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PROJECTIONS INDUCING AUTOMORPHISMS OF STABLE UHF-ALGEBRAS

KAZUNORI KODAKA

coy Department of Mathematical Sciences, College of Science, Ryukyu University, Nishihara-cho, Okinawa, 903-0213 Japan

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Abstract. Let A be a UHF-algebra and $\mathbf{K}$ the C*-algebra of all compact operators on a countably infinite-dimensional Hilbert space. In this note we shall find all projections p in A with $pAp \cong A$ and, using these projections, we shall determine the group of automorphisms of $K_0(A \otimes \mathbf{K})$ induced by those of $A \otimes \mathbf{K}$ in some cases.

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0. Introduction. Let A be a UHF-algebra and $\mathbf{K}$ the C*-algebra of all compact operators on a countably infinite-dimensional Hilbert space. Let p be a projection in $A \otimes \mathbf{K}$ with $p(A \otimes \mathbf{K})p \cong A$. In [**9**] we showed that we can construct any automorphism of $A \otimes \mathbf{K}$ using the projection p above, an automorphism of A and a unitary element in $M(A \otimes \mathbf{K})$, where $M(A \otimes \mathbf{K})$ is the multiplier algebra of $A \otimes \mathbf{K}$. But since A is a UHF-algebra, it suffices to find all projections p in A with $pAp \cong A$ in order to determine the group of automorphisms of $K_0(A \otimes \mathbf{K})$ induced by those of $A \otimes \mathbf{K}$. By the above result we can compute the Picard group of A in some cases. Furthermore let β be an automorphism of $A \otimes \mathbf{K}$ with $\beta_* \neq \mathrm{id}$ on $K_0(A \otimes \mathbf{K})$. Then, by Rørdam [**12**] a crossed product $A \otimes \mathbf{K} \times_\beta \mathbf{Z}$ is a purely infinite simple C*-algebra and its isomorphism class can be determined by Elliott, Evans and Kishimoto [**5**] if the automorphism β_* of $K_0(A \otimes \mathbf{K})$ is known to us.

Since $A \otimes \mathbf{K}$ is an AF-algebra, we can determine the group of automorphisms of $K_0(A \otimes \mathbf{K})$ induced by those of $A \otimes \mathbf{K}$ by Blackadar [**2**, Theorem 7.3.2]. In fact if $A = M_{2^\infty}$, we can easily do it, where M_{2^∞} is the UHF-algebra of type 2^∞. However, it seems difficult in general to determine order-preserving automorphisms of the dimension group $K_0(A \otimes \mathbf{K})$ and so we apply the method above to determine projections p in A with $pAp \cong A$.

1. Preliminaries. For each $n \in \mathbf{N}$, let M_n be the C*-algebra of $n \times n$-matrices over $\mathbf{C}$. For positive integers $m(1), m(2) \geq 2$ let ι be a monomorphism of $M_{m(1)}$ into $M_{m(1)m(2)}$ such that $\iota(I_{m(1)}) = I_{m(1)m(2)}$, where $I_{m(1)}$ and $I_{m(1)m(2)}$ are the unit elements in $M_{m(1)}$ and $M_{m(1)m(2)}$ respectively. Given a sequence $\{m(n)\}_{n=1}^\infty$ of positive integers greater than 1, let $m(n)! = \prod_{k=1}^n m(k)$. We consider the inductive system

$$M_{m(1)!} \xrightarrow{\iota} M_{m(2)!} \xrightarrow{\iota} \cdots \xrightarrow{\iota} M_{m(n)!} \xrightarrow{\iota} \cdots \quad .$$

We call the C*-algebra generated by the inductive system above a UHF-algebra of type $\{m(n)!\}$.

Let A be a UHF-algebra and τ the unique tracial state on A. Then by Blackadar [**2**], $K_0(A)$ is a simple dimension group which is a dense subgroup of $\mathbf{Q}$ containing $\mathbf{Z}$.

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Let τ_* be the homomorphism of $K_0(A)$ to $\mathbf{R}$ induced by τ. By Blackadar [**1**, Theorem 3.9] τ_* is injective and the positive cone of $K_0(A)$ is given by the formula

$$K_0(A)_+ = \{x \in K_0(A) | \tau_*(x) \geq 0\}.$$

We identify $K_0(A)$ with $\tau_*(K_0(A))$. Since $K_0(A)$ is a dense subgroup of $\mathbf{Q}$, an automorphism of $K_0(A)$ is multiplication by a positive rational number.

Lemma 1.1. *For any automorphism α of A, $\alpha_* = \mathrm{id}$ on $K_0(A)$.*

Proof. This can easily be proved using the facts that, by the uniqueness of trace, α preserves the trace τ and the homomorphism $\tau_* : K_0(A) \to \mathbf{R}$ is injective. Q.E.D.

