The problem of the in-and-circumscribed polygon for a plane quartic curve

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1. Consider a plane curve C of order n and class X; it is to be supposed throughout that C has only ordinary Plücker singularities, *i.e.* nodes, cusps, inflections and bitangents. Through any point P_1 of C there pass, apart from the tangent at P_1 itself, $X - 2$ lines which touch C; let T_{12} be the point of contact of any one of these tangents and P_2 any one of the $n - 3$ further intersections of $P_1 T_{12}$ with C. Through P_2 there pass, apart from the tangent at P_2 itself and the line $P_2 P_1$, $X - 3$ lines which touch C; let T_{23} be the point of contact of any one of these with C and P_3 any one of its $n - 3$ further intersections with C. Proceeding in this way we obtain points $P_4, P_5, \ldots, P_{m+1}$, each line $P_{i-1} P_i$ being a tangent of C. If we can so arrange matters that P_{m+1} coincides with P_1 we obtain a polygon of m sides whose vertices all lie on C and whose sides all touch C, each of the m points of contact being, it must be understood, distinct from the vertices; this polygon is both inscribed and circumscribed to C, and is called an *in-and-circumscribed m-gon* of C. The number of in-and-circumscribed triangles of a plane curve was found by Cayley.[1]

The determination of the number of in-and-circumscribed m-gons of a curve is one of those problems which, as soon as they have been propounded, seem immediately to suggest that a solution will be forthcoming by application of the theory of correspondence. In fact, given a point P_1 of C there are $X - 2$ tangents $P_1 T_{12}$ each of which meets C in $n - 3$ further points—corresponding to P_1 there are $(X - 2)(n - 3)$ positions of P_2. Similarly, to each position of P_2 there correspond $(X - 3)(n - 3)$ positions of P_3, so that to any position of P_1 on C there correspond $(X - 2)(X - 3)(n - 3)^2$ positions of P_3. Proceeding in this manner we find that to any position of P_1 there correspond $(X - 2)(X - 3)^{m-1}(n - 3)^m$ positions of P_{m+1}. We will

[1] *Phil. Trans. Roy. Soc*, 161 (1871), 369-412; or *Papers*, 8, 212-257.

denote the correspondence between the points P_1 and P_{m+1} by S_m; it is clearly a symmetrical correspondence, and if γ_m is its valency and p the genus of C the number of united points of S_m is[1]

$$2(X-2)(X-3)^{m-1}(n-3)^m + 2p\gamma_m.$$

These united points include all the vertices of all the in-and-circumscribed m-gons of C; indeed they include each vertex twice over. For if $A_1 A_2 \ldots . A_m$ is any in-and-circumscribed m-gon we may take P_1 at any vertex and proceed round the polygon in either direction; if P_1 is, for example, at A_1 we may take P_2 to be either of the two vertices A_2, A_m which are contiguous to A_1, and in either case we obtain a position of P_{m+1} at A_1. Thus, if N_m is the number of in-and-circumscribed m-gons of C we have the relation

$$2mN_m = 2(X-2)(X-3)^{m-1}(n-3)^m + 2p\gamma_m - H_m,$$

where H_m is the number of points of C which are united points of S_m without being vertices of in-and-circumscribed m-gons, each of these points being included *according to its proper degree of multiplicity*. We say, following Cayley, that the problem has H_m *heterotypic solutions*. This much is easy; the whole difficulty, and it is not an inconsiderable one, lies in calculating H_m. In order to calculate H_m we have first to discover all those points of C which are united points of S_m without being vertices of in-and-circumscribed m-gons; secondly we have to decide how often each of these points is to be included in the number H_m.

2. Cayley solved the problem in the case when $m = 3$ not by means of correspondence theory but by means of his functional method; he gave indications of the solution by correspondence theory but he was unable satisfactorily to account for the heterotypic solutions, and this matter remained unsettled until it was cleared up later by Zeuthen.[2] We shall consider in this present paper the case when C is a curve of the fourth order; this simplifies the problem somewhat, for although heterotypic solutions can be numerous enough for a quartic curve they are by no means so numerous as for curves of higher orders.

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[1] This is the well-known Cayley-Brill correspondence theorem, the result being first stated by Cayley and afterwards proved by Brill. For a proof see Zeuthen's text-book, referred to below, pp. 205-210.

[2] *Lehrbuch der abzählenden Methoden der Geometrie* (Leipzig, 1914), 249-253.

The problem of the in-and-circumscribed polygon for a plane quartic without multiple points has been solved recently[1]; in this present paper the problem is solved for any plane quartic with only ordinary singularities. The number N_m of in-and-circumscribed m-gons is calculated for values of m up to 10; the detailed work of calculating the number of heterotypic solutions becomes tedious for the larger values of m, but the aim has been to work out the problem to such a stage that the calculation of N_m for larger values of m offers no further theoretical difficulty. The work proceeds step by step; the value of N_3 being known already we first calculate N_4, then N_5 and so on. The results are tabulated at the end of the paper. The curve being a quartic we have $n = 4$ and $2p = X - 6 + \kappa$, where κ is the number of cusps; hence the equation for N_m is

$$2m\, N_m = 2\,(X-2)\,(X-3)^{m-1} + (X - 6 + \kappa)\,\gamma_m - H_m.$$

By considering the relations connecting the successive correspondences S_m we find, as in *C. P.* § 21, that the valency γ_m satisfies the difference equation

$$\gamma_m + (X-6)\,\gamma_{m-1} + (X-3)\,\gamma_{m-2} = 0.$$

Since (*cf.* *C. P.* § 5) $\gamma_1 = X - 6$ and $\gamma_2 = -(X^2 - 13\,X + 38)$, the values of $\gamma_3, \gamma_4, \ldots,$ can be calculated *seriatim* from this difference equation; the fact that $\gamma_1 = X - 6$ causes every γ_m to have $X - 6$ as a factor when m is odd.

