element of G centralizes some conjugate of P_w.

Let h be any p'-element of G, which can be assumed to belong to $H \subseteq \boldsymbol{C}_G(w)$. We factorize $h = h_r h_q$ with $h_r \in R$ and $h_q \in Q$. As we proved in Step 3, h_r has index 1 or $p^a m$. Assume first that h_r has index $p^a m$. Since $h_r \in H \subseteq \boldsymbol{C}_G(w)$, we conclude that $\boldsymbol{C}_G(wh_r) = \boldsymbol{C}_G(h_r) = \boldsymbol{C}_G(h)$, which implies that $\boldsymbol{C}_G(h) \subseteq \boldsymbol{C}_G(w)$. But $|\boldsymbol{C}_G(w) : \boldsymbol{C}_G(h)|$ is m and we obtain that $\boldsymbol{C}_G(h)$ contains some Sylow p-subgroup of $\boldsymbol{C}_G(w)$, and consequently h centralizes some conjugate of P_w. Therefore, we may assume that h_r is central in G, whence h can be assumed to be a q-element. Thus, by applying Step 4, h has index p^a. Since by Step 3 any r-element of G has index 1 or $p^a m$, we can choose an r-element $t \in \boldsymbol{C}_G(h)$ of index $p^a m$. By minimality of the order of the centralizer of t in G, we have $\boldsymbol{C}_G(th) = \boldsymbol{C}_G(t)$, so $\boldsymbol{C}_G(t) \subseteq \boldsymbol{C}_G(h)$. On the other hand, t lies in some p-complement $H^g \subseteq \boldsymbol{C}_G(w^g)$ and similarly $\boldsymbol{C}_G(t) \subseteq \boldsymbol{C}_G(w^g)$. Moreover, $|\boldsymbol{C}_G(w^g) : \boldsymbol{C}_G(t)|$ is necessarily m, so some conjugate of P_w must lie in $\boldsymbol{C}_G(t)$ and, therefore, also in $\boldsymbol{C}_G(h)$ and this case is finished.

Step 6 ($\boldsymbol{O}^{\boldsymbol{p}}(\boldsymbol{G}) = \boldsymbol{G}$). Suppose that $\boldsymbol{O}^p(G) < G$. Let ρ be a p-regular element of $\boldsymbol{O}^p(G)$ such that $|\rho^G| = p^a$. We have

$$\frac{|G|}{|\boldsymbol{O}^p(G)|}\frac{|\boldsymbol{O}^p(G)|}{|\boldsymbol{C}_{\boldsymbol{O}^p(G)}(\rho)|} = \frac{|G|}{|\boldsymbol{C}_G(\rho)|}\frac{|\boldsymbol{C}_G(\rho)|}{|\boldsymbol{C}_{\boldsymbol{O}^p(G)}(\rho)|}.$$

Let P_ρ be a Sylow p-subgroup of $\boldsymbol{C}_G(\rho)$. The fact that $|\boldsymbol{C}_G(\rho) : \boldsymbol{C}_{\boldsymbol{O}^p(G)}(\rho)|$ is a p-number implies that

$$\frac{|G|}{|\boldsymbol{O}^p(G)|}\frac{|\boldsymbol{O}^p(G)|}{|\boldsymbol{O}^p(G) \cap \boldsymbol{C}_G(\rho)|} = \frac{|G|}{|\boldsymbol{C}_G(\rho)|}\frac{|P_\rho|}{|P_\rho \cap \boldsymbol{O}^p(G)|} = \frac{|G|}{|\boldsymbol{C}_G(\rho)|}\frac{|P_\rho \boldsymbol{O}^p(G)|}{|\boldsymbol{O}^p(G)|},$$

and thus

$$\frac{|\boldsymbol{O}^p(G)|}{|\boldsymbol{C}_{\boldsymbol{O}^p(G)}(\rho)|} = p^a \frac{|P_\rho \boldsymbol{O}^p(G)|}{|G|} = p^k,$$

where $k \geqslant 0$.

By Step 5 there exists $g \in G$ such that $P_w^g \subseteq \boldsymbol{C}_G(\rho)$. Since $|\boldsymbol{C}_G(\rho)|_p = |\boldsymbol{C}_G(w)|_p$, that is, P_ρ is G-conjugate to P_w^g, we deduce that p^k is constant in the above equation for any element ρ of index p^a.

