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RESEARCH ARTICLE

Optimal defined-contribution pension management with financial and mortality risks

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Abstract
This paper studies optimal defined-contribution (DC) pension management under stochastic interest rates and expected inflation. In addition to financial risk, we consider the risk of pre-retirement death and introduce life insurance to the pension account as an option to manage this risk. We formulate this pension management problem as a random horizon utility maximization problem and derive its explicit solution under the assumption of constant relative risk aversion utility. We calibrate our model to the U.S. data and demonstrate that the pension member's demand for life insurance has a hump-shaped pattern with age and a U-shaped pattern with the real interest rate and expected inflation. The optimal pension account balance in our model resembles a variable annuity, wherein the death benefits are endogenously determined and depend on various factors including age, mortality, account balance, future contributions, preferences, and market conditions. Our study suggests that offering variable annuities with more flexible death benefits within the DC account could better cater to the bequest demands of its members.

1. Introduction

Population aging has posed a major challenge for actuaries in pension management. Most pension funds can be classified into two schemes: defined-benefit (DB) and defined-contribution (DC) plans. In a DB plan, retirees receive a guaranteed retirement benefit based on their salary histories, years of service, and age. By contrast, a DC plan requires its members to contribute a predetermined amount during the accumulation phase, and their retirement benefit is based on the investment returns generated by those contributions. The prevalence of DC plans has been increasing globally, with less than 50% of pension assets being managed through DB schemes in 28 of the 33 reporting jurisdictions according to OECD (2020).

The long-term investment horizon is a key feature of DC plan management. As accumulation periods typically last for 20–40 years, managing time-varying interest and inflation rates is of great significance. The existing literature on DC plan management extensively explores these long-term risks. For instance, Boulier et al. (2001) study a DC pension plan with a retirement guarantee dependent on stochastic interest rates. They show that the optimal fund composition should include a loan linked to future contributions, a contingent claim delivering the guarantee, and a hedging portfolio. Battocchio and Menoncin (2004) investigate the utility maximization problem of a DC pension member under stochastic interest rates, salary rates, and inflation rates. They emphasize that the addition of inflation rates makes the previously riskless asset risky and the lack of tools to hedge against inflation risk results in heightened return risk. Other pertinent studies in this field include Han and Hung (2012), Yao et al. (2013), Guan and Liang (2014), Chen and Delong (2015), Menoncin and Vigna (2017), Tang et al. (2018), Dong and

Zheng (2020), Xu et al. (2020), Forsyth et al. (2020), Wang et al. (2021), Chen et al. (2023), and Wei and Yang (2023), among others.

In addition to financial risks, DC plan members also face the risk of pre-retirement death. To address this risk, most DC plans include death benefit clauses during the accumulation period. The current literature primarily focuses on two types of death benefits: return of premiums and return of account value. The return of premiums clause entails returning the contributed premiums to the beneficiary, with or without predetermined interest, during the accumulation phase (e.g., He and Liang, 2013; Sun et al., 2016; Li et al., 2017; Bian et al., 2018). On the other hand, the return of account value clause involves returning the pension account value to the beneficiary (e.g., Blake et al., 2008; Yao et al., 2014; Konicz and Mulvey, 2015; Wu and Zeng, 2015).The latter clause is more commonly observed in practice because the investment revenue is part of the estate that can be inherited by the designated beneficiary (IRS, 2020). However, the existing literature only considers exogenously determined death benefits, leaving the mortality risk unhedged.

This research presents a study on the DC pension plan management problem that encompasses both financial and mortality risks. The financial market is modeled using a two-factor model by Koijen et al. (2011), which incorporates time variations in real interest rates, inflation rates, and risk premia. A representative pension plan member can invest a part of the account balance in a stock index, nominal and inflation-linked bonds, and a cash account. Furthermore, the individual can utilize the other part of the account balance to purchase life insurance to manage the individual's risk of pre-retirement death. Specifically, the individual continuously pays the insurance premium to the insurer while alive. In the event of the member's pre-retirement death, the beneficiary is entitled to receive a death benefit comprising the account balance and the life insurance payment. We formulate this DC plan management problem as a utility maximization problem with a random horizon and derive the corresponding Hamilton–Jacobi–Bellman (HJB) equation. Specifically, for the standard constant relative risk aversion (CRRA) utility function, we obtain the optimal strategy explicitly up to the solution to a Hermitian matrix Riccati differential equation (HRDE). The global existence of the HRDE's solution depends on the relative risk aversion parameter and model parameters, and we provide a detailed analysis of the HRDE along with sufficient conditions for its global existence. Additionally, we prove a verification theorem by demonstrating that the candidate strategy is indeed optimal to the stochastic control problem.

We calibrate our model to the U.S. data and numerically illustrate the optimal investment and insurance strategies of a representative male plan member. In particular, we observe that the expected life insurance premium exhibits a hump-shaped pattern with respect to age, reaching its peak at age 59. This pattern is influenced by four key factors: the force of mortality, the surplus process, the bequest-wealth ratio, and future contributions. During early ages, the increasing force of mortality and the accumulation of the surplus process contribute to a higher demand for life insurance. However, as the individual transitions into the mid-to-retirement periods, the decreasing bequest-wealth ratio and the depletion of future contributions lead to a decline in the demand for insurance. Furthermore, we discover that the individual's demand for life insurance follows a U-shaped pattern in relation to the real short rate and expected inflation. Specifically, the individual tends to purchase more life insurance when the real short rate and expected inflation are either exceptionally high or extremely low. This U-shaped pattern can be attributed to the combined effects of the bequest-wealth ratio and future contributions.

This research contributes to the literature by providing a comprehensive analysis of the DC pension plan management problem with consideration of both financial and mortality risks. Unlike the existing literature that focuses only on investment risk, we take into account the risk of pre-retirement death and introduce life insurance to the pension account. Our model allows the DC plan member to purchase life insurance from the account balance to manage the individual's risk of pre-retirement death. Our findings reveal that the demand for life insurance exhibits a hump-shaped pattern with age and a U-shaped pattern with real interest rates and expected inflation. These insights have important implications for pension plan management and product design. Specifically, our model suggests that the DC account of an individual resembles a variable annuity with a flexible death benefit, which differs from traditional variable annuities that have exogenously determined death benefits. In our model, the optimal death benefit within the DC account is endogenously determined and depends on various factors such as age, force of mortality, account balance, future contributions, preferences, and market conditions (including interest rates and inflation). Our results recommend the offering of variable annuities with flexible death benefits to better accommodate the bequest demands of DC plan members.

2. Economic setting

2.1. Financial market

We consider a financial market similar to that presented in Koijen et al. (2011), which accommodates time variations in real interest rates, inflation rates, and risk premia. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space. The financial risk is described by Z_t, a four-dimensional vector of independent Brownian motions, which is adapted to the filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$.

In the financial market, the real short rate is driven by a single factor, X_1,

$$r_t = \delta_r + X_{1,t}, \quad \delta_r > 0,$$

and expected inflation is affine in a second factor, X_2,

$$\pi_t^e = \delta_{\pi^e} + X_{2,t}, \quad \delta_{\pi^e} > 0.$$

Following the literature, we assume the two factors satisfy the following Ornstein–Uhlenbeck (OU) process

$$dX_t = -K_X X_t dt + \Sigma_X dZ_t, \tag{2.1}$$

where $X_t = (X_{1,t}, X_{2,t})^\top$, $K_X = diag(\kappa_1, \kappa_2)$, $\kappa_i > 0$, $i = 1, 2$, $\Sigma_X = (\sigma_1, \sigma_2)^\top$, $\sigma_i \in \mathbb{R}^4$, $i = 1, 2$. The use of the OU process captures mean reversion in the real short rate and expected inflation. It is worth noting that the OU process can lead to negative values of the real short rate and expected inflation, which are not uncommon in the U.S. Negative values of the real short rate occur when inflation exceeds the nominal interest rate, while negative expected inflation captures instances of deflation.

The realized inflation is then given by

$$\frac{d\Pi_t}{\Pi_t} = \pi_t^e dt + \sigma_\Pi^\top dZ_t, \ \Pi_0 = 1, \tag{2.2}$$

where Π_t denotes the level of the (consumer) price index at time t and $\sigma_\Pi \in \mathbb{R}^4$.

The equity index S_t satisfies the following dynamics

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_S^\top dZ_t,$$

where $\mu_t = R_t + \mu_0 + \mu_1^\top X_t$ and R_t denotes the instantaneous nominal short rate that is derived in (2.3) below. For identification purposes, we assume the volatility matrix $(\sigma_1, \sigma_2, \sigma_\Pi, \sigma_S)^\top$ is lower triangular.

We assume the nominal state price density ϕ satisfies

$$\frac{d\phi_t}{\phi_t} = -R_t dt - \Lambda_t^\top dZ_t, \ \phi_0 = 1,$$

in which the market prices of risk, Λ_t, are affine in the term-structure variables, that is,

$$\Lambda_t = \Lambda_0 + \Lambda_1 X_t.$$

We follow Koijen et al. (2011) to impose restrictions on Λ_0 and Λ_1

$$\Lambda_0 = \begin{pmatrix} \Lambda_{0(1)} \\ \Lambda_{0(2)} \\ 0 \\ \Lambda_{0(4)} \end{pmatrix}, \ \Lambda_1 = \begin{pmatrix} \Lambda_{1(1,1)} & 0 \\ 0 & \Lambda_{1(2,2)} \\ 0 & 0 \\ \Lambda_{1(4,1)} & \Lambda_{1(4,2)} \end{pmatrix},$$

with $\sigma_S^\top \Lambda_0 = \mu_0^\top$ and $\sigma_S^\top \Lambda_1 = \mu_1^\top$. The real state price density $\phi_t^R = \phi_t \Pi_t$ then satisfies

$$\frac{d\phi_t^R}{\phi_t^R} = -(R_t - \pi_t^e + \sigma_\Pi^\top \Lambda_t)dt - (\Lambda_t^\top - \sigma_\Pi^\top)dZ_t = -r_t dt - (\Lambda_t^\top - \sigma_\Pi^\top)dZ_t, \quad \phi_0^R = 1,$$

which implies for the instantaneous nominal short rate

$$R_t = \delta_R + (\iota_2^\top - \sigma_\Pi^\top \Lambda_1)X_t, \tag{2.3}$$

where $\delta_R = \delta_r + \delta_{\pi^e} - \sigma_\Pi^\top \Lambda_0$ and $\iota_2 = (1, 1)^\top$.

