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THEORETICAL PEARL
Applications of Plotkin-terms: partitions and morphisms for closed terms

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Abstract

This theoretical pearl is about the closed term model of pure untyped lambda-terms modulo β-convertibility. A consequence of one of the results is that for arbitrary distinct combinators (closed lambda terms) M, M', N, N' there is a combinator H such that

$$HM = HM' \neq HN = HN'.$$

The general result, which comes from Statman (1998), is that uniformly r.e. partitions of the combinators, such that each 'block' is closed under β-conversion, are of the form $\{H^{-1}\{M\}\}_{M\in\Lambda^{\Phi}}$. This is proved by making use of the idea behind the so-called Plotkin-terms, originally devised to exhibit some global but non-uniform applicative behaviour. For expository reasons we present the proof below. The following consequences are derived: a characterization of morphisms and a counter-example to the perpendicular lines lemma for β-conversion.

1 Introduction

We use notations from recursion theory and lambda calculus (see Rogers (1987) and Barendregt (1984)).

Notation.

(i) φ_e is the e-th partial recursive function of one argument.
(ii) $W_e = \mathrm{dom}(\varphi_e) \subseteq \mathbb{N}$ is the r.e. set with index e.
(iii) Λ is the set of lambda-terms and Λ^{Φ} is the set of closed-lambda terms (combinators).
(iv) $\mathscr{W}_e = \{M \in \Lambda^{\Phi} \mid \# M \in W_e\} \subseteq \Lambda^{\Phi}$; here $\#M$ is the code of the term M.

Definition 1.1

(i) Inspired by Visser (1980), we define a *Visser-partition* (V-partition) of Λ^Φ to be a family $\{\mathscr{W}_e\}_{e\in S}$ such that

(1) $S \subseteq \mathbb{N}$ is an r.e. set.
(2) $\forall e \in S \forall M, N\ (M \in \mathscr{W}_e\ \&\ N = M) \Rightarrow N \in \mathscr{W}_e$.
(3) $\mathscr{W}_e \cap \mathscr{W}_{e'} \neq \emptyset \Rightarrow \mathscr{W}_e = \mathscr{W}e'$.

(ii) A family $\{\mathscr{W}_e\}_{e\in S}$ is a *pseudo*-V-partition if it satisfies just (1) and (2).

Definition 1.2

Let $\{We\}e\in s$ be a V-partition:

1. The partition is said to be *covering* if $\bigcup_{e\in S} \mathscr{W}_e = \Lambda^\Phi$.
2. The partition is said to be *inhabited* if $\forall e \in S\, \mathscr{W}_e \neq \emptyset$.
3. A V-partition $\{\mathscr{W}_e\}_{e\in S'}$ is said to be (*extensionally*) *equivalent* with $\{\mathscr{W}_e\}$ if these families define the same collection of non-empty sets, i.e. if

$$\{\mathscr{W}_e \mid e \in S\ \&\ \mathscr{W}_e \neq \emptyset\} = \{\mathscr{W}_e \mid e \in S'\ \&\ \mathscr{W}_e \neq \emptyset\}.$$

Example 1.3

Let H be some given combinator. Define

$$\mathscr{W}_{e(M,H)} = \{N \in \Lambda^\Phi \mid HN = HM\}.$$

Then $\{\mathscr{W}_e\}_{e\in S_H}$, with $S_H = \{e(M,H) \mid M \in \Lambda^\Phi\}$, is an example of a covering and inhabited V-partition. We denote this V-partition by $\{\mathscr{W}_{e(M,H)}\}_{M\in\Lambda^\Phi}$.

Proposition 1.4

(i) Every V-partition is effectively equivalent to an inhabited one.
(ii) Every V-partition can effectively be extended to a covering one.

Proof

(i) Given $\{\mathscr{W}_e\}_{e\in S}$, define $S' = \{e \in S \mid \mathscr{W}_e \neq \emptyset\}$. Then $\{\mathscr{W}_e\}_{e\in S'}$ is the required modified partition.

(ii) Given $\{\mathscr{W}_e\}_{e\in S}$, define

$$\mathscr{W}_{e(M)} = \{N \mid N = M \vee \exists e \in S\, M, N \in \mathscr{W}_e\}.$$

Then $\{\mathscr{W}_{e(M)}\}_{M\in\Lambda^\Phi}$ is the required V-partition. □

The main theorem comes in two versions. The second, more sharp version is needed for the construction of so-called inevitably consistent equations, see Statman (1999).