Now, let ρ be a p-regular element of $\boldsymbol{O}^p(G)$ with $|\rho^G| = p^a m$. So we can write $\rho = \rho_r \rho_q$, where ρ_r and ρ_q are the r-part and q-part of ρ, respectively. By Step 4, ρ_r cannot be central. Thus it is easy to see that $\boldsymbol{C}_G(\rho) = \boldsymbol{C}_G(\rho_r)$. Hence, we may assume that ρ is an

r-element. There exists $g \in G$ such that $\rho \in \boldsymbol{C}_G(w^g)$ and hence $\boldsymbol{C}_G(\rho) \subseteq \boldsymbol{C}_G(w^g)$. Then $|\boldsymbol{C}_G(w^g) : \boldsymbol{C}_G(\rho)| = m$. As $(m, p) = 1$, we have $\boldsymbol{C}_G(w^g) = \boldsymbol{C}_G(\rho)\boldsymbol{C}_{\boldsymbol{O}^p(G)}(w^g)$ and

$$\begin{aligned} |\boldsymbol{O}^p(G) : \boldsymbol{C}_{\boldsymbol{O}^p(G)}(\rho)| &= |\boldsymbol{O}^p(G) : \boldsymbol{C}_{\boldsymbol{O}^p(G)}(w^g)|\,|\boldsymbol{C}_{\boldsymbol{O}^p(G)}(w^g) : \boldsymbol{C}_{\boldsymbol{O}^p(G)}(\rho)| \\ &= p^k|\boldsymbol{C}_G(w^g) : \boldsymbol{C}_G(\rho)| \\ &= p^k m. \end{aligned}$$

Therefore, the set of p-regular class sizes of $\boldsymbol{O}^p(G)$ is $\{1, p^k, p^k m\}$. If $k \neq 0$, then by minimal counterexample m is a prime power, which is a contradiction. Thus, $k = 0$ and $\{1, m\}$ are the p-regular conjugacy class sizes of $\boldsymbol{O}^p(G)$. This forces m to be a $\{p, q\}$-number by Theorem 2.5, which is a contradiction by Step 1.

Step 7. There exists N a proper normal subgroup of G such that the index $|G : N|$ is a p'-number and $\boldsymbol{Z}(G) \subseteq \boldsymbol{O}_{pp'}(G) \subseteq N$.

First we show that $\boldsymbol{O}_{pp'}(G) < G$. Otherwise, G has a normal Sylow p-subgroup P. Then $G = PH$, and it is easy to see that $\boldsymbol{C}_G(h) = \boldsymbol{C}_P(h)\boldsymbol{C}_H(h)$ for all $h \in H$. This implies that

$$|G : \boldsymbol{C}_G(h)| = |P : \boldsymbol{C}_P(h)|\,|H : \boldsymbol{C}_H(h)|,$$

which is 1, p^a or $p^a m$. Therefore, $|H : \boldsymbol{C}_H(h)|$ is 1 or m for every $h \in H$. By Itô's Theorem on groups with two class sizes [**9**, Theorem 33.6], m is a prime power, which is a contradiction. Hence, $\boldsymbol{O}_{pp'}(G) < G$.

Take N to be the maximal proper subgroup in the upper pp'-series of G and note that the index $|G : N|$ is a p'-number, since $\boldsymbol{O}^{p'}(G) < G$ by Step 6. Moreover, it is obvious that $\boldsymbol{Z}(G) \subseteq \boldsymbol{O}_{pp'}(G) \subseteq N < G$.

Step 8. If Q is a Sylow q-subgroup of H, then $Q\boldsymbol{O}_p(G) \trianglelefteq G$, $Q \trianglelefteq H$ and Q is abelian. Moreover, $\bar{R} = R/\boldsymbol{Z}(G)_r$ has exponent r, where R is a Sylow r-subgroup of G.

