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INTEGRAL MEANS AND DIRICHLET INTEGRAL FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract

For a normalized analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the estimate of the integral means

$$L_1(r, f) := \frac{r^2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2}$$

is an important quantity for certain problems in fluid dynamics, especially when the functions $f(z)$ are nonvanishing in the punctured unit disk $\mathbb{D}\backslash\{0\}$. Let $\Delta(r, f)$ denote the area of the image of the subdisk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ under f, where $0 < r \leq 1$. In this paper, we solve two extremal problems of finding the maximum value of $L_1(r, f)$ and $\Delta(r, z/f)$ as a function of r when f belongs to the class of m-fold symmetric starlike functions of complex order defined by a subordination relation. One of the particular cases of the latter problem includes the solution to a conjecture of Yamashita, which was settled recently by Obradović *et al.* ['A proof of Yamashita's conjecture on area integral', *Comput. Methods Funct. Theory* **13** (2013), 479–492].

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1. Introduction

Let $\mathcal{H}$ denote the family of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the subfamily of $\mathcal{H}$ consisting of the functions $f(z)$ normalized by $f(0) = 0 = f'(0) - 1$. Any function $f(z)$ belonging to the class $\mathcal{A}$ has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{for } z \in \mathbb{D}. \tag{1.1}$$

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A function $f(z)$ is said to be univalent in a domain $\Omega \subseteq \mathbb{C}$ if it is one-to-one in Ω. Denote by $\mathcal{S}$ the class of all univalent functions in the class $\mathcal{A}$. A function $f \in \mathcal{H}$ is said to be m-fold symmetric ($m = 1, 2, 3, \ldots$) if

$$f(e^{2\pi i/m} z) = e^{2\pi i/m} f(z).$$

The study of m-fold symmetric functions was initiated by the work of Golusin [5], Noshiro [13] and Robertson [19]. If $f \in \mathcal{A}$ is an m-fold symmetric function, then $f(z)$ has the following representation:

$$f(z) = z + \sum_{k=1}^{\infty} a_{km+1} z^{km+1}. \tag{1.2}$$

For two functions $f, g \in \mathcal{H}$, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that $f(z) = g(w(z))$ for $z \in \mathbb{D}$. Furthermore, if g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ (see [4]).

For $A, B \in \mathbb{C}$ with $|B| \le 1$ and $A \ne B$, let $\mathcal{S}^*(A, B)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the subordination relation

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad \text{for } z \in \mathbb{D}.$$

Without loss of generality, we can assume that B is real. Also, it is easy to see that $\mathcal{S}^*(A, B) = \mathcal{S}^*(-A, -B)$ and hence we assume that $-1 \le B \le 0$. For $-1 \le B < A \le 1$, the class $\mathcal{S}^*(A, B)$ was introduced and investigated by Janowski [7].

In this paper, we pay attention to the class $\mathcal{S}_m^*(A, B)$ of m-fold symmetric functions of the form (1.2) which satisfy

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az^m}{1 + Bz^m}, \quad z \in \mathbb{D}, \tag{1.3}$$

where $A \in \mathbb{C}$, $-1 \le B \le 0$ with $A \ne B$. We note that functions in the class $\mathcal{S}_m^*(A, B)$ need not be univalent. For suitable choice of the parameters m, A and B, we can obtain different subclasses studied by various authors. For instance, we list some of the subclasses for certain parameters:

(1) for $0 \le \alpha < 1$, the class $\mathcal{S}_m^*(\alpha) := \mathcal{S}_m^*(1 - 2\alpha, -1)$ denotes the family of m-fold symmetric starlike functions of order α and $\mathcal{S}^*(\alpha) := \mathcal{S}_1^*(\alpha)$ is the class of starlike functions of order α which was introduced by Robertson [19]. Further, $\mathcal{S}^* := \mathcal{S}^*(0)$ is the class of starlike functions which was introduced by Nevanlinna [12];

(2) for $\gamma \in \mathbb{C}\backslash\{0\}$, the class $\mathcal{S}^*(\gamma) := \mathcal{S}_1^*(2\gamma - 1, -1)$ is the class of starlike functions of complex order which was introduced by Nasr and Aouf [11];

(3) the class $\mathcal{S}_1^*(1, 0)$ was introduced by Singh [23];

(4) for $0 < \alpha \le 1$, the class $\mathcal{S}(\alpha) := \mathcal{S}_1^*(\alpha, -\alpha)$ was introduced by Padmanabhan [16];

(5) for $\alpha \ge \frac{1}{2}$, the class $\mathcal{S}_1^*(1, 1/\alpha - 1)$ was introduced by Singh and Singh [24];

(6) for $a + b \geq 1, b \leq a \leq 1 + b$, the class $\mathcal{S}_1^*((b^2 - a^2 + a)/b, (1 - a)/b)$ was introduced by Silverman [22];

(7) for $-1 \leq B < A \leq 1$, the class $\mathcal{S}_m^*(A, B)$ denotes the family of m-fold symmetric starlike functions which was introduced by Anh [1].

Moreover, for $A = e^{i\alpha}(e^{i\alpha} - 2\beta\cos\alpha)$, $B = -1$ and $m = 1$ with $\beta < 1$, the class $\mathcal{S}_m^*(A, B)$ reduces to the class of α-spiral-like functions of order β which is denoted by $\mathcal{S}_\alpha(\beta)$ (see [8]). Further, functions in the class $\mathcal{S}_\alpha(\beta)$ are univalent for $\beta \in [0, 1)$ and $\alpha \in (-\pi/2, \pi/2)$ (see [8]). In particular, functions in the class $\mathcal{S}_\alpha(0)$ are called α-spiral-like. The class $\mathcal{S}_\alpha(0)$ was introduced by Špaček [25].

For $A \in \mathbb{C}$, $-1 \leq B \leq 0$ with $A \neq B$, we define

$$k_{A,B}(z) = \begin{cases} ze^{Az} & \text{for } B = 0, \\ z(1 + Bz)^{A/B-1} & \text{for } B \neq 0 \end{cases} \tag{1.4}$$

and

$$k_{A,B}^{(m)}(z) = (k_{A,B}(z^m))^{1/m}. \tag{1.5}$$

It is easy to see that for every $m \in \mathbb{N}$, the function $k_{A,B}^{(m)}(z)$ belongs to the class $\mathcal{S}_m^*(A, B)$.