Let $\mathbf{K}$ be the C*-algebra of all compact operators on a countably infinite-dimensional Hilbert space and $\{e_{ij}\}_{i,j\in\mathbf{Z}}$ matrix units of $\mathbf{K}$. Let Tr be the canonical trace on $\mathbf{K}$. Then $\tau \otimes \mathrm{Tr}$ is a densely defined lower semi-continuous trace on $A \otimes \mathbf{K}$ and, as described in Elliott, Evans and Kishimoto [**5**], it is unique up to a constant multiple. Let β be an automorphism of $A \otimes \mathbf{K}$. We define $s(\beta) \in \mathbf{Q}$ by $(\tau \otimes \mathrm{Tr}) \circ \beta = s(\beta)(\tau \otimes \mathrm{Tr})$. Then an automorphism β_* of $K_0(A \otimes \mathbf{K})$ is multiplication by the positive rational number $s(\beta)$.

Let $M_n(A)$ be the C*-algebra of $n \times n$-matrices over A, for any $n \in \mathbf{N}$; we identify $M_n(A)$ with $A \otimes M_n$. Let p be a projection in $\cup_{n=1}^{\infty} M_n(A) \subset A \otimes \mathbf{K}$ with $p(A \otimes \mathbf{K})p \cong A$. We denote by χ_p an isomorphism of A onto $p(A \otimes \mathbf{K})p$. By Brown [**3**, Lemma 2.5], there is a partial isometry $z \in M(A \otimes \mathbf{K} \otimes \mathbf{K})$ such that $z^* z = p \otimes 1$ and $zz^* = 1 \otimes 1 \otimes 1$. Let ψ be an isomorphism of $\mathbf{K} \otimes \mathbf{K}$ onto $\mathbf{K}$ with $\psi_* = \mathrm{id}$ of $K_0(\mathbf{K} \otimes \mathbf{K})$ onto $K_0(\mathbf{K})$. Let β_p be the automorphism of $A \otimes \mathbf{K}$ defined by

$$\beta_p = (\mathrm{id} \otimes \psi) \circ \mathrm{Ad}(z) \circ (\chi_p \otimes \mathrm{id}).$$

Lemma 1.2. *With the notations above the automorphism β_{p*} of $K_0(A \otimes \mathbf{K})$ is multiplication by* $(\tau \otimes \mathrm{Tr})(p)$.

Proof. It suffices to show that $s(\beta_p) = (\tau \otimes \mathrm{Tr})(p)$. Let $(\tau \otimes \mathrm{Tr})_*$ be the homomorphism of $K_0(A \otimes \mathbf{K})$ to $\mathbf{R}$ induced by $\tau \otimes \mathrm{Tr}$. We note that $\beta_p(1 \otimes e_{00})$ is in the ideal of definition of $\tau \otimes \mathrm{Tr}$ by [**7**, Lemma 1]. Hence

$$\begin{aligned}(\tau \otimes \mathrm{Tr}) \circ \beta_p(1 \otimes e_{00}) &= (\tau \otimes \mathrm{Tr})_* \circ \beta_{p*}([1 \otimes e_{00}]) \\ &= (\tau \otimes \mathrm{Tr})_* \circ (\mathrm{id} \otimes \psi)_*([z(p \otimes e_{00})z^*]) \\ &= (\tau \otimes \mathrm{Tr})_* \circ (\mathrm{id} \otimes \psi)_*([p \otimes e_{00}]) \\ &= (\tau \otimes \mathrm{Tr})(p).\end{aligned}$$

Since $(\tau \otimes \mathrm{Tr}) \circ \beta_p = s(\beta_p)(\tau \otimes \mathrm{Tr})$ and $(\tau \otimes \mathrm{Tr})(1 \otimes e_{00}) = 1$ it follows that $s(\beta_p) = (\tau \otimes \mathrm{Tr})(p)$. Q.E.D.

Corollary 1.3. *Let β_p be as above. If $(\tau \otimes \mathrm{Tr})(p) > 1$, there is a projection $q \in A$ with $qAq \cong A$ such that $\beta_{p*}^{-1} = \beta_{q \otimes e_{00}*}$ on $K_0(A \otimes \mathbf{K})$.*

Proof. By [**9**, Theorem 4.5 and Remark 2.1], there are an $n \in \mathbf{N}$, a projection $q_1 \in M_n(A)$, an automorphism α of A and a unitary element $w \in M(A \otimes \mathbf{K})$ such that

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$$q_1(A \otimes \mathbf{K})q_1 \cong A, \quad \beta_p^{-1} = \mathrm{Ad}(w) \circ \beta_{q_1} \circ (\alpha \otimes \mathrm{id}),$$

where $M(A \otimes \mathbf{K})$ is the multiplier algebra of $A \otimes \mathbf{K}$. By Lemma 1.1 and [**9**, Lemma 1.1] $\beta_{p*}^{-1} = \beta_{q_1 *}$. Hence, by Lemma 1.2, $(\tau \otimes \mathrm{Tr})(q_1)(\tau \otimes \mathrm{Tr})(p) = 1$. We note that $\tau(\mathrm{Proj}A) = \tau_*(K_0(A)) \cap [0, 1]$, where $\mathrm{Proj}A$ is the set of all projections in A. Since A has cancellation, there is a projection $q \in A$ such that $q \otimes e_{00}$ is unitarily equivalent to q_1 in $(A \otimes \mathbf{K})^+$, where $(A \otimes \mathbf{K})^+$ is the unitized C*-algebra of $A \otimes \mathbf{K}$. Thus $qAq \cong A$ and $\beta_{p*}^{-1} = \beta_{q \otimes e_{00} *}$ on $K_0(A \otimes \mathbf{K})$. Q.E.D.

