Can. J. Math., Vol. XXXI, No. 1, 1979, pp. 87–92

COMMENTS ON A DISCRETENESS CONDITION FOR SUBGROUPS OF SL(2, C)

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1. Introduction. $SL(2, \mathbf{C})$ is the group of all complex unimodular 2×2 matrices. A subgroup of $SL(2, \mathbf{C})$ is said to be *discrete* if it does not contain any convergent sequence of distinct elements. A subgroup is said to be *elementary* if the commutator of any two elements of infinite order has trace 2.

The discreteness condition which this note relates to is the following:

PROPOSITION 1. *If two complex, unimodular 2×2 matrices X and Y generate a non-elementary, discrete group, then*

$$|\text{ trace } X^2 - 2| + |\text{ trace } XYX^{-1}Y^{-1} - 2| \geqq 1.$$

This was proved in [**1**]. There are many examples in which equality holds (see [**2**]), so in some sense the condition is best possible. However, in connection with this result, I have often been asked whether already the commutator-trace is uniformly bounded away from 2. In other words: Does there exist a positive real number K, such that the inequality $|\text{trace } XYX^{-1}Y^{-1} - 2| \geqq K$ holds whenever X and Y generate a non-elementary, discrete group? Another question has been whether the condition as stated is a consequence of stronger inequalities such as

$$|\text{trace } X^2 + \text{trace } XYX^{-1}Y^{-1} - 4| \geqq 1$$

or

$$|\text{trace } X^2 - \text{trace } XYX^{-1}Y^{-1}| \geqq 1$$

by means of "unnecessary" use of the triangle-inequality. The answer to each of these questions is in the negative.

PROPOSITION 2. *Each of the three functions* $|\text{trace } XYX^{-1}Y^{-1} - 2|$, $|\text{trace } X^2 + \text{trace } XYX^{-1}Y^{-1} - 4|$ *and* $|\text{trace } X^2 - \text{trace } XYX^{-1}Y^{-1}|$ *has the infinum* 0 *over the set of all pairs* (X, Y) *of generators of non-elementary, discrete subgroups of* $SL(2, \mathbf{C})$.

The proof to be given is by means of explicit examples. The discreteness of these groups will be established by a method which goes back to F. Schottky, F. Klein and H. Poincaré: One views X and Y as Möbius transformations and shows that the group they generate acts discontinuously somewhere in the extended complex plane (in fact, by exhibiting a fundamental polygon).

Received August 9, 1977 and in revised form, January 13, 1978. This work was supported by The National Science Foundation.

The third function spoken of in Proposition 2 is the same as

$$|\text{trace}\, XYXY^{-1} - 2|.$$

The following will be proved, using Proposition 1 and some simple observations (mentioned in [**3**]) about Lie-products of 2×2 matrices:

PROPOSITION 3. *If X and Y generate a non-elementary discrete group, then*

$$|\text{trace}\, X^2 - 2| + |\text{trace}\, XYXY^{-1} - 2| \geqq 1.$$

A more symmetric inequality, namely,

$$|\text{trace}\, (XY)^2 - 2| + |\text{trace}\, X^2Y^2 - 2| \geqq 1$$

can be used as well. It is easily obtained, using XY and Y instead of X and Y.

It is possible to prove many similar discreteness conditions. For the applications which I am aware of (see for instance [**1**; **4**; and **5**]), it is enough to know just one of these. However, there may be still other conditions which could lead to new results.

2. Examples. Let us denote by τ the function which to an element of $SL(2, \mathbf{C})$ assigns its trace. It satisfies the identitites $\tau(AB) = \tau(BA)$ and $\tau(A)\tau(B) = \tau(AB) + \tau(AB^{-1})$. The unit element has trace 2. Using these properties, which can be said to characterize τ, it is easy to derive the formula

$$\tau(ABA^{-1}B^{-1}) + 2 = \tau^2(A) + \tau^2(B) + \tau^2(AB) - \tau(A)\tau(B)\tau(AB).$$

It goes back to R. Fricke and will be referred to as "Fricke's formula" in the following.

Example 1. Let λ be a real number, strictly greater than 1. Consider the matrices

$$X = \sqrt{-1}\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{bmatrix}$$

$$Y = (\lambda^2 - \lambda^{-2})^{-1}\sqrt{-1}\begin{bmatrix} \lambda^2 + \lambda^{-2} & -2 \\ 2 & -\lambda^2 - \lambda^{-2} \end{bmatrix}.$$

Clearly, both determinants are equal to 1. The traces of X, Y and XY are easily seen to be

$$\tau(X) = (\lambda - \lambda^{-1})\sqrt{-1}$$
$$\tau(Y) = 0$$
$$\tau(XY) = -(\lambda - \lambda^{-1})^{-1}(\lambda^2 + \lambda^{-2}).$$

Using Fricke's formula, one obtains

$$\tau(XYX^{-1}Y^{-1}) - 2 = 4(\lambda - \lambda^{-1})^{-2}.$$

https://doi.org/10.4153/CJM-1979-010-5 Published online by Cambridge University Press

In particular, we have $\tau(XYX^{-1}Y^{-1}) \neq 2$. Since

$$XYX^{-1}Y^{-1} = X(YX)X^{-1}(YX)^{-1}$$

and both X and YX have infinite order (namely traces not equal to $2 \cos \pi r$ for any rational number r), the group generated by X and Y is non-elementary. It remains to show that it is discrete.