Finally, we present the prices of nominal and inflation-linked bonds. The derivation is standard in the literature (e.g., Duffie and Kan, 1996). The time-t price of a nominal bond with maturity s is

$$P(X_t, t, s) = \exp\{A_0(s-t) + [A_1(s-t)]^\top X_t\},$$

where A_0 and A_1 satisfy the following ODE system

$$\frac{\partial A_0(\tau)}{\partial \tau} = \frac{1}{2}[A_1(\tau)]^\top \Sigma_X \Sigma_X^\top A_1(\tau) - [A_1(\tau)]^\top \Sigma_X \Lambda_0 - \delta_R, \; A_0(0) = 0, \tag{2.4}$$

$$\frac{\partial A_1(\tau)}{\partial \tau} = -[K_X^\top + \Lambda_1^\top \Sigma_X^\top]A_1(\tau) - \iota_2 + \Lambda_1^\top \sigma_\Pi, \; A_1(0) = 0. \tag{2.5}$$

In addition, the dynamics of $P(X_t, t, s)$ satisfy

$$\frac{dP(X_t, t, s)}{P(X_t, t, s)} = \{R_t + [A_1(s-t)]^\top \Sigma_X \Lambda_t\}dt + [A_1(s-t)]^\top \Sigma_X dZ_t.$$

Similarly, the time-t real price of an inflation-linked bond with maturity s is

$$P^R(X_t, t, s) = \exp\{A_0^R(s-t) + [A_1^R(s-t)]^\top X_t\},$$

where A_0^R and A_1^R satisfy the ODE system

$$\frac{\partial A_0^R(\tau)}{\partial \tau} = \frac{1}{2}[A_1^R(\tau)]^\top \Sigma_X \Sigma_X^\top A_1^R(\tau) - [A_1^R(\tau)]^\top \Sigma_X(\Lambda_0 - \sigma_\Pi) - \delta_r, \; A_0^R(0) = 0,$$

$$\frac{\partial A_1^R(\tau)}{\partial \tau} = -(K_X^\top + \Lambda_1^\top \Sigma_X^\top)A_1^R(\tau) - e_1, \; A_1^R(0) = 0,$$

in which e_i represents the i-th unit vector in $\mathbb{R}^2$. Then, the nominal price of the inflation-linked bond $\Pi_t P^R(X_t, t, s)$ satisfies

$$\frac{d(\Pi_t P^R(X_t, t, s))}{\Pi_t P^R(X_t, t, s)} = \{R_t + [A_1^R(s-t)]^\top \Sigma_X \Lambda_t + \sigma_\Pi^\top \Lambda_t\}dt + \{[A_1^R(s-t)]^\top \Sigma_X + \sigma_\Pi^\top\}dZ_t.$$

2.2. Mortality

In this subsection, we introduce mortality risk. Denote by T_x the future lifetime of an individual aged x, which is a nonnegative random variable independent of the financial market (i.e., T_x is independent of the filtration $\mathbb{F}$ associated with the financial market). We can define the following probabilities

$${}_tp_x = \mathbb{P}[T_x > t], \quad {}_tq_x = \mathbb{P}[T_x \le t] = 1 - {}_tp_x, \quad \lim_{t\to\infty} {}_tp_x = 0, \quad \lim_{t\to\infty} {}_tq_x = 1,$$

where ${}_tp_x$ is the probability that the individual alive at age x survives to at least age $x+t$ and ${}_tq_x$ is the probability that the individual dies before age $x+t$. In actuarial science, it is common to work with the instantaneous force of mortality (or hazard rate)

$$\mu_{x+t} = \frac{1}{{}_tp_x}\frac{d}{dt}{}_tq_x = -\frac{1}{{}_tp_x}\frac{d}{dt}{}_tp_x,$$

and we have

$$ {}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}ds\right\}, \quad {}_tq_x = \int_0^t {}_sp_x\mu_{x+s}ds. $$

The probability density function of T_x is then given by $f_{T_x}(t) = {}_tp_x\mu_{x+t}$, for $t > 0$.

2.3. Wealth process

The individual enters the DC pension plan at age x at time 0 and retires at time T (so the retirement age is $x+T$). The future lifetime of the individual is denoted by T_x. Before retirement or death, the individual contributes a fixed percentage of labor income continuously to the fund. We assume that real labor income is deterministic and thus the real contribution rate, $C_t = C_t^{\$}\Pi_t^{-1}$, satisfies

$$ \frac{dC_t}{C_t} = g_t^R dt, \quad 0 \le t < T \wedge T_x, \tag{2.6} $$

where $\wedge$ is the minimum of the two variables, $C_t^{\$}$ is the nominal contribution rate, and g_t^R is the growth rate of the real contribution rate (which is also the growth rate of labour income).

During the accumulation period, the individual allocates his or her wealth dynamically to the stock index, two nominal bonds, and an inflation-linked bond. In particular, the individual uses the "rolling bond" strategy for purchasing bonds, as outlined in Boulier et al. (2001). Denote by α_t the proportions of wealth invested in these assets at time t. The rest of the wealth is invested in the cash account. Additionally, the individual can purchase (term) life insurance that is available continuously to manage mortality risk. This assumption is unrealistic but necessary to obtain the closed-form solution. Suppose the individual pays the life insurance premium at a rate of $I_t^{\$}$ (in nominal terms) continuously to the insurer while alive. If the individual dies at time $T_x = t$ prior to retirement, the beneficiary receives the death benefit (the face value of life insurance) $I_t^{\$}/\mu_{x+t}$ in addition to the account balance. The individual's DC account balance then evolves according to the following equation:

$$ dW_t = W_t(\alpha_t^\top \Sigma\Lambda_t + R_t)dt + C_t^{\$}dt + W_t\alpha_t^\top \Sigma dZ_t - I_t^{\$}dt, \quad 0 \le t < T \wedge T_x, $$

where $W_0 = 0$ and Σ is the volatility matrix of tradable assets. We assume the two nominal bonds have maturities T_1 and T_2, and the inflation-linked bond has maturity T_3. Consequently,

$$ \Sigma = \begin{pmatrix} [A_1(T_1)]^\top \Sigma_X \\ [A_1(T_2)]^\top \Sigma_X \\ [A_1^R(T_3)]^\top \Sigma_X + \sigma_\Pi^\top \\ \sigma_S^\top \end{pmatrix}. $$

We can then derive the dynamics of the real wealth $W_t^R = W_t/\Pi_t$

$$ dW_t^R = W_t^R[r_t + (\alpha_t^\top \Sigma - \sigma_\Pi^\top)(\Lambda_t - \sigma_\Pi)]dt + C_t dt + W_t^R(\alpha_t^\top \Sigma - \sigma_\Pi^\top)dZ_t - I_t dt, \tag{2.7} $$

where $0 \le t < T \wedge T_x$, $W_0^R = 0$, and $I_t = I_t^{\$}/\Pi_t$ is the real insurance premium rate.

If the individual dies before retirement, then the death benefit is added to the account balance

$$ W_t^R = W_{t-}^R + B_t = W_{t-}^R + \frac{I_t}{\mu_{x+t}}, \text{ if } T_x = t < T. $$

2.4. Preference

We assume the individual chooses investment and insurance strategies (α, I) to maximize the expected utility of account balance at retirement or death, whichever occurs first, that is,

$$ \sup_{\alpha, I} E[U(W_{T\wedge T_x}^R)]. $$

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Because T_x is independent of financial risks, we can show

$$\sup_{\alpha,I} E[U(W^R_{T\wedge T_x})] = \sup_{\alpha,I} E\left[\int_0^T {}_tp_x\mu_{x+t}U\left(W^R_t + \frac{I_t}{\mu_{x+t}}\right)dt + {}_Tp_xU(W^R_T)\right]. \tag{2.8}$$

From a technical point of view, Equation (2.8) allows us to convert the random horizon optimization problem to a problem with a fixed terminal time.

3. Optimization problem

3.1. Dynamic programming

Following Deelstra et al. (2003), we introduce the surplus process $W^{\widetilde{C}}_t$

$$W^{\widetilde{C}}_t = W^R_t + \widetilde{C}(t, X_t), \tag{3.1}$$

where $\widetilde{C}(t, X_t)$ is the time-t value of (discounted) future contributions

$$\widetilde{C}(t, X_t) = \int_t^T {}_{s-t}p_{x+t}P^R(X_t, t, s)C_s ds.$$

Next, by Ito's formula, we have

$$d\widetilde{C}(t, X_t) = -C_t dt + (r_t + \mu_{x+t})\widetilde{C}(t, X_t)dt + \frac{\partial \widetilde{C}(t, X_t)}{\partial X}\Sigma_X(\Lambda_t - \sigma_\Pi)dt + \frac{\partial \widetilde{C}(t, X_t)}{\partial X}\Sigma_X dZ_t. \tag{3.2}$$

Assume that there exists a process ξ_t such that

$$\begin{aligned} d\widetilde{C}(t, X_t) = &-C_t dt + \widetilde{C}(t, X_t)[r_t + (\xi_t^\top\Sigma - \sigma_\Pi^\top)(\Lambda_t - \sigma_\Pi)]dt + \mu_{x+t}\widetilde{C}(t, X_t)dt \\ &+\widetilde{C}(t, X_t)(\xi_t^\top\Sigma - \sigma_\Pi^\top)dZ_t, \end{aligned} \tag{3.3}$$

then we can obtain ξ by comparing the relevant terms in (3.2) and (3.3)

$$\xi_t = \frac{1}{\widetilde{C}(t, X_t)}(\Sigma^\top)^{-1}\Sigma_X^\top\frac{\partial \widetilde{C}(t, X_t)}{\partial X^\top} + (\Sigma^\top)^{-1}\sigma_\Pi.$$

Furthermore, adding (2.7) and (3.3), we derive the SDE for the surplus process

$$dW^{\widetilde{C}}_t = W^{\widetilde{C}}_t\{r_t + (\beta_t^\top\Sigma - \sigma_\Pi^\top)(\Lambda_t - \sigma_\Pi)\}dt + W^{\widetilde{C}}_t(\beta_t^\top\Sigma - \sigma_\Pi^\top)dZ_t + \mu_{x+t}\widetilde{C}(t, X_t)dt - I_t dt, \tag{3.4}$$

where $W^{\widetilde{C}}_t = W^R_t + \widetilde{C}(t, X_t)$, $0 \le t < T \wedge T_x$, and $\beta_t^\top = [W^R_t\alpha_t^\top + \widetilde{C}(t, X_t)\xi_t^\top]/W^{\widetilde{C}}_t$. The SDE (3.4) models the investment in the financial market, and the purchase of life insurance with premium $-\mu_{x+t}\widetilde{C}(t, X_t) + I_t$. When the individual dies before retirement, the surplus process has the following jump

$$W^{\widetilde{C}}_t = W^{\widetilde{C}}_{t-} - \widetilde{C}(t, X_t) + \frac{I_t}{\mu_{x+t}}, \text{ if } T_x = t < T.$$

Then, by definition (3.1), and given that $W^{\widetilde{C}}_T = W^R_T$ at T, the objective function (2.8) can be transformed to

$$\sup_{\beta,I} E\left[\int_0^T {}_tp_x\mu_{x+t}U\left(W^{\widetilde{C}}_t - \widetilde{C}(t, X_t) + \frac{I_t}{\mu_{x+t}}\right)dt + {}_Tp_xU(W^{\widetilde{C}}_T)\right].$$

Define the value function

$$V(t, w^{\widetilde{C}}, X) = \sup_{\beta,I} E_{t,w^{\widetilde{C}},X}\left[\int_t^T {}_{s-t}p_{x+t}\mu_{x+s}U\left(W^{\widetilde{C}}_s - \widetilde{C}(s, X_s) + \frac{I_s}{\mu_{x+s}}\right)ds + {}_{T-t}p_{x+t}U(W^{\widetilde{C}}_T)\right],$$

where $X = (x_1, x_2)^\top$ and $E_{t,w^{\widetilde{C}},X}[\,\cdot\,]$ is short for $E[\,\cdot\,|W^{\widetilde{C}}_t = w^{\widetilde{C}}, X_t = X]$. Then, by the dynamic programming principle, we derive the following HJB

$$\sup_{\beta_t,I_t}\left\{\mu_{x+t}U\left(w^{\widetilde{C}} - \widetilde{C}(t, X) + \frac{I_t}{\mu_{x+t}}\right) - \mu_{x+t}V(t, w^{\widetilde{C}}, X) + \mathcal{D}^{\beta,I}V(t, w^{\widetilde{C}}, X)\right\} = 0, \tag{3.5}$$

where

$$\mathcal{D}^{\beta,I}V(t,w^{\widetilde{C}},X)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial w^{\widetilde{C}}}\{w^{\widetilde{C}}[r_t+(\beta_t^\top\Sigma-\sigma_\Pi^\top)(\Lambda_t-\sigma_\Pi)]+\mu_{x+t}\widetilde{C}(t,X)-I_t\}$$
$$-\frac{\partial V}{\partial X}K_XX+\frac{1}{2}\frac{\partial^2 V}{(\partial w^{\widetilde{C}})^2}(w^{\widetilde{C}})^2(\beta_t^\top\Sigma\Sigma^\top\beta_t-2\beta_t^\top\Sigma\sigma_\Pi+\sigma_\Pi^\top\sigma_\Pi)$$
$$+w^{\widetilde{C}}(\beta_t^\top\Sigma-\sigma_\Pi^\top)\Sigma_X^\top\frac{\partial^2 V}{\partial w^{\widetilde{C}}\partial X^\top}+\frac{1}{2}\mathrm{Tr}\left(\Sigma_X^\top\frac{\partial^2 V}{\partial X^\top\partial X}\Sigma\right),\tag{3.6}$$