As we proved in Step 4, every q-element of G has class size 1 or p^a. So, by using Theorem 2.9, G has an abelian Sylow q-subgroup Q, and $Q\boldsymbol{O}_p(G) \trianglelefteq G$. Also, by using the fact that $Q \subseteq H$, it easily follows that $Q \trianglelefteq H$, as required.

Now we shall show that $\bar{R} = R/\boldsymbol{Z}(G)_r$ has exponent r. Let $x \in H \setminus Q\boldsymbol{Z}(G)_r$. Then we factorize $x = x_r x_q$, where x_r and x_q are the r-part and q-part of x, respectively. Note that $x_r \notin \boldsymbol{Z}(G)_r$. So $\boldsymbol{C}_G(x) \subseteq \boldsymbol{C}_G(x_r)$, and if we also take into account that x_r has index $p^a m$ in G, we conclude that $\boldsymbol{C}_G(x) = \boldsymbol{C}_G(x_r)$. Therefore, $\boldsymbol{C}_H(x) = \boldsymbol{C}_H(x_r)$, whence $|x^H| = m$, by using Step 1. Now we apply Isaacs's Theorem on groups having a normal subgroup such that the class sizes of the elements not in the normal subgroup are equal (see [**10**]). So we conclude that $H/(Q\boldsymbol{Z}(G)_r)$, which is isomorphic to $\bar{R}$, is cyclic, or has exponent r. However, $\boldsymbol{Z}(R) = \boldsymbol{Z}(G)_r$ and R cannot be abelian by Lemma 2.1, so we conclude that $\bar{R}$ has exponent r.

Step 9. If η is a non-central r-element of G, then $\boldsymbol{C}_G(\eta) = P_\eta \times \langle\eta\rangle\boldsymbol{Z}(G)_r \times Q_\eta$, where P_η and Q_η are the Sylow p-subgroup and q-subgroup of $\boldsymbol{C}_G(\eta)$, respectively.

We may assume that H is a p-complement of G, such that $\boldsymbol{C}_H(\eta)$ is a p-complement of $\boldsymbol{C}_G(\eta)$. So by Step 3, $\boldsymbol{C}_H(\eta) = R_\eta \times Q_\eta$ for some Sylow r-subgroup R_η and Sylow q-subgroup Q_η of $\boldsymbol{C}_G(\eta)$. Hence by the fact that $Q \trianglelefteq H$, R_η acts on Q. Since Q is abelian

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and this action is coprime, it follows that $Q = [Q, R_\eta] \times \boldsymbol{C}_Q(R_\eta)$ (see, for example, [**9**, Theorem 14.5]). On the other hand, we consider the action of $\bar{R}_\eta = R_\eta/\boldsymbol{Z}(G)_r$ on $[Q, R_\eta]$, and we claim that this action has no fixed points. Otherwise, there exist $x \in [Q, R_\eta]$ and $y \in R_\eta$ such that $x^{\bar{y}} = x$. Therefore, $x^y = x$, and as a consequence $x \in \boldsymbol{C}_G(y) = P_y R_y \times Q_y$, where P_y and R_y are some Sylow p-subgroup and Sylow r-subgroup of $\boldsymbol{C}_G(y)$, respectively, and Q_y is abelian, by Step 3. Since x is a q-element, it is obvious that $x \in Q_y$. On the other hand, from the fact that $y \in R_\eta$, we conclude that $Q_\eta \subseteq \boldsymbol{C}_G(y)$, and so $Q_\eta = Q_y$, by considering the order equality. Thus $x \in Q_\eta$, and consequently $x \in \boldsymbol{C}_Q(R_\eta)$. Hence, $x \in [Q, R_\eta] \cap \boldsymbol{C}_Q(R_\eta) = 1$, and our claim is proved. So it is well known that $\bar{R}_\eta$ is cyclic or is a generalized quaternion group. By considering Step 8, $\bar{R}_\eta$ is cyclic of order r, and therefore $R_\eta = \langle \eta \rangle \boldsymbol{Z}(G)_r$, so the result follows by the obtained fact in Step 3, that is, $\boldsymbol{C}_G(\eta) = P_\eta R_\eta \times Q_\eta$, where P_η is a Sylow p-subgroup of $\boldsymbol{C}_G(\eta)$.

