A closed dimensionless linear set

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1. The problem which we discuss in this paper can be easily settled for a closed plane set; after a brief introduction giving the definitions and theorems used later we indicate how this may be done; we then proceed to the main problem.

2. We are concerned with the measure and dimension theory of sets of points formulated by Hausdorff[1]. Let A be any set of points in a Euclidean space of n dimensions (n any integer); and let $\lambda(t)$ be any function of the real variable t, defined in some interval $0 \leqq t \leqq t_0$ in which it is continuous, concave and strictly increasing, and such that $\lambda(0) = 0$. Given a positive number $\rho \leqq t_0$, denote by $U(A, \rho)$ any set of spheres $\{U_r\}$ of respective diameters $\{d_r\}$ such that (i) $d_r < \rho$ for each r, and (ii) each point of A is interior to at least one sphere U_r. Then

$$\text{lower bound} \sum_{U(A,\,\rho)} \lambda(d_r)$$

is denoted by $\lambda - m_\rho A$, and $\lim\limits_{\rho \to 0} (\lambda - m_\rho A)$ (which obviously exists) is denoted by $\lambda - m^*E$ and is called the *exterior* λ-*measure* of A. This exterior measure satisfies all the standard requirements of a measure function[2]; if A is *measurable*, in particular if it is closed, we speak of its λ-*measure* and denote the latter by $\lambda - mA$. We have further that

(1) The union V of a finite or enumerable sequence of measurable sets $A_1, A_2, \ldots.$ is measurable;

(2) The measure of the sum of a finite or enumerable sequence of measurable sets $A_1, A_2, \ldots.$ no two of which contain a common point is equal to the sum of the measures of those sets, *i.e.* if $A_i A_j = 0$ for all i, j, then

$$\sum_{i=0}^{\infty} (\lambda - mA_i) = \lambda - m\left(\sum_{i=1}^{\infty} A_i\right).$$

[1] F. Hausdorff, *Dimension und äusseres Mass*, Math. Annalen, **79** (1919), 157-179.

[2] C. Carathéodory, *Über das lineare Mass von Punktmengen*, Göttingen Nachrichten, 1914, 404-426.

We define the *dimension* of a given set A to be $[\lambda(x)]$ if and only if

$$0 < \lambda - mA < \infty$$

(strict inequality must be preserved).

3. We consider first the non-linear problem. Suppose we take the set $\{L_r\}$ of segments of lines where L_r is the segment between the points $(1/r, 0)$ and $(1/r, 1)$ on the line $x = 1/r$. Then take a similar set of points A_r on each segment—similar in the sense that if, for any r, $(1/r, y)$ belongs to the set, this holds for all r. We make each A_r closed and call the similar set on the y-axis, the set A_0. Then we define

$$A = \sum_{r=0}^{\infty} A_r.$$

If now $\lambda - mA_r = 0$ then obviously $\lambda - mA = 0$; and if $\lambda - mA_r > 0$ then $\lambda - mA = \infty$. Hence the set A is dimensionless.

4. Hausdorff proves that if we define a set A in the interval $(0, 1)$ by first omitting a length $1 - 2\xi$ $(\xi < 1/2)$ from the centre of the interval, and then by omitting a length proportional to $1 - 2\xi$ from the centres of the two remaining intervals, and so on, in a manner similar to the derivation of the Cantor ternary set then $\lambda - mA = 1$ where $\lambda(t) = t^a$ and $a = -(\log 2)/\log \xi$. It is easy to prove that if we shrink this set A uniformly into an interval of length d then its measure is d^a with the same measure function; this follows directly from the definition of the measure. For the new covering sets will have diameters $\{dd_r\}$ and

$$\lambda(dd_r) = d^a \lambda(d_r)$$

and we shall have a factor d^a coming outside the whole expression for the lower bound, giving our result.