$\frac{\partial}{\partial X}(\cdot)=(\frac{\partial}{\partial x_1}(\cdot),\frac{\partial}{\partial x_2}(\cdot))$ is the gradient operator with the factor vector $X=(x_1,x_2)^\top$, and $\frac{\partial}{\partial X^\top}(\cdot)=(\frac{\partial}{\partial x_1}(\cdot),\frac{\partial}{\partial x_2}(\cdot))^\top$ is the gradient operator with $X^\top=(x_1,x_2)$. The first-order condition with respect to β_t and I_t yields that

$$\beta_t^*=-\frac{(\Sigma^\top)^{-1}}{w^{\widetilde{C}}\frac{\partial^2 V}{(\partial w^{\widetilde{C}})^2}}\left[\frac{\partial V}{\partial w^{\widetilde{C}}}(\Lambda_t-\sigma_\Pi)+\Sigma_X^\top\frac{\partial^2 V}{\partial w^{\widetilde{C}}\partial X^\top}\right]+(\Sigma^T)^{-1}\sigma_\Pi,\tag{3.7}$$

$$I_t^*=\mu_{x+t}(U')^{-1}\left(\frac{\partial V}{\partial w^{\widetilde{C}}}\right)-\mu_{x+t}(W_t^{\widetilde{C}})^*+\mu_{x+t}\widetilde{C}(t,X_t).\tag{3.8}$$

Under the optimal strategy, the individual's DC account evolves as a variable annuity with an endogenously determined death benefit. This is in contrast to traditional variable annuities that have a fixed death benefit, typically defined as the maximum of the account value and a guaranteed minimum. The death benefit in the DC account of our model is endogenously determined and adapts based on various factors such as age, mortality, account balance, future contributions, personal preferences, and market conditions like interest rates and inflation. In particular, numerical examples in Section 4 indicate that the optimal face value is hump-shaped and peaks at age 50. These insights suggest that pension plan sponsors should offer variable annuities with more flexible death benefits based on individual circumstances and market conditions, to meet the diverse needs of their members. For more details of the variable annuities, see SEC (2009).

3.2. Solution under the CRRA utility

We proceed by solving the optimization problem under the assumption of constant relative risk aversion (CRRA) utility.

$$U(x)=\frac{x^{1-\gamma}}{1-\gamma},$$

where $\gamma>0$ and $\gamma\neq 1$ is the Arrow-Pratt coefficient of relative risk aversion.

Proposition 3.1. *The candidate solution (value function) to the HJB Equation (3.5) is given by*

$$G(t,W_t^{\widetilde{C}},X_t)=\frac{1}{1-\gamma}(W_t^{\widetilde{C}})^{1-\gamma}f_1(t,X_t)^\gamma,\tag{3.9}$$

where

$$f_1(t,X_t)=\int_t^T {}_{s-t}p_{x+t}\mu_{x+s}f(X_t,s-t)ds+{}_{T-t}p_{x+t}f(X_t,T-t),\tag{3.10}$$

$$f(X_t,\tau)=\exp\left[\Gamma_0(\tau)+\Gamma_1^\top(\tau)X_t+\frac{1}{2}X_t^\top\Gamma_2(\tau)X_t\right],\ \tau\in[0,T-t].\tag{3.11}$$

Functions $\Gamma_0(\tau)\in\mathbb{R}$, $\Gamma_1(\tau)\in\mathbb{R}^2$ and $\Gamma_2(\tau)\in\mathbb{R}^2\times\mathbb{R}^2$ are given by the following ODE system

$$\frac{\partial\Gamma_2(\tau)}{\partial\tau}-\Gamma_2(\tau)Z_2\Gamma_2(\tau)-Z_1^\top\Gamma_2(\tau)-\Gamma_2(\tau)Z_1-Z_0=0,\ \Gamma_2(0)=0,\tag{3.12}$$

$$\frac{\partial \Gamma_1(\tau)}{\partial \tau} - \Gamma_2(\tau)B_2\Gamma_1(\tau) - \Gamma_2(\tau)B_{11} - B_{12}\Gamma_1(\tau) - B_0 = 0,\ \Gamma_1(0) = 0, \tag{3.13}$$

$$\frac{\partial \Gamma_0(\tau)}{\partial \tau} - \Gamma_1^\top(\tau)D_2\Gamma_1(\tau) - \Gamma_1^\top(\tau)D_1 - \frac{1}{2}\mathrm{Tr}\{\Sigma_X^\top \Gamma_2(\tau)\Sigma_X\} - D_0 = 0,\ \Gamma_0(0) = 0, \tag{3.14}$$

in which

$$Z_2 = \Sigma_X\Sigma_X^\top, Z_1 = \frac{1-\gamma}{\gamma}\Sigma_X\Lambda_1 - K_X, Z_0 = \frac{1-\gamma}{\gamma^2}\Lambda_1^\top\Lambda_1,$$
$$B_2 = Z_2, B_{11} = \frac{1-\gamma}{\gamma}\Sigma_X(\Lambda_0 - \sigma_\Pi), B_{12} = Z_1^\top, B_0 = \frac{1-\gamma}{\gamma^2}\Lambda_1^\top(\Lambda_0 - \sigma_\Pi) + \frac{1-\gamma}{\gamma}e_1,$$
$$D_2 = \frac{1}{2}Z_2, D_1 = B_{11}, D_0 = \frac{1-\gamma}{\gamma}\delta_r + \frac{1-\gamma}{2\gamma^2}(\Lambda_0^\top - \sigma_\Pi^\top)(\Lambda_0 - \sigma_\Pi).$$

The candidate strategies are given by

$$\beta_t^* = \frac{(\Sigma^\top)^{-1}}{\gamma}(\Lambda_t - \sigma_\Pi) + (\Sigma^\top)^{-1}\Sigma_X^\top\frac{1}{f_1(t,X_t)}\frac{\partial f_1(t,X_t)}{\partial X^\top} + (\Sigma^\top)^{-1}\sigma_\Pi, \tag{3.15}$$

$$I_t^* = \mu_{x+t}\left(\frac{1}{f_1(t,X_t)} - 1\right)(W_t^{\widetilde{C}})^* + \mu_{x+t}\widetilde{C}(t,X_t). \tag{3.16}$$

Next, we prove the candidate solution's global existence and verify it is indeed optimal.

3.3. The global existence and verification theorem

Among the ODEs determining the candidate solution, (3.13) and (3.14) are linear ODEs. Their solutions are unique and exist globally (see Theorem 1.1.1. in Abou-Kandil et al., 2012). However, the ODE (3.12) is a Hermitian matrix Riccati differential equation (HRDE), whose existence requires special treatment. The HRDE has the following matrix representation

$$\frac{\partial \Gamma_2(\tau)}{\partial \tau} = (\widetilde{I}_2, \Gamma_2(\tau))JH(\tau)\begin{pmatrix}\widetilde{I}_2\\ \Gamma_2(\tau)\end{pmatrix} := \mathcal{H}(\Gamma_2; H), \tau \in [0,T], \tag{3.17}$$

where $\widetilde{I}_2$ is the 2nd-order identity matrix,

$$J := \begin{pmatrix}0_{2\times2} & \widetilde{I}_2\\ -\widetilde{I}_2 & 0_{2\times2}\end{pmatrix} \in \mathbb{R}^2\times\mathbb{R}^2,\ \text{and } H := \begin{pmatrix}-Z_1 & -Z_2\\ Z_0 & Z_1^\top\end{pmatrix} \in \mathbb{R}^2\times\mathbb{R}^2,$$

which is called the Hamiltonian matrix. The global existence of the HRDE (3.17) largely depends on the relative risk aversion coefficient γ, which is also the case for the verification theorem. Inspired by Honda and Kamimura (2011), we divide the proofs in this subsection into two cases $\gamma > 1$ and $0 < \gamma < 1$.

Proposition 3.2. *For $\gamma > 1$, define the admissible set as*

$$\mathcal{A}_\gamma(0,T) := \left\{(\beta, I)\,\middle|\, \begin{array}{c}\beta(t,X_t) : [0,T]\times\mathbb{R}^2 \to \mathbb{R}^4\\ \textit{grows linearly with respect to } X_t,\\ \textit{and SDE (3.4) has a unique strong solution.}\end{array}\right\}.$$

If $\Sigma_X\Sigma_X^\top > 0$ and $\Lambda_1^\top\Lambda_1 > 0$, then the candidate solution $G(t, W_t^{\widetilde{C}}, X_t)$ exists in [0, T] and satisfies $G(t, W_t^{\widetilde{C}}, X_t) = V(t, W_t^{\widetilde{C}}, X_t)$. The strategy (β^*, I^*) given by (3.15) and (3.16) is the optimal portfolio and insurance strategy. For matrices, "$>$" ("$<$") indicates positive (negative) definite.

For $0 < \gamma < 1$, we can prove the existence of (3.12) by Radon's Lemma with additional conditions. Denote $(Q,P)^\top$ as a solution to the linear system of differential equations

$$\frac{d}{d\tau}\begin{pmatrix}Q(\tau)\\ P(\tau)\end{pmatrix} = H\begin{pmatrix}Q(\tau)\\ P(\tau)\end{pmatrix}, Q(0) = \widetilde{I}_2, P(0) = \Gamma_2(0)Q(0) = 0. \tag{3.18}$$

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By Radon's Lemma (see Theorem 3.1.1. in Abou-Kandil et al., 2012), we can represent the solution to (3.12) as $\Gamma_2(\tau) = P(\tau)/Q(\tau)$. Next, we only need $\Gamma_2(\tau) < 0$ to guarantee the candidate solution's global existence. For tractability, we follow Abou-Kandil et al. (2012) and assume H is diagonalizable, that is, there exists a 4-dimensional basis of eigenvectors

$$v_1, \ldots, v_4 \in \mathbb{C}^4,$$

where $\mathbb{C}^4$ denotes the complex vector space of 4×1 complex vectors, and the corresponding eigenvalues are $\lambda_1, \ldots, \lambda_4$ sorted by their real parts

$$\mathcal{R}(\lambda_1) \leq \mathcal{R}(\lambda_2) \leq \mathcal{R}(\lambda_3) \leq \mathcal{R}(\lambda_4).$$

Denote $V = (v_1, \ldots, v_4) \in \mathbb{C}^{4\times 4}$, in which $\mathbb{C}^{4\times 4}$ denotes the complex vector space of 4×4 complex matrices, then we have that the solution to (3.18) satisfies

$$\begin{pmatrix} Q(\tau) \\ P(\tau) \end{pmatrix} = Ve^{\Delta\tau}V^{-1}\begin{pmatrix} Q(0) \\ P(0) \end{pmatrix} = Ve^{\Delta\tau}V^{-1}\begin{pmatrix} \widetilde{I}_2 \\ 0 \end{pmatrix},$$

where $\Delta := V^{-1}HV = \text{diag}(\lambda_1, \ldots, \lambda_4)$.

Furthermore, define

$$f_\lambda(\lambda) = |\lambda \widetilde{I}_4 - H| = \lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + j, \tag{3.19}$$

we can finally prove the following proposition for global existence and verification.

Proposition 3.3. *For $0 < \gamma < 1$, define the admissible set as*

$$\mathcal{A}_\gamma(0, T) := \left\{ (\beta, I) \middle| \begin{array}{c} (\beta, I) \text{ such that } W_t^{\widetilde{C}} > 0, \\ \text{and SDE (3.4) has a unique strong solution.} \end{array} \right\}. \tag{3.20}$$

If

$$\widetilde{\Delta} > 0, \; q < 0, \; s < \frac{q^2}{4}, \tag{3.21}$$

$$det|Q(\tau)| \neq 0 \text{ and } P(\tau)/Q(\tau) < 0 \text{ for } \forall \tau \in (0, T], \tag{3.22}$$

then the candidate solution $G(t, W_t^{\widetilde{C}}, X_t)$ exists in $[0, T]$ and equals $V(t, W_t^{\widetilde{C}}, X_t)$. Moreover, the strategy (β^*, I^*) given by (3.15) and (3.16) is the optimal portfolio and insurance strategy. The expressions of $\widetilde{\Delta}$, q, and s are given in Appendix C.