For $f \in \mathcal{H}$ the functional, called integral means,

$$M(r, f, \lambda_1, \lambda_2) := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda_1}|f'(re^{i\theta})|^{\lambda_2}\, d\theta \quad (z = re^{i\theta} \in \mathbb{D}),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, was introduced and investigated by Gromova and Vasil'ev [6]. The integral means

$$I_1(r, f) := M(r, f, -2, 0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2}$$

and

$$L_1(r, f) := r^2 I_1(r, f)$$

have many important applications in fluid dynamics (see [26, 27]). In 2014, Ponnusamy and Wirths [18] proved that for $f \in \mathcal{S}^*(\alpha)$,

$$L_1(r, f) \leq \frac{\Gamma(5 - 4\alpha)}{\Gamma^2(3 - 2\alpha)} \tag{1.6}$$

and the inequality (1.6) is sharp. This has settled the open problem of Gromova and Vasil'ev [6]. In the same paper the authors discussed a similar problem for the class of α-spiral-like functions of order β and also for the class $\mathcal{S}^*(A, B)$ with $-1 \leq B < A \leq 1$. Also, in 2014, Obradović *et al.* [15] considered a similar problem for some subclasses of the class $\mathcal{A}$. Except for these few recent results, the estimates of $L_1(r, f)$ for many geometric subclasses of the class $\mathcal{S}$ are not known.

For $g \in \mathcal{H}$, we denote the area of the multi-sheeted image of $|z| < r$ under $w = g(z)$ by $\Delta(r, g)$, where $0 < r \leq 1$. Thus, for $g(z) = \sum_{n=0}^{\infty} b_n z^n$,

$$\Delta(r, g) = \iint_{|z|<r} |g'(z)|^2\, dx\, dy = \pi\sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \quad (z = x + iy). \tag{1.7}$$

Computation of the area (1.7) for an analytic function g is known as the area problem. We call g a Dirichlet-finite function whenever $\Delta(1, g)$, the area covered by the mapping $z \mapsto g(z)$ for $z \in \mathbb{D}$, is finite. For example, all polynomials and more generally all functions $f \in \mathcal{A}$ for which $f'(z)$ is bounded on the unit disk $\mathbb{D}$ are Dirichlet-finite functions. In 1990, Yamashita [28] conjectured that

$$\max_{f \in C} \Delta\Big(r, \frac{z}{f(z)}\Big) = \pi r^2,$$

where C denotes the class of convex univalent functions in the unit disk $\mathbb{D}$ of the form (1.1). The maximum is attained only by rotations of the function $f_0(z) = z/(1-z)$. In 2013, Yamashita's conjecture was settled in a more general setting for functions in the class $\mathcal{S}^*(\alpha)$ by Obradović *et al.* [14]. In 2014, Ponnusamy and Wirths [18], Obradović *et al.* [15] and Sahoo and Sharma [21] discussed the maximum area problem for functions of type $z/f(z)$ when f belongs to certain subclasses of the class $\mathcal{S}$. Moreover, recently, Ponnusamy *et al.* [17] solved the same problem for the class $\mathcal{S}^*(A, B)$, where $-1 \le B < A \le 1$.

Our first aim of this paper is to estimate $L_1(r, f)$ for functions in the class $\mathcal{S}^*_m(A, B)$, where $A \in \mathbb{C}$, $-1 \le B \le 0$ with $A \ne B$. Our second aim is to investigate Yamashita's conjecture (or the maximum area problem for functions of type $z/f(z)$) for the class $\mathcal{S}^*_m(A, B)$, where $A \in \mathbb{C}$, $-1 \le B \le 0$ with $A \ne B$.

Before we state our main results, we recall that for $a, b, c \in \mathbb{C}$ with $c \ne 0, -1, -2, -3, \ldots$ the function

$$F(a, b; c; z) := {}_2F_1(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is called the Gaussian hypergeometric function, which is analytic in the unit disk $\mathbb{D}$. Here $(a)_0 = 1$ for $a \ne 0$ and $(a)_n$ denotes the Pochhammer symbol $(a)_n = a(a+1) \cdots (a+n-1)$ for $n \in \mathbb{N}$. Clearly, the shifted function $zF(a, b; c; z)$ belongs to the class $\mathcal{A}$. The asymptotic behaviour of $F(a, b; c; z)$ near $z = 1$ gives that

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty \quad \text{for } \operatorname{Re} c > \operatorname{Re}(a+b).$$

Similarly, the function ${}_0F_1(a; z)$ is defined by

$${}_0F_1(a; z) = 1 + \sum_{n=1}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!},$$

which is analytic throughout the finite complex plane.

2. Main results

THEOREM 2.1. *Let $f \in \mathcal{S}^*_m(A, B)$ for some $A \in \mathbb{C}$, $-1 \le B \le 0$ with $A \ne B$. Then, for $0 < r \le 1$,*

$$L_1(r, f) \le \begin{cases} F(\delta, \overline{\delta}; 1; B^2 r^{2m}) & \text{for } B \ne 0, \\ {}_0F_1\Big(1; \dfrac{|A|^2 r^{2m}}{m^2}\Big) & \text{for } B = 0, \end{cases} \tag{2.1}$$

where $\delta = 1/m(A/B - 1)$. The inequality (2.1) is sharp.

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PROOF. Let $f \in \mathcal{S}^*_m(A, B)$ and $p(z) = z/f(z)$. Then $p(z)$ is analytic in the unit disk $\mathbb{D}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{D}$. In view of (1.3),

$$\frac{zf'(z)}{f(z)} = 1 - \frac{zp'(z)}{p(z)} \prec \frac{1 + Az^m}{1 + Bz^m}, \quad z \in \mathbb{D}$$

or, equivalently,

$$\frac{zp'(z)}{p(z)} \prec \frac{(B - A)z^m}{1 + Bz^m} =: \phi(z), \quad z \in \mathbb{D}. \tag{2.2}$$

Since $\phi(z)$ is convex in $\mathbb{D}$ and $\phi(0) = 0$, it follows that (see for example [10, Corollary 3.1d.1, page 76])

$$\frac{z}{f(z)} = p(z) \prec \exp\left(\int_0^z \frac{\phi(t)}{t}\, dt\right) = q^{(m)}_{A,B}(z),$$

where

$$q^{(m)}_{A,B}(z) = \begin{cases} (1 + Bz^m)^{(1/m)(A/B-1)} = F(1, \delta; 1; Bz^m) & \text{for } B \neq 0,\\ e^{-(A/m)z^m} & \text{for } B = 0, \end{cases}$$

with $\delta = (1/m)(A/B - 1)$. If we write the series representation of $q^{(m)}_{A,B}(z)$ by

$$q^{(m)}_{A,B}(z) = 1 + \sum_{k=1}^{\infty} c_{km} z^{km},$$

then the Littlewood subordination theorem (see [4, 9]) yields

$$\begin{aligned} L_1(r, f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2}{|f(re^{i\theta})|^2}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{z}{f(z)}\right|^2 d\theta\\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |q^{(m)}_{A,B}(z)|^2\, d\theta\\ &= \sum_{k=0}^{\infty} |c_{km}|^2 r^{2km} \quad \text{where } c_{0m} = 1. \end{aligned} \tag{2.3}$$

