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SEPARATION OF CONVEX SETS IN EXTENDED NORMED SPACES

G. BEER and J. VANDERWERFF✉

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Abstract

We give continuous separation theorems for convex sets in a real linear space equipped with a norm that can assume the value infinity. In such a space, it may be impossible to continuously strongly separate a point p from a closed convex set not containing p, that is, closed convex sets need not be weakly closed. As a special case, separation in finite-dimensional extended normed spaces is considered at the outset.

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1. Introduction

We will consider extended normed spaces as introduced in [3]. That is, given a vector space X over a field of scalars $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$) we let 0_X denote the origin of X and adopt the standard convention that $0 \cdot \infty = 0$, where ∞ means $+\infty$ throughout this paper. We say that a function $\|\cdot\| : X \to [0, \infty]$ is an *extended norm* provided it satisfies the following properties:

(i) $\|x\| = 0$ if and only if $x = 0_X$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for each $x \in X$, $\alpha \in \mathbb{F}$;
(iii) $\|x + y\| \le \|x\| + \|y\|$ for each $x, y \in X$.

When X is a vector space and $\|\cdot\|$ is an extended norm on X, we refer to $\langle X, \|\cdot\|\rangle$ as an *extended normed space*. When the extended norm is understood in the context, we may simply refer to X as an extended normed space. In this paper, we will restrict our attention to *real* extended normed spaces, that is, where the scalar field is $\mathbb{R}$.

The papers [1, 3, 4] present various arguments for using extended metrics and extended norms. Indeed, in much of analysis natural norm constructions often fail to be finite-valued on a vector space. For example, the supremum norm on $C(K)$,

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the continuous real-valued functions on a compact Hausdorff space K, would assume infinite values if K were not compact. On the other hand, defining a truncated supremum metric loses information on the large structure of the space, and a metric does not take advantage of the linear structure. Further discussion in this direction can also be found in [1, 3, 4]. With this in mind, [3] demonstrates that, taking appropriate care, considerable information concerning an extended normed space $\langle X, \|\cdot\|\rangle$ can be obtained from the structure of $\langle X_{\text{fin}}, \|\cdot\|\rangle$, where $X_{\text{fin}} = \{x \in X : \|x\| < \infty\}$ is a linear subspace of X and is itself a conventional normed space. We may occasionally say that an extended norm is *nontrivial* if $\{0_X\} \neq X_{\text{fin}} \neq X$.

Let X be an extended normed space, and let ϕ be a nonconstant linear functional on X. As in conventional normed linear spaces, ϕ is continuous if and only if $\text{Ker}(\phi)$ is closed [3, Corollary 4.6], and since translation is a homeomorphism, this occurs if and only if some/all level sets of ϕ are closed hyperplanes. On the other hand, in the extended norm setting, such hyperplanes can be open as well; this occurs if and only if $X_{\text{fin}} \subseteq \text{Ker}(\phi)$ [3, Corollary 3.9]. Also, as in the conventional setting, a nonconstant linear functional ϕ is continuous as soon as it is either bounded above or below on some nonempty open set. From either property, it is clear that ϕ is bounded on the unit ball of the space so that ϕ is continuous at 0_X, from which global continuity follows [3, Theorem 4.3].

When $\langle X, \|\cdot\|\rangle$ is an extended normed space, we let $B_X = \{x : \|x\| \leq 1\}$ and we use X^* to denote the continuous linear functionals on X. However, [3] shows that the natural 'operator norm' on X^* need not be a norm, but rather it is a seminorm on X^*. Nevertheless, we follow [3] and denote

$$\|\phi\|_{\text{op}} = \sup\{|\phi(x)| : x \in B_X\} \quad \text{for } \phi \in X^*.$$

Also, it is not hard to show, but important to keep in mind, that as soon as X is not a conventional normed space, X endowed with the extended norm topology is not a topological vector space in that scalar multiplication fails to be jointly continuous [3, Proposition 3.2]. On the other hand, X endowed with the weak topology remains a locally convex topological vector space.

The goal of this paper is to continue and build upon the broad study of extended normed spaces as initiated in [3]. Our particular focus centers on convex sets and separation; our results ultimately reflect an altered relationship between core and interior. We have chosen this focus because many of the important applications of convex analysis, such as Fenchel duality, sandwich theorems, and subdifferential analysis, rely upon separation theorems [6, 8, 9, 11]. We anticipate that comprehensive versions of separation theorems are necessarily the key to the development of convex analysis in extended normed spaces; however, this development is beyond the scope of the current paper.

As in [3], considerable information can be derived efficiently by taking advantage of the well-established topological and algebraic theory for conventional normed spaces; however, differences do arise and often they may be quite delicate and perhaps unexpected when first encountered.