4. Numerical results

4.1. Model calibration

For the financial market, we use monthly U.S. data from June 1961 to December 2020 to estimate the parameters. We use zero-coupon nominal yields from Gürkaynak et al. (2007) with eight maturities: three months, six months, one year, two years, three years, five years, seven years, and ten years. The realized inflation index is obtained from CRSP's Consumer Price Index for All Urban Consumers (CPI-U NSA index). The equity index is based on the CRSP's value-weighted NYSE/Amex/Nasdaq index, which includes the dividend payments.

We estimate the parameters with the help of a Kalman filter (see Appendix D for details) and present the results in Table 1 and Figure 1. Similar to Koijen et al. (2011), we have $\kappa_1 > \kappa_2$, which means that expected inflation is more persistent than the real short rate. For the innovations, we capture the negative correlation between the real short rate and expected inflation ($\sigma_{2(1)} < 0$). For the equity index process, we find the risk premium is decreasing with the real short rate and expected inflation ($\mu_{1(1)}, \mu_{1(2)} < 0$). Moreover, the unconditional price of risk, Λ_0, is negative for the real short rate and expected inflation but positive for the equity index. Finally, all the parameters in the conditional price of risk, Λ_1, are negative,

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Table 1: *Estimation results for the financial market.*

Parameter	Estimate	Parameter	Estimate	Parameter	Estimate
Average short rate & average expected inflation					
δ_r	0.01256	δ_R	0.05166	δ_{π^e}	0.03879
Two-factor process					
κ_1	0.62591	κ_2	0.19710	$\sigma_{1(1)}$	0.02056
$\sigma_{2(1)}$	−0.00665	$\sigma_{2(2)}$	0.01476		
Realized inflation process					
$\sigma_{\Pi(1)}$	0.00033	$\sigma_{\Pi(2)}$	0.00181	$\sigma_{\Pi(3)}$	0.01286
Equity index process					
μ_0	0.04660	$\mu_{1(1)}$	−1.97908	$\mu_{1(2)}$	−1.41777
$\sigma_{S(1)}$	−0.02016	$\sigma_{S(2)}$	−0.01799	$\sigma_{S(3)}$	−0.00799
$\sigma_{S(4)}$	0.15400				
Prices of risk of real short rate, inflation, and equity					
$\Lambda_{0(1)}$	−0.00390	$\Lambda_{0(2)}$	−0.17056	$\Lambda_{0(4)}$	0.28216
$\Lambda_{1(1,1)}$	−9.92622	$\Lambda_{1(2,2)}$	−9.98032	$\Lambda_{1(4,1)}$	−14.15060
$\Lambda_{1(4,2)}$	−10.37218				

The parameters in the table are annualized. $\Lambda_0(1)$, $\Lambda_0(2)$, $\Lambda_1(4, 1)$, and $\Lambda_1(4, 2)$ can be obtained by solving three equations: $\delta_R = \delta_r + \delta_{\pi^e} - \sigma_\Pi^\top \Lambda_0$, $\sigma_S^\top \Lambda_0 = \mu_0$, $\sigma_S^\top \Lambda_1 = \mu_1$. So, there are 21 parameters in total to be estimated. More details can be found in Appendix D.

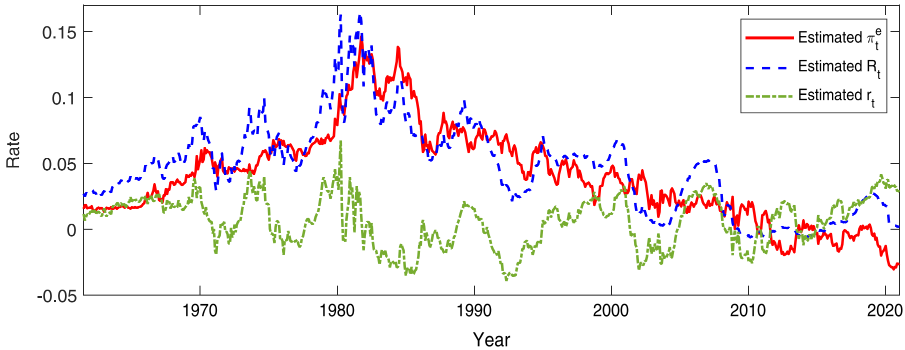

Figure 1. *Estimated short rate and expected inflation process. The solid line is the estimated expected inflation π_t^e. The dashed line is the estimated nominal short rate R_t. The dash-dotted line is the estimated real short rate r_t.*

which means the price of risk is decreasing with two factors X_t. Figure 1 shows the estimated short rates and expected inflation.

We assume that the pension member enters at age 22 and retires at age 66, which implies $T = 44$. The pension member allocates his or her wealth among 3-year nominal bonds, 10-year nominal bonds, 10-year inflation-linked bonds, the equity index, and cash ($T_1 = 3, T_2 = T_3 = 10$), and also purchases life insurance. Moreover, the risk-aversion coefficient γ equals 5 for the pension member.

Similar to Koijen et al. (2011), we suppose that the growth rate g_t^R in the real contribution rate (2.6) follows

$$g_t^R = 0.1682 - 0.00646(22 + t) + 0.00006(22 + t)^2,$$

which corresponds to an individual with a high school education in the estimates of Cocco et al. (2005) and Munk and Sorensen (2010). The initial real contribution rate C_0 is set to be \$1 kUSD (per annum).

For the individual mortality rate, we use the U.S. data of males in the "2017 Period Life Table for the Social Security area population." Following Forfar et al. (1988), we assume the force of mortality

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Table 2: *Estimation results for the force of mortality.*

Model	$GM_a^{3,3}(x)$	BIC	254255.08
Parameters	a_1	a_2	a_3
values	-1.196773×10^{-3}	-1.406588×10^{-4}	-1.568144×10^{-5}
Parameters	a_4	a_5	a_6
values	-5.956450×10^{0}	9.006499×10^{-2}	-4.710629×10^{-4}

μ_x follows a general form of the Gompertz-Makeham approach

$$\mu_x = GM_a^{s_1,s_2}(x) = \sum_{i=1}^{s_1} a_i(x-22)^{i-1} + \exp\left\{\sum_{i=s_1+1}^{s_1+s_2} a_i(x-22)^{i-s_1-1}\right\}, \quad 22 \le x \le 67,$$

where the change of location $x-22$ is used to improve the significance of parameters. After some trials, we find that a few parameters become insignificant when estimating $GM_a^{4,0}$ or $GM_a^{0,5}$. Therefore, we test all the combinations of s_1 and s_2 in 3×4 and pick up the $GM_a^{s_1,s_2}$ model with the lowest Bayesian information criterion (BIC) with all parameters significant. Table 2 shows the estimation results for the force of mortality.

4.2. Sensitivity of optimal strategies with respect to the age

In this subsection, we demonstrate how the individual's optimal strategies change with age. We use the Monte-Carlo method with 10,000,000 simulations and a time-step of one year. We plot the expected annual optimal investment strategies $E[\beta_t^*]$ and insurance strategy $E[I_t^*]$ and the corresponding confidence intervals in Figures 2 and 3, respectively.

For the investment strategies, the first two graphs in Figure 2 illustrate the individual's allocations to 3-year nominal bonds and 10-year nominal bonds. Specifically, the individual shorts 3-year nominal bonds and longs 10-year nominal bonds. He gradually reduces the absolute exposure to these bonds before reaching age 60 after which he increases the exposure to nominal bonds up to retirement. The individual holds long positions in the stock and 10-year inflation-linked bonds and the allocations are relatively insensitive to the age. Furthermore, the individual holds a smaller proportion of stocks compared to bonds. It should be noted that Figure 2 is based on the expected optimal investment strategy. The 50% confidence intervals are relatively wide, indicating significant uncertainty around the optimal investment strategy. We perform a sensitivity analysis of the individual's strategy with respect to the two factors X_t in the next subsection.

To gain some intuition, we decompose the optimal investment strategy in (3.15) into the following form

$$\beta_t^* = \underbrace{\frac{(\Sigma^\top)^{-1}}{\gamma}\Lambda_t}_{\text{standard myopic demand}} + \underbrace{\left(1-\frac{1}{\gamma}\right)(\Sigma^\top)^{-1}\sigma_\Pi}_{\text{inflation hedging demand}} + \underbrace{(\Sigma^\top)^{-1}\Sigma_X^\top \frac{1}{f_1(t,X_t)}\frac{\partial f_1(t,X_t)}{\partial X^\top}}_{\text{intertemporal hedging demand}}. \tag{4.1}$$

Among three components, the standard myopic demand (SMD) exploits the risk-return trade-off of the assets. The inflation hedging demand (IFHD) accounts for the individual's desire to hedge against realized inflation Π_t in (2.2) (σ_Π is the volatility term of Π_t). Lastly, the intertemporal hedging demand (ITHD) depends on the investment horizon and reflects the individual's desire to hedge against changes in future investment opportunity sets.

We present the unconditional expectations of SMD and IFHD in Table 3. The expected SMD is a constant vector because X_t in (2.1) follows a normal distribution $N(0, \Sigma_t)$, where $\Sigma_t = \int_0^t e^{-K_x(t-s)}\Sigma_X\Sigma_X^\top e^{-K_X^\top(t-s)}ds$. Moreover, IFHD_4 is zero as the fourth entry of σ_Π is zero, which is implied by the assumption that $(\sigma_1, \sigma_2, \sigma_\Pi, \sigma_S)^\top$ is lower triangular.

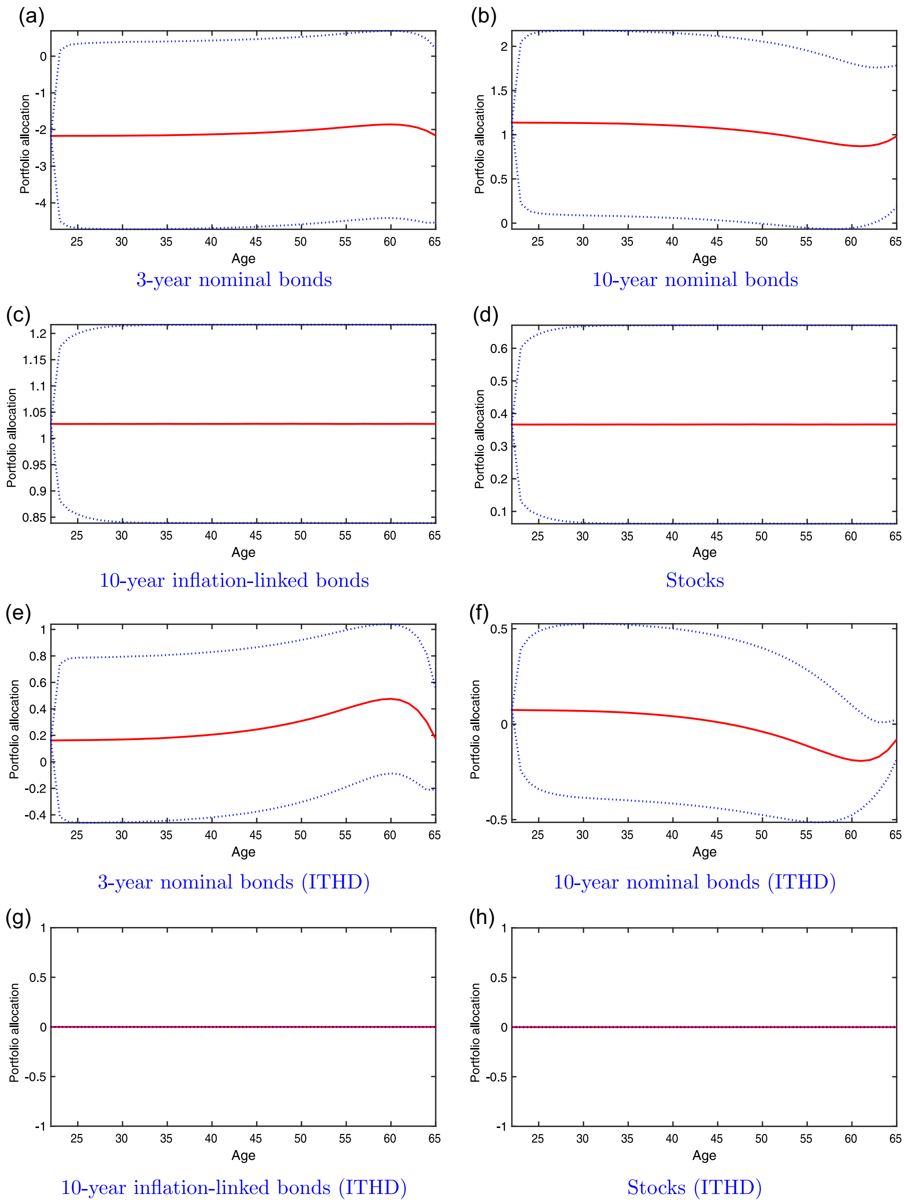

Figure 2. *Expected annual optimal investment strategies (optimal investment strategy v.s. ITHD). The red solid lines are expected values. The blue dotted lines are 50% confident intervals.*

We also plot the unconditional expectations of ITHD in Figure 2. The $IFHD_3$ and $IFHD_4$ are zero since the third and fourth rows of Σ_X are zero. Compared with the optimal investment strategy in Figure 2, we observe that it is ITHD that mainly affects the evolution of the individual's expected investment strategies.