3. Before proceeding further one or two remarks must be made concerning the number of times a point of C^4 must be included in H_m when this point is a united point of S_m and is not a vertex of an in-and-circumscribed m-gon. Let P_0 be such a point of C^4; then of those points which correspond to P_0 in S_m a certain number, ν say, coincide with P_0. If then P_1 is taken to be a point of C^4 near to P_0 there are ν points $P^{(1)}_{m+1}, P^{(2)}_{m+1}, \ldots, P^{(\nu)}_{m+1}$ which correspond to P_1 in S_m and which are also near P_0. Suppose now that the coordinates of the points of C^4 in the neighbourhood of P_0 are expressed in terms of a parameter, the point P_0 itself being given by the zero value of the parameter. The parameter of P_1 will then be an infinitesimal. Taking this parameter of P_1 as the principal infinitesimal the ν parameters of the points $P^{(1)}_{m+1}, P^{(2)}_{m+1}, \ldots, P^{(\nu)}_{m+1}$ will be infinitesimals of certain

[1] Edge: "Cayley's problem of the in-and circumscribed triangle"; *Proc. London Math. Soc.* (2), 36 (1933), 142-171. This paper will be referred to as *C. P.*

orders; in all the cases with which we shall be concerned these ν parameters will be infinitesimals of the same order, say of order α. Then *the point* P_0 *makes a contribution* $\alpha\nu$ *to the number* H_m. This rule is due to Zeuthen.[1] When $\alpha = 1$ the number of times that P_0 is to be reckoned as a contribution to H_m is equal to the number of points that correspond to P_0 in S_m and at the same time coincide with P_0; this often happens, but care should always be taken to see that Zeuthen's rule is properly applied. We will, however, in order to shorten the work, not allude to Zeuthen's rule in those cases when $\alpha = 1$.

In-and-circumscribed quadrilaterals.

4. The valency of the correspondence S_4 on the quartic curve C^4 is found, by use of the difference equation, to be

$$\gamma_4 = -\{X^4 - 27X^3 + 261X^2 - 1073X + 1590\};$$

hence the total number of united points of the correspondence S_4 is

$$\begin{aligned} 2(X-2)(X-3)^3 - (X-6+\kappa)(X^4 - 27X^3 + 261X^2 - 1073X + 1590) \\ = -X^5 + 35X^4 - 445X^3 + 2729X^2 - 8190X + 9648 \\ - \kappa(X^4 - 27X^3 + 261X^2 - 1073X + 1590). \end{aligned}$$

We have now to account for the heterotypic solutions.

Let us first consider those heterotypic solutions which are associated with the nodes of C^4. From each node there are $X-4$ tangents to the curve; let the points of contact of those from a particular node D be $d^{(1)}, d^{(2)}, \ldots, d^{(X-4)}$. Suppose P_1 is at D; then any one of the $X-4$ tangents from D, say $Dd^{(1)}$, gives a position of P_2 on the *other* branch of the curve at D. To obtain P_3 we may take any one of the $X-4$ tangents from D other than $Dd^{(1)}$; each of these $X-5$ tangents gives a position of P_3 coinciding with P_1. Then we have a choice of $X-5$ tangents each of which gives a position of P_4 on the other branch of the curve at D, while a final choice of $X-5$ tangents gives P_5 coinciding with P_1. Thus of the $(X-2)(X-3)^3$ points which correspond to P_1 in the correspondence S_4, $(X-4)(X-5)^3$ coincide with P_1. This is true if P_1 is on *either* branch of the curve at D, so that there arise in this way $2\delta(X-4)(X-5)^3$ heterotypic solutions associated with the nodes of C^4, where δ is the number of nodes.

[1] *Loc. cit.*, p. 186. See also Enriques: *Teoria geometrica delle equazioni*, Vol. 1 (Bologna 1929), 160. The statement of this rule in *C. P.* (pp. 151-152) is not as accurate as it might have been; it is not the lengths of infinitesimal arcs that must be considered, but infinitesimal differences of parameters.