Now, for $B \neq 0$,

$$\begin{aligned} \sum_{k=0}^{\infty} |c_{km}|^2 r^{2km} = \sum_{k=0}^{\infty}\left|\frac{(\delta)_k}{k!}\right|^2 (Br^m)^{2k} &= \sum_{k=0}^{\infty} \frac{(\delta)_k(\overline{\delta})_k}{(1)_k}\frac{(Br^m)^{2k}}{k!}\\ &= F(\delta, \overline{\delta}; 1; B^2r^{2m}). \end{aligned} \tag{2.4}$$

Also, for $B = 0$,

$$\sum_{k=0}^{\infty} |c_{km}|^2 r^{2km} = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left|\frac{Ar^m}{m}\right|^{2k} = {}_0F_1\left(1; \frac{|A|^2 r^{2m}}{m^2}\right). \tag{2.5}$$

Finally, the desired conclusion follows from (2.3) to (2.5). Equality occurs for both cases in (2.1) for the function $k^{(m)}_{A,B}(z)$ defined by (1.5). □

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REMARK 2.2. We observe that for $B = -1$ and $r = 1$, the series

$$F(\delta, \overline{\delta}; 1; B^2 r^{2m}) = \sum_{k=0}^{\infty} \left|\frac{(\delta)_k}{k!}\right|^2 (Br^m)^{2k}$$

converges if

$$1 > \mathrm{Re}(\delta + \overline{\delta}) = \frac{2}{m}\,\mathrm{Re}\Big(\frac{A}{B} - 1\Big) \quad \text{that is, if } \mathrm{Re}(A+1) > -\frac{m}{2}.$$

REMARK 2.3. For $-1 \le B < A \le 1$, by substituting $m = 1$ in Theorem 2.1, we obtain the result proved by Ponnusamy and Wirths [18, Theorem 1]. Again, if we substitute $m = 1$, $A = e^{i\alpha}(e^{i\alpha} - 2\beta\cos\alpha)$ and $B = -1$ with $\beta < 1$, $\alpha \in (-\pi/2, \pi/2)$ in Theorem 2.1, we obtain the result proved by Ponnusamy and Wirths [18, Theorem 2].

LEMMA 2.4. *Let $f \in \mathcal{S}_m^*(A, B)$ for some $A \in \mathbb{C}$, $-1 \le B \le 0$ with $A \ne B$. If*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{km} z^{km} \quad \textit{for } z \in \mathbb{D}, \tag{2.6}$$

then

$$\sum_{k=1}^{\infty} ((km)^2 - |B - A - kmB|^2)|b_{km}|^2 \le |A - B|^2.$$

PROOF. For $f \in \mathcal{S}_m^*(A, B)$, set $p(z) = z/f(z)$. In view of the relation (2.2), it immediately follows that there exists an analytic function $w : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\frac{zp'(z)}{p(z)} = \frac{(B - A)z^m w(z)}{1 + Bz^m w(z)}, \quad z \in \mathbb{D},$$

which is equivalent to

$$p'(z) = w(z)z^{m-1}((B - A)p(z) - Bzp'(z)). \tag{2.7}$$

From (2.6) and (2.7),

$$\sum_{k=1}^{\infty} kmb_{km}z^{km-1} = w(z)z^{m-1}\Big[(B - A) + \sum_{k=1}^{\infty}(B - A - kmB)b_{km}z^{km}\Big]$$

or, equivalently,

$$\sum_{k=1}^{n} kmb_{km}z^{km-1} + \sum_{k=n+1}^{\infty} c_{km}z^{km-1}$$
$$= w(z)z^{m-1}\Big[(B - A) + \sum_{k=1}^{n-1}(B - A - kmB)b_{km}z^{km}\Big]$$

for certain coefficients c_{km}. By Clunie's method [2] (see also [3, 20]),

$$\sum_{k=1}^{n}(km)^2|b_{km}|^2 r^{2(km-1)} - \sum_{k=1}^{n-1}|B - A - kmB|^2|b_{km}|^2 r^{2((k+1)m-1)}$$
$$\le |A - B|^2 r^{2(m-1)}. \tag{2.8}$$

The required result follows if we take $r \to 1^-$ and allow $n \to \infty$. □

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Lemma 2.5. *Let $f \in \mathcal{S}_m^*(A,0)$ for some $A \in \mathbb{C}\backslash\{0\}$. If*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{km} z^{km} \quad \textit{for } z \in \mathbb{D}$$

and

$$q_{A,0}^{(m)}(z) = e^{-(A/m)z^m} = 1 + \sum_{k=1}^{\infty} c_{km} z^{km} \quad \textit{for } z \in \mathbb{D},$$

then, for each $N \in \mathbb{N}$ and for $|z| < r$, $r \in (0,1]$,

$$\sum_{k=1}^{N} (km)|b_{km}|^2 r^{2km} \le \sum_{k=1}^{N} (km)|c_{km}|^2 r^{2km}. \tag{2.9}$$

Proof. By considering the inequality (2.8) for $B = 0$ and then multiplying by r^2 on both sides of it, we obtain

$$\sum_{k=1}^{n-1} \{(km)^2 - |A|^2 r^{2m}\}|b_{km}|^2 r^{2km} + (nm)^2 |b_{nm}|^2 r^{2nm} \le |A|^2 r^{2m}. \tag{2.10}$$

Since the function $q_{A,0}^{(m)}(z)$ satisfies the following differential equation:

$$\frac{d}{dz}(q_{A,0}^{(m)}(z)) = -Az^{m-1} q_{A,0}^{(m)}(z) \quad \text{for } z \in \mathbb{D},$$

it is clear that the equality in (2.10) is attained for $b_{km} = c_{km}$.