Step 10. $\bar{R} = R/\boldsymbol{Z}(G)_r$ has order r^2, and consequently it is elementary abelian.

Let N be the normal subgroup introduced in Step 7 and let M be a maximal normal subgroup containing N. Recall that $|G : N|$ is a p'-number. We shall show that $|G/M| = r$. In Step 8 we proved that $Q\boldsymbol{O}_p(G) \trianglelefteq G$. So it is easy to conclude that $Q\boldsymbol{O}_p(G) \subseteq \boldsymbol{O}_{pp'}(G) \subseteq N$. As a consequence, $|G : N|$ is an r-number. Therefore, $|G : M|$ is an r-number, and since G/M is simple, it follows that $|G/M| = r$.

In the following we shall show that $m_r = r$, and so, by using Step 9, it is obvious that $|\bar{R}| = r^2$, whence $\bar{R}$ is abelian and, as a consequence of Step 8, elementary, as desired.

Let x be a non-central p-regular element of M. Then

$$\frac{|G|}{|M|}\frac{|M|}{|\boldsymbol{C}_M(x)|} = \frac{|G|}{|\boldsymbol{C}_G(x)|}\frac{|\boldsymbol{C}_G(x)|}{|\boldsymbol{C}_M(x)|}.$$

Let us consider a Sylow r-subgroup R_x of $\boldsymbol{C}_G(x)$; the above equality then becomes

$$\frac{|G|}{|M|}\frac{|M|}{|\boldsymbol{C}_M(x)|} = \frac{|G|}{|\boldsymbol{C}_G(x)|}\frac{|R_x|}{|R_x \cap M|} = \frac{|G|}{|\boldsymbol{C}_G(x)|}\frac{|R_x M|}{|M|}$$

and we have the following equality:

$$\frac{|M|}{|\boldsymbol{C}_M(x)|} = \frac{|G|}{|\boldsymbol{C}_G(x)|}\frac{|R_x M|}{|G|}.$$

First suppose that $|x^G| = p^a$. Therefore, R_x is a Sylow r-subgroup of G, whence $G = R_x M$. So the above equation implies that $|x^M| = |x^G| = p^a$.

Now suppose that $|x^G| = p^a m$. We factorize $x = x_r x_q$, where x_r and x_q are the r-part and q-part of x, respectively. By Step 4, x_r is a non-central element. Then $\boldsymbol{C}_G(x) = \boldsymbol{C}_G(x_r) \cap \boldsymbol{C}_G(x_q) \subseteq \boldsymbol{C}_G(x_r)$, and by the fact that x_r has class size $p^a m$ by Step 3, it follows that $\boldsymbol{C}_G(x) = \boldsymbol{C}_G(x_r)$. Also, by Step 9, the Sylow r-subgroup of $\boldsymbol{C}_G(x_r)$ is $R_{x_r} = \langle x_r \rangle \boldsymbol{Z}(G)_r$, which is a subgroup of M, since $x_r \in M$. On the other hand, the equality $\boldsymbol{C}_G(x) = \boldsymbol{C}_G(x_r)$ implies that R_{x_r} is the Sylow r-subgroup of $\boldsymbol{C}_G(x)$, that is, R_x. So $R_x M = M$ and, consequently, $|x^M| = p^a m/r$.

Thus $\mathrm{cs}_{p'}(M) = \{1, p^a, p^a m/r\}$, and by minimal counterexample it follows that m/r must be a prime power, whence $m_r = r$, and this completes the step.

Step 11. $\boldsymbol{N}_G(P_x) = \boldsymbol{C}_G(P_x)P_x$, where P_x is the Sylow p-subgroup of $\boldsymbol{C}_G(x)$ for every non-central r-element x of G.

In the following, we will show that $\boldsymbol{N}_G(P_w) = \boldsymbol{C}_G(P_w)P_w$, where P_w is a Sylow p-subgroup of $\boldsymbol{C}_G(w)$. Then by using the fact that there exists some $t \in G$ such that $P_x = P_w^t$, for every r-element x of G, which is a consequence of Step 5, our claim will be proved.