Now the tangents to C^4 at a node have each one further intersection with the curve; let the two tangents of the node D meet C^4 again in d_1 and d_2 respectively. Then d_1 and d_2 are also united points of S_4. For suppose P_1 is at d_1. One of the $X - 2$ tangents from d_1 to the curve is d_1D; the remaining intersection of this tangent with C^4 is at D, on the branch which it does not touch; taking this intersection as P_2 there are $X - 4$ tangents from it to C^4 each giving a position of P_3 on the other branch at D; from P_3 there is then a choice of $X - 5$ tangents each of which gives a position of P_4 coinciding with P_2. From P_4 we can then return to P_1 along the tangent Dd_1, thus giving a position of P_5 coinciding with P_1. It is important to notice that, of the $X - 2$ tangents from P_4 to the curve, *two* coincide with the tangent to that branch at the node on which P_4 does not lie. Thus, when P_1 is at d_1, $2(X-4)(X-5)$ of its corresponding points in S_4 coincide with it. In this way there arise $4\delta(X-4)(X-5)$ heterotypic solutions. But, further, each of the tangents, other than d_1D, from d_1 to C^4 meets C^4 in a point which is also a united point of S_4. For let d_{11} be an intersection of C^4 with a tangent $d_1 d_{11}$, other than d_1D, from d_1. Then if P_1 is at d_{11} we may take P_2 at d_1, P_3 at D on that branch of the curve which d_1D does not touch, P_4 again at d_1 and P_5 at d_{11}. We are justified in saying that P_4 may be at d_1 because, in order to pass from P_3 to P_4 we must choose one of the $X - 2$ tangents, other than $P_3 P_2$, from P_3; this condition is not violated here, although $P_3 P_2$ and $P_3 P_4$ are the same tangent, because in this case P_3 is at a node and, as has already been remarked, *two* of the $X - 2$ tangents from P_3 coincide with $P_3 d_1$. Since there are two points d_1, d_2 associated with each node D, and since there are, apart from the tangent at the node, $X - 3$ tangents of C^4 passing through each of them, the number of heterotypic solutions arising in this way is $2\delta(X-3)$. The total number of heterotypic solutions associated with the nodes of C^4 is therefore

$$2\delta(X-4)(X-5)^3 + 4\delta(X-4)(X-5) + 2\delta(X-3).$$

Suppose now that I is a point of inflection of C^4; the tangent at I has one remaining intersection j with C^4, and there are $X - 3$ other tangents from I to the curve. Let P_1 be the remaining intersection of one of these $X - 3$ tangents with C^4; then we may take P_2 at I and P_3 at j. Since, I being an inflection, two of the $X - 2$ tangents from j to C^4 coincide with jI, we may take P_4 to be at I, and then P_5 at P_1, which is therefore a united point of S_4. Hence we have,

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associated with the inflections of C^4, $(X-3)\iota$ heterotypic solutions, ι being the number of inflections of C^4.

5. It remains now to consider those heterotypic solutions associated with the cusps of C^4; here it is a little more difficult to arrive at the result because the application of Zeuthen's rule has to be considered. Let K be a cusp of C^4; there are $X-3$ tangents, other than the cuspidal tangent, of C^4 which pass through K. Suppose P_1 is at K; then any one of these $X-3$ tangents has its remaining intersection P_2 also at K; there are then $X-4$ tangents which may be used for passing from P_2 to P_3, P_3 being also at K; we have then a choice of $X-4$ tangents for P_3P_4 and of $X-4$ tangents for P_4P_5, both P_4 and P_5 being at K. Hence K is a united point of S_4, and the number of corresponding points which coincide with it is $(X-3)(X-4)^3$. To find how many times K is to be counted among the heterotypic solutions we apply Zeuthen's rule: if P_1 is taken near K we also have a position of P_5 near K; when the points near K on the curve are expressed in terms of a parameter in such a way that the value of the parameter at the cusp itself is zero the parameters of P_1 and P_5 will both be infinitesimal. If the parameter of P_1 is taken as the principal infinitesimal the difference between the parameters of P_1 and P_5 will be an infinitesimal of a certain order α and, in order to find how many times K must be reckoned among the heterotypic solutions, it is necessary to take the product of α and the number of points which correspond to K in the correspondence S_4 and coincide with it.

In order to calculate α it will be sufficient to take a particular quartic curve; let us therefore, as on a previous occasion,[1] take the curve for which

$$x : y : 1 = am^2\lambda^3 : a^2m\lambda^2(\lambda^2+1) : m^2\lambda^2 + a^2(\lambda^2+1)^2.$$

Referred to ordinary rectangular Cartesian coordinates this is a bicircular quartic with a cusp at the origin, the parameter of the cusp being $\lambda = 0$. If then we take P_1 to have the parameter $\lambda = \mu$ we find a point P_5 whose parameter is $\lambda = \mu - 8\mu^2$ as far as the second order of μ; the difference between the parameters of P_1 and P_5 is thus $8\mu^2$, and is an infinitesimal of the second order. Hence $\alpha = 2$. Wherefore the number of times that K is to be counted among the heterotypic solutions is $2(X-3)(X-4)^3$.

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[1] *C. P.*, p. 160.

The tangent at the cusp K meets C^4 in one further point, say t, and t occurs among the united points of S_4. Indeed if P_1 is at t the tangent tK has its fourth intersection with C^4 at K, and so P_2 is at K. Any one of the $X-3$ tangents from K then gives P_3 also at K, while any one of the remaining $X-4$ tangents from K gives P_4 at K; we may then take the tangent P_4P_5 to be Kt, giving a position of P_5 at t. It appears then that when P_1 is at t there are $(X-3)(X-4)$ of its corresponding points in S_4 also at t. Further it is found that, when the application of Zeuthen's rule is considered, we have to multiply this number by 2 in order to obtain the number of times which t must be reckoned among the united points of S_4. The total number of heterotypic solutions associated with the cusps of C^4 is therefore

$$2\kappa(X-3)(X-4)^3 + 2\kappa(X-3)(X-4).$$

The total number of heterotypic solutions of the problem is therefore

$$H_4 = 2\delta(X-4)(X-5)^3 + 4\delta(X-4)(X-5) + 2\delta(X-3)$$
$$+ \iota(X-3) + 2\kappa(X-3)(X-4)^3 + 2\kappa(X-3)(X-4).$$