Theorem 1.5 (Main theorem)

(i) Let $\{\mathscr{W}_e\}_{e\in S}$ be a V-partition. Then one can construct effectively a combinator H such that for all M, $N \in \Lambda^\Phi$

$$HM = HN \Leftrightarrow M = N \vee \exists e \in S\, M, N \in \mathscr{W}_e. \qquad (*)$$

The construction of H is effective in the code of the underlying r.e. set S.

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(ii) Let $\{\mathscr{W}_e\}_{e\in S}$ be a pseudo-V-partition. Then one can construct effectively a combinator H such that if $\{W_e\}_{e\in S}$ is an actual V-partition, then $(*)$ holds.

The theorem will be proved in section 2. It has several consequences. To state these we have to formulate the notion of morphism on Λ^Φ and the so-called perpendicular lines lemma.

Definition 1.6
Let $\varphi : \Lambda^\Phi \to \Lambda^\Phi$ be a map. Then φ is a *morphism* if

1. $\varphi(M) = \mathsf{E}\mathbf{c}_{f(\# M)}$, for some recursive function f.
2. $M = N \Rightarrow \varphi(M) = \varphi(N)$.

Lemma 1.7
(i) Let F be a combinator and define $\varphi_H(M) \equiv HM$. Then φ_H is a morphism.

(ii) Let F, G be combinators such that for all $M \in \Lambda^\Phi$ there exists a unique $N \in \Lambda^\Phi$ with $FM = GN$. Then there is a map $\varphi_{F,G}$ such that $FM = G_{\varphi F,G}(M)$, for all M, which is a morphism.

Proof
(i) For the coding # let app be the recursive function such that $\#(PQ) = \text{app}(\#P, \#Q)$. Define $f(m) = \text{app}(\#H, m)$. Then $\varphi_H(M) = \mathsf{E}\mathbf{c}_{f(\# M)}$. It is obvious that φ_H preserves β-equality.

(ii) Let $R(m,n)$ be an r.e. relation. Then we have $R(m,n) \Leftrightarrow \exists z\, T(m,n,z)$, for some recursive T. Let $\langle n,z\rangle$ be a recursive pairing with recursive inverses $\langle n,z\rangle.0 = n$, $\langle n,z\rangle.1 = z$. Define (μ is the least number operator)

$$\iota_n . R(m,n) = (\mu p . T(m, p.0, p.1)).0.$$

Then $\exists n \in \mathbb{N}\, R(m,n) \Rightarrow R(m, \iota_n . R(m,n))$. To construct the morphism $\varphi_{F,G}$, define

$$f(m) = \iota_n . F(\mathsf{E}\mathbf{c}_m) = G(\mathsf{E}\mathbf{c}_n).$$

By the assumption (existence) f is total. Define $\varphi_{F,G}(M) = \mathsf{E}\mathbf{c}_{f(\# M)}$. Now

$$f(\# M) = n \Rightarrow F(\mathsf{E}\mathbf{c}_c) = G(\mathsf{E}\mathbf{c}_n).$$

Therefore, $FM = G\varphi_{F,G}(M)$, for all M. The condition

$$M = M' \Rightarrow \varphi_{F,G}(M) = \varphi_{F,G}(M')$$

holds by the assumption (unicity). □

One may wonder if by dropping the unicity condition in Lemma 1.7(ii) one may obtain a morphism by making a right uniformization. This is not the case.

Proposition 1.8
There exist combinators F, G such that $\forall M \exists N\, FM = GN$ but without any morphism satisfying $\forall M\, FM = G\varphi(N)$.

Proof
Let $\Delta = \mathsf{Y}\Omega$ and define $F = \lambda x.\langle x, \Delta, \mathsf{I}\rangle$ and $G = \lambda y.\langle \mathsf{E}y, y\Omega\Delta, y\mathsf{I}\rangle$. Then (see Statman, 1986)

$$FM =_\beta GN \Leftrightarrow (N =_\beta c_n \vee N =_\beta \mathsf{I})\ \&\ \mathsf{E}N =_\beta M. \qquad (1)$$

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Any morphism φ such that $FM = G\varphi(M)$ would solve the convertibility problem recursively: one has by (1)

$$M = M' \Leftrightarrow \varphi(M) = \varphi(M'), \tag{2}$$

and since $\varphi(M), \varphi(M')$ we have nf's by (1), the RHS of (2) is decidable. □

Proposition 1.9
Not every morphism is of the form φ_H.