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2. Separation in finite-dimensional spaces

Before we examine separation results, we provide a natural example of a finite-dimensional extended normed space.

Example 2.1. Let X be the collection polynomials on the real line whose degree is less than or equal to n. Then $\langle X, \|\cdot\|_\infty\rangle$ is an extended normed space where for $p \in X$, we define $\|p\|_\infty = \sup_{x\in\mathbb{R}} |p(x)|$. Letting $X = \operatorname{span}(\{1, x, x^2, \ldots, x^n\})$, we write $p \in X$ as $p = a_0 + a_1 x + \cdots + a_n x^n$, and then

$$\|p\|_\infty = \begin{cases} |a_0| & \text{if } p = a_0, \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, if we consider the subspace $Z = \operatorname{span}(\{x, x^2, \ldots, x^n\})$, then $\|p_1 - p_2\|_\infty = \infty$ whenever $p_1, p_2 \in Z$ and $p_1 \neq p_2$. Consequently, the extended norm topology is discrete when restricted to Z.

Many examples used in this paper, and indeed in [3], are based on this type of extended norm where the specific vectors $\{1, x, x^2, \ldots, x^n\}$ are replaced with an algebraic basis in an arbitrary finite-dimensional space. While this particular extended normed space does not appear to be very rich in structure, remember that it is just one of many subspaces of $\langle C(\mathbb{R}), \|\cdot\|_\infty\rangle$, the continuous functions on the real line endowed with the extended supremum norm defined by $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. Moreover, the extended normed space $\langle C(\mathbb{R}), \|\cdot\|_\infty\rangle$ is very rich, because the Banach space $C[0, 1]$ embeds naturally into it, and according to the Banach–Mazur theorem, $C[0, 1]$ isometrically contains every separable Banach space; see [7, Theorem 5.17]. On the other hand, no nontrivial separable extended normed space exists, as in such a space there must be a vector subspace other than $\{0_X\}$ whose relative topology is discrete.

Let A and B be nonempty convex subsets of an extended normed space X. Adopting terminology from conventional functional analysis, we say that a nonconstant linear functional ϕ on X *properly separates* A and B if $\sup_A \phi \leq \inf_B \phi$ and ϕ assumes at least two values on $A \cup B$; see [9, page 95]. Note that ϕ is not assumed continuous in the definition. Equivalently, there exists $\alpha \in \mathbb{R}$ such that $A \subseteq \{x : \phi(x) \leq \alpha\}$, $B \subseteq \{x : \phi(x) \geq \alpha\}$, and either $A \cap \{x : \phi(x) < \alpha\} \neq \emptyset$ or $B \cap \{x : \phi(x) > \alpha\} \neq \emptyset$, or both. Of course, $\{x : f(x) = \alpha\}$ is then called a *properly separating hyperplane* for A and B.

We adopt the following additional notation relative to a nonempty subset A of an extended normed space X:

- $\operatorname{int}(A)$ is the interior of A with respect to the norm topology;
- $\overline{A}$ is the closure of A with respect to the norm topology;
- $\operatorname{conv}(A)$ is the convex hull of A;
- $\operatorname{span}(A)$ is the smallest linear subspace containing A;
- $\operatorname{aff}(A)$ is the *affine hull* of A, that is, the smallest flat containing A.

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Given a nonconstant $\phi \in X^*$ and $\alpha \in \mathbb{R}$, the halfspaces $\{x : \phi(x) \leq \alpha\}$ and $\{x : \phi(x) \geq \alpha\}$ are weakly closed but not weakly open. Otherwise, the connectedness of X equipped with the locally convex weak topology would be violated. Dually, the halfspaces $\{x : \phi(x) > \alpha\}$ and $\{x : \phi(x) < \alpha\}$ are weakly open but not weakly closed. On the other hand, such halfspaces can be strongly closed and strongly open, that is, *strongly clopen*, as made precise by our next result, which is anticipated by [3, Corollary 3.9].

Proposition 2.2. *Let X be an extended normed space, and let $\phi \in X^*$ be nonconstant. The following conditions are equivalent:*

(a) *there exists $\alpha \in \mathbb{R}$ such that $\{x : \phi(x) > \alpha\}$ is norm closed;*
(b) *for all $\alpha \in \mathbb{R}$, $\{x : \phi(x) > \alpha\}$ is norm closed;*
(c) *there exists $\alpha \in \mathbb{R}$ such that $\{x : \phi(x) \geq \alpha\}$ is norm open;*
(d) *for all $\alpha \in \mathbb{R}$, $\{x : \phi(x) \geq \alpha\}$ is norm open;*
(e) $X_{\text{fin}} \subseteq \text{Ker}(\phi)$.