Consider the interval $(1/(r+1), 1/r)$ on the x-axis. We form in it the set considered in the last paragraph. Call the set E_r; then its measure will be

$$\left[\frac{1}{r(r+1)}\right]^{-(\log 2)/\log \xi}.$$

We define, adding in the origin to make the set closed,

$$E = \sum_{1}^{\infty} E_r;$$

then by the theorems of § 2, E is measurable and

$$a - mE = \sum_{1}^{\infty} (a - mE_r),$$

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where we denote by $\alpha - mF$ the measure of the set F with measure function t^α. Hence

$$\alpha - mE = \sum_1^\infty \left[\frac{1}{r(r+1)}\right]^{-(\log 2)/\log \xi}.$$

If we now choose ξ so that

$$-\frac{\log 2}{\log \xi} \leqq \tfrac{1}{2},$$

that is, so that $\xi \leqq \frac{1}{4}$, then $\alpha - mE = \infty$, since the series is divergent.

5. We have now to prove that if we take any function $h(t)$ then either $h - mE = \infty$ or $h - mE = 0$. There are two cases to consider.

$$\text{(i)} \quad \varlimsup_{t \to 0} \frac{h(t)}{t^\alpha} = k \text{ where } 0 < k \leqq \infty;$$

then by the general properties of this measure theory, since $\alpha - mE = \infty$, $h - mE = \infty$ also.

$$\text{(ii)} \quad \varliminf_{t \to 0} \frac{h(t_r)}{t^\alpha} = 0.$$

We consider this case in detail.

6. We can determine a sequence $t_1, t_2, t_3 \ldots\ldots$ for which

$$\frac{h(t_r)}{t_r^\alpha} < \epsilon, \quad \epsilon > 0$$

and $t_1 > t_2 > t_3 > \ldots\ldots$ and t_n tends to 0 as n tends to ∞. We can cover E_r with 2^n intervals each of length $\xi^n/r(r+1)$ where we always have

$$\frac{2^n \xi^{n\alpha}}{[r(r+1)]^\alpha} = \frac{1}{[r(r+1)]^\alpha}$$

or $2^n \xi^{n\alpha} = 1$, for $\alpha = -(\log 2)/\log \xi$ by definition.

We are always able to select a sub-sequence $s_1, s_2 \ldots\ldots$ of the sequence $t_1, t_2, t_3 \ldots\ldots$ such that

$$\xi^{n_1+1} \leqq s_1 < \xi^{n_1}$$
$$\xi^{n_2+1} \leqq s_2 < \xi^{n_2}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\xi^{n_p+1} \leqq s_p < \xi^{n_p}.$$
$$\ldots\ldots\ldots\ldots\ldots$$

The necessity of selecting a sub-sequence arises because more than one t_p may lie in between any ξ^n and the succeeding ξ^{n+1}.

Then to form the h-measure of the set E we must first consider

$$h(t) - m_\rho E_r = \text{lower bound} \sum_{U(A,\rho)} h(d_n).$$

In this case the spheres $\{U_r\}$ become intervals of equal length and the diameters become their length. Take $\rho = \xi^{n_p - 1}$; then

$$h - m_\rho E_r \leqq 2^{n_p+1} h(\xi^{n_p+1}) \leqq 2^{n_p+1} h(s_p)$$

since $h(t)$ increases with t for small t. Now by the definition of s_r

$$h(s_p) < \epsilon.\, s_p^\alpha < \epsilon\, \xi^{n_p \alpha}$$

since t^α increases with t. Hence

$$h - m_\rho E_r < \epsilon \,.\, 2^{n_p+1} \xi^{n_p \alpha} = 2\epsilon \,.\, (2\xi^\alpha)^n = 2\epsilon.$$

Hence $$h - mE_r = \lim_{\rho \to 0} h - m_\rho E_r = 0$$

since ϵ is arbitrary.

The theorems of § 2 give us

$$h - mE = \sum_{r=1}^{\infty} (h - mE_r) = 0.$$

No other case can arise since $h(t)$ is always positive and therefore $h - mE$ is zero or infinite for all $h(t)$. *Hence the set E has no dimensions.*

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