Let $\mathrm{Aut}(K_0(A \otimes \mathbf{K}))$ be the group of automorphisms of $K_0(A \otimes \mathbf{K})$ and let

$$S = \{\beta_{p \otimes e_{00} *} \in \mathrm{Aut}(K_0(A \otimes \mathbf{K})) | p \text{ is a projection in } A \text{ with } pAp \cong A\}.$$

Corollary 1.4. *With the notations above, S is a semigroup of automorphisms of $K_0(A \otimes \mathbf{K})$ with the unit element.*

Proof. Since S is a subset of the group $\mathrm{Aut}(K_0(A \otimes \mathbf{K}))$, it suffices to show that S is invariant under the product of $\mathrm{Aut}(K_0(A \otimes \mathbf{K}))$ and that S has the unit element in $\mathrm{Aut}(K_0(A \otimes \mathbf{K}))$. Since $\tau(1) = 1$, $\beta_{1 \otimes e_{00} *}$ is the unit element in $\mathrm{Aut}(K_0(A \otimes \mathbf{K}))$. Thus S has the unit element in $\mathrm{Aut}(K_0(A \otimes \mathbf{K}))$. For $j = 1, 2$, let p_j be a projection in A with $p_j A p_j \cong A$. Then, in the same way as in the proof of Corollary 1.3, we see that there is a projection p_3 in A such that

$$\tau(p_3) = \tau(p_1)\tau(p_2), \quad p_3 A p_3 \cong A.$$

Since $\tau(p_3) = \tau(p_1)\tau(p_2)$, by Lemma 1.2 we deduce that $\beta_{p_3 *} = \beta_{p_1 *} \circ \beta_{p_2 *}$. Hence $\beta_{p_1 *} \circ \beta_{p_2 *} \in S$. Therefore we obtain the conclusion. Q.E.D.

Remark 1.5. Let A be a UHF-algebra of type $\{m(n)!\}$. By Corollary 1.3 and [**9**], the group of automorphisms of $K_0(A \otimes \mathbf{K})$ induced by those of $A \otimes \mathbf{K}$ is generated by S and, since an automorphism of $K_0(A \otimes \mathbf{K})$ is multiplication by a positive rational number, by Lemma 1.2 and Corollary 1.3 we have

$$S = \{\tau(p) \in \mathbf{Q} | p \text{ is a projection in } A \text{ with } pAp \cong A\}.$$

Furthermore, by Blackadar [**2**, Proposition 4.6.6],

$$S = \{\tau(p) \in \mathbf{Q} | p \text{ is a projection in } \cup_{n=1}^{\infty} M_{m(n)!} \text{ with } pAp \cong A\}.$$

2. Projections p in A with pAp isomorphic to A. Let A be a UHF-algebra of type $\{m(n)!\}$. Following Glimm [**6**] we define a function $f(\{m(n)!\})$ whose domain is the prime numbers. For each prime number r, let

$$f(\{m(n)!\}(r) = sup\{k \in \mathbf{N} | \text{there is an } n \in \mathbf{N} \text{ such that } r^k \text{ divides } m(n)!\}.$$

Also, for each subset N of $\mathbf{N}$ we denote by $\#(N)$ the number of elements in N.

Lemma 2.1. *Let $f(\{m(n)!\})$ be as above and r a prime number. Then the following conditions hold:*

(1) *$f(\{m(n)!\})(r) = \infty$ if and only if $\#\{n \in N | r$ divides $m(n)\} = \infty$,*
(2) *$f(\{m(n)!\})(r) = 0$ if and only if r does not divide $m(n)$ for any $n \in N$,*
(3) *$f(\{m(n)!\})(r) < \infty$ if and only if there is an $n_0 \in N$ such that r does not divide $m(n)$ for any $n \geq n_0$.*

Proof. (1)$\Rightarrow$: We suppose that $\#\{n \in \mathbf{N} | r$ divides $m(n)\} < \infty$. Then there is an $n_0 \in \mathbf{N}$ such that r does not divide $m(n)$ for any $n \geq n_0$. Thus

$$\begin{aligned} f(\{m(n)!\})(r) &= \sup\{k \in \mathbf{N} | \text{there is an } n \in \mathbf{N} \text{ such that } r^k \text{ divides } m(n)!\} \\ &= \sup\{k \in \mathbf{N} | \text{there is an integer } n \text{ with } 1 \leq n \leq n_0 - 1 \text{ such that } r^k \\ &\quad \text{divides } m(n)!\} \\ &< \infty. \end{aligned}$$

This is a contradiction. Therefore $\#\{n \in \mathbf{N} | r$ divides $m(n)\} = \infty$.

$\Leftarrow$: For any $k \in \mathbf{N}$ there is a set $\{n_1, n_2, \ldots, n_k\} \subset \{n \in \mathbf{N} | r$ divides $m(n)\}$ with $n_1 < n_2 < \ldots < n_k$. Since r divides $m(n_j)$, for $j = 1, 2, \ldots, k$, r^k divides $m(n_k)!$. Thus $f(\{m(n)!\}(r) \geq k$. Since k is an arbitrary positive integer, $f(\{m(n)!\})(r) = \infty$.