For the insurance strategy, the first graph in Figure 3 depicts the expected insurance premium paid by the individual, which is hump-shaped and peaks at age 59. While our model does not restrict $I_t^* > 0$, the 95% confidence interval for I_t^* is positive. The second graph illustrates that the expected face value $E[I_t^*]/\mu_{x+t}$ is also hump-shaped but peaks at age 50. Equation (3.16) shows that the optimal insurance

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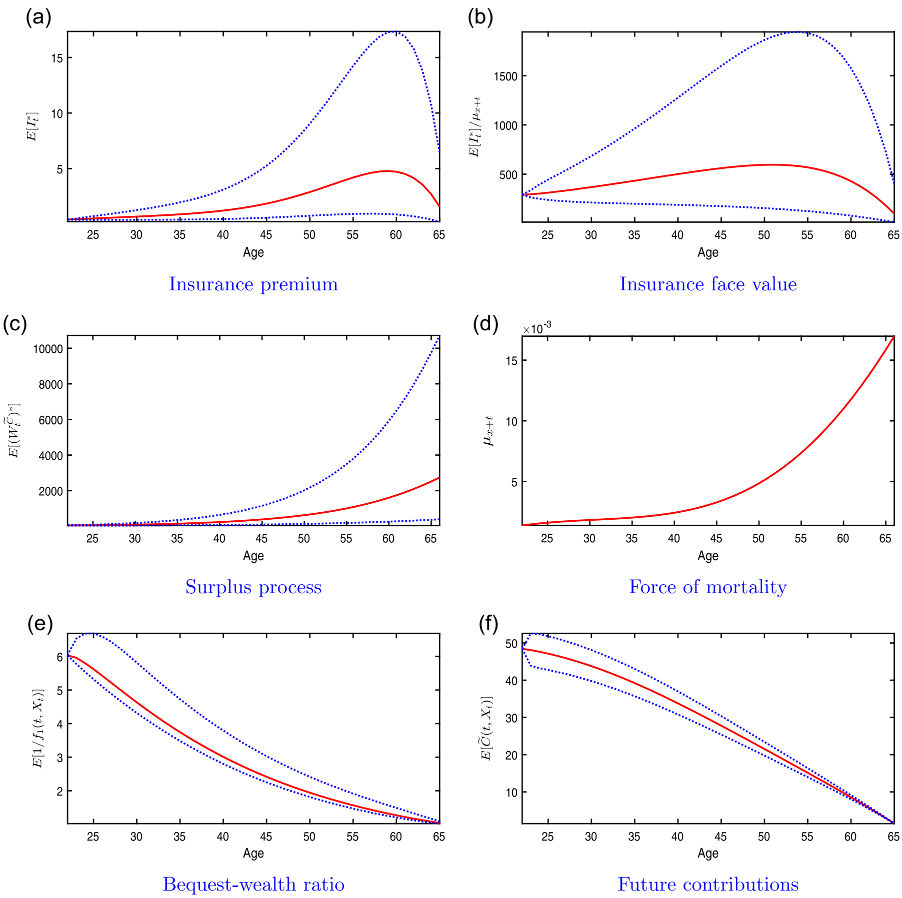

Figure 3. *Expected annual optimal insurance strategy and its components. The red solid lines are expected values. The blue dotted lines are 95% confident intervals.*

premium depends on four components, the force of mortality μ_{x+t}, the optimal surplus process $(W_t^{\widetilde{C}})^*$, the bequest-wealth ratio $1/f_1(t, X_t)$, and the future contributions $\widetilde{C}(t, X_t)$. Specifically, rearranging terms in (3.16) leads to

$$\frac{1}{f_1(t, X_t)} = \frac{(W_t^R)^* + I_t^*/\mu_{x+t}}{(W_t^R)^* + \widetilde{C}(t, X_t)},$$

which is the ratio of the bequest to the current account balance plus future contributions, and thus termed the bequest-wealth ratio.

The last four graphs of Figure 3 demonstrate that the expected surplus process $E[(W_t^{\widetilde{C}})^*]$ and the force of mortality μ_{x+t} are increasing with age, while the bequest-wealth ratio $1/f_1(t, X_t)$ and the future contributions $\widetilde{C}(t, X_t)$ are decreasing with age. At early ages, the bequest-wealth ratio is high and thus the insurance demand is primarily driven by the increase in the surplus process. The bequest-wealth ratio decreases to 1 when the individual is near retirement, and as a result of this, the effect of the surplus process vanishes and the depletion of future contributions reduces the insurance demand. Therefore, the optimal face value is humped-shaped with a peak at age 60. Moreover, because the force of mortality grows at increasing rates (especially during mid-to-retirement ages), the expected insurance face value peaks earlier than the insurance premium.

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Table 3: *Expected standard myopic demand and inflation hedging demand.*

	SMD_1	SMD_2	SMD_3	SMD_4
Values	−0.534693	0.425144	0.227675	0.366445
	$IFHD_1$	$IFHD_2$	$IFHD_3$	$IFHD_4$
Values	−1.80268	0.637749	0.800000	0.000000

SMD_i is the ith entry of the standard myopic demand vector. $IFHD_i$ is the ith entry of the inflation hedging demand vector.

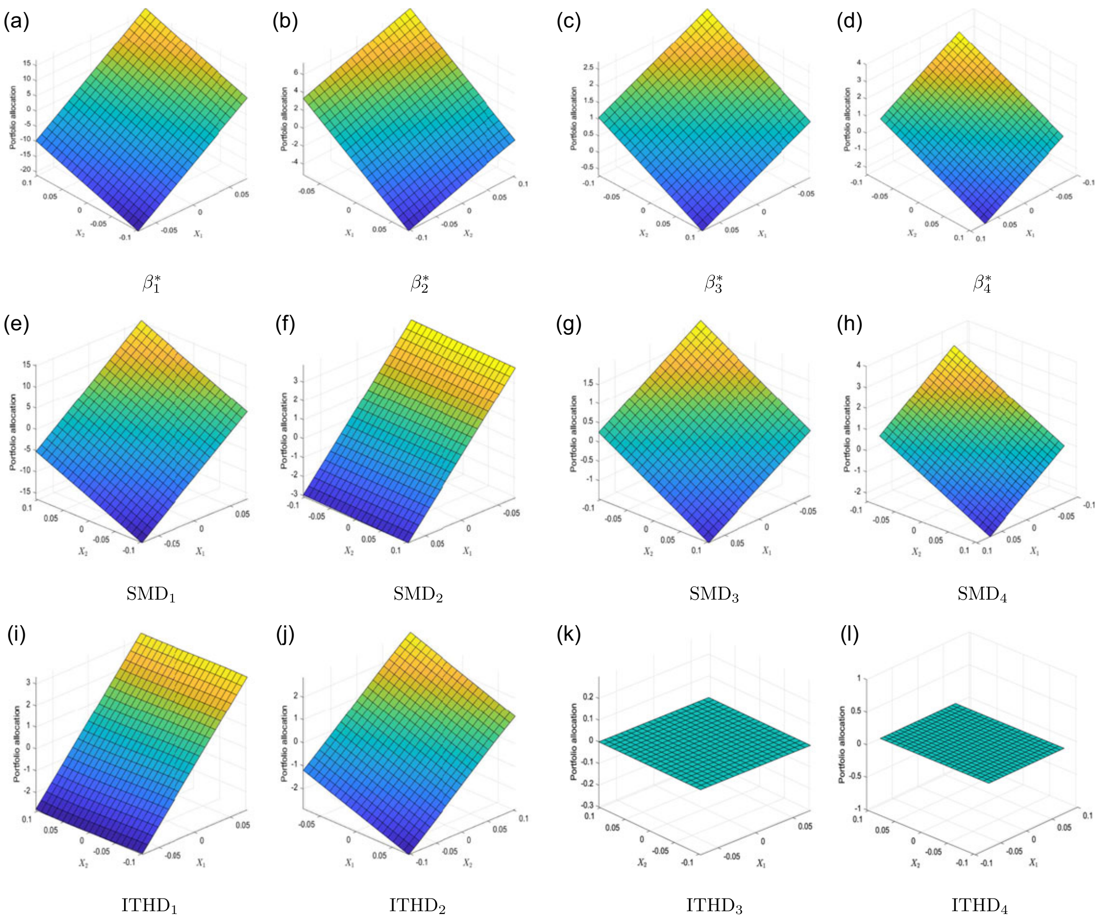

Figure 4. *Individual's investment strategy at $t = T/2$ with respect to two factors X_t.*

4.3. Sensitivity of optimal strategies with respect to the two factors X_t

This section conducts the static analysis of optimal strategies with two factors X_t. For all the figures in this section, we set the range of X_1 as $[-0.0736, 0.0736]$ and the range of X_2 as $[-0.1032, 0.1032]$, which cover eight standard deviations of $X_{1,T}$ and $X_{2,T}$, respectively.