The Möbius transformation corresponding to (plus and minus) X amounts to multiplication by $-\lambda^2$. The Möbius transformation determined by Y is the elliptic element (rotation) of order 2 with fixed points λ^2 and λ^{-2}.

By X, the disk with centre $\frac{1}{2}(\lambda^{-2} - 1)$ and radius $\frac{1}{2}(\lambda^{-2} + 1)$ is mapped onto the disk with centre $\frac{1}{2}(\lambda^2 - 1)$ and radius $\frac{1}{2}(\lambda^2 + 1)$, and Y interchanges the interior and the exterior (including ∞) of the disk with centre $\frac{1}{2}(\lambda^2 + \lambda^{-2})$ and radius $\frac{1}{2}(\lambda^2 - \lambda^{-2})$.

Consider the open set P lying between the three circles (which two and two are tangent as shown on Figure 1). It is easy to see that every element different from the identity in the group generated by X and Y maps P onto a set which is disjoint from P. Therefore, no sequence chosen among these mappings can converge to the identity (or, in fact, converge at all) without eventually being constant. It follows at once that the matrix-group generated by X and Y is discrete.

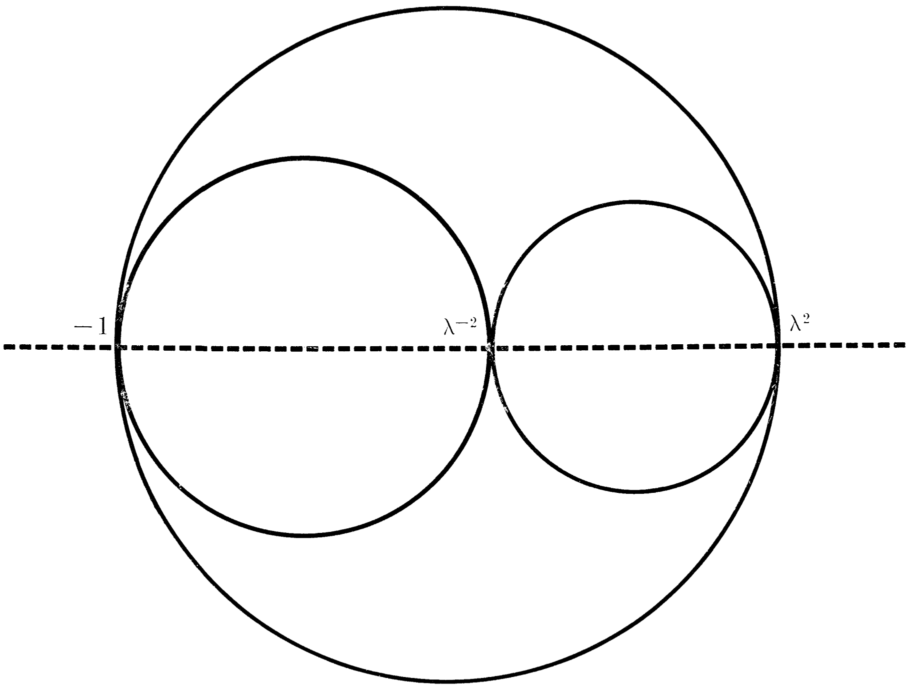

FIGURE 1.

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Example 2. Whenever three non-zero numbers a, b and c have their product equal to the sum of their squares, there exist unimodular matrices X and Y such that a, b and c are the traces of X, Y and XY. Fricke's formula shows that the commutator of X and Y has trace -2. One possible choice of matrices is:

$$X = \begin{bmatrix} a - bc^{-1} & ac^{-2} \\ a & bc^{-1} \end{bmatrix}$$

$$Y = \begin{bmatrix} b - ac^{-1} & -bc^{-2} \\ -b & ac^{-1} \end{bmatrix}$$

$$XY = \begin{bmatrix} c & -c^{-1} \\ c & 0 \end{bmatrix}$$

Then one has

$$YX = \begin{bmatrix} c & c^{-1} \\ -c & 0 \end{bmatrix} \quad \text{and} \quad XYX^{-1}Y^{-1} = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}.$$

Two elements X and Y of $SL(2, \mathbf{R})$ which have non-zero traces and whose commutator has trace -2 always generate a discrete (and free) group. Clearly such a group is non-elementary, too.

To prove the discreteness, consider the corresponding Möbius transformations. It is easy to see that the fixed points of the parabolic elements $XYX^{-1}Y^{-1}$, $YX^{-1}Y^{-1}X$, $X^{-1}Y^{-1}XY$ and $Y^{-1}XYX^{-1}$ lie in the given cyclic order on the extended real axis. Hence, in succession, they are the points of tangency of four circles, α, β, γ and δ (see Figure 2) which are perpendicular to the real axis and paired in the sense that X maps γ onto α and Y maps δ onto β. Clearly, the open region P lying between these circles cannot be mapped onto a set intersecting P by any transformation (other than the identity) in the group generated by X and Y. This implies that the group of matrices is discrete.