(2)$\Rightarrow$: If there is an $n_0 \in \mathbf{N}$ such that r divides $m(n_0)$, then r divides $m(n_0)!$. Hence $f(\{m(n)!\})(r) \geq 1$. This is a contradiction. Thus r does not divide $m(n)$, for any $n \in \mathbf{N}$.

$\Leftarrow$: If $f(\{m(n)!\})(r) \geq 1$, then there is an $n_0 \in \mathbf{N}$ such that r divides $m(n_0)!$. Hence there is an $n_1 \in \mathbf{N}$ such that r divides $m(n_1)$. This is a contradiction. Thus $f(\{m(n)!\})(r) = 0$.

(3) is equivalent to (1). Q.E.D.

Let A be a UHF-algebra of type $\{m(n)!\}$. We suppose that $f(\{m(n)!\})(r) = 0$ or ∞, for any prime number r. If $f(\{m(n)!\})(r) = 0$, for any prime number r, then $A \cong \mathbf{C}$ and so we also suppose that

$$\#\{r | r \text{ is a prime number with } f(\{m(n)!\})(r) = \infty\} \geq 1.$$

LEMMA 2.2. *With the notations and assumptions above, let n_0 be a positive integer and p a projection in $M_{m(n_0)!}$ with $\tau(p) = \frac{k}{m(n_0)!}$. Then the following conditions hold.*

(1) *If $k = 1$, then $pAp \cong A$,*

We suppose that $k \neq 1$. Let $k = c_1^{d_1} \ldots c_h^{d_h}$ be the decomposition of k by prime factors with $d_j \neq 0$ for $j = 1, 2, \ldots, h$.

(2) *If $f(\{m(n)!\})(c_j) = \infty$, for $j = 1, 2, \ldots, h$, then $pAp \cong A$.*
(3) *If there is an integer j_0 with $1 \leq j_0 \leq h$ such that $f(\{m(n)!\})(c_{j_0}) = 0$, then pAp is not isomorphic to A.*

Proof. (1) For the UHF-algebra pAp we have the inductive system

$$M_{m(n_0+1)} \longrightarrow M_{m(n_0+1)m(n_0+2)} \longrightarrow \ldots \longrightarrow M_{m(n_0+1)\ldots m(n_0+n)} \longrightarrow \cdots \quad .$$

For any prime number r with $f(\{m(n)!\})(r) = \infty$, $\#\{n \in \mathbf{N} | r$ divides $m(n)\} = \infty$, by Lemma 2.1. Hence $f(\{m(n_0+1)\ldots m(n_0+n)\})(r) = \infty$. Also, for any prime number r with $f(\{m(n)!\})(r) = 0$, $\#\{n \in \mathbf{N} | r$ divides $m(n)\} = 0$, by Lemma 2.1. Hence $f(\{m(n_0+1)\ldots m(n_0+n)\})(r) = 0$. Thus

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$$f(\{m(n)!\}) = f(\{m(n_0+1)\dots m(n_0+n)\}).$$

Therefore, by Glimm [**6**, Theorem 1.12], we have $pAp \cong A$.

(2) For the UHF-algebra pAp we have the inductive system

$$M_k \longrightarrow M_{km(n_0+1)} \longrightarrow \dots \longrightarrow M_{km(n_0+1)\dots m(n_0+n-1)} \longrightarrow \cdots \quad .$$

For any prime number r with $f(\{m(n)!\})(r) = \infty$, we have

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r) = \infty,$$

by Lemma 2.1. For any prime number r with $f(\{m(n)!\})(r) = 0$, we have $f(\{km(n_0+1)\dots m(n_0+n-1)\})(r) = 0$ since r does not divide k and $m(n)$ for any $n \in \mathbf{N}$. Therefore by Glimm [**6**, Theorem 1.12] $pAp \cong A$.

(3) Since c_{j_0} divides k and does not divide $m(n)$, for any $n \in \mathbf{N}$,

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(c_{j_0}) = d_{j_0} \geq 1.$$

On the other hand $f(\{m(n)!\})(c_{j_0}) = 0$. Hence by Glimm [**6**, Theorem 1.12] pAp is not isomorphic to A. Q.E.D.

THEOREM 2.3. *With the same assumptions as in Lemma 2.2, let n_0 be a positive integer and p a projection in $M_{m(n_0)!}$ with $\tau(p) = \frac{k}{m(n_0)!}$. Then $pAp \cong A$ if and only if $k = 1$ or $k = c_1^{d_1}\dots c_h^{d_h}$ with $f(\{m(n)!\})(c_j) = \infty$ and $d_j \neq 0$, for $j = 1, 2, \dots, h$.*

Proof. This is immediate, by Lemma 2.2. Q.E.D.