We can now substitute for δ and ι in this expression for H_4 from Plücker's equations, which give

$$2\delta = 12 - X - 3\kappa, \qquad \iota = 3X - 12 + \kappa.$$

We then find, after some reduction,

$$H_4 = -X^5 + 31X^4 - 365X^3 + 2089X^2 - 5862X + 6480$$
$$+ \kappa(-X^4 + 27X^3 - 241X^2 + 897X - 1206).$$

When this is subtracted from the number, already found, of united points of S_4 we obtain

$$8N_4 = 4X^4 - 80X^3 + 640X^2 - 2328X + 3168 - \kappa(20X^2 - 176X + 384)$$

or

$$2N_4 = (X-4)\{X^3 - 16X^2 + 96X - 198 - \kappa(5X-24)\}.$$

It is simpler, especially for higher values of m, to work with $y = X - 4$ instead of with X, and this we shall do. In terms of y we have

$$2N_4 = y\{y^3 - 4y^2 + 16y - 6 - \kappa(5y-4)\}$$

and

$$\gamma_4 = -(y^4 - 11y^2 + 33y^2 - 25y + 2).$$

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In-and-circumscribed pentagons.

6. The valency of S_5 is

$$\gamma_5 = (y-2)(y^4 - 12y^3 + 38y^2 - 20y + 1);$$

hence the total number of united points of S_5 is

$$\begin{aligned} 2(X-2)(X-3)^4 &+ (X-6+\kappa)\gamma_5 \\ &= 2(y+2)(y+1)^4 + (y-2+\kappa)\gamma_5 \\ &= y^6 - 14y^5 + 102y^4 - 192y^3 + 265y^2 - 66y + 8 + \kappa\gamma_5. \end{aligned}$$

We must now enumerate those united points of S_5 which are not vertices of in-and-circumscribed pentagons.

Consider first heterotypic solutions associated with the bitangents of C^4. Through a point of contact of a bitangent there pass $X-3$ tangents of C^4, other than the bitangent itself; let h_1 be the remaining intersection of any one of these $X-3$ tangents with C^4. Then through h_1 there pass $X-3$ further tangents of C^4; let h_2 be the remaining intersection of any one of these tangents with C^4. It is easily seen that h_2 is a united point of S_5; for let P_1 be at h_2. Then we obtain a position of P_2 at h_1 and a position of P_3 at the point of contact of the bitangent. We can then choose the bitangent itself as the tangent P_3P_4, so that P_4 coincides with P_3: we can then take P_5 at h_1 and P_6 at h_2. Thus we have P_6 coinciding with P_1. Since each bitangent gives rise to $2(X-3)$ points h_1 and each point h_1 to $(X-3)$ points h_2 we obtain $2\tau(X-3)^2$ heterotypic solutions associated with the bitangents of C^4, τ being the number of bitangents.

Consider now heterotypic solutions associated with the cusps of C^4. If K is a cusp of C^4 we see, arguing as in the case of the correspondence S_4, that $(X-3)(X-4)^4$ of those points which correspond to K in S_5 coincide with K. In this case however the application of Zeuthen's rule does not lead to the introduction of any further numerical factor; for if P_1 is a point near K any point P_6 which corresponds to P_1 in the correspondence S_5, and which is also near K, is on the opposite side of K to P_1; thus the difference of the two infinitesimal parameters which give the two points P_1 and P_6 must be an infinitesimal of the *same* order as the parameter of P_1. We see also that t, the intersection of C^4 with its tangent at K, is a united point of S_5, and that $(X-3)(X-4)^2$ of its corresponding points coincide with it; here again it is not necessary to multiply this by any numerical factor. Further: if any one of the $X-3$ tangents, other than tK, from t to C^4 meets C^4 again in a point t_1, t_1 is also a united point of S_5 and coincides with $X - 3$ of its corresponding points; in this way we have $(X - 3)^2$ heterotypic solutions associated with each cusp. The aggregate of the heterotypic solutions associated with the cusps of C^4 is therefore

$$\kappa \{(X-3)(X-4)^4 + (X-3)(X-4)^2 + (X-3)^2\}.$$

The total number of heterotypic solutions now obtained is

$$2\tau (X-3)^2 + \kappa \{(X-3)(X-4)^4 + (X-3)(X-4)^2 + (X-3)^2\}$$
$$= 2\tau (y+1)^2 + \kappa \{y^4 (y+1) + y^2 (y+1) + (y+1)^2\}.$$

These are in fact all the heterotypic solutions associated with the singularities of C^4. If we now, using Plücker's equations, substitute

$$2\tau = X^2 - 10X + 32 - 3\kappa = y^2 - 2y + 8 - 3\kappa,$$

and subtract this total number of heterotypic solutions from the number of united points of S_5, the result is

$$y^6 - 14y^5 + 101y^4 - 192y^3 + 260y^2 - 80y + \kappa(-15y^4 + 61y^3 - 95y^2 + 45y).$$