We consider the inequalities (2.10) for $n = 1, 2, \ldots, N$ and multiply the corresponding nth inequality by a factor $\lambda_{n,N}^{(m)}$. These factors are chosen in such a way that the addition of the left-hand side of the modified inequalities gives the left-hand side of (2.9). Therefore, the factors $\lambda_{n,N}^{(m)}$ can be obtained from the following system of linear equations:

$$km = (km)^2 \lambda_{k,N}^{(m)} + \sum_{n=k+1}^{N} \lambda_{n,N}^{(m)} \{(km)^2 - |A|^2 r^{2m}\}, \quad k = 1, 2, \ldots, N. \tag{2.11}$$

Since the matrix representation of the system of Equations (2.11) is an upper triangular matrix with positive integers as diagonal elements, the solution of the system (2.11) is uniquely determined. Hence, by Cramer's rule, the solution of the system (2.11) can be written as

$$\lambda_{n,N}^{(m)} = \frac{((n-1)!)^2 m^{2(n-1)}}{(N!)^2 m^{2N}} \operatorname{Det}(A_{n,N}),$$

where $A_{n,N}$ is an $(N-n+1) \times (N-n+1)$ matrix given by

$$A_{n,N} = \begin{bmatrix} nm & n^2m^2 - |A|^2 r^{2m} & \cdots & n^2m^2 - |A|^2 r^{2m} \\ (n+1)m & (n+1)^2 m^2 & \cdots & (n+1)^2 m^2 - |A|^2 r^{2m} \\ \vdots & \vdots & \vdots & \vdots \\ Nm & 0 & \cdots & N^2 m^2 \end{bmatrix}.$$

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Determinants of these matrices can be obtained by expanding according to Laplace's rule with respect to the last row, wherein the first element is Nm, the last element is N^2m^2 and the remaining entries are zeros. This expansion and a mathematical induction give the following recurrence formula for $k \le N-1$:

$$\lambda_{k,N}^{(m)} = \lambda_{k,N-1}^{(m)} - \frac{1}{Nm}\Big(1 - \frac{|A|^2 r^{2m}}{k^2m^2}\Big)\prod_{l=k+1}^{N-1}\frac{|A|^2r^{2m}}{l^2m^2},$$

which can be written as

$$\lambda_{k,N}^{(m)} = \lambda_{k,N-1}^{(m)} - \frac{1}{Nm}(1 - a(k))\prod_{l=k+1}^{N-1} a(l), \tag{2.12}$$

where $a(k) = |A|^2r^{2m}/(km)^2$. We note that $1 - a(k)$ may be positive as well as negative for $k \in \mathbb{N}$. Now we shall prove that the multipliers $\lambda_{k,N}^{(m)}$ are positive for all $N \in \mathbb{N}, 1 \le k \le N$. Indeed, this proves our required result, since, as we noted at the beginning of the proof, equality is attained for $b_{km} = c_{km}$.

Case (i). Suppose that $1 - a(k) \le 0$. Then, from (2.12), we see that for fixed $k \in \mathbb{N}, N \ge k$, the sequence $\{\lambda_{k,N}^{(m)}\}$ is an increasing sequence. Thus,

$$\lambda_{k,N}^{(m)} \ge \lambda_{k,N-1}^{(m)} \ge \cdots \ge \lambda_{k,k}^{(m)} = \frac{1}{km} > 0$$

and hence the multipliers $\lambda_{k,N}^{(m)}$ are positive.

Case (ii). Suppose that $1 - a(k) > 0$. Then, for fixed $k \in \mathbb{N}, N \ge k$, the sequence $\{\lambda_{k,N}^{(m)}\}$ is a strictly decreasing sequence with

$$\begin{aligned}\lambda_k^{(m)} := \lim_{N\to\infty}\lambda_{k,N}^{(m)} &= \frac{1}{km} - (1 - a(k))\sum_{n=k+1}^{\infty}\frac{1}{nm}\prod_{l=k+1}^{n-1} a(l)\\ &= (1 - a(k))^{-1}\Big(\sum_{n=0}^{\infty}\frac{1}{km}(a(k))^n - \sum_{n=k+1}^{\infty}\frac{1}{nm}\prod_{l=k+1}^{n-1} a(l)\Big).\end{aligned}$$

To prove that $\lambda_{k,N}^{(m)} > 0$ for all $N \in \mathbb{N}, 1 \le k \le N$, it suffices to prove that $\lambda_k^{(m)} \ge 0$ for $k \in \mathbb{N}$. Since $a(k) > a(l)$ for $l \ge k+1$, each term in the first summation is greater than the corresponding term in the second summation. Hence, $\lambda_k^{(m)} > 0$, which completes the proof. □

LEMMA 2.6. *Let $f \in \mathcal{S}_m^*(A, B)$ for some $A \in \mathbb{C}$, $-1 \le B < 0$ with $A \ne B$. If*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_{km}z^{km} \quad \text{for } z \in \mathbb{D}$$

and

$$q_{A,B}^{(m)}(z) = (1 + Bz^m)^{(1/m)(1-A/B)} = 1 + \sum_{k=1}^{\infty} c_{km}z^{km} \quad \text{for } z \in \mathbb{D},$$

then, for each $N \in \mathbb{N}$ and $|z| < r$, $r \in (0, 1]$,

$$\sum_{k=1}^{N} (km)|b_{km}|^2 r^{2km} \le \sum_{k=1}^{N} (km)|c_{km}|^2 r^{2km}. \tag{2.13}$$

PROOF. By considering the inequality (2.8) for $B \neq 0$ and then multiplying by r^2 on both sides of it, we can rewrite the inequality (2.8) as

$$\sum_{k=1}^{n-1} \{(km)^2 - |km - \phi|^2 B^2 r^{2m}\}|b_{km}|^2 r^{2km} + (nm)^2|b_{nm}|^2 r^{2nm} \le B^2|\phi|^2 r^{2m}, \tag{2.14}$$

where $\phi := 1 - A/B$. Since the function $q_{A,B}^{(m)}(z)$ satisfies the following differential equation:

$$\frac{d}{dz}(q_{A,B}^{(m)}(z)) = z^{m-1}\Big[(B - A)q_{A,B}^{(m)}(z) - Bz\frac{d}{dz}(q_{A,B}^{(m)}(z))\Big] \quad \text{for } z \in \mathbb{D},$$

it is clear that the equality in (2.14) is attained for $b_{km} = c_{km}$.