Let A be a UHF-algebra of type $\{m(n)!\}$. We suppose that

$$1 \leq \#\{r|r \text{ is a prime number with } 1 \leq f(\{m(n)!\})(r) < \infty\} < \infty,$$

$$\#\{r|r \text{ is a prime number with } f(\{m(n)!\})(r) = \infty\} = \infty.$$

Let $\{r_j\}_{j=1}^{l}$ be the set of all prime numbers with $1 \leq f(\{m(n)!\})(r_j) < \infty$. We put $t_j = f(\{m(n)!\})(r_j)$, for $j = 1, 2, \dots, l$. By the assumptions above there is an $n_0 \in \mathbf{N}$ such that $r_1^{t_1}\dots r_l^{t_l}$ divides $m(n_0)!$ and, for any $n \geq n_0$ and $j = 1, 2, \dots, l$, r_j does not divide $m(n)$. Let n_1 be any positive integer with $n_1 \geq n_0$ and p a projection in $M_{m(n_1)!}$ with $\tau(p) = \frac{k}{m(n_1)!}$. We note that, for the UHF-algebra pAp, we have the inductive system

$$M_k \longrightarrow M_{km(n_0+1)} \longrightarrow \dots \longrightarrow M_{km(n_0+1)\dots m(n_0+n-1)} \longrightarrow \cdots \quad .$$

LEMMA 2.4. *With the notations and assumptions above, the following conditions hold.*

(1) *If $k = r_1^{t_1}\dots r_l^{t_l}$, then $pAp \cong A$,*
(2) *If $r_1^{t_1}\dots r_l^{t_l}$ does not divide k, then pAp is not isomorphic to A.*

Proof. (1) Since $f(\{m(n)!\})(r_j) = t_j$, for $j = 1, 2, \dots, l$, and r_j does not divide $m(n)$ for any $n \geq n_1$ and $j = 1, 2, \dots, l$, we have

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$$f(\{km(n_1+1)\dots m(n_1+n-1)\})(r_j)=t_j.$$

Also, by Lemma 2.1,

$$f(\{km(n_1+1)\dots m(n_1+n-1)\})(r)=\infty,\quad \text{if } f(\{m(n)!\})(r)=\infty,$$
$$f(\{km(n_1+1)\dots m(n_1+n-1)\})(r)=0,\quad \text{if } f(\{m(n)!\})(r)=0.$$

Hence, by Glimm [**6**, Theorem 1.12], $pAp\cong A$.

(2) Since $r_1^{t_1}\dots r_l^{t_l}$ does not divide k, there is a $j_0\in\mathbf{N}$ with $1\le j_0\le l$ such that $r_{j_0}^{t_{j_0}}$ does not divide k. Hence

$$f(\{km(n_1+1)\dots m(n_1+n-1)\})(r_{j_0})<t_{j_0}=f(\{m(n)!\}))(r_{j_0}).$$

Thus pAp is not isomorphic to A by Glimm [**6**, Theorem 1.12]. Q.E.D.

By Lemma 2.4 (2), if $pAp\cong A$, there is a $k_1\in\mathbf{N}$ such that $k=r_1^{t_1}\dots r_l^{t_l}k_1$.

Lemma 2.5. *With the same notations as in Lemma 2.4, we suppose that there is a $k_1\in\mathbf{N}$ such that $k=r_1^{t_1}\dots r_l^{t_l}k_1$. Let $k_1=c_1^{d_1}\dots c_h^{d_h}$ be the decomposition of k_1 by prime factors with $d_j\neq 0$, for $j=1,2,\dots,h$. Then the following conditions hold.*

(1) *If there is a $j_0\in\mathbf{N}$ with $1\le j_0\le h$ such that $f(\{m(n)!\})(c_{j_0})=0$, then pAp is not isomorphic to A.*
(2) *If $f(\{m(n)!\})(c_j)=\infty$ for $j=1,2,\dots,h$, then $pAp\cong A$.*

Proof. (1) Since $f(\{km(n_1+1)\dots m(n_1+n-1)\})(c_{j_0})\ge 1$, we have

$$f(\{km(n_1+1)\dots(n_1+n-1)\})\neq f(\{m(n)!\}).$$

Thus pAp is not isomorphic to A, by Glimm [**6**, Theorem 1.12].

(2) By Lemma 2.1, for any prime number r with $f(\{m(n)!\})(r)=\infty$, we have

$$f(\{km(n_1+1)\dots m(n_1+n-1)\})(r)=\infty.$$

Let r be a prime number with $1\le f(\{m(n)!\})(r)<\infty$. Then there is a $j_0\in\mathbf{N}$ with $1\le j_0\le l$ such that $r=r_{j_0}$ and that $f(\{m(n)!\})(r)=t_{j_0}$. Since $r_{j_0}^{t_{j_0}}$ divides k and r_{j_0} does not divide $m(n)$, for any $n\ge n_1$, $f(\{km(n_1+1)\dots m(n_1+n-1)\})(r)=t_{j_0}$. Let r be a prime number with $f(\{m(n)!\})(r)=0$. Then $r\neq r_j$, for $j=1,2,\dots,l$, and $r\neq c_j$, for $j=1,2,\dots,h$, since $f(\{m(n)!\})(c_j)=\infty$, for $j=1,2,\dots,h$. Hence r does not divide k. Hence

$$f(\{km(n_1+1)\dots m(n_1+n-1)\})(r)=0.$$

Thus $pAp\cong A$, by Glimm [**6**, Theorem 1.12]. Q.E.D.