First we show that $G = \bigcup_{h\in G}(\boldsymbol{C}_G(P_w)P_w)^h \cup N$, where N is the subgroup that is mentioned in Step 7. Let g be a non-central element of G and write $g = g_p g_{p'}$. If $g_{p'} \in \boldsymbol{Z}(G) \subseteq N$, then, since $g_p \in N$, it follows that $g \in N$, as required. If $|g_{p'}^G| = p^a$, then by applying Lemma 2.4 we get $g_{p'} \in N$, and similarly we conclude that $g \in N$. So we may assume that $|g_{p'}^G| = p^a m$ and write $g_{p'} = g_q g_r$, where g_q and g_r are the q-part and r-part of g, respectively. Therefore, $g_r \notin \boldsymbol{Z}(G)$, by Step 4, and since $\boldsymbol{C}_G(g_{p'}) = \boldsymbol{C}_G(g_q) \cap \boldsymbol{C}_G(g_r) \subseteq \boldsymbol{C}_G(g_r)$, we conclude that $\boldsymbol{C}_G(g_{p'}) = \boldsymbol{C}_G(g_r)$. By using Step 5, there exists $h \in G$ such that $P_w^h \subseteq \boldsymbol{C}_G(g_r) = \boldsymbol{C}_G(g_{p'})$, whence $g_{p'} \in \boldsymbol{C}_G(P_w)^h$. On the other hand, $g_p \in \boldsymbol{C}_G(g_r)$, and by the fact that P_w^h is the only Sylow p-subgroup of $\boldsymbol{C}_G(g_r)$ by Step 9, we conclude that $g_p \in P_w^h$. Thus we have $g \in (\boldsymbol{C}_G(P_w)P_w)^h$, as required.

The above equality implies that

$$|G| \leqslant |G : \boldsymbol{N}_G(\boldsymbol{C}_G(P_w)P_w)|(|\boldsymbol{C}_G(P_w)P_w| - 1) + |N|,$$

and as a consequence

$$1 \leqslant \frac{|\boldsymbol{C}_G(P_w)P_w| - 1}{|\boldsymbol{N}_G(\boldsymbol{C}_G(P_w)P_w)|} + \frac{|N|}{|G|}.$$

We set $|\boldsymbol{N}_G(\boldsymbol{C}_G(P_w)P_w)| = n$. If $\boldsymbol{C}_G(P_w)P_w < \boldsymbol{N}_G(\boldsymbol{C}_G(P_w)P_w)$, then

$$1 \leqslant \frac{1}{2} - \frac{1}{n} + \frac{1}{2},$$

which is a contradiction. Therefore, $\boldsymbol{N}_G(\boldsymbol{C}_G(P_w)P_w) = \boldsymbol{C}_G(P_w)P_w$, and so it is easy to obtain $\boldsymbol{N}_G(P_w) = \boldsymbol{C}_G(P_w)P_w$, as desired.

Step 12. Let R be a Sylow r-subgroup of H. Then there exists a Sylow p-subgroup P_w of $\boldsymbol{C}_G(w)$ such that $R \subseteq \boldsymbol{C}_G(P_w)$.

Let $x \in R$ be a non-central r-element. Since $R \subseteq \boldsymbol{C}_G(w)$, we obtain $\boldsymbol{C}_G(wx) = \boldsymbol{C}_G(w) \cap \boldsymbol{C}_G(x)$, so we conclude that $\boldsymbol{C}_G(x) \subseteq \boldsymbol{C}_G(w)$. Therefore, there exists a Sylow p-subgroup of $\boldsymbol{C}_G(w)$, say P_w, such that $P_w \in \mathrm{Syl}_p(\boldsymbol{C}_G(x))$.

Now let $\alpha \in R$ be a non-central element. Since $R/\boldsymbol{Z}(G)_r$ is abelian, we have $[x, \alpha] \in \boldsymbol{Z}(G)$. It follows that $x^\alpha = xz$ for some element $z \in \boldsymbol{Z}(G)$. Therefore, $\boldsymbol{C}_G(x)^\alpha = \boldsymbol{C}_G(x)$ and so $\alpha \in \boldsymbol{N}_G(\boldsymbol{C}_G(x))$ and we deduce that $\alpha \in \boldsymbol{N}_G(P_w)$. Therefore, by using the previous step we get $\alpha \in \boldsymbol{C}_G(P_w)P_w$. By the fact that $\boldsymbol{C}_G(P_w)$ is a normal subgroup of $\boldsymbol{N}_G(P_w) = \boldsymbol{C}_G(P_w)P_w$ whose index is a p-number, we conclude that it contains all p'-elements of $\boldsymbol{N}_G(P_w)$. In particular, $\alpha \in \boldsymbol{C}_G(P_w)$, and so $R \subseteq \boldsymbol{C}_G(P_w)$, as required.