Figure 4 illustrates the investment strategy β^* of the pension member with respect to two factors. Notably, the individual prioritizes 3-year nominal bonds among all three types of bonds. Moreover, an increase in the real short-rate factor X_1 prompts the individual to sell more 10-year nominal and 10-year inflation-linked bonds and buy more 3-year nominal bonds. On the other hand, when the inflation factor X_2 increases, the individual purchases more 3-year nominal bonds but fewer 10-year nominal and

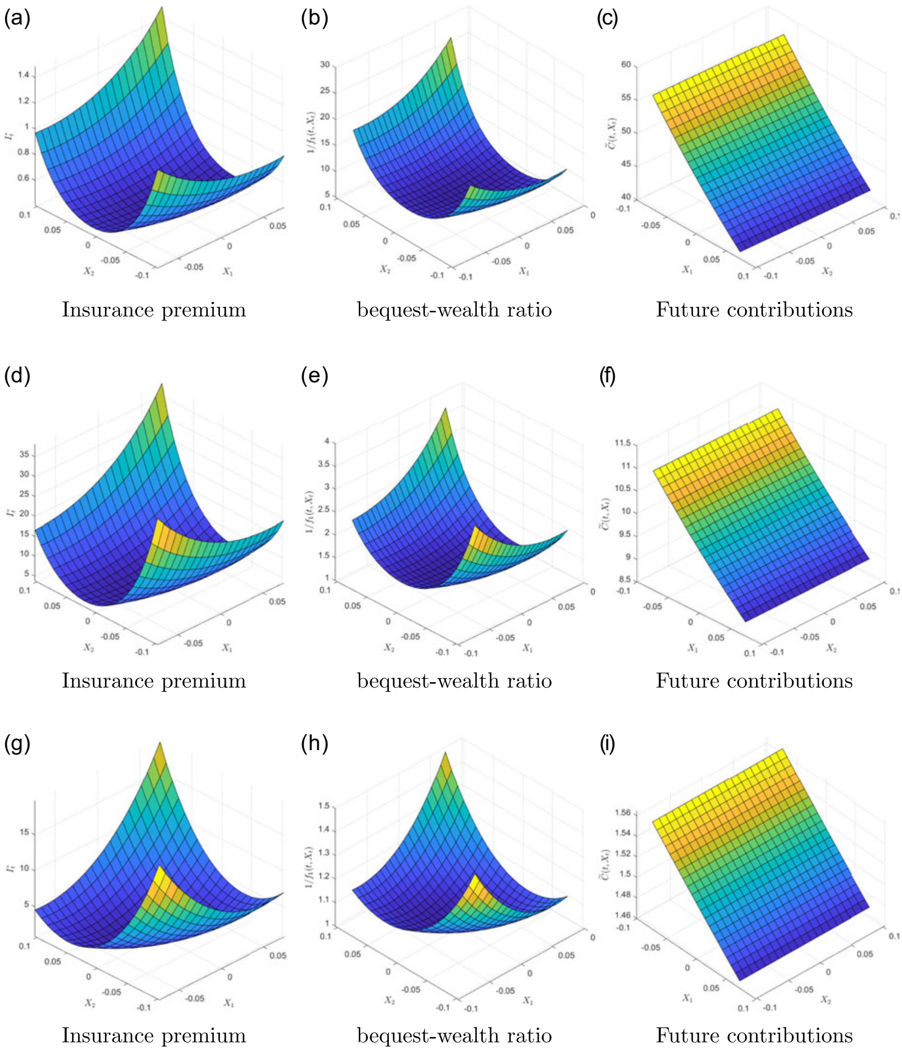

Figure 5. *Individual's optimal insurance with respect to two factors X_t. The figures in the same row share the same age, 22, 59, and 65, respectively. In each row, the left figure is the optimal insurance premium I_t^*. The middle figure is the bequest-wealth ratio $1/f_1(t, X_t)$. The right figure is the future contributions $\widetilde{C}(t, X_t)$.*

inflation-linked bonds. Finally, the individual shorts more equity when either the real short-rate factor X_1 or the inflation factor X_2 increases.

In addition to the investment strategies β^*, we have decomposed them into three components: standard myopic demand (SMD), inflation hedging demand (IFHD), and intertemporal hedging demand (ITHD), based on (4.1). While the IFHD is a constant vector as shown in Table 2, SMD and ITHD are represented in Figure 4. Our analysis reveals that SMD plays a more prominent role in determining an individual's investment strategies, as evidenced by the larger range of SMD as compared to ITHD. Specifically, SMD significantly influences an individual's allocation to 3-year nominal bonds, 10-year inflation-linked bonds, and equity. Notably, the allocations to inflation-linked bonds and equity due to ITHD are both zero, as the third and fourth rows of $\Sigma_X^\top$ in ITHD from (4.1) are zero.

Figure 5 reveals that the individual's demand for life insurance follows a U-shaped pattern. In other words, the individual purchases more life insurance when the real short rate and expected inflation are

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both extraordinarily high or both extremely low. This phenomenon is due to the combined effects of the two components in the optimal insurance strategy (3.16). One component is the bequest-wealth ratio $1/f_1(t, X_t)$. Since $\Gamma_2(\tau) < 0$ (guaranteed by Proposition 3.2 and 3.3), we know that $f(t, X_t)$ in $f_1(t, X_t)$ (see (3.10) and (3.11)) follows a quadratic form opening downwards for X_t on the exponential. Therefore, the bequest-wealth ratio $1/f_1(t, X_t)$ exhibits a U-shaped pattern. The other component is the future contributions $\widetilde{C}(t, X_t)$. It decreases with the real short-rate factor X_1 and is insensitive to the inflation factor X_2. We plot the insurance premium with respect to these two components in three ages (22, 59, and 65) in Figure 5. At each age, the optimal surplus process in the insurance premium (see expression (3.16)) is set to be the expected surplus process $E[(W_t^{\widetilde{C}})^*]$, which is 48.42, 1,462.00, and 2,516.00, respectively. We observe that the bequest effect dominates the insurance demand throughout the individual's lifetime when comparing $E[(W_t^{\widetilde{C}})^*]/f_1(t, X_t)$ with $\widetilde{C}(t, X_t)$.

4.4. Additional results

In this section, we present additional numerical results to complement our main findings. Details are available upon request.

4.4.1. Low risk aversion

While our numerical examples have assumed a risk aversion parameter of $\gamma = 5$, reasonable for pension management, we investigate the impact of varying risk aversion. Under our calibrated parameters (Table 1), we find that for $0 < \gamma < 1$, the condition for the existence of a well-defined solution is violated. Specifically, the function $\Gamma_2(\tau) = P(\tau)/Q(\tau)$ given by (3.12) is positive definite throughout the time horizon, violating condition (3.22) in Proposition 3.3.

4.4.2. Ignoring the risk of pre-retirement death

We consider an individual who ignores the risk of pre-retirement death and thus opts out of life insurance. The "no insurance" individual solves the following optimization problem

$$\begin{aligned} &\sup_{\beta} \ E[U(W_T^{\widetilde{C}})] \\ &s.t. \ dW_t^{\widetilde{C}} = W_t^{\widetilde{C}}[r_t + (\beta_t^\top \Sigma - \sigma_\Pi^\top)(\Lambda_t - \sigma_\Pi)]dt + W_t^{\widetilde{C}}(\beta_t^\top \Sigma - \sigma_\Pi^\top)dZ_t, \end{aligned}$$

which can be viewed as a limiting case of the original optimization problem (2.8), where the risk of pre-retirement death is ignored ($\mu_{x+t} \equiv 0$). The optimal investment strategy in this scenario can be derived from (4.1) by setting $\mu_{x+t} \equiv 0$, that is,

$$\beta_t = \underbrace{\frac{(\Sigma^\top)^{-1}}{\gamma}\Lambda_t}_{\text{standard myopic demand}} + \underbrace{\left(1 - \frac{1}{\gamma}\right)(\Sigma^\top)^{-1}\sigma_\Pi}_{\text{inflation hedging demand}} + \underbrace{(\Sigma^\top)^{-1}\Sigma_X^\top[\Gamma_1(T-t) + \Gamma_2(T-t)X_t]}_{\text{intertemporal hedging demand}}. \tag{4.2}$$

Compared with (4.1), the only difference is the intertemporal hedging demand, which is quantitatively small.

Figure 6(a) plots densities of wealth upon retirement, conditional on no pre-retirement death. Because the individual who recognizes the risk of pre-retirement death purchases life insurance, the accumulated wealth at retirement will be lower if the individual survives to retirement. Figure 6(a) shows the expected payoff upon pre-retirement death (account balance plus insurance payout), conditional on death at a given age. The protection provided by life insurance is advantageous (in terms of expected payoff) if the individual dies before age 59.

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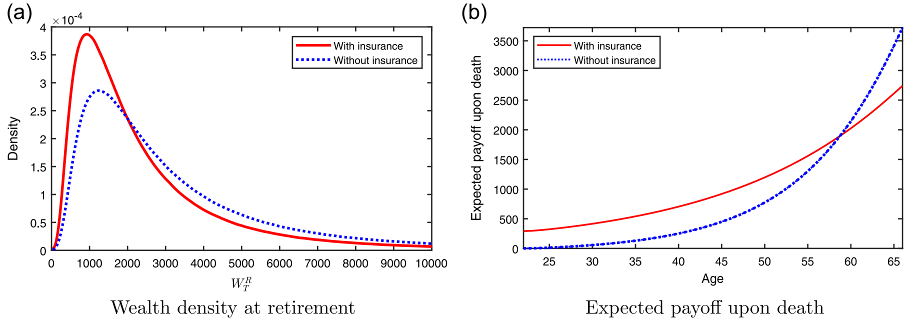

Figure 6. *Expected payoff upon retirement/death.*

4.4.3. Mortality improvement

We have employed a relatively simple Gompertz-Makeham model and ignored the fact that mortality rates may change over time. We conduct additional analyses comparing the Gompertz-Makeham model with and without mortality improvement to see how mortality projection affects optimal investment and insurance strategies. We use mortality data from the Society of Actuaries' Mortality Improvement Model (MIM-2021-v4) to incorporate mortality projections from 2017 to 2061. We design two experiments. The first one is "base year 2017" without mortality improvement. The second is "mortality projection 2017–2061" with mortality improvement.

We find that mortality improvement only marginally affects the optimal investment strategy, the optimal face value of the life insurance, and the expected surplus (accumulated account balance). Mortality improvement increases the optimal insurance premium in early ages but reduces it in mid-to-retirement ages, and the change is mostly due to the force of mortality, that is, how the life insurance is priced.

5. Conclusion

This research studies a DC pension plan management problem under both financial and mortality risks. The pension member can invest their account balance in a stock index, nominal and inflation-linked bonds, and a nominal cash account. Additionally, they can purchase life insurance to hedge against the risk of pre-retirement death. We formulate this pension management problem as a random horizon utility maximization problem and derive its explicit solution under the assumption of constant relative risk aversion (CRRA) utility. We calibrate our model to the U.S. data via a Kalman filter and demonstrate that the pension member's demand for life insurance has a hump-shaped pattern with age and a U-shaped pattern with the real interest rate and expected inflation. The optimal pension account balance resembles a variable annuity with death benefits that are endogenously determined based on various factors, including age, mortality, account balance, future contributions, preferences, and market conditions. Our model offers insights into the design of new insurance products that variable annuities with more flexible death benefits should be offered in the DC account to cater to its member's bequest demand.

The model presented in this research has several limitations that warrant further exploration. First, we assume the pension plan member can purchase (term) life insurance that is available continuously. This assumption is unrealistic but necessary to obtain the closed-form solution. It would be interesting to investigate other types of life insurance, such as single premium and annual renewable term life insurance. Second, our model focuses on term life insurance during the accumulation phase. It is an interesting direction to consider the decumulation phase and explore the role of whole life insurance. Third, we assume that the force of mortality is deterministic. We can consider stochastic mortality models that explicitly account for the uncertainty in future mortality rates. We leave these problems for future research.

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Appendix A

A. *Proof of Proposition 3.1*

Proof. We substitute the candidate solution $G(t, W_t^{\widetilde{C}}, X_t)$ into the HJB equation (3.5) to verify the result. The derivatives of the candidate solution are given by

$$\frac{\partial G}{\partial t} = \frac{\gamma}{1-\gamma}\left(\frac{f_1}{W_t^{\widetilde{C}}}\right)^{\gamma-1}\frac{\partial f_1}{\partial t},\ \frac{\partial G}{\partial w^{\widetilde{C}}} = \left(\frac{f_1}{W_t^{\widetilde{C}}}\right)^{\gamma},\ \frac{\partial G}{\partial X^\top} = \frac{\gamma}{1-\gamma}\left(\frac{f_1}{W_t^{\widetilde{C}}}\right)^{\gamma-1}\frac{\partial f_1}{\partial X^\top},$$

$$\frac{\partial^2 G}{(\partial w^{\widetilde{C}})^2} = -\gamma (W_t^{\widetilde{C}})^{-\gamma-1} f_1^{\gamma},\ \frac{\partial^2 G}{\partial w^{\widetilde{C}}\partial X^\top} = \gamma (W_t^{\widetilde{C}})^{-\gamma} f_1^{\gamma-1}\frac{\partial f_1}{\partial X^\top},$$

$$\frac{\partial^2 G}{\partial X^\top \partial X} = -\gamma (w^{\widetilde{C}})^{1-\gamma} f_1^{\gamma-2}\frac{\partial f_1}{\partial X^\top}\frac{\partial f_1}{\partial X} + \frac{\gamma}{1-\gamma}(w^{\widetilde{C}})^{1-\gamma} f_1^{\gamma-1}\frac{\partial^2 f_1}{\partial X^\top \partial X}.$$

Substitute these derivatives into the HJB Equation (3.5), we can verify that the equality holds. Therefore, $G(t, W_t^R, X_t)$ is the candidate solution to the HJB equation (3.5). Finally, plug $G(t, W_t^R, X_t)$ into (3.7) and (3.8), we can derive the optimal strategies (3.15) and (3.16).