Proof
Let $F, G \in \Lambda^{\Phi}$ be such that $F \circ G = \mathsf{I}$. Then F, G determine a so-called *inner model* $[\![\ ]\!] = [\![\ ]\!]^{F,G}$ as follows:

$$[\![x]\!] = x;$$
$$[\![PQ]\!] = F[\![P]\!]\,[\![Q]\!];$$
$$[\![\lambda x, P]\!] = G(\lambda x\,.\,[\![P]\!]).$$

Using the condition on F, G it can be proved that

$$M =_{\beta} N \Rightarrow [\![M]\!] = [\![N]\!].$$

Therefore, defining $\varphi(M) = [\![M]\!]$ we obtain a morphism.

Now take $F \equiv \lambda y\,.\,u\mathsf{I}$, $\Gamma \equiv \lambda xy\,.\,yx$. Then, indeed, $F \circ G = \mathsf{I}$, and for the resulting inner model one has $[\![I]\!] = \lambda y\,.\,y\mathsf{I}$ and $[\![\Omega]\!] = (\lambda y\,.\,y(\lambda z\,.\,z\mathsf{I}z))\mathsf{I}(\lambda y\,.\,y(\lambda z\,.\,z\mathsf{I}z))$.

Suppose towards a contradiction that the resulting φ is of the form φ_H. Then $H\mathsf{I} = \lambda y\,.\,\lambda\mathsf{I}$, so H is solvable, and hence has a hnf $\lambda x_1 \ldots x_n\,.\,{}_i M_1 \ldots M_m$. However, $H\Omega = (\lambda y\,.\,y(\lambda z\,.\,z\mathsf{I}z))\mathsf{I}(\lambda y\,.\,y(\lambda z\,.\,z\mathsf{I}z))$, which is unsolvable. Therefore, the head-variable x_i is x_1, but then $H\Omega = \lambda x_2 \ldots x_n\,.\,\Omega M_1^* \ldots M_m^*$, which is not of the correct form. □

The following is a corollary to the main theorem.

Corollary 1.10
Every morphism φ is of the form $\varphi_{F,G}$.

Proof
Let φ be a given morphism. Define

$$\mathscr{W}_{e(N)} = \{Z \mid \exists M \in \Lambda^{\Phi}[\varphi(M) = N \,\&\, [Z = \langle \mathbf{c}_0, M\rangle \vee Z = \langle \mathbf{c}_1, N\rangle)]\}.$$

Then $\{\mathscr{W}_{e(N)}\}$ is a V-partition. By the main theorem, there exists an H such that

$$H\langle \mathbf{c}_0, M\rangle = H\langle \mathbf{c}_1, N\rangle \Leftrightarrow \langle \mathbf{c}_0, M\rangle = \langle \mathbf{c}_1, N\rangle \vee N = \varphi(M)$$
$$\Leftrightarrow N = \varphi(M).$$

Define

$$F = \lambda m.\, H\langle \mathbf{c}_0, m\rangle;$$
$$G = \lambda n\,.\,H\langle \mathbf{c}_1, n\rangle.$$

Then $FM = GN \Leftrightarrow N = \varphi(M)$. Therefore, $\varphi = \varphi_{F,G}$ □

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Note that for a given morphism φ, one can define

$$\mathscr{W}_{e(M,\varphi)} = \{N \in \Lambda^{\Phi} \mid \varphi(M) = \varphi(N)\}.$$

This is an inhabited V-partition. It is not difficult to show that each V-partition is equivalent to one of the form $\{\mathscr{W}_{e(M,\varphi)}\}$. Note that $\{\mathscr{W}_{e(M,H)}\} = \{\mathscr{W}_{e(M,\varphi_H)}\}$, see Lemma 1.7. The following result shows that covering V-partitions are always of this more restricted form.

Corollary 1.11
If $\{\mathscr{W}_e\}$ is a covering V-partition, then $\{\mathscr{W}_e\}$ is equivalent to $\{\mathscr{W}_{e(M,H)}\}_{M \in \Lambda^{\Phi}}$ for some H, effectively found from $\{\mathscr{W}_e\}$.

Proof
Let H be the combinator constructed effectively from $\{\mathscr{W}_e\}$. We will show that $\mathscr{W}_{e(M,H)} = \{N \mid HN = HM\}$ is equivalent to $\{\mathscr{W}_e\}$.