3. Proof of Propositions 2 and 3.

Let X and Y be two elements of $SL(2, \mathbf{C})$. One can show by careful, but not difficult, computations that *X and Y have a common fixed point in the extended complex plane if and only if their commutator has trace* 2.

The determinant of $XY - YX$ is equal to $2 - \tau(XYX^{-1}Y^{-1})$. Therefore, if X and Y have no common fixed points, as we shall assume, then their Lie-product determines a Möbius transformation φ which is elliptic of order 2 (since it has trace 0). An important property of φ is that it transforms X into X^{-1} and Y into Y^{-1} [**3**]. Consequently, the group generated by X and Y is a subgroup of index at most 2 in its extension by φ. Thus, if X and Y generate a discrete group, then so do X, Y and φ together and, in particular, so do X and $Y\varphi$. In this case, we know from [**1**] that

$$|\tau X^2 - 2| + |\tau(XYX^{-1}Y^{-1}) - 2| \geqq 1,$$

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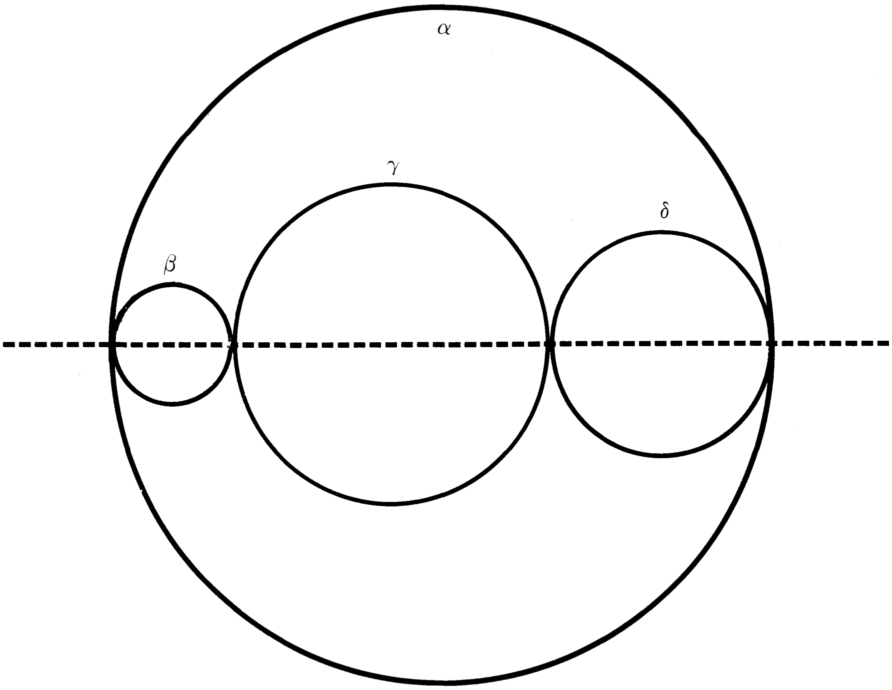

FIGURE 2.

except in certain cases of elementary groups where, in fact, the set of points fixed under X is mapped onto itself by Y. Applying this to X and $Y\varphi$, we obtain

$$|\tau X^2 - 2| + |\tau(XYXY^{-1}) - 2| \geqq 1.$$

This is true because

$$X(Y\varphi)X^{-1}(Y\varphi)^{-1} = XY\varphi X^{-1}\varphi^{-1}Y^{-1} = XYXY^{-1}.$$

To conclude the proof of Proposition 3, we remark that if $Y\varphi$ maps the set of fixed points for X onto itself, then so does Y (since φ does) and thus, the group generated by X and Y is elementary.

To prove Proposition 2, first we refer to Example 1 and observe that $\tau(XYX^{-1}Y^{-1})$ becomes arbitrarily close to 2 if λ is chosen sufficiently large. Secondly, we have

$$\begin{aligned}\tau((XY)^2) - \tau(XYX^{-1}Y^{-1}) &= \tau^2(XY) - \tau(XYX^{-1}Y^{-1}) - 2\\ &= (\lambda - \lambda^{-1})^{-2}(\lambda^2 + \lambda^{-2})^2 - 4(\lambda - \lambda^{-1})^{-2} - 4\\ &= (\lambda - \lambda^{-1})^2.\end{aligned}$$

It is clear that (XY, Y) and (X, Y) generate the same group and have the same commutator. Therefore, letting λ approach 1, we see that the last function spoken of in the proposition becomes arbitrarily small.

Finally, if in Example 2 we choose $\tau(X) = \tau(Y) = 2\sqrt{2}$ and $\tau(XY) = 4$, then $\tau(X^2) + \tau(XYX^{-1}Y^{-1}) - 4 = 0$. This completes the proof.

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https://doi.org/10.4153/CJM-1979-010-5 Published online by Cambridge University Press