We consider the inequalities (2.14) for $n = 1, 2, \ldots, N$ and multiply the corresponding nth inequality by a factor $\lambda_{n,N}^{(m)}$. These factors are chosen in such a way that the addition of the left-hand side of the modified inequalities gives the left-hand side of (2.13). Therefore, the factors $\lambda_{n,N}^{(m)}$ can be obtained from the following system of linear equations:

$$km = (km)^2\lambda_{k,N}^{(m)} + \sum_{n=k+1}^{N} \lambda_{n,N}^{(m)}\{(km)^2 - |km - \phi|^2 B^2 r^{2m}\}, \quad k = 1, 2, \ldots, N. \tag{2.15}$$

Since the matrix representation of the system of equations (2.15) is an upper triangular matrix with positive integers as diagonal elements, the solution of the system (2.15) is uniquely determined. Hence, by Cramer's rule, the solution of the system (2.15) can be written as

$$\lambda_{n,N}^{(m)} = \frac{((n-1)!)^2 m^{2(n-1)}}{(N!)^2 m^{2N}} \operatorname{Det}(A_{n,N}),$$

where $A_{n,N}$ is an $(N - n + 1) \times (N - n + 1)$ matrix given by

$$A_{n,N} = \begin{bmatrix} nm & n^2m^2 - |nm - \phi|^2 B^2 r^{2m} & \cdots & n^2m^2 - |nm - \phi|^2 B^2 r^{2m} \\ (n+1)m & (n+1)^2m^2 & \cdots & (n+1)^2m^2 - |(n+1)m - \phi|^2 B^2 r^{2m} \\ \vdots & \vdots & \vdots & \vdots \\ Nm & 0 & \cdots & N^2m^2 \end{bmatrix}.$$

Determinants of these matrices can be obtained by expanding according to Laplace's rule with respect to the last row, wherein the first element is Nm, the last element is N^2m^2 and the remaining entries are zeros. This expansion and a mathematical induction give the following recurrence formula for $k \le N - 1$:

$$\lambda_{k,N}^{(m)} = \lambda_{k,N-1}^{(m)} - \frac{1}{Nm}\bigg(1 - \bigg|1 - \frac{\phi}{km}\bigg|^2 B^2 r^{2m}\bigg) \prod_{l=k+1}^{N-1} \bigg|1 - \frac{\phi}{lm}\bigg|^2 B^2 r^{2m}$$

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or, equivalently,

$$\lambda_{k,N}^{(m)} = \lambda_{k,N-1}^{(m)} - \frac{1}{Nm}(1 - a(k)) \prod_{l=k+1}^{N-1} a(l), \tag{2.16}$$

where

$$a(k) = \left|1 - \frac{\phi}{km}\right|^2 B^2 r^{2m}.$$

Here we note that $1 - a(k)$ may be positive as well as negative for $k \in \mathbb{N}$. Now we shall prove that the multipliers $\lambda_{k,N}^{(m)}$ are positive for all $N \in \mathbb{N}, 1 \leq k \leq N$. Indeed, this proves our required result, since, as we noted at the beginning of the proof, equality is attained for $b_{km} = c_{km}$.

Case (i). Suppose that $1 - a(k) \leq 0$. Then, from (2.16), we see that for fixed $k \in \mathbb{N}, N \geq k$, the sequence $\{\lambda_{k,N}^{(m)}\}$ is an increasing sequence. Thus,

$$\lambda_{k,N}^{(m)} \geq \lambda_{k,N-1}^{(m)} \geq \cdots \geq \lambda_{k,k}^{(m)} = \frac{1}{km} > 0$$

and hence the multipliers $\lambda_{k,N}^{(m)}$ are positive.

Case (ii). Suppose that $1 - a(k) > 0$. Then, for fixed $k \in \mathbb{N}, N \geq k$, the sequence $\{\lambda_{k,N}^{(m)}\}$ is a strictly decreasing sequence with

$$\lambda_k^{(m)} := \lim_{N\to\infty} \lambda_{k,N}^{(m)} = \frac{1}{km} - (1 - a(k)) \sum_{n=k+1}^{\infty} \frac{1}{nm} \prod_{l=k+1}^{n-1} a(l). \tag{2.17}$$

To prove that $\lambda_{k,N}^{(m)} > 0$ for all $N \in \mathbb{N}, 1 \leq k \leq N$, it suffices to prove that $\lambda_k^{(m)} \geq 0$ for $k \in \mathbb{N}$. Now one can verify that

$$\frac{1}{n(1 - a(n))} > \frac{1}{(n+1)(1 - a(n+1))} \tag{2.18}$$

and

$$\frac{1}{n(1 - a(n))} = \frac{1}{n} + \frac{a(n)}{n(1 - a(n))} \tag{2.19}$$

are valid for every $n \in \mathbb{N}$. By repeated application of (2.18) and (2.19) for $n = k, k+1, \ldots, P$,

$$\frac{1}{km(1 - a(k))} > \sum_{n=k+1}^{P} \frac{1}{nm} \prod_{l=k+1}^{n-1} a(l) + \frac{\prod_{l=k+1}^{P} a(l)}{m(P+1)(1 - a(P+1))} \quad \text{for } k \leq P.$$

By taking $P \to \infty$,

$$\frac{1}{km(1 - a(k))} \geq \sum_{n=k+1}^{\infty} \frac{1}{nm} \prod_{l=k+1}^{n-1} a(l).$$

Therefore, from (2.17), it follows that $\lambda_k^{(m)} \geq 0$. This completes the proof of the lemma. □

The following result settles Yamashita's conjecture for functions in the class $\mathcal{S}_m^*(A, B)$.

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Theorem 2.7. *Let $f \in \mathcal{S}_m^*(A,B)$ for some $A \in \mathbb{C}$, $-1 \le B \le 0$ with $A \ne B$. Then, for $0 < r \le 1$,*

$$\max_{f \in \mathcal{S}_m^*(A,B)} \Delta\Big(r, \frac{z}{f(z)}\Big) = E_{A,B}^{(m)}(r),$$

where

$$E_{A,B}^{(m)}(r) = \begin{cases} \dfrac{\pi}{m}|A-B|^2 r^{2m} F(\delta+1, \overline{\delta}+1; 2; B^2 r^{2m}) & \text{for } B \ne 0, \\ \dfrac{\pi}{m}|A|^2 r^{2m} {}_0F_1(2; |Ar^m/m|^2) & \text{for } B = 0, \end{cases} \tag{2.20}$$

with $\delta = (1/m)(A/B - 1)$. The maximum is attained by the rotation of the function $k_{A,B}^{(m)}(z)$ defined by (1.5).