Claim. For $N \in \mathscr{W}_e$ one has $\mathscr{W}_e = \mathscr{W}_{e(M,H)}$. Indeed,

$$\begin{aligned} N \in \mathscr{W}_e &\Leftrightarrow M = N \vee M, N \in \mathscr{W}_e \\ &\Leftrightarrow HN = HM \\ &\Leftrightarrow N \in \mathscr{W}_{e(M,H)}. \end{aligned}$$

Therefore, noting that $M \in \mathscr{W}_{e(M,H)}$,

$$\{\mathscr{W}_e \mid M \in \Lambda^{\Phi}, \mathscr{W}_e \neq \emptyset\} \subseteq \{\mathscr{W}_{e(M,H)} \mid \mathscr{W}_{e(M,H)} \neq \emptyset, M \in \Lambda^{\Phi}\}.$$

The converse inclusion also holds, since every M belongs to some $\mathscr{W}_e$, and hence $\mathscr{W}_{e(M,H)} = \mathscr{W}_e$ for this e. □

The following theorem states that if a combinator, seen as function of n arguments, is constant – modulo Böhm-tree equality – on n perpendicular lines, then it is constant everywhere.

Theorem 1.12 (Perpendicular lines lemma)
Let F be a combinator. Suppose that for $n \in \mathbb{N}$ there are combinators M_{ij}, $1 \leqslant i \neq j \leqslant n$, and $N_1, \ldots, N_n$ such that for all terms $Z \in \Lambda$ one has ($\cong$ denotes Böhm-tree equality, i.e. $M \cong N \Leftrightarrow BT(M) = BT(N)$)

$$\begin{array}{ccccccc} F & Z & M_{12} & \ldots & M_{1n-1} & M_{1n} & \cong & N_1; \\ F & M_{21} & Z & \ldots & M_{2n-1} & M_{2n} & \cong & N_2; \\ & & & \ldots & & & & \\ & & & \ldots & & & & \\ F & M_{n1} & M_{n2} & \ldots & M_{nn-1} & Z & \cong & N_n. \end{array}$$

Then for all $P_1 \ldots, P_n \in \Lambda^{\Phi}$ one has

$$FP_1 \ldots P_n \cong N_1 (\cong N_2 \cong \ldots \cong N_n).$$

Proof
This is proved in Barendregt (1984, Theorem 14.4.12). □

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Proposition 1.13
If the perpendicular lines lemma is restricted to closed terms and if $\cong$ is replaced by $=_\beta$, then the perpendicular lines lemma is false for $n > 1$.

Proof
(For $n = 1$ the perpendicular lines lemma is trivially true for $=_\beta$.) Assume $n > 1$. For notational simplicity we assume $n = 2$, and give a counter example. Define

$$\mathscr{W}_{e_1} = \{N \in \Lambda^\Phi \mid N = \langle \mathsf{S}, \mathsf{S} \rangle\}$$
$$\mathscr{W}_{e_2} = \{N \in \Lambda^\Phi \mid \exists Z \in \Lambda^\Phi [N = \langle \mathsf{I}, Z \rangle \vee N = \langle Z, \mathsf{I} \rangle]\}.$$

Then $\{\mathscr{W}_e\}_{e \in \{e_1, e_2\}}$ is a V-partition. Let H be the combinator obtained from this partition by the main theorem. Then for all $Z \in \Lambda^\Phi$

$$H\langle \mathsf{S}, \mathsf{S} \rangle \neq H\langle \mathsf{I}, Z \rangle = H\langle Z, \mathsf{I} \rangle.$$

Now define $F \equiv \lambda xy.H\langle x, y \rangle$. Then for all $Z \in \Lambda^\Phi$

$$F\mathsf{SS} \neq F\mathsf{I}Z = FZ\mathsf{I}.$$

This is indeed a counter-example. □

We conjecture that the perpendicular lines lemma does hold for closed terms. We formulate this for $n = 3$.

Conjecture 1.14
Let F, M_{12}, M_{13}, M_{21}, M_{23}, M_{31}, M_{32}, N_1, N_2, $N_3 \in \Lambda^\Phi$ and suppose that for all $Z \in \Lambda^\Phi$ one has

$$\begin{array}{ccccccc} F & Z & M_{12} & M_{13} & \cong & N_1; \\ F & M_{21} & Z & M_{23} & \cong & N_2; \\ F & M_{31} & M_{32} & Z & \cong & N_3. \end{array}$$

Then for all X, Y, $Z \in \Lambda^\Phi$ one has $FXYZ \cong N_1 (\cong N_2 \cong N_3)$.

We also believe the conjecture in Barendregt (1984), stating that the perpendicular line lemma with $\cong$ replaced by $=_\beta$ is correct for open terms.