Step 13. G is r-nilpotent.

Set $\bar{G} = G/\boldsymbol{Z}(G)_r$. Also, in the following we use $\bar{T} = T/Z(G)_r$. Take R to be a Sylow r-subgroup of G. We shall show that

$$\bar{G} = \bigcup_{\bar{h} \in \bar{G}} \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}}) \cup \bar{N},$$

where N is the normal subgroup mentioned in Step 7.

Let $g = g_p g_{p'}$ be an element of G. If $g_{p'} \in \boldsymbol{Z}(G)$, then $\bar{g} \in \bar{N}$. So assume that $g_{p'} \notin \boldsymbol{Z}(G)$. If $|g_{p'}^G| = p^a$, then by Lemma 2.4 we have $g_{p'} \in \boldsymbol{O}_{pp'}(G) \subseteq N$, so $\bar{g} \in \bar{N}$. Thus we assume that $|g_{p'}^G| = p^a m$ with $g_{p'} = g_q g_r$, where g_q and g_r are the q-part and r-part of g, respectively. So $g_r \notin \boldsymbol{Z}(G)$, and we deduce that $\boldsymbol{C}_G(g_{p'}) = \boldsymbol{C}_G(g_r) \subseteq \boldsymbol{C}_G(g_q)$. There then exists $h \in G$ such that $g_r \in R^h \subseteq \boldsymbol{C}_G(g_q)$, whence $\bar{g}_q \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$. Moreover, $\bar{g}_r \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$, since $\bar{R}^{\bar{h}}$ is abelian. We conclude that $\bar{g}_{p'} \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$. On the other hand, there exists a Sylow p-subgroup P_w of $\boldsymbol{C}_G(w)$ such that $R \subseteq \boldsymbol{C}_G(P_w)$, by Step 12. So $g_r \in R^h \subseteq \boldsymbol{C}_G(P_w)^h$, which implies that P_w^h is the Sylow p-subgroup of $\boldsymbol{C}_G(g_r)$, and by Step 9 we have $g_p \in \boldsymbol{C}_G(R^h)$, and hence $\bar{g}_p \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$. So $\bar{g} \in \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}})$, as desired. Thus, we have proved that

$$\bar{G} = \bigcup_{\bar{h} \in \bar{G}} \boldsymbol{C}_{\bar{G}}(\bar{R}^{\bar{h}}) \cup \bar{N}.$$

This implies that

$$|\bar{G}| \leqslant |\bar{G} \colon \boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))|(|\boldsymbol{C}_{\bar{G}}(\bar{R})| - 1) + |\bar{N}|,$$

and hence

$$1 \leqslant \frac{|\boldsymbol{C}_{\bar{G}}(\bar{R})| - 1}{|\boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))|} + \frac{|\bar{N}|}{|\bar{G}|}.$$

We set $|\boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))| = n$. If we assume that $\boldsymbol{C}_{\bar{G}}(\bar{R}) < \boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R}))$, then we obtain the following contradiction:

$$1 \leqslant \frac{1}{2} - \frac{1}{n} + \frac{1}{2}.$$

Therefore, $\boldsymbol{N}_{\bar{G}}(\boldsymbol{C}_{\bar{G}}(\bar{R})) = \boldsymbol{C}_{\bar{G}}(\bar{R})$ and consequently $\boldsymbol{N}_{\bar{G}}(\bar{R}) = \boldsymbol{C}_{\bar{G}}(\bar{R})$. Now, by using Burnside's Theorem (see, for example, [**12**, 10.1.8]), we get that $\bar{G}$ is r-nilpotent. So G is r-nilpotent too, as required.

Step 14. Final contradiction.