B. *Proof of Proposition 3.2*

Proof. We can apply Theorems 4.1.4. and 4.1.6 in Abou-Kandil et al. (2012) to prove the global existence of $\Gamma_2(\tau)$. It is not difficult to restrict their comparison theorem from a semi-definite matrix case to a definite matrix case. Because $\Gamma_2(\tau)$ exists and $\Gamma_2(\tau) < 0$ for $\forall \tau \in (0, T]$, the existence of the candidate solution $G(t, W_t^{\widetilde{C}}, X_t)$ in (3.9) follows.

To prove the verification theorem, we define the value process for any $(\beta_t, I_t) \in \mathcal{A}_\gamma(0, T)$

$$g^{\beta,I}(s, W_s^{\widetilde{C}}, X_s) := \int_t^s {}_{u-t}p_{x+t}\mu_{x+u} U\left(W_u^{\widetilde{C}} - \widetilde{C}(u, X_u) + \frac{I_u}{\mu_{x+u}}\right)du + {}_{s-t}p_{x+t} G(s, W_s^{\widetilde{C}}, X_s), \tag{B1}$$

where $s \in [t, T]$. By Ito's formula, we have

$$dg^{\beta,I}(s, W_s^{\widetilde{C}}, X_s) = {}_{s-t}p_{x+t}\left\{\mu_{x+s} U\left(W_s^{\widetilde{C}} - \widetilde{C}(s, X_s) + \frac{I_s}{\mu_{x+s}}\right) - \mu_{x+s} G(s, W_s^{\widetilde{C}}, X_s)\right.$$
$$\left. + \mathcal{D}^{\beta,I} G(s, W_s^{\widetilde{C}}, X_s)\right\} ds + g^{\beta,I}(s, W_s^{\widetilde{C}}, X_s) h^{\beta,I}(s, W_s^{\widetilde{C}}, X_s) dZ_s, \tag{B2}$$

where $\mathcal{D}^{\beta,I}$ is the infinitesimal generator defined in (3.6) and $h^{\beta,I}$ satisfies

$$h^{\beta,I}(s, W_s^{\widetilde{C}}, X_s) = \frac{{}_{s-t}p_{x+t} G(s, W_s^{\widetilde{C}}, X_s)}{g^{\beta,I}(s, W_s^{\widetilde{C}}, X_s)}\left[(1-\gamma)(\beta_s^\top \Sigma - \sigma_\Pi^\top) + \gamma \frac{1}{f_1}\frac{\partial f_1}{\partial X}\Sigma_X\right]. \tag{B3}$$

Next, fix $(t, w^R, X) \in [0, T] \times [0, \infty) \times \mathbb{R}^2$ and denote the conditional expectation of the value process as

$$J(t, w^{\widetilde{C}}, X) := E_{t, w^{\widetilde{C}}, X}\left[\int_t^T {}_{s-t}p_{x+t}\mu_{x+s} U\left(W_s^{\widetilde{C}} - \widetilde{C}(s, X_s) + \frac{I_s}{\mu_{x+s}}\right)ds + {}_{T-t}p_{x+t} U(W_T^{\widetilde{C}})\right],$$

where $E_{t,w^{\widetilde{C}},X}[\,\cdot\,]$ is short for $E[\,\cdot\,|W_t^{\widetilde{C}} = w^{\widetilde{C}}, X_t = X]$. Then, we have

$$V(t, W_t^{\widetilde{C}}, X_t) = \sup_{(\beta,I)\in\mathcal{A}_\gamma(0,T)} J(t, W_t^{\widetilde{C}}, X_t). \tag{B4}$$

Finally, we can prove the verification theorem by the following three steps:

Step 1: Verify the optimal strategy (β^*, I^*) is in the admissible set $\mathcal{A}_\gamma(0, T)$.

Recall from (3.15)

$$\beta_t^* = \frac{(\Sigma^\top)^{-1}}{\gamma}(\Lambda_t - \sigma_\Pi) + (\Sigma^\top)^{-1}\Sigma_X^\top \frac{1}{f_1}\frac{\partial f_1}{\partial X^\top} + (\Sigma^\top)^{-1}\sigma_\Pi,$$

which satisfies a linear growth with X_t due to the linear growth of $(\Lambda_t - \sigma_\Pi)$ and $\frac{1}{f_1}\frac{\partial f_1}{\partial X^\top}$. Moreover, substitute (3.15) and (3.16) into (3.4), we have

$$d(W_t^{\widetilde{C}})^* = (W_t^{\widetilde{C}})^* \left\{ r_t + \mu_{x+t}\left(1 - \frac{1}{f_1(t, X_t)}\right) + (\eta_t)^\top(\Lambda_t - \sigma_\Pi)\right\} dt + (W_t^{\widetilde{C}})^*(\eta_t)^\top dZ_t, \tag{B5}$$

where $(\eta_t)^\top = \frac{1}{\gamma}(\Lambda_t^\top - \sigma_\Pi^\top) + \frac{1}{f_1}\frac{\partial f_1}{\partial X}\Sigma_X$. Since the drift term and volatility term of SDE (B5) are almost surely sample continuous, then by Proposition 1.1 in Kraft (2004), we can show that the SDE (3.4) has a unique strong solution under (β^*, I^*). Consequently, $(\beta_t^*, I_t^*) \in \mathcal{A}_\gamma(0, T)$.

Step 2: Verify $J(t, W_t^{\widetilde{C}}, X_t) \le G(t, W_t^{\widetilde{C}}, X_t)$ for any $(\beta, I) \in \mathcal{A}_\gamma(0, T)$.

Before we proceed, we need the following useful lemma.

Lemma B.1. *Assume a n-dimensional stochastic process $\widetilde{X}_t$ is driven by a m-dimensional Brownian motion $\widetilde{Z}$*

$$d\widetilde{X}_t = \mu(t, \widetilde{X}_t)dt + \sigma(t)d\widetilde{Z}_t,\ \ \widetilde{X}_0 = \widetilde{x}_0,$$

where $\widetilde{x}_0$ is a constant n-dimensional vector, $\mu(t, \widetilde{X}) : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a borel function and $\sigma(t){:}(0, \infty) \to \mathbb{R}^n \times \mathbb{R}^m$ a continuous function satisfying

$$||\mu(t, \widetilde{X}_t) - \mu(t, \widetilde{Y}_t)||_2 \le k||\widetilde{X}_t - \widetilde{Y}_t||_2,\ ||\mu(\cdot, 0)||_2 + ||\sigma(\cdot)||_2 \in L^2(0, T; \mathbb{R}), \forall T > 0,$$

where $||\cdot||_2$ is the Euclidean norm and $L^2(0, T; \mathbb{R})$ represents the set of Lebesgue measurable function $\psi{:}[0, T] \to \mathbb{R}$, such that $\int_0^T |\psi(t)|^2 dt < \infty$. If a stochastic process $\widetilde{g}(t, \widetilde{X}_t)$, $\widetilde{g} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, grows linearly with respect to $\widetilde{X}_t$ ($||\widetilde{g}(t, \widetilde{X}_t)||_2 \le c_0 + c_1||\widetilde{X}_t||_2$), then we have

$$E[\mathcal{E}(T, \widetilde{g})] = 1,$$

where

$$\mathcal{E}(t, \widetilde{g}) := \exp\left\{ \int_0^t [\widetilde{g}(s, \widetilde{X}_s)]^\top d\widetilde{Z}_s - \frac{1}{2}\int_0^t ||\widetilde{g}(s, \widetilde{X}_s)||_2^2 ds \right\}.$$

Proof. The proof is an extension of Lemma 4.1.1. in Bensoussan (2004) to the case where $\widetilde{X}_t$ and $\mathcal{E}(t, \widetilde{g})$ share the same Brownian motion. □

Next, following (3.5) and (B2), we have

$$g^{\beta,I}(T, W_T^{\widetilde{C}}, X_T) \le g^{\beta,I}(t, W_t^{\widetilde{C}}, X_t)\frac{\mathcal{E}(T, h^{\beta,I})}{\mathcal{E}(t, h^{\beta,I})}, \tag{B6}$$

Recall from (B3), for $s \in [t, T]$,

$$h^{\beta,I}(s, W_s^{\widetilde{C}}, X_s) = \frac{{}_{s-t}p_{x+t}G(s, W_s^{\widetilde{C}}, X_s)}{g^{\beta,I}(s, W_s^{\widetilde{C}}, X_s)}\left[(1-\gamma)(\beta_s^\top\Sigma - \sigma_\Pi^\top) + \gamma\frac{1}{f_1}\frac{\partial f_1}{\partial X}\Sigma_X\right].$$

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It is easy to prove $h^{\beta,I}(s, W_s^{\widetilde{C}}, X_s)$ is subject to a linear growth with respect to X_t. By Lemma 7.1, $\mathcal{E}(t,h)$ is a martingale. Hence, for $\forall(\beta, I) \in \mathcal{A}_\gamma(0,T)$, we have

$$J(t, w^{\widetilde{C}}, X) = E_{t,w^{\widetilde{C}},X}\left[\int_t^T {}_{s-t}p_{x+t}\mu_{x+s}U\left(W_s^{\widetilde{C}} - \widetilde{C}(s,X_s) + \frac{I_s}{\mu_{x+s}}\right)ds + {}_{T-t}p_{x+t}U(W_T^{\widetilde{C}})\right]$$

$$= E_{t,w^{\widetilde{C}},X}[g^{\beta,I}(T, W_T^{\widetilde{C}}, X_T)] \le E_{t,w^{\widetilde{C}},X}\left[g^{\beta,I}(t, w^{\widetilde{C}}, X)\frac{\mathcal{E}(T, h^{\beta,I})}{\mathcal{E}(t, h^{\beta,I})}\right] = G(t, w^{\widetilde{C}}, X). \tag{B7}$$

Step 3: Verify $V(t, W_t^{\widetilde{C}}, X_t) = G(t, W_t^{\widetilde{C}}, X_t)$ under the optimal strategy (β^*, I^*).

Since (β_t^*, I_t^*) maximizes the HJB (3.5) and $G(t, W_t^{\widetilde{C}}, X_t)$ is the solution to (3.5), the equality in (B6) holds

$$g^{\beta^*,I^*}(s', (W_{s'}^{\widetilde{C}})^*, X_{s'}) = g^{\beta^*,I^*}(s, (W_s^{\widetilde{C}})^*, X_s)\frac{\mathcal{E}(s', h^{\beta^*,I^*})}{\mathcal{E}(s, h^{\beta^*,I^*})}, s' \in [s,T],$$

where

$$h^{\beta^*,I^*}(s, (W_s^{\widetilde{C}})^*, X_s) = \frac{{}_{s-t}p_{x+t}G(s, (W_s^{\widetilde{C}})^*, X_s)}{g^{\beta^*,I^*}(s, (W_s^{\widetilde{C}})^*, X_s)}\left[\frac{1-\gamma}{\gamma}(\Lambda_s^\top - \sigma_\Pi^\top) + \frac{1}{f_1}\frac{\partial f_1}{\partial X}\Sigma_X\right].$$

It is easy to verify that h^{β^*,I^*} satisfies the linear growth condition. Therefore, by Lemma 7.1, $\mathcal{E}(t, h^{\beta^*,I^*})$ is a martingale. Then, we have the inequality for $G(t, w^{\widetilde{C}}, X)$

$$V(t, w^{\widetilde{C}}, X) \ge E_{t,w^{\widetilde{C}},X}\left[\int_t^T {}_{s-t}p_{x+t}\mu_{x+s}U\left((W_s^{\widetilde{C}})^* - \widetilde{C}(s,X_s) + \frac{I_s^*}{\mu_{x+s}}\right)ds + {}_{T-t}p_{x+t}U((W_T^{\widetilde{C}})^*)\right]$$

$$= E_{t,w^{\widetilde{C}},X}[g^{\beta^*,I^*}(T, (W_T^{\widetilde{C}})^*, X_T)] = E_{t,w^{\widetilde{C}},X}\left[g^{\beta^*,I^*}(t, w^{\widetilde{C}}, X)\frac{\mathcal{E}(T, h^{\beta^*,I^*})}{\mathcal{E}(t, h^{\beta^*,I^*})}\right] = G(t, w^{\widetilde{C}}, X). \tag{B8}$$

Collecting (B7), (B8), and (B4), we show that $G(t, W_t^{\widetilde{C}}, X_t) = V(t, W_t^{\widetilde{C}}, X_t)$, and (β^*, I^*) given by (3.15) and (3.16) is the optimal portfolio and insurance strategy. The proof is complete.