2 Proof of the main theorem

To prove the main Theorem 1.5, let a V-partition determined by S be fixed in this section. By Proposition 1.4 it may be assumed that the partition is inhabited.

Lemma 2.1
Let $\{\mathscr{W}_e\}_{e \in S}$ be an inhabited V-partition.

(i) There exists a total recursive function $f = f_S$ such that

$$\forall e \in S\ W_e = \{f((2e+1)2^n) \mid n \in \mathbb{N}\}.$$

(ii) There exists a combinator E^S such that

$$\forall e \in S\ \mathscr{W}_e = \{\mathsf{E}^S \mathbf{c}_{(2e+1)2^n} \mid n \in \mathbb{N}\}.$$

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Proof

(i) By elementary recursion theory there exists a recursive function h such that $W_e = \mathrm{Range}(\varphi_{h(e)})$ and $\varphi_{h(e)}$ is total, for all $e \in S$. Observing that e, n are uniquely determined by $k = (2e+1)2^n$, define f by $f(0) = 0$, $f((2e+1)2^n) = \varphi_{h(e)}(n)$.

(ii) Take $\mathsf{E}^S = \mathsf{E} \circ F_S$, where F_S lambda defines f_S and $\mathsf{E}\mathbf{c}_{\#M} = M$ for all $M \in \Lambda^{\Phi}$. □

Definition 2.2

(i) Define

$$\mathtt{odd}(0) = 0;$$

$$\mathtt{odd}((2e+1)2^n) = 2e+1.$$

(ii) Define $M \sim N$ iff $M = N \vee M = \mathsf{E}_m$, $N = \mathsf{E}_n$ and $\mathtt{odd}(m) = \mathtt{odd}(n)$, for some m, n.

Notice that $M \sim N$ iff $M = N$ or $\exists e \in SM, N \in \mathscr{W}_e$. Therefore, we have to prove that there exists a combinator H such that

$$HM = HN \Leftrightarrow M \sim N.$$

The proof consists in constructing a combinator $H = H^S$ such that

1. $M \sim N \Rightarrow HM = HN$, Proposition 2.4;
2. $HM = HN \Rightarrow M \sim N$, Proposition 2.9.

The second part of the main theorem easily follows by inspecting the proof.

Definition 2.3

(i) Define

$$T \equiv \lambda xyz\,.\,xy(xyz);$$

$$A \equiv \lambda fgxyz\,.\,fx(a(\mathsf{E}x))\,[f(\mathsf{S}^+x)y(g(\mathsf{S}^+x))z];$$

$$B \equiv \lambda fgx\,.\,f(\mathsf{S}x)\,(a(\mathsf{E}(Tx))\,(g(\mathsf{S}^+x))\,(gx).$$

(ii) By the double fixed-point theorem there exists terms F, G such that

$$F \twoheadrightarrow AFG;$$

$$G \twoheadrightarrow BFG.$$

To be explicit, write

$$D \equiv (\lambda xy\,.\,y(xxy));$$

$$Y \equiv DD;$$

$$G \equiv Y(\lambda u\,.\,B(Y(\lambda v\,.\,Auv))u);$$

$$F \equiv Y(\lambda u\,.\,AuG).$$

(iii) Finally, define

$$H \equiv \lambda xa\,.\,F\mathbf{c}_1(ax)\,(G\mathbf{c}_1).$$

Notation
Write

$$F_k \equiv F\mathbf{c}_k;$$
$$G_k \equiv G\mathbf{c}_k;$$
$$E_k \equiv E\mathbf{c}_k;$$
$$a_k \equiv a\mathsf{E}_k;$$
$$H_k[\ ] \equiv F_k[\ ]G_k;$$
$$C_k[\ ] \equiv F_k\,a_k([\ ]G_k).$$

Note that, by construction,

$$F_k\,MN \twoheadrightarrow F_k\,a_k(F_{k+1}\,MG_{k+1}\,N);$$
$$G_k \twoheadrightarrow F_{k+1}\,a_{2k}\,G_{k+1}\,G_k.$$

By reducing F, respectively G, it follows that

$$H_k[a_p] \equiv F_k\,a_p\,G_k \twoheadrightarrow C_k[H_{k+1}[a_p]] \tag{1}$$
$$H_k[a_k] \equiv F_k\,a_k\,G_k \twoheadrightarrow C_k[H_{k+1}[a_{2k}]]. \tag{2}$$