Theorem 2.6. *With the notations and assumptions above, let n_1 be an integer with $n_1\ge n_0$ and p a projection in $M_{m(n_1)!}$ with $\tau(p)=\frac{k}{m(n_1)!}$. Then $pAp\cong A$ if and only if there is a $k_1\in\mathbf{N}$ such that $k=r_1^{t_1}\dots r_l^{t_l}k_1$ and $k_1=1$ or $k_1=c_1^{d_1}\dots c_h^{d_h}$ with $f(\{m(n)!\})(c_j)=\infty$ and $d_j\neq 0$ for $j=1,2,\dots,h$.*

Proof. This is immediate by Lemmas 2.4 and 2.5 Q.E.D.

Let A be a UHF-algebra of type $\{m(n)!\}$. We suppose that

$$\#\{r | r \text{ is a prime number with } 1 \leq f(\{m(n)!\})(r) < \infty\} = \infty,$$

$$\#\{r | r \text{ is a prime number with } f(\{m(n)!\})(r) = \infty\} \geq 1.$$

By the assumptions above we may assume that, for any $n \in \mathbf{N}$, $m(n)!$ has a prime number r as a factor with $1 \leq f(\{m(n)!\})(r) < \infty$. Let p be a projection in $M_{m(n_0)!}$ with $\tau(p) = \frac{k}{m(n_0)!}$. For the UHF-algebra pAp we have the inductive system

$$M_k \longrightarrow M_{km(n_0+1)} \longrightarrow \ldots \longrightarrow M_{km(n_0+1)\ldots m(n_0+n-1)} \longrightarrow \cdots \quad .$$

By Lemma 2.1 we can easily see that

$$f(\{km(n_0+1)\ldots m(n_0+n-1)\})(r) = \infty,$$

for any prime number r with $f(\{m(n)!\})(r) = \infty$. Let $r_1^{s_1}\ldots r_l^{s_l}$ be a factor of $m(n_0)!$ with $1 \leq f(\{m(n)!\})(r_j) = t_j < \infty$ and $1 \leq s_j \leq t_j$, for $j = 1, 2, \ldots, l$, such that r_j does not divide $\frac{m(n_0)!}{r_1^{s_1}\ldots r_l^{s_l}}$, for $j = 1, 2, \ldots, l$, and r does not divide $m(n_0)!$, for any prime number r with $r \neq r_j$ for $j = 1, 2, \ldots, l$ and $1 \leq f(\{m(n)!\})(r) < \infty$.

Lemma 2.7. *With the notations and assumptions above, if $r_1^{s_1}\ldots r_l^{s_l}$ does not divide k, then pAp is not isomorphic to A.*

Proof. Since $f(\{m(n)!\})(r_j) = t_j$, for $j = 1, 2, \ldots, l$, there is an $n_j \in \mathbf{N}$ with $n_j \geq n_0 + 1$ such that $r_j^{t_j - s_j}$ divides $\frac{m(n_j)!}{m(n_0)!}$. Since $r_1^{s_1}\ldots r_l^{s_l}$ does not divide k, there is a $j_0 \in \mathbf{N}$ with $1 \leq j_0 \leq l$ such that $r_{j_0}^{s_{j_0}}$ does not divide k. Thus

$$f(\{km(n_0+1)\ldots m(n_0+n-1)\})(r_{j_0}) < s_{j_0} + (t_{j_0} - s_{j_0}) = t_{j_0}.$$

On the other hand $f(\{m(n)!\})(r_{j_0}) = t_{j_0}$. Thus pAp is not isomorphic to A by Glimm [**6**, Theorem 1.12]. Q.E.D.

By Lemma 2.7, if $pAp \cong A$, then there is a $k_1 \in \mathbf{N}$ such that $k = r_1^{s_1}\ldots r_l^{s_l}k_1$. So we suppose that there is a $k_1 \in \mathbf{N}$ such that $k = r_1^{s_1}\ldots r_l^{s_l}k_1$.

Lemma 2.8. *With the notations and assumptions above, if there is a prime number r_0 with $f(\{m(n)!\})(r_0) < \infty$ such that r_0 divides k_1, then pAp is not isomorphic to A.*

Proof. If $f(\{m(n)!\})(r_0) = 0$, then $f(\{km(n_0+1)\ldots m(n_0+n-1)\})(r_0) \geq 1$, since r_0 divides k. Thus pAp is not isomorphic to A by Glimm [**6**, Theorem 1.12]. We suppose that $f(\{m(n)!\})(r_0) \geq 1$. Furthermore, we suppose that there is a $j_0 \in \mathbf{N}$ with $1 \leq j_0 \leq l$ such that $r_0 = r_{j_0}$. If $s_{j_0} = t_{j_0}$, then $r_{j_0}^{t_{j_0}+1}$ divides k, since

$$k = r_1^{s_1}\ldots r_{j_0}^{t_{j_0}+1}\ldots r_l^{s_l}\frac{k_1}{r_{j_0}}.$$

Hence

$$f(\{km(n_0+1)\ldots m(n_0+n-1)\})(r_{j_0}) = t_{j_0} + 1.$$

Since $f(\{m(n)!\})(r_{j_0}) = t_{j_0}$, pAp is not isomorphic to A, by Glimm [**6**, Theorem 1.12]. If $s_{j_0} < t_{j_0}$, then $r_{j_0}^{s_{j_0}+1}$ divides k, since