C. Proof of Proposition 3.3

Proof. Substituting $y = \lambda - \frac{b}{4}$ into (3.19), we have

$$f_y(y) = y^4 + qy^2 + ry + s,$$

where $q = (8c - 3b^2)/8$, $r = (b^3 - 4bc + 8d)/8$, and $s = (-3b^4 + 256j - 64bd + 16b^2c)/256$. Moreover, the discriminant of $f_y(y)$ is given by

$$\widetilde{\Delta} = -4q^3r^2 - 27r^4 + 256s^3 + 16q^4s + 144qr^2s - 128q^2s^2.$$

According to Rees (1922), once condition (3.21) is satisfied, (3.19) has four distinct real roots, that is, the Hamiltonian matrix H has four different real eigenvalues, which guarantees its diagonalizability and the full rank of its eigenvector matrix V. By Radon's lemma (see Theorem 3.1.1. in Abou-Kandil et al., 2012), we we have $\Gamma_2(\tau) = P(\tau)/Q(\tau)$ and the existence and negative definiteness of $\Gamma_2(\tau)$ from (3.22). Since $\Gamma_2(\tau)$ exists and $\Gamma_2(\tau) < 0$ for $\tau \in (0,T]$, the candidate solution $G(t, W_t^{\widetilde{C}}, X_t)$ in (3.9) exists globally. □

Similar to Appendix B, we can prove the verification theorem in three steps. There are only two differences: first, in Step 1, we need to check $(W_t^{\widetilde{C}})^* > 0$. It is easy to see the solution to (B5) satisfies

$$(W_t^{\widetilde{C}})^* = W_0^{\widetilde{C}} \exp\left\{\int_0^t \left[r_s + \mu_{x+s}\left(1 - \frac{1}{f_1(s,X_s)}\right) + \eta_s^\top(\Lambda_s - \sigma_\Pi) - \frac{1}{2}\eta_s^\top\eta_s\right]ds + \int_0^t \eta_s^\top dZ_s\right\},$$

which is positive and satisfies the requirement of the admissible set (3.20). The argument for the strong solution is the same as Step 1 in Appendix B.

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Second, in Step 2, we can use Fatou's lemma rather than Lemma B.1 to prove the inequality (B7), as the value process (B1) is bounded below by zero when $0 < \gamma < 1$. Define

$$\Psi(s) := \int_t^s ||g^{\beta,I}(u, W_u^{\widetilde{C}}, X_u)h^{\beta,I}(u, W_u^{\widetilde{C}}, X_u)||_2^2 du,$$

and $\tau_n := T \wedge \inf\{s \in [t, T] | \Psi(s) \geq n\}, n \in \mathbb{N}$. For $s \in [t, \tau_n]$, the stochastic integral $\int_t^s g^{\beta,I}(u, W_u^{\widetilde{C}}, X_u)h^{\beta,I}(u, W_u^{\widetilde{C}}, X_u)dZ_u$ is a martingale. Then, following (3.5) and (B2), we have

$$g^{\beta,I}(\tau_n, W_{\tau_n}^{\widetilde{C}}, X_{\tau_n}) \leq g^{\beta,I}(t, W_t^{\widetilde{C}}, X_t) + \int_t^{\tau_n} g^{\beta,I}(s, W_s^{\widetilde{C}}, X_s)h^{\beta,I}(s, W_s^{\widetilde{C}}, X_s)dZ_s. \tag{C1}$$

Because $\lim_{n\to\infty} \tau_n = T$ and $g^{\beta,I}(t, W_t^{\widetilde{C}}, X_t) \geq 0$ for any $t \in [0, T]$ under $0 < \gamma < 1$, we can show

$$\begin{aligned} J(t, w^{\widetilde{C}}, X) &= E_{t,w^{\widetilde{C}},X}\left[\int_t^T {}_{s-t}p_{x+t}\mu_{x+s} U\left(W_s^{\widetilde{C}} - \widetilde{C}(s, X_s) + \tfrac{I_s}{\mu_{x+s}} \right) ds + {}_{T-t}p_{x+t} U(W_T^{\widetilde{C}}) \right] \\ &= E_{t,w^{\widetilde{C}},X}[g^{\beta,I}(T, W_T^{\widetilde{C}}, X_T)] \leq \lim_{n\to\infty} E_{t,w^{\widetilde{C}},X}[g^{\beta,I}(\tau_n, W_{\tau_n}^{\widetilde{C}}, X_{\tau_n})] \leq g^{\beta,I}(t, w^{\widetilde{C}}, X) = G(t, w^{\widetilde{C}}, X), \end{aligned}$$

for $\forall(\beta, I) \in \mathcal{A}_\gamma(0, T)$, where the first inequality is by Fatou's lemma and the second inequality can be obtained by taking conditional expectation on both sides of (C1). The proof is complete.

D. Estimation details for financial market

Denote $K_t = (X_{1,t}, X_{2,t}, \log \Pi_t, \log S_t)^\top$, then the underlying states in the financial market are given by

$$dK_t = (\theta_0 + \theta_1 K_t)dt + \Sigma_K dZ_t,$$

where

$$\theta_0 = \begin{pmatrix} 0_{2\times1} \\ \delta_{\pi^e} - \frac{1}{2}\sigma_\Pi^\top\sigma_\Pi \\ \delta_R + \mu_0 - \frac{1}{2}\sigma_S^\top\sigma_S \end{pmatrix}, \theta_1 = \begin{pmatrix} -K_X & 0_{2\times2} \\ e_2^\top & 0_{1\times2} \\ \iota_2^\top - \sigma_\Pi^\top\Lambda_1 + \mu_1^\top & 0_{1\times2} \end{pmatrix}, \Sigma_K = \begin{pmatrix} \Sigma_X \\ \sigma_\Pi^\top \\ \sigma_S^\top \end{pmatrix},$$

e_i represents the ith unit vector in $\mathbb{R}^2$ and $\iota_2 = (1, 1)^\top$. By Ito's formula, the transition equation for states follows

$$K_{t+\Delta t} = \Upsilon_1 + \Psi_1 K_t + \epsilon_{t+\Delta t}, \;\; \epsilon_{t+\Delta t} \overset{i.i.d.}{\sim} N(0_{4\times1}, \Sigma_\epsilon), \tag{D1}$$

where

$$\Upsilon_1 = \int_0^{\Delta t} e^{\theta_1(\Delta t - s)}\theta_0 ds, \;\; \Psi_1 = e^{\theta_1 \Delta t}, \;\; \Sigma_\epsilon = \int_0^{\Delta t} e^{\theta_1(\Delta t - s)}\Sigma_K\Sigma_K^\top(e^{\theta_1(\Delta t - s)})^\top ds.$$

For monthly data, we set $\Delta t = 1/12$. Every month, there are 10 observations in the financial market: inflation index, equity index, and yield rate of nominal zero-coupon bonds with eight maturities. Following Koijen et al. (2011), we also assume that the yield rates are observed with independent errors. Let $R^Y(t, \tau_i), i = 1, 2, ..., 8$ denote the yield rates of nominal zero-coupon bonds at time t with maturity $\tau_i, i = 1, 2, ..., 8$, then we have the measurement equation for the states

$$L_t = \Upsilon_2 + \Psi_2 K_t + \eta_t, \; \eta_t \overset{i.i.d.}{\sim} N(0_{10\times1}, \Sigma_\eta), \tag{D2}$$

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where $L_t = (R^Y(t, \tau_i)_{i=1,2,...,8}, \log \Pi_t, \log S_t)^\top$ is the observation vector. Moreover, the coefficients in (D2) are

$$\Upsilon_2 = \begin{pmatrix} -A_0(\tau_1)/\tau_1 \\ \vdots \\ -A_0(\tau_8)/\tau_8 \\ 0_{2\times 1} \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} -A_1^\top(\tau_1)/\tau_1 & 0_{1\times 2} \\ \vdots & \vdots \\ -A_1^\top(\tau_8)/\tau_8 & 0_{1\times 2} \\ 0_{2\times 2} & \widetilde{I}_2 \end{pmatrix}, \quad \Sigma_\eta = \begin{pmatrix} \chi_1 & & & & \\ & \ddots & & & \\ & & \chi_8 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

where A_0 and A_1 are given by (2.4) and (2.5), respectively, $\widetilde{I}_2$ is the 2nd-order identity matrix, and χ_i, $i = 1, 2, ..., 8$ are measurement errors in yields to be estimated.

Let $\widetilde{L}_t$ denote the set of the past observations $\{L_1, L_2, ..., L_t\}$ for $t = 1, 2, ..., n$ and define the conditional means and variances

$$\bar{K}_{t|t} = E[K_t|\widetilde{L}_t], \;\; \bar{K}_{t+1} = E[K_{t+1}|\widetilde{L}_t], \;\; P_{t|t} = Var(K_t|\widetilde{L}_t), \;\; P_{t+1} = Var(K_{t+1}|\widetilde{L}_t),$$

$$v_t = L_t - E[L_t|\widetilde{L}_{t-1}] = L_t - \Upsilon_2 - \Psi_2\bar{K}_t, \;\; F_t = Var(v_t|\widetilde{L}_{t-1}) = \Psi_2 P_t \Psi_2^\top + \Sigma_\eta.$$

We can then derive the Kalman filter for (D1) and (D2)

$$v_t = L_t - \Upsilon_2 - \Psi_2\bar{K}_t, \; F_t = \Psi_2 P_t \Psi_2^\top + \Sigma_\eta, \; \bar{K}_{t|t} = \bar{K}_t + P_t\Psi_2^\top F_t^{-1} v_t,$$

$$P_{t|t} = P_t - P_t\Psi_2^\top F_t^{-1}\Psi_2 P_t, \; \bar{K}_{t+1} = \Upsilon_1 + \Psi_1\bar{K}_{t|t}, \; P_{t+1} = \Psi_1 P_{t|t}\Psi_1^\top + \Sigma_\epsilon.$$

Denote the vector of all model parameters by ψ, which contains 21 parameters in Table 1 and eight parameters in Σ_η. Then, we can derive the log-likelihood function via "prediction error decomposition" (see Chapter 3.4 of Harvey, 1990).

$$\text{logL}(\widetilde{L}_n|\psi) = \sum_{t=1}^{n} \log p(L_t|\widetilde{L}_{t-1}, \psi) = -\frac{10n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}(\log|F_t| + v_t^\top F_t^{-1} v_t).$$

Finally, we can maximize $\text{logL}(\widetilde{L}_n|\psi)$ to obtain the maximum likelihood estimate for the unknown parameters ψ. Alternative estimation approaches include the expectation-maximization (EM) algorithm and Markov chain Monte Carlo (MCMC) algorithm (see Chapters 7.3.4 and 13.4 of Durbin and Koopman, 2012).

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