Let R be a Sylow r-subgroup of H. By Step 12 there exists a Sylow p-subgroup P_w of $\boldsymbol{C}_G(w)$ such that $R \subseteq \boldsymbol{C}_G(P_w)$, whence $R \subseteq \boldsymbol{N}_G(P_w)$. On the other hand, by Step 13, G has a normal r-complement K, and so it is obvious that $K \cap \boldsymbol{N}_G(P_w)$ is normal in $\boldsymbol{N}_G(P_w)$. Hence, R acts coprimely on $K \cap \boldsymbol{N}_G(P_w)$. By coprime action properties, there exists an R-invariant Sylow p-subgroup of $\boldsymbol{N}_G(P_w)$, say P_1. Note that P_w is a normal subgroup of $\boldsymbol{N}_G(P_w)$ and so P_w is contained in P_1. Hence, $P_w \subseteq P_1 \subseteq P$ for some Sylow p-subgroup P of G, and consequently $P_1 = \boldsymbol{N}_P(P_w)$.

Note that $\boldsymbol{N}_P(P_w)/P_w$ is non-trivial. Otherwise $\boldsymbol{N}_P(P_w) = P_w$, and P_w would therefore be a Sylow p-subgroup of G, which is impossible because $|w^G| = p^a$ and $a > 0$. We claim that $\bar{R} = R/\boldsymbol{Z}(G)_r$ acts fixed-point-freely on

$$\widetilde{\boldsymbol{N}_P(P_w)} = \boldsymbol{N}_P(P_w)/P_w,$$

and so, by a well-known result, $\bar{R}$ is either cyclic (which is impossible) or a generalized quaternion group, which contradicts Step 10.

Suppose that $\tilde{x}^{\bar{t}} = \tilde{x}$ for some $x \in \boldsymbol{N}_P(P_w)$ and some $t \in R$. We can assume that x belongs to $\boldsymbol{C}_P(P_w)$ since, using Step 11, we have $\boldsymbol{N}_P(P_w) = \boldsymbol{C}_P(P_w)P_w$. Then $[x,t] \in P_w$. In particular, $[x,t]$ centralizes x and t. Moreover, as x is a p-element and t is an r-element, we have $1 = [x, t^{o(t)}] = [x^{o(t)}, t] = [x,t]^{o(t)}$. However, $[x,t]$ is a p-element, and this implies that $[x,t] = 1$, that is, $x \in \boldsymbol{C}_G(t)$. By the fact that $t \in R \subseteq \boldsymbol{C}_G(P_w)$, we deduce that P_w is the only Sylow p-subgroup of $\boldsymbol{C}_G(t)$, and so $x \in P_w$, that is, $\tilde{x} = 1$, and the action is fixed-point-free, as desired.

□

Examples. In the following we give some examples of the cases of Theorems A and B.

- Let $G = \boldsymbol{Z}_5 \rtimes Q_8$ be the semidirect product of the group $\boldsymbol{Z}_5 = \langle x \rangle$ acted on by the quaternion group $Q_8 = \langle y, z \colon y^4 = 1,\ y^2 = z^2,\ y^z = y^{-1} \rangle$ such that $x^y = x^{-1}$ and $x^z = x$. Then it is easy to see that the set of 5-regular conjugacy class sizes of G is equal to $\{1, 2, 10\}$. This provides an example of a group described in Theorem A.

- Let $G = (\boldsymbol{Z}_7 \times Q_8) \rtimes \boldsymbol{Z}_3$, and further let $\boldsymbol{Z}_7 = \langle x \rangle$, $Q_8 = \langle y, z \colon y^4 = 1,\ y^2 = z^2,\ y^z = y^{-1} \rangle$ and $\boldsymbol{Z}_3 = \langle w \rangle$, where $x^w = x^2$, $y^w = z^5$ and $z^w = z^3 y$. One can easily check that the set of the conjugacy class sizes of 3-regular elements of G is $\{1, 3, 6\}$, which is an example of case (i) of Theorem B.

- The group $\Gamma(8)$, whose set of 7-regular class sizes is exactly $\{1, 7, 28\}$ (see, for example, [**9**, p. 147]), provides an example of case (ii) of Theorem B.

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