$$k = r_1^{s_1} \dots r_{j_0}^{s_{j_0}+1} \dots r_l^{s_l} \frac{k_1}{r_{j_0}}.$$

Furthermore, since $f(\{m(n)!\})(r_{j_0}) = t_{j_0}$, there is an $n_{j_0} \in \mathbf{N}$ with $n_{j_0} \geq n_0 + 1$ such that $r^{t_{j_0} - s_{j_0}}$ divides $\frac{m(n_{j_0})!}{m(n_0)!}$. Thus

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r_{j_0}) \geq s_{j_0} + 1 + t_{j_0} - s_{j_0} = t_{j_0} + 1.$$

Hence, by Glimm [**6**, Theorem 1.12], pAp is not isomorphic to A.

Next, we suppose that $r_0 \neq r_j$, for $j = 1, 2, \dots, l$. Then, since r_0 does not divide $m(n_0)!$, we have

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r_0) = 1 + f(\{m(n)!\})(r_0).$$

Thus pAp is not isomorphic to A, by Glimm [**6**, Theorem 1.12]. Q.E.D.

By Lemmas 2.7 and 2.8, if $pAp \cong A$, then there is a $k_1 \in \mathbf{N}$ such that $k = r_1^{s_1} \dots r_l^{s_l} k_1$ and $k_1 = 1$ or $k_1 = c_1^{d_1} \dots c_h^{d_h}$, where c_j is a prime number with $f(\{m(n)!\})(c_j) = \infty$ and $d_j \neq 0$ for $j = 1, 2, \dots, h$.

LEMMA 2.9. *With the same assumptions as in Lemma 2.8, we suppose that there is a* $k_1 \in \mathbf{N}$ *such that* $k = r_1^{s_1} \dots r_l^{s_l} k_1$ *and* $k_1 = 1$ *or* $k_1 = c_1^{d_1} \dots c_h^{d_h}$, *where* c_j *is a prime number with* $f(\{m(n)!\})(c_j) = \infty$ *and* $d_j \neq 0$ *for* $j = 1, 2, \dots, h$. *Then* $pAp \cong A$.

Proof. We suppose that r is a prime number such that $r = r_{j_0}$, for some $j_0 \in \mathbf{N}$, with $1 \leq j_0 \leq l$. Then, since $f(\{m(n)!\})(r_{j_0}) = t_{j_0}$, there is an $n_{j_0} \in \mathbf{N}$ with $n_{j_0} \geq n_0 + 1$ such that $r^{t_{j_0} - s_{j_0}}$ divides $\frac{m(n_{j_0})!}{m(n_0)!}$. Since r_{j_0} does not divide k_1, we have

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r_{j_0}) = s_{j_0} + t_{j_0} - s_{j_0} = t_{j_0} = f(\{m(n)!\})(r_{j_0}).$$

Next, we suppose that r is a prime number with $r \neq r_j$, for $j = 1, 2, \dots, l$. In this case we divide a proof into three subcases to show that

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r) = f(\{m(n)!\})(r).$$

(i) *Case of* $1 \leq f(\{m(n)!\})(r) < \infty$. Then r does not divide $m(n_0)!$. Hence there is an $n_1 \in \mathbf{N}$ with $n_1 \geq n_0 + 1$ such that r^{t_0} divides $\frac{m(n_1)!}{m(n_0)!}$, where $t_0 = f(\{m(n)!\})(r)$. Thus

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r) = t_0 = f(\{m(n)!\})(r).$$

(ii) *Case of* $f(\{m(n)!\})(r) = 0$. Then r does not divide k and $m(n)$, for any $n \in \mathbf{N}$. Thus

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r) = 0 = f(\{m(n)!\})(r).$$

(iii) *Case of* $f(\{m(n)!\})(r) = \infty$. Then, by Lemma 2.1, there are countably many $n \in \mathbf{N}$ with $n \geq n_0 + 1$ such that r divides $m(n)$. Thus

$$f(\{km(n_0+1)\dots m(n_0+n-1)\})(r) = \infty = f(\{m(n)!\})(r).$$

Therefore, since $f(\{km(n_0+1)\dots m(n_0+n-1)\}) = f(\{m(n)!\})$, by Glimm [**6**, Theorem 1.12] $pAp \cong A$. Q.E.D.

THEOREM 2.10. *Let n_0 be a positive integer and p a projection in $M_{m(n_0)!}$ with $\tau(p) = \frac{k}{m(n_0)!}$. Let $r_1^{s_1}\dots r_l^{s_l}$ be a factor of $m(n_0)!$ with $1 \le f(\{m(n)!\})(r_j) = t_j < \infty$, for $j = 1, 2, \dots, l$, such that r_j does not divide $\frac{m(n_0)!}{r_1^{s_1},\dots,r_l^{s_l}}$ and r does not divide $m(n_0)!$, for any prime number r with $r \neq r_j$ for $j = 1, 2, \dots, l$ and $1 \le f(\{m(n)!\})(r) < \infty$. Then $pAp \cong A$ if and only if there is a $k_1 \in \mathbf{N}$ such that $k = r_1^{s_1}\dots r_l^{s_l}k_1$ and $k_1 = 1$ or $k_1 = c_1^{d_1}\dots c_h^{d_h}$ with $f(\{m(n)!\})(c_j) = \infty$ and $d_j \neq 0$, for $j = 1, 2, \dots, h$.*

Proof. This is immediate, by Lemmas 2.7, 2.8 and 2.9. Q.E.D.