7. We have not yet however arrived at the formula giving ten times the number of in-and-circumscribed pentagons, for there are now heterotypic solutions other than those associated with the singularities of C^4. For suppose *efg* is any in-and-circumscribed triangle of C^4; through any one of its vertices, say through e, there pass, apart from the two sides of the triangle which meet in that vertex, $X - 4$ tangents of C^4; if v_1 is the remaining intersection of any one of these tangents with C^4 then v_1 occurs twice among the united points of S_5 (*cf.* *C. P.*, p. 170). Thus we have, associated with each of the N_3 triangles *efg*, $6y$ heterotypic solutions. Hence the number which we have just obtained by subtracting the number of heterotypic solutions associated with the singularities of C^4 from the number of united points of S_5 is equal to $10N_5 + 6yN_3$. We know that

$$6N_3 = y\{y^3 - 9y^2 + 38y - 24 - 3\kappa(3y - 5)\};$$

hence we obtain

$$10N_5 = y\{y^5 - 15y^4 + 110y^3 - 230y^2 + 284y - 80 - 5\kappa(3y^3 - 14y^2 + 22y - 9)\}.$$

The number of in-and-circumscribed pentagons of any given plane quartic is obtained at once from this formula by substituting the appropriate values for y and κ; the results are given in the table at the end of the paper.

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In-and-circumscribed hexagons.

8. We pass now to the consideration of the correspondence S_6 and its united points. It is found that

$$-\gamma_6 = y^6 - 17y^5 + 100y^4 - 242y^3 + 225y^2 - 61y + 2,$$

and that the total number of united points of S_6 is

$$-y^7 + 21y^6 - 120y^5 + 482y^4 - 649y^3 + 561y^2 - 102y + 8 + \kappa\gamma_6.$$

Let us now enquire as to the nature of the heterotypic solutions that are associated with the singularities of C^4.

We commence by finding the heterotypic solutions that are associated with the nodes of C^4. Suppose, exactly as in the case of the correspondence S_4, that D is a node of C^4; let, again, d_1 be the intersection of C^4 with either of its two tangents at D and d_{11} the remaining intersection of C^4 with any of its tangents from d_1 other than d_1D. We must now introduce also the point d_{111}, this being the remaining intersection of C^4 with any of its tangents other than $d_{11}\,d_1$ from any of the points d_{11}. Associated with each node D of C^4 there are two points d_1, $2\,(X-3)$ points d_{11} and $2\,(X-3)^2$ points d_{111}. All these points are united points of S_6. A discussion similar to that above concerning the correspondence S_4 explains that each branch of the node at D is to be counted $(X-4)\,(X-5)^5$ times among the united points of S_6, each point d_1 is to be counted $2\,(X-4)\,(X-5)^3$ times, each point d_{11} $2\,(X-4)\,(X-5)$ times and each point d_{111} once. Hence the total number of heterotypic solutions associated with the nodes of C^4 is

$$2\delta\,\{(X-4)\,(X-5)^5+2(X-4)\,(X-5)^3+2\,(X-4)\,(X-5)\,(X-3)+(X-3)^2\}$$
$$= 2\delta\,\{y\,(y-1)^5 + 2y\,(y-1)^3 + 2y\,(y^2-1) + (y+1)^2\}.$$

Next there are heterotypic solutions associated with the inflections of C^4. Let I be an inflection, p_1 the remaining intersection of C^4 with any one of the $X-3$ tangents (other than the inflectional tangent itself) from I to the curve; through each point p_1 there pass $X-3$ other tangents of C^4, apart from p_1I; let p_{11} be the remaining intersection of C^4 with any one of those tangents. There are $(X-3)^2$ points p_{11} associated with each inflection I of C^4, and each of them is a united point of S_6; if we take a position of P_1 at p_{11} we can take P_2, P_3, P_4, P_5, P_6, P_7 respectively to be at p_1, I, j, I, p_1, p_{11}, where j is, as before, the remaining intersection of C^4 with its inflectional tangent at I. Hence we have, when P_1 is at p_{11}, a

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position of P_7 coinciding with it; wherefore p_{11} is a united point of S_6. Hence we have, associated with the inflections of C^4, a number of heterotypic solutions equal to $(X-3)^2\iota$ or $(y+1)^2\iota$.

Lastly, in order to obtain the total number of heterotypic solutions associated with the singularities of C^4, we must consider those associated with the cusps. As in the discussion of the heterotypic solutions belonging to S_4, let K be a cusp of C^4 and t the intersection of C^4 with its cuspidal tangent at K. We have now also to introduce the points t_1, where t_1 is the remaining intersection of C^4 with any one of its $X-3$ tangents, other than tK, which pass through t. Arguing as we did for the correspondence S_4 we find that K, t, t_1 are all united points of S_6; of those points which correspond to K in the correspondence S_6 there are $(X-3)(X-4)^5$ which coincide with K; of those which correspond to t there are $(X-3)(X-4)^3$ which coincide with t and of those which correspond to t_1 there are $(X-3)(X-4)$ which coincide with t_1. Moreover, in order to find how many solutions these points contribute to the number H_6 we must in each case multiply by 2 as we see on appealing to Zeuthen's rule. Hence, as there are $X-3$ points t_1 associated with K, the number of heterotypic solutions associated with the cusps of C^4 is

$$2\kappa(X-3)(X-4)\{(X-4)^4+(X-4)^2+X-3\}=2\kappa y(y+1)(y^4+y^2+y+1).$$

9. We have now obtained the total number of heterotypic solutions associated with the singularities of C^4; it is

$$2\delta\{y(y-1)^5+2y(y-1)^3+2y(y^2-1)+(y+1)^2\}$$
$$+\iota(y+1)^2+2\kappa y(y+1)(y^4+y^2+y+1).$$