Proposition 2.4

$$M \sim N \Rightarrow HM = HN.$$

Proof
By Lemma 2.1, it suffices to show $H\mathsf{E}_k = H\mathsf{E}_{2k}$ for all k:

$$\begin{aligned} H\mathsf{E}_k &= \lambda a.\,H_1[a_k] \\ &= \lambda a.\,C_1[C_2[\ldots\,C_{k-1}[H_k[a_k]]\,..]], && \text{by (1)}, \\ &= \lambda a.\,C_1[C_2[\ldots\,C_{k-1}[C_k[H_k[a_{2k}]]]\,..]], && \text{by (2)}, \\ H\mathsf{E}_{2k} &= \lambda a.\,H_1[a_{2k}] \\ &= \lambda a.\,C_1[C_2[\ldots\,C_{k-1}[C_k[H_k[a_{2k}]]]\,..]], && \text{by (1)}. \quad \square \end{aligned}$$

As a piece of art we exhibit in more detail the reduction flow (contracted redexes are underlined).

$$\underline{H\mathsf{E}_k}$$
$$\lambda a.\,\underline{F_1\,a_k\,G_1}$$
$$\lambda a.\,F_1\,a_1(\underline{F_2\,a_2\,G_2}\,G_1)$$
$$\lambda a.\,F_1\,a_1(F_2\,a_2(\underline{F_3\,a_k\,G_3}\,G_2)G_1)$$
$$\ldots$$
$$\lambda a.\,F_1\,a_1(F_2\,a_2(F_3\,a_3(\ldots\,(F_k\,a_k\,G_k\,G_{k-1})\,\ldots)G_2)G_1) \equiv$$
$$\lambda a.\,F_1\,a_1(F_2\,a_2(F_3\,a_3(\ldots(F_k\,a_k \qquad \underline{G_k} \qquad G_{k-1})\,\ldots)G_2)G_1)$$
$$\lambda a.\,F_1\,a_1(F_2\,a_2(F_3\,a_3(\ldots\,(F_k\,a_k(F_{k+1}\,a_{2k}\,G_{k+1}\,G_k)G_{k-1})\,\ldots)G_2)G_1),$$

and also

$$H\mathsf{E}_{2k} \twoheadrightarrow \ldots \twoheadrightarrow$$
$$\lambda a.\,F_1\,a_1(F_2\,a_2(F_3\,a_3(\ldots(F_k\,a_k(F_{k+1}\,a_{2k}G_{k+1}\,G_k)G_{k-1})\,\ldots)G_2)G_1).$$

For the converse implication we need the fine structure of the reduction.

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Definition 2.5
Define

$$
\begin{aligned}
D_k^0[M] &\equiv F_x(aM) \equiv Y(\lambda u\,.\,AuG)\mathbf{c}_k(aM)\\
D_k^1[M] &\equiv (\lambda y\,.\,y(DDy))\,(\lambda u\,.\,AuG)\mathbf{c}_k(aM)\\
D_k^2[M] &\equiv (\lambda u\,.\,AuG)\,F_k(aM)\\
D_k^3[M] &\equiv AFG\mathbf{c}_k(aM)\\
D_k^4[M] &\equiv (\lambda gxyz\,.\,F_x(a\mathsf{E}_x)\,(F_{\mathsf{S}^+x}\,y(g(\mathsf{S}^+x))z))G\mathbf{c}_k(aM)\\
D_k^5[M] &\equiv (\lambda xyz\,.\,F_x(a\mathsf{E}_x)\,(F_{\mathsf{S}^+x}\,yGG_{\mathsf{S}^+x}\,z))\mathbf{c}_k(aM)\\
D_k^6[M] &\equiv (\lambda yz\,.\,F_k(a\mathsf{E}_k)\,(F_{\mathsf{S}^+\mathbf{c}_k}\,yG_{\mathsf{S}^+\mathbf{c}_k}\,z))\,(aM)\\
D_k^7[M] &\equiv (\lambda z\,.\,F_k(a\mathsf{E}_k)\,(F_{\mathsf{S}^+\mathbf{c}_k}(aM)G_{\mathsf{S}^+\mathbf{c}_k}\,z)).
\end{aligned}
$$

Lemma 2.6
Let $F_k(aM)N$ head-reduce in $8p+q$ steps to W. Then

$$
\begin{aligned}
W &\equiv D_k^q[M]N, && \text{if } p=0;\\
&\equiv D_k^q[\mathsf{E}_k]\,((H_{k+1}[\mathsf{E}_k])^{p-1}(H_{k+1}[M]N)), && \text{else.}
\end{aligned}
$$