Let A be a UHF-algebra of type $\{m(n)!\}$. We suppose that

$$\#\{r | r \text{ is a prime number with } 1 \le f(\{m(n)!\})(r) < \infty\} = \infty,$$

$$\#\{r | r \text{ is a prime number with } f(\{m(n)!\})(r) = \infty\} = 0.$$

In this case k_1 in the statement of Theorem 2.10 is always equal to 1, and so $pAp \cong A$ if and only if $\tau(p) = 1$. By Remark 1.5 we obtain the following theorem.

THEOREM 2.11. *With the assumptions above, for any automorphism β of $A \otimes \boldsymbol{K}$, we have $\beta_* = \mathrm{id}$ on $K_0(A \otimes \boldsymbol{K})$.*

3. Examples. Let B be a C*-algebra and $M(B)$ its multiplier algebra. Let Aut(B) be the group of all automorphisms of B. For each unitary element $w \in M(B)$, let Ad(w) denote the automorphism of B defined by $\mathrm{Ad}(w)(b) = wbw^*$, for any $b \in B$. We call Ad(w) a *generalized inner automorphism* of B, and we denote by Int(B) the group of all generalized inner automorphisms of B. It is easily seen that Int(B) is a normal subgroup of Aut(B). We note that if B is unital, Int(B) is the group of all inner automorphisms of B, since $M(B) = B$. Let Pic(B) be the Picard group of B. We note that $\mathrm{Pic}(B) \cong \mathrm{Aut}(A \otimes \mathbf{K})/\mathrm{Int}(A \otimes \mathbf{K})$.

Let A be a UHF-algebra of type $\{m(n)!\}$ and S the semigroup of automorphisms of $K_0(A \otimes \mathbf{K})$ defined in Section 1.

EXAMPLE 3.1. We suppose that $m(n) = k \in \mathbf{N}$ with $k \geq 2$, for any $n \in \mathbf{N}$; that is, A is a UHF-algebra of type k^∞.

(1) If k is a prime number, then by Theorem 2.3 we have

$$S = \left\{\frac{1}{k^t} | t \in \mathbf{Z} \quad \text{with} \quad t \geq 0\right\}.$$

Hence the group of automorphisms of $K_0(A \otimes \mathbf{K})$ induced by those of $A \otimes \mathbf{K}$ is $\{\frac{1}{k^t} | t \in \mathbf{Z}\} \cong \mathbf{Z}$. Also, by Lemma 1.1 and [**8**, Proposition 4], Pic(A) is isomorphic to a semidirect product of $\mathrm{Aut}(A)/\mathrm{Int}(A)$ with $\mathbf{Z}$.

(2) If $k = 6$, then by Theorem 2.3 we have

$$S = \{\frac{2^{d_1} \cdot 3^{d_2}}{6^t} | 1 \leq 2^{d_1} \cdot 3^{d_2} \leq 6^t, \quad d_1, d_2, t = 0, 1, \ldots \quad \}.$$

EXAMPLE 3.2 We suppose that

$$\#\{r | r \text{ is a prime number with } 1 \leq f(\{m(n)!\})(r) < \infty\} = \infty,$$

$$\#\{r | r \text{ is a prime number with } f(\{m(n)!\})(r) = \infty\} = 0.$$

Then, by Lemma 2.1, $S = \{1\}$. Hence the group of automorphisms of $K_0(A \otimes \mathbf{K})$ induced by those of $A \otimes \mathbf{K}$ is $\{1\}$. Therefore $\mathrm{Pic}(A) \cong \mathrm{Aut}(A)/\mathrm{Int}(A)$, by Lemma 1.1 and [**8**, Proposition 4].

REMARK 3.3. Let A be an AF-algebra by an inductive limit of finite dimensional C*-algebras for which the corresponding limit of K_0-groups is

$$\cdots \longrightarrow \mathbf{Z}^2 \xrightarrow{\phi_n} \mathbf{Z}^2 \longrightarrow \cdots \quad ,$$

where each $\mathbf{Z}^2$ is endowed with its natural ordering and

$$\phi_n = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix},$$

where $[a_1, a_2, \ldots, a_n, \ldots]$ is the continued fraction expansion of an irrational number θ. Then, in the same way as in [**8**], we see that if θ is not quadratic, $\mathrm{Pic}(A) \cong \mathrm{Aut}(A)/\mathrm{Int}(A)$ and that if θ is quadratic, $\mathrm{Pic}(A)$ is isomorphic to a semi-direct product of $\mathrm{Aut}(A)/\mathrm{Int}(A)$ with $\mathbf{Z}$.

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