Since Plücker's equations give

$$2\delta=8-y-3\kappa, \qquad \iota=3y+\kappa,$$

this total number of heterotypic solutions is

$$-y^7+13y^6-52y^5+110y^4-121y^3+105y^2-22y+8$$
$$-\kappa(y^6-17y^5+34y^4-46y^3+31y^2-13y+2).$$

When this number is subtracted from the total number of united points of S_6 the result is

$$8y^6-68y^5+372y^4-528y^3+456y^2-80y-\kappa(66y^4-196y^3+194y^2-48y).$$

Any further united points of S_6 which are not vertices of in-and-circumscribed hexagons are associated with in-and-circumscribed

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polygons with a lesser number of sides. In the first place a vertex of an in-and-circumscribed triangle counts twice among the united points of S_6; if *efg* is an in-and-circumscribed triangle and we take P_1 to be at e then we have two positions of P_7 also at e; we can take the sequence of points $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ to be either e, f, g, e, f, g, e or e, g, f, e, g, f, e. Again: through each vertex of an in-and-circumscribed quadrilateral there pass $X - 4$ tangents of C^4 apart from the two sides of the quadrilateral which meet in that vertex; if u_1 is the remaining intersection of C^4 with any such tangent then u_1 occurs twice among the united points of S_6 (*cf. C. P.*, p. 170). Thus the number just obtained by subtracting the heterotypic solutions associated with the singularities of C^4 from the total number of united points of S_6 is equal to $12N_6 + 6N_3 + 8yN_4$. Since

$$6N_3 = y\{y^3 - 9y^2 + 38y - 24 - 3\kappa(3y - 5)\}$$

and

$$2N_4 = y\{y^3 - 4y^2 + 16y - 6 - \kappa(5y - 4)\},$$

we obtain finally

$$12N_6 = y\{8y^5 - 72y^4 + 387y^3 - 583y^2 + 442y - 56 - \kappa(66y^3 - 216y^2 + 201y - 33)\}.$$

In-and-circumscribed heptagons.

10. For the correspondence S_7 it is found that

$$\gamma_7 = (y - 2)(y^6 - 18y^5 + 111y^4 - 268y^3 + 207y^2 - 42y + 1),$$

while the total number of united points is

$$y^8 - 20y^7 + 203y^6 - 730y^5 + 1823y^4 - 1832y^3 + 1069y^2 - 146y + 8 + \kappa\gamma_7.$$

As in the case of the correspondence S_5 there are heterotypic solutions associated with bitangents and heterotypic solutions associated with cusps. Through each of the points h_2 introduced in considering the correspondence S_5 there pass, apart from the tangent $h_2 h_1$, $X - 3$ further tangents of C^4; if h_3 is the remaining intersection of any one of these tangents with C^4 then it is easily seen that h_3 is a united point of S_7. We thus obtain $2\tau(X - 3)^3$ heterotypic solutions associated with the bitangents of C^4.

If K is a cusp of C^4, t the remaining intersection of C^4 with its cuspidal tangent at K, t_1 the remaining intersection of C^4 with any one of the $X - 3$ tangents other than tK which pass through t, t_{11} the remaining intersection of C^4 with any one of the $X - 3$ tangents other than $t_1 t$ which pass through t_1, then the points K, t, t_1, t_{11} are united points of S_7. Of those points which correspond to K in

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S_7 there are $(X-3)(X-4)^6$ which coincide with K; of the points which correspond to t, $(X-3)(X-4)^4$ coincide with t; of the points which correspond to t_1, $(X-3)(X-4)^2$ coincide with t_1 and of the points which correspond to t_{11}, $X-3$ coincide with t_{11}. Since there are $X-3$ points t_1 and $(X-3)^2$ points t_{11} associated with each cusp of C^4 the number of heterotypic solutions associated with the cusps of C^4 is

$$\kappa\{(X-3)(X-4)^6+(X-3)(X-4)^4+(X-3)^2(X-4)^2+(X-3)^3\}.$$

The total number of heterotypic solutions associated with the singularities of C^4 is therefore

$$2\tau(y+1)^3+\kappa\{y^6(y+1)+y^4(y+1)+y^2(y+1)^2+(y+1)^3\}$$

which, since $2\tau=y^2-2y+8-3\kappa$, is equal to

$$y^5+y^4+5y^3+19y^2+22y+8+\kappa(y^7+y^6+y^5+2y^4-5y^2-6y-2).$$

When this is subtracted from the total number of united points of S_7 the result is

$$y^8-20y^7+203y^6-731y^5+1822y^4-1837y^3+1050y^2-168y \\ -\kappa(21y^6-146y^5+492y^4-743y^3+451y^2-91y).$$

This result includes, as well as the vertices of the in-and-circumscribed heptagons all counted twice, certain heterotypic solutions associated with in-and-circumscribed polygons with a lesser number of sides. In view of the discussions which have already taken place it will be sufficient merely to state that the value of this last expression is $14N_7+10yN_5+6y(y+1)N_3$. On substituting their known values for $10N_5$ and $6N_3$ we find after calculation that

$$14N_7=y\{y^7-21y^6+217y^5-833y^4+2023y^3-2135y^2+1154y-168 \\ -7\kappa(3y^5-23y^4+79y^3-121y^2+73y-13)\},$$

and the different values of N_7 can now be tabulated forthwith.