Proof
Note that $F_k(aM)N \equiv D_k^0[M]N$. Moreover,

$$
\begin{aligned}
D_k^q[M]N &\rightarrow_h D_k^{q+1}[M]N, && \text{for } q<7;\\
D_k^7[M]N &\rightarrow_h D_k^0[\mathsf{E}_k]\,(H_{k+1}[M]N).
\end{aligned}
$$

The rest is clear. At steps 16, 24 we obtain, for example,

$$D_k^7[\mathsf{E}_k]\,(H_{k+1}[M]N) \rightarrow_h D_k^0[\mathsf{E}_k]\,((H_{k+1}[\mathsf{E}_k])\,(H_{k+1}[M]G_k)).$$

$$D_k^7[\mathsf{E}_k]\,((H_{k+1}[\mathsf{E}_k])(H_{k+1}[M]G_k))\,D_k^0[\mathsf{E}_k]\,((H_{k+1}[\mathsf{E}_k])^2(H_{k+1}[M]G_k)). \qquad \square$$

Remember that a standard reduction $\sigma: M \twoheadrightarrow_s N$ always consists of a head reduction followed by an internal reduction:

$$\sigma: M \twoheadrightarrow_h W \twoheadrightarrow_i N.$$

Notation
Write $M =_{s\leqslant n} N$ if there are standard reductions of length $\leqslant n$ from M (respectively N) to a common reduct Z. Similarly, $M =_{i\leqslant n} N$ for internal standard reductions. Also, the notations $=_{s<n}$ and $=_{i<n}$ will be used.

Lemma 2.7
(i) $D_k^q[M]N =_{i\leqslant n} D_k^{q'}[M']N' \Rightarrow q = q'$ & $N =_{s\leqslant n} N'$.
(ii) $D_k^q[M]N =_{i\leqslant n} D_k^q[M']N'$ & $q \leqslant 7 \Rightarrow M =_{s\leqslant n} M'$.
(iii) $D_k^7[M]N =_{i\leqslant n} D_k^7[M']N' \Rightarrow H_{k+1}[M] =_{s\leqslant n} H_{k+1}[M']$.

Proof
(i) Suppose $D_k^q[M]N =_{i\leqslant n} D_k^{q'}[M']N'$. Then by observing where the free variable a occurs, one can conclude that $q = q'$. Since the reductions to a common reduct are internal, the positions of N, N' are not changed, and hence $N =_{s\leqslant n} N'$.

(ii) Obvious from the definition of D^q_k.

(iii) In this case it follows that

$$D^0_k[\mathsf{E}_k](H_{k+1}[M]z) =_{i \leqslant n} D^0_k[\mathsf{E}_k](H_{k+1}[M']z).$$

The conclusion $H_{k+1}[M] =_{s \leqslant n} H_{k+1}[M']$ depends upon the fact that there are the free variables z to mark the residuals. □

Lemma 2.8

Suppose $G_k =_{s \leqslant n} (H_{k+1}[\mathsf{E}_k])^d (H_{k+1}[M]G_k)$. Then

$$H_{k+1}[\mathsf{E}(T\mathbf{c}_k)] =_{s<n} H_{k+1}[M].$$

Proof

By induction on d. If $d = 0$, then we have $G_k =_{s \leqslant n} H_{k+1}[M]G_k$. So there are standard reductions of these two terms to a common reduct. Observe that the head-reduction starting with G_k begins as follows:

$$\begin{aligned} G_k &\equiv Y(\lambda u\,.\,B(Y(\lambda v\,.\,Avu))u)\mathbf{c}_k \\ &\to_h (\lambda x\,.\,x(Yx))(\lambda u\,.\,B(Y(\lambda v\,.\,Avu))u)\mathbf{c}_k \\ &\to_h (\lambda u\,.\,B(Y(\lambda v\,.\,Avu))u)G\mathbf{c}_k \\ &\to_h BFG\mathbf{c}_k \\ &\to_h (\lambda gx\,.\,F(\mathsf{S}^+k)(a(\mathsf{E}^S(Tx)))(g(\mathsf{S}^+k))(gx)G\mathbf{c}_k \\ &\to_h (\lambda x\,.\,F(\mathsf{S}^+k)(a(\mathsf{E}^S(Tx)))(G(\mathsf{S}^+k))(Gx))\mathbf{c}_k \\ &\to_h F(\mathsf{S}^+k)(a(\mathsf{E}^S(T\mathbf{c}_k)))(G(\mathsf{S}^+k))(G\mathbf{c}_k). \end{aligned}$$

The hands of these terms are not of order 0 except the last one, but $H_{k+1}[X]$ is always of order 0. Therefore, the mentioned standard reduction of G_k goes at least to this last term $H_{k+1}[\mathsf{E}^S(T\mathbf{c}_k)]G_k$, but then $H_{k+1}[\mathsf{E}^S(T\mathbf{c}_k)] -_{s<n} H_{k+1}[M]$.