Proof. Let $f \in \mathcal{S}_m^*(A,B)$. If we set

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_{nm} z^{nm} \quad \text{and} \quad \frac{z}{k_{A,B}^{(m)}(z)} = 1 + \sum_{n=1}^{\infty} c_{nm} z^{nm} \quad \text{for } z \in \mathbb{D},$$

then, from Lemmas 2.5 and 2.6, it follows that for each $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} (nm)|b_{nm}|^2 r^{2nm} \le \sum_{n=1}^{N} (nm)|c_{nm}|^2 r^{2nm}, \quad r \in (0,1]. \tag{2.21}$$

In view of (1.7) and (2.21),

$$\begin{aligned} \Delta\Big(r, \frac{z}{f(z)}\Big) &= \pi \sum_{n=1}^{\infty} (nm)|b_{nm}|^2 r^{2nm} \\ &\le \pi \sum_{n=1}^{\infty} (nm)|c_{nm}|^2 r^{2nm} = \Delta\Big(r, \frac{z}{k_{A,B}^{(m)}(z)}\Big). \end{aligned}$$

To complete the proof, we have to show that

$$\pi \sum_{n=1}^{\infty} (nm)|c_{nm}|^2 r^{2nm} = E_{A,B}^{(m)}(r), \tag{2.22}$$

where $E_{A,B}^{(m)}(r)$ is defined by (2.20). Now, for $B \ne 0$, a formal computation gives

$$\begin{aligned} \pi \sum_{n=1}^{\infty} (nm)|c_{nm}|^2 r^{2nm} &= \pi \sum_{n=1}^{\infty} (nm) \Big|\frac{(\delta)_n}{n!}\Big|^2 (Br^m)^{2n} \\ &= \pi \sum_{n=1}^{\infty} (nm) \frac{(\delta)_n (\overline{\delta})_n}{(1)_n (1)_n} (Br^m)^{2n} \\ &= \pi m |\delta|^2 B^2 r^{2m} \sum_{n=0}^{\infty} \frac{(\delta+1)_n (\overline{\delta}+1)_n}{(2)_n (1)_n} (Br^m)^{2n} \\ &= \frac{\pi}{m} |A-B|^2 r^{2m} F(\delta+1, \overline{\delta}+1; 2; B^2 r^{2m}) \\ &= E_{A,B}^{(m)}(r). \end{aligned} \tag{2.23}$$

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TABLE 1. Approximate values of $E_{A,B}^{(m)}(1)$.

m	A	Approximate values of $E_{A,0}^{(m)}(1)$	B	Approximate values of $E_{A,B}^{(m)}(1)$
1	$1+i/2$	6.949 42	$-2/3$	15.826 4
1	$3+i/4$	591.462	$-3/4$	451.334
2	$1+2i$	13.898 8	$-1/10$	14.110 8
2	$5+3i$	960.441 9	$-3/4$	482.049

Also, for $B=0$,

$$\begin{aligned}\pi\sum_{n=1}^{\infty}(nm)|c_{nm}|^2r^{2nm} &= \pi\sum_{n=1}^{\infty}(nm)\frac{1}{(n!)^2}\left|\frac{Ar^m}{m}\right|^{2n}\\ &= \pi m\left|\frac{Ar^m}{m}\right|^2\sum_{n=0}^{\infty}\frac{1}{(2)_n(1)_n}\left|\frac{Ar^m}{m}\right|^{2n}\\ &= \frac{\pi}{m}|A|^2r^{2m}{}_0F_1(2;|Ar^m/m|^2)\\ &= E_{A,0}^{(m)}(r). \end{aligned}\tag{2.24}$$

Finally, the desired conclusion follows from (2.22) to (2.24). This completes the proof. □

REMARK 2.8. When $B=-1$ and $r=1$, the series

$$F(\delta+1,\overline{\delta}+1;2;B^2r^{2m})=\sum_{n=0}^{\infty}\frac{(\delta+1)_n(\overline{\delta}+1)_n}{(2)_n(1)_n}(Br^m)^{2n}$$

converges (finitely) if $2>\mathrm{Re}(\delta+1+\overline{\delta}+1)$, that is, if $\mathrm{Re}(A+1)>0$.

Before we proceed further, it is worth mentioning certain basic properties of the functional $E_{A,B}^{(m)}(r)$ given by (2.20). Since the series expansion of $E_{A,B}^{(m)}(r)$ (in either case) has positive coefficients, $E_{A,B}^{(m)}(r)$ is a nondecreasing and convex function of the real variable r. Thus, $E_{A,B}^{(m)}(r)\le E_{A,B}^{(m)}(1)$. In order to see the bounds for the Dirichlet-finite function, we write

$$E_{A,B}^{(m)}(1)=\begin{cases}\dfrac{\pi}{m}|A-B|^2\displaystyle\sum_{n=0}^{\infty}\frac{(\delta+1)_n(\overline{\delta}+1)_n}{(2)_n(1)_n}B^{2n} & \text{for } B\neq 0,\\[2ex] \dfrac{\pi}{m}|A|^2\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2)_n(1)_n}\left|\frac{A}{m}\right|^{2n} & \text{for } B=0.\end{cases}$$

For certain values of the parameters m, A and B, the numerical values of $E_{A,B}^{(m)}(1)$ and the images of the unit disk under the extremal functions $q_{A,B}^{(m)}(z)=z/k_{A,B}^{(m)}(z)$ are described in Table 1 and Figures 1–8, respectively.

If we choose $A=1-2\beta$ and $B=-1$ in Theorem 2.7, then we obtain the following result.

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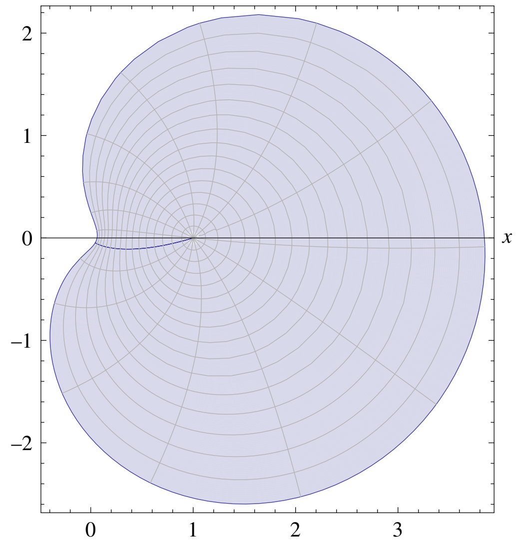

FIGURE 1. Image of the unit disk under $q^{(m)}_{A,B}(z)$ with $m = 1$, $A = 1 + i/2$, $B = -2/3$.