In-and-circumscribed polygons for which $m>7$.

11. Whatever the length of the calculations required to evaluate the number N_m of in-and-circumscribed m-gons of C^4, the general lines on which the work proceeds should now be clear enough. There are two types of heterotypic solutions occurring in the number H_m which is to be subtracted from the total number of united points of the correspondence S_m; heterotypic solutions of one type are associated

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with the singularities of C while heterotypic solutions of the other type are associated with in-and-circumscribed polygons of a lesser number of sides. The nature of the heterotypic solutions that are associated with the singularities of C^4 depends on the parity of m. If m is odd there is a set of heterotypic solutions associated with the bitangents and a chain of heterotypic solutions associated with the cusps; if $m = 2p + 1$ it is found that the number of heterotypic solutions that arise in this way is

$$2\tau (X-3)^p + \kappa (X-3) \{(X-4)^{2p} + \sum_{\nu=0}^{p-1} (X-3)^\nu (X-4)^{2p-2\nu-2}\}$$
$$= (y^2 - 2y + 8 - 3\kappa)(y+1)^p + \kappa (y+1) \{y^{2p} + \sum_{\nu=0}^{p-1} (y+1)^\nu y^{2p-2\nu-2}\}.$$

If however m is even, there is a set of heterotypic solutions associated with the inflections, a chain of heterotypic solutions associated with the nodes and also a chain of heterotypic solutions associated with the cusps; if $m = 2p$ the total number of these solutions is found to be

$$\iota(X-3)^{p-1} + 2\delta\{(X-4)(X-5)^{2p-1} + (X-3)^{p-1} + 2(X-4)\sum_{\nu=0}^{p-2}(X-3)^\nu(X-5)^{2p-2\nu-3}\}$$
$$+ 2\kappa (X-3)(X-4)\{(X-4)^{2p-2} + \sum_{\nu=0}^{p-2} (X-3)^\nu (X-4)^{2p-2\nu-4}\},$$

the factor 2 in front of κ being demanded by the rule of Zeuthen. This expression may be written

$$(3y+\kappa)(y+1)^{p-1} + (8-y-3\kappa)\{y(y-1)^{2p-1} + (y+1)^{p-1} + 2y\sum_{\nu=0}^{p-2}(y+1)^\nu(y-1)^{2p-2\nu-3}\}$$
$$+ 2\kappa y (y+1)\{y^{2p-2} + \sum_{\nu=0}^{p-2} (y+1)^\nu y^{2p-2\nu-4}\}.$$

It is not so easy to enumerate precisely those heterotypic solutions which are associated with in-and-circumscribed polygons whose sides are less than m in number, as these solutions depend on the divisors of the numbers $m, m-2, m-4, \ldots$. But when the number of heterotypic solutions associated with the singularities of C^4 is subtracted from the total number of united points of S_m the result is the sum of a certain number of terms. Among this sum is always included the expression

$$2mN_m + 2y \Sigma (m-2r)(y+1)^{r-1} N_{m-2r},$$

the summation being with respect to r from 1 to the integral part of $\frac{1}{2}(m-3)$. Also if μ is any divisor of m greater than or equal to 3 there occurs a term $2\mu N_\mu$ in addition to those just enumerated; if $\mu\,(\geqq 3)$ is any divisor of $m-2$ there occurs a term $2\mu y N_\mu$; if $\mu\,(\geqq 3)$ is any divisor of $m-4$ there occurs a term $2\mu y\,(y+1)\,N_\mu$; if $\mu\,(\geqq 3)$ is any divisor of $m-6$ there occurs a term $2\mu y\,(y+1)^2 N_\mu$; and so on. This process accounts for all the terms of the sum.

12. Without going into the details of the arithmetical calculations we now give the salient points in the calculation of the numbers of in-and-circumscribed polygons of eight and nine sides.

The correspondence S_8 has valency γ_8 where

$$-\gamma_8 = y^8 - 23y^7 + 203y^6 - 867y^5 + 1865y^4 - 1925y^3 + 833y^2 - 113y + 2,$$

while the total number of its united points is

$$-y^9 + 27y^8 - 231y^7 + 1343y^6 - 3445y^5 + 5865y^4 - 4501y^3 + 1877y^2 - 198y + 8 + \kappa\gamma_8.$$

The number of heterotypic solutions associated with the singularities of C^4 is

$$-y^9 + 15y^8 - 79y^7 + 227y^6 - 397y^5 + 465y^4 - 317y^3 + 189y^2 - 30y + 8 - \kappa\,(y^8 - 23y^7 + 67y^6 - 133y^5 + 153y^4 - 123y^3 + 57y^2 - 17y + 2),$$

and when this number is subtracted from the total number of united points of S_8 we obtain the equation

$$16N_8 + 12yN_6 + 8y\,(y+1)\,N_4 + 8N_4 + 6yN_3 = 12y^8 - 152y^7 + 1116y^6 - 3048y^5 + 5400y^4 - 4184y^3 + 1688y^2 - 168y - \kappa\,(136y^6 - 734y^5 + 1712y^4 - 1802y^3 + 776y^2 - 96y).$$