If $d > 0$, then start the same argument as above, but at the intermediate conclusion

$$H_{k+1}[\mathsf{E}^S(T\mathbf{c}_k)]G_k =_{s<n} (H_{k+1}[\mathsf{E}_k])^d (H_{k+1}[M]G_k),$$

one proceeds by concluding that

$$G_k =_{s<n} H_{k+1}[\mathsf{E}_k]^{d-1}(H_{k+1}[M]G_k)$$

and uses the induction hypotheses. □

Proposition 2.9

$$H_k[M] = H_k[N] \Rightarrow M \sim N.$$

Proof

By the standardization theorem, it suffices to show for all n that

$$\forall k \in \mathbb{N}[H_k[M] =_{s \leqslant n} H_k[N] \Rightarrow M \sim N].$$

This will be done by induction on n. From $H_k[M] =_{s \leqslant n} H_k[N]$, it follows that

$$\begin{aligned} H_k[M] &\twoheadrightarrow_h W_M \twoheadrightarrow_i Z \\ H_k[N] &\twoheadrightarrow_h W_N \twoheadrightarrow_i Z \end{aligned}$$

for some W_M, W_N, Z.

Case 1. W_M, W_N are both reached after < 8 steps. Then by Lemma 2.6, $W_M \equiv D_k^q[M]G_k$, $W_N \equiv D_k^{q'}[N]G_k$. By Lemma 2.7(i), it follows that $q = q'$. If $q < 7$, then by Lemma 2.7(ii) one has $M = N$, so $M \sim N$. If $q = 7$, then by Lemma 2.7(iii) one has $H_{k+1}[M] =_{s<n} H_{k+1}[N]$, and by the induction hypothesis one has $M \sim N$.

Case 2. W_M is reached after $p \geqslant 8$ steps and W_N after $q < 8$ steps. Then $p = 8d + q$ and, keeping in mind Lemma 2.7(i), it follows that $W_M \equiv D_k^q[M]G_k$, $W_N \equiv D_k^q[E_k]R$, $G_k =_{s<n} R$, where $R \equiv (H_{k+1}[\mathsf{E}_k])^{d-1}(H_{k+1}[N]G_k)$. Then as in case 1, it follows that $M \sim \mathsf{E}_k$. Moreover, by Lemma 2.8 $H_{k+1}[\mathsf{E}_{2k}] =_{s<n} H_{k+1}[N]$, so by the induction hypothesis $\mathsf{E}_{2k} \sim N$. So $M \sim \mathsf{E}_k \sim \mathsf{E}_{2k} \sim N$.

Case 3. Both W_M, W_N are reached after $\geqslant 8$ steps. Then

$$W_M \equiv D_k^j[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M]G_k));$$
$$W_N \equiv D_k^j[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^{d'}(H_{k+1}[N]G_k)).$$

If $d = d'$, then by Lemma 2.7

$$(H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M]G_k) =_{s<n} (H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[N]G_k),$$

so

$$H_{k+1}[M] =_{s<n} H_{k+1}[N],$$

since $H_{k+1}[X]$ is always of order 0. Therefore, by the induction hypothesis $M \sim N$.

If, on the other hand, say, $d < d'$, then (writing $d' = d + e$)

$$W_M \equiv D_k^j[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[M] \qquad G_k \qquad));$$
$$W_N \equiv D_k^k[\mathsf{E}_k]((H_{k+1}[\mathsf{E}_k])^d(H_{k+1}[\mathsf{E}_k]\,\underline{((H_{k+1}[\mathsf{E}_k])^{e-1}(H_{k+1}[N]G_k))}));$$

so

$$H_{k+1}[M] =_{s<n} H_{k+1}[\mathsf{E}_k]$$
$$G_k =_{s<n} (H_{k+1}(\mathsf{E}_k])^{e-1}(H_{k+1}[N]G_k),$$

since $H_{k+1}[X]$ is always of order 0. Therefore, by Lemma 2.8

$$H_{k+1}[\mathsf{E}_{2k}] =_{s<n} H_{k+1}[N].$$

Therefore, by the induction hypothesis, twice we obtain $M \sim \mathsf{E}_k \sim \mathsf{E}_{2k} \sim N$. □

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