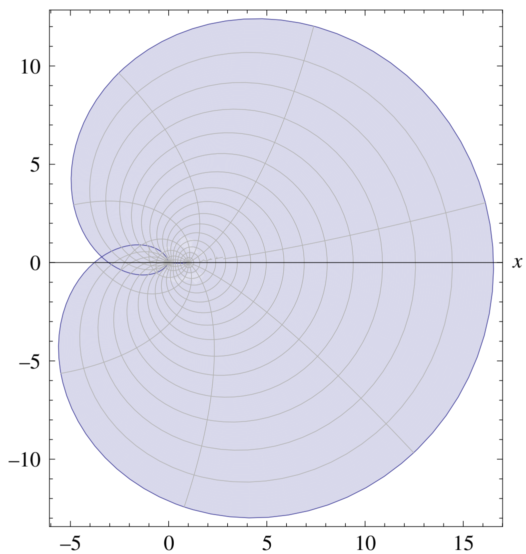

FIGURE 2. Image of the unit disk under $q^{(m)}_{A,B}(z)$ with $m = 1$, $A = 3 + i/4$, $B = -3/4$.

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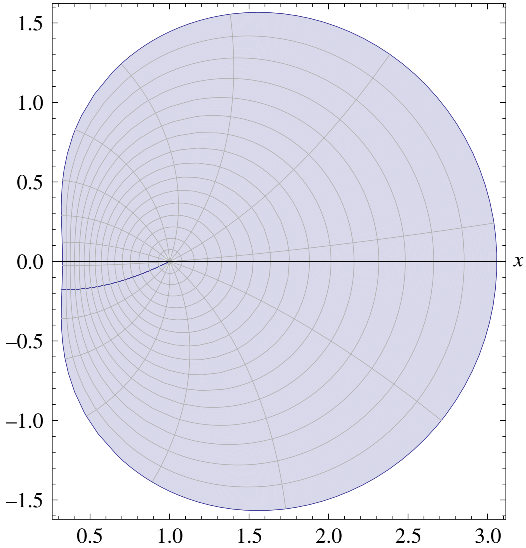

FIGURE 3. Image of the unit disk under $q^{(m)}_{A,B}(z)$ with $m = 1$, $A = 1 + i/2$, $B = 0$.

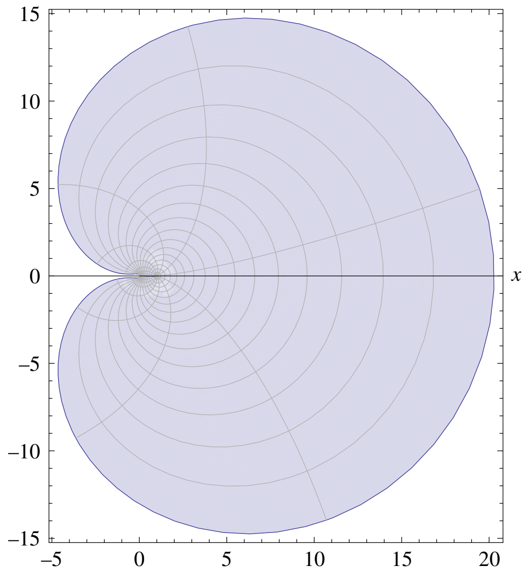

FIGURE 4. Image of the unit disk under $q^{(m)}_{A,B}(z)$ with $m = 1$, $A = 3 + i/4$, $B = 0$.

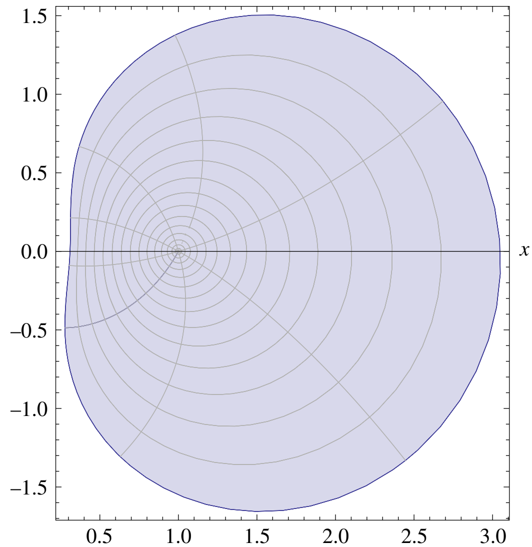

FIGURE 5. Image of the unit disk under $q_{A,B}^{(m)}(z)$ with $m = 2$, $A = 1 + 2i$, $B = -1/10$.

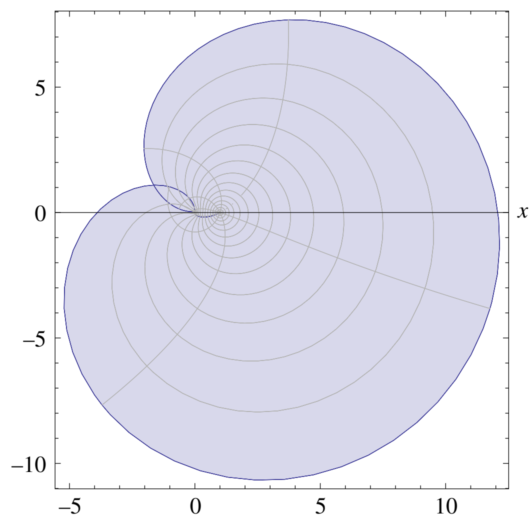

FIGURE 6. Image of the unit disk under $q_{A,B}^{(m)}(z)$ with $m = 2$, $A = 5 + 3i$, $B = -3/4$.

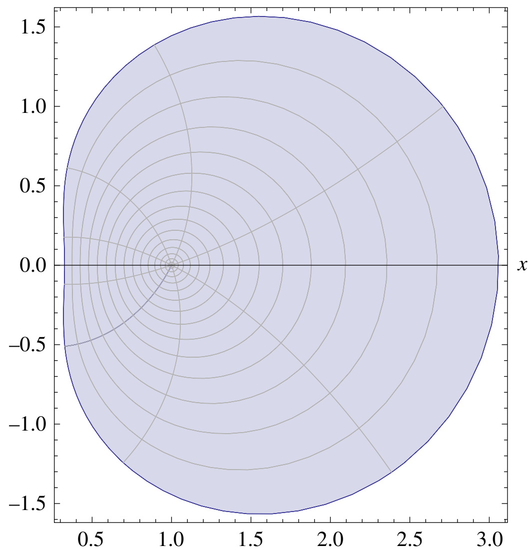

FIGURE 7. Image of the unit disk under $q_{A,B}^{(m)}(z)$ with $m = 2$, $A = 1 + 2i$, $B = 0$.