The values of N_3, N_4 and N_6 have already been obtained, so that this equation gives the value of N_8. Notice, to shorten the actual calculations somewhat, that the value of $12N_6 + 6N_3 + 8yN_4$ is given explicitly in § 9. The final result is

$$4N_8 = y\,\{3y^7 - 40y^6 + 296y^5 - 856y^4 + 1485y^3 - 1172y^2 + 432y - 36 - \kappa\,(34y^5 - 200y^4 + 477y^3 - 504y^2 + 205y - 20)\}.$$

The valency of S_9 is

$$\gamma_9 = (y-2)\,(y^8 - 24y^7 + 220y^6 - 960y^5 + 2022y^4 - 1864y^3 + 668y^2 - 72y + 1),$$

and the total number of its united points is

$$y^{10} - 26y^9 + 340y^8 - 1848y^7 + 6966y^6 - 13428y^5 + 16604y^4 - 9920y^3 + 3089y^2 - 258y + 8 + \kappa\gamma_9.$$

The number of heterotypic solutions associated with the singularities of C^4 is

$$y^6 + 2y^5 + 6y^4 + 24y^3 + 41y^2 + 30y + 8$$
$$+ \kappa (y^9 + y^8 + y^7 + 2y^6 + 3y^5 + 2y^4 - 5y^3 - 11y^2 - 8y - 2),$$

so that we obtain the equation

$$18N_9 + 14yN_7 + 10y(y+1)N_5 + 6y(y+1)^2 N_3 + 6N_3$$
$$= y^{10} - 26y^9 + 340y^8 - 1848y^7 + 6956y^6 - 13430y^5 + 16598y^4 - 9944y^3 + 3048y^2 - 288y$$
$$- \kappa (27y^8 - 267y^7 + 1402y^6 - 3939y^5 + 5910y^4 - 4401y^3 + 1397y^2 - 153y).$$

This gives finally

$$18N_9 = y\{y^9 - 27y^8 + 360y^7 - 2052y^6 + 7710y^5 - 15354y^4 + 18635y^3 - 11283y^2$$
$$+ 3282y - 264 - 3\kappa(y-1)(9y^6 - 87y^5 + 429y^4 - 1053y^3 + 1185y^2 - 467y + 46)\}.$$

13. The work may be continued to any length. For the number of in-and-circumscribed decagons we obtain the equation

$$20N_{10} + 16yN_8 + 12y(y+1)N_6 + 8y(y+1)^2 N_4 + 10N_5 + 8yN_4 + 6y(y+1)N_3$$
$$= 16y^{10} - 268y^9 + 2508y^8 - 10396y^7 + 28708y^6 - 43456y^5$$
$$+ 40992y^4 - 19392y^3 + 4528y^2 - 288y - \kappa(230y^8 - 1824y^7$$
$$+ 6926y^6 - 14222y^5 + 15822y^4 - 8850y^3 + 2150y^2 - 160y),$$

which gives

$$20N_{10} = y\{16y^9 - 280y^8 + 2660y^7 - 11520y^6 + 31823y^5 - 49217y^4 + 45610y^3 - 21370y^2$$
$$+ 4516y - 208 - 5\kappa(46y^7 - 392y^6 + 1532y^5 - 3200y^4 + 3561y^3 - 1954y^2 + 440y - 23)\}.$$

Table of numerical results.

14. In conclusion we give a table of the numbers of in-and-circumscribed m-gons of plane quartics, for $3 \leqq m \leqq 10$. The values of N_3 are of course already known, as are also those of N_m, for all the values of m tabulated, in the case when the curve has no multiple points, *i.e.* in the case $y = 8$.

There are two types of plane quartic curves which do not appear in the table. The tricuspidal quartic, for which $y = -1$ and $\kappa = 3$, does not appear since, being only of class 3, it cannot have any in-and-circumscribed polygons. Nor does the quartic with two cusps and one node, for which $y = 0$ and $\kappa = 2$, appear, since the value of N_m for this curve is always zero. The problem however for a plane quartic with two cusps and one node is poristic; there may be special curves, with two cusps and one node, having an infinite number of in-and-circumscribed polygons. Indeed such curves have been obtained by Roberts and Hilton[1]; Hilton's method of obtaining them is particularly simple, the problem being reduced by him to that of polygons circumscribed to one conic and inscribed in another. It is not possible, however, to obtain a plane quartic with a node and two cusps that has an infinity of in-and-circumscribed *triangles*.

[1] Roberts: *Proc. London Math. Soc.*, 23 (1892), 202.
Hilton: *Plane Algebraic Curves* (Oxford, 1920), 287.

$y = X - 4$	κ	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}
8	0	288	1512	12096	87696	685152	5375160	43059744	348636960
6	0	96	486	3264	17048	117792	670518	4486496	27264912
5	1	30	195	1230	5055	34710	160680	1010740	5072403
4	0	32	116	640	2304	11168	47260	216736	964384
3	1	12	33	192	544	2148	7350	28116	98586
2	0	8	18	48	116	312	810	2184	5880
2	2	6	6	42	105	294	732	2128	5727
1	1	2	3	6	9	18	30	56	99

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