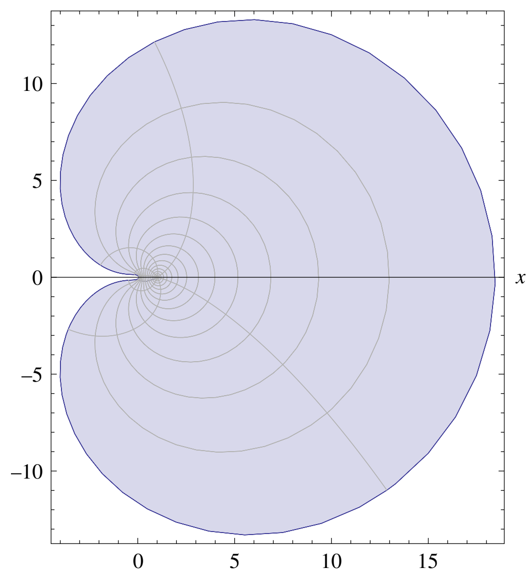

FIGURE 8. Image of the unit disk under $q_{A,B}^{(m)}(z)$ with $m = 2$, $A = 5 + 3i$, $B = 0$.

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COROLLARY 2.9. *Let $f \in \mathcal{S}_m^*(\beta)$ for some $0 \le \beta < 1$. Then, for $0 < r \le 1$,*

$$\max_{f\in\mathcal{S}_m^*(\beta)} \Delta\Big(r, \frac{z}{f(z)}\Big) = \frac{4\pi}{m}(1-\beta)^2 r^{2m} F\Big(\frac{2}{m}(\beta-1)+1, \frac{2}{m}(\beta-1)+1; 2; r^{2m}\Big).$$

The maximum is attained for the function $k_\beta^{(m)}(z) = z(1-z^m)^{(2/m)(\beta-1)}$.

In particular, if we choose $m = 1$ in Corollary 2.9, then we obtain the result of Obradović *et al.* [14, Theorem 3]. For the choice $A = 1 - 2\gamma$ and $B = -1$, where $\gamma \in \mathbb{C}\backslash\{0\}$, Theorem 2.7 reduces to the following maximal area problem for the class of m-fold starlike functions of complex order.

COROLLARY 2.10. *Let $f \in \mathcal{S}_m^*(\gamma) := \mathcal{S}_m^*(1-2\gamma, -1)$ for some $\gamma \in \mathbb{C}\backslash\{0\}$. Then, for $0 < r \le 1$,*

$$\max_{f\in\mathcal{S}_m^*(1-2\gamma,-1)} \Delta\Big(r, \frac{z}{f(z)}\Big) = \frac{4\pi}{m}|\gamma|^2 r^{2m} F\Big(-\frac{2\gamma}{m}+1, -\frac{2\overline{\gamma}}{m}+1; 2; r^{2m}\Big).$$

The maximum is attained for the function $k_\gamma^{(m)}(z) = z(1-z^m)^{-2\gamma/m}$.

If we put $m = 1$ in Corollary 2.10, then it reduces to the maximal area problem for the class $\mathcal{S}^*(\gamma) := \mathcal{S}_1^*(1-2\gamma, -1)$ of starlike functions of complex order which was introduced by Nasr and Aouf [11]. If we choose $A = e^{i\alpha}(e^{i\alpha} - 2\beta\cos\alpha)$, $B = -1$ and $m = 1$ with $\beta < 1$ in Theorem 2.7, then we obtain the result of Ponnusamy and Wirths [18, Theorem 3] which solves Yamashita's conjecture for functions in the class $\mathcal{S}_\alpha(\beta)$. If we choose $A = (1-2\beta)\alpha$, $B = -\alpha$ and $m = 1$ in Theorem 2.7, then we obtain the following result.

COROLLARY 2.11 [21, Theorem 1.3]. *Let $f \in \mathcal{S}^*((1-2\beta)\alpha, -\alpha)$ for some $0 < \alpha \le 1$ and $0 \le \beta < 1$. Then, for $0 < r \le 1$,*

$$\max_{f\in\mathcal{S}^*((1-2\beta)\alpha,-\alpha)} \Delta\Big(r, \frac{z}{f(z)}\Big) = 4\pi\alpha^2(\beta-1)^2 r^2 F(2\beta-1, 2\beta-1; 2; \alpha^2 r^2).$$

The maximum is attained for the function $k_{(1-2\beta)\alpha,-\alpha}(z)$ defined by (1.4).

If we choose $\beta = 0$ in Corollary 2.11, then we obtain the result of Sahoo and Sharma [21, Theorem 3.1]. This solves Yamashita's conjecture for functions in the class $\mathcal{S}(\alpha)$ which was introduced by Padmanabhan [16]. More generally, if we choose $m = 1$ in Theorem 2.7, then it reduces to the following maximal area problem for the class $\mathcal{S}^*(A, B)$.

COROLLARY 2.12 [17, Theorems 2.1 and 2.3]. *Let $f \in \mathcal{S}_1^*(A, B, 1) = \mathcal{S}^*(A, B)$ for some $-1 \le B \le 0$, $A \ne B$ and $A \in \mathbb{C}$. Then, for $0 < r \le 1$,*

$$\max_{f\in\mathcal{S}^*(A,B)} \Delta\Big(r, \frac{z}{f(z)}\Big) = E_{A,B}(r),$$

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where

$$E_{A,B}(r) = \begin{cases} \pi|A-B|^2 r^2 F\Big(\dfrac{A}{B}, \dfrac{\overline{A}}{B}; 2; B^2 r^2\Big) & \text{for } B \neq 0, \\ \pi|A|^2 r^2 {}_0F_1(2; |A|^2 r^2) & \text{for } B = 0. \end{cases}$$

The maximum is attained by the rotation of the function $k_{A,B}(z)$ defined by (1.4).

Finally, if we choose $A = (b^2 - a^2 + a)/b$, $B = (1-a)/b$ with $a + b \geq 1$, $a \in [b, 1+b]$ in Corollary 2.12, then we obtain the result of Ponnusamy *et al.* [17, Corollary 2.7]. This solves the maximal area problem for functions in the class $\mathcal{S}^*((b^2 - a^2 + a)/b, (1-a)/b)$ which was introduced by Silverman [22].

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