Proof. Conditions (a) and (b) are equivalent because translation is a homeomorphism, as are conditions (c) and (d). Conditions (b) and (d) are equivalent because $\{x : \phi(x) > \alpha\}$ is the complement of $\{x : -\phi(x) \geq -\alpha\}$.

Suppose that condition (e) fails. We can then pick $x_0 \in X_{\text{fin}}$ with $\phi(x_0) > 0$. We compute

$$0_X \in \overline{\{\alpha x_0 : 0 < \alpha \leq 1\}} \subseteq \overline{\{x : \phi(x) > 0\}},$$

and so condition (b) fails as well.

If condition (e) holds, put $E = \{x : \phi(x) \geq 0\}$. Evidently, $E = \bigcup_{x \in E}(x + X_{\text{fin}})$, expressing E as a union of norm open sets, so that E is norm open and (c) holds. □

We denote the *algebraic dual* of X by X', that is, the collection of all linear functionals on X. The weakest topology on X making each element of X' continuous is a locally convex Hausdorff topology that we call the *intrinsic topology* of X. If $A \neq \emptyset$, we denote the interior of A with respect to the relative topology that aff(A) inherits from the intrinsic topology on X by i-int(A) and call this the *intrinsic interior* of A; see [8, page 8]. When $A \subset \mathbb{R}^n$ endowed with a conventional norm, the intrinsic interior, i-int(A), is the same as ri(A), the *relative interior* as defined in [9, page 44]. We have chosen to use the term 'intrinsic interior' because different extended norms will change the topology of the linear space, even in the finite-dimensional case, but they will not affect the intrinsic interior. The following is a classical separation theorem for finite-dimensional spaces.

Theorem 2.3 (See [9], Theorem 11.3). *Let A and B be nonempty convex sets in $\mathbb{R}^n$ (or any finite-dimensional real vector space). Then A and B can be separated properly by some $\phi \in \mathbb{R}^n = (\mathbb{R}^n)'$ if and only if* $\text{ri}(A) \cap \text{ri}(B) = \emptyset$.

The next observation discusses the continuous linear functionals on a finite-dimensional space.

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Fact 2.4. *Let X be a finite-dimensional extended normed space. Then X^* is just the set of linear functionals on X. In particular, if X is a finite-dimensional vector space, it has the same set of continuous linear functionals irrespective of the norm or extended norm with which it is endowed.*

Proof. Let ϕ be any linear functional on X. Then, X_{fin} is a conventional finite-dimensional normed space, and so $\phi|_{X_{\text{fin}}}$ is continuous because it is a linear functional on X_{fin}. The result follows because [3, Theorem 4.2] shows that ϕ is continuous if and only if $\phi|_{X_{\text{fin}}}$ is continuous. □

One may think of the above fact in other ways. For example, consider a finite-dimensional conventional normed space X. Because all conventional norms on X are equivalent, the norm topology is independent of the particular equivalent norm in use. Now let $\|\cdot\|_1$ be any extended norm on X. Then $\|\cdot\|_1$ restricted to X_{fin} can be extended to an equivalent standard norm, say $\|\cdot\|_2$ on X. Because $\|\cdot\|_2 \le \|\cdot\|_1$ on X, any real-valued function that is continuous with respect to $\|\cdot\|_2$ will be continuous with respect to $\|\cdot\|_1$. Expressed yet another way, the topology generated by $\|\cdot\|_1$ is stronger than the topology generated by $\|\cdot\|_2$ (or any other equivalent conventional norm on X), because X_{fin} is a proper clopen subset of X as soon as the extended norm is not a conventional norm. In light of Fact 2.4, the following is then a recasting of Theorem 2.3.

Fact 2.5. *Suppose that X is a finite-dimensional extended normed space and suppose that A and B are nonempty convex subsets of X. Then A and B can be properly separated by some $\phi \in X^*$ if and only if* $\text{i-int}(A) \cap \text{i-int}(B) = \emptyset$.

Proof. Suppose that A and B can be properly separated by $\phi \in X^*$. By Fact 2.4, ϕ is continuous on X equipped with any conventional norm, and by Theorem 2.3 we have $\text{i-int}(A) \cap \text{i-int}(B) = \emptyset$.

Conversely, suppose that $\text{i-int}(A) \cap \text{i-int}(B) = \emptyset$. By Theorem 2.3, A and B can be properly separated by a linear functional ϕ on X. By Fact 2.4, $\phi \in X^*$. □

The equivalence in the previous fact would no longer be valid if the intrinsic interior were replaced by a relative interior considered with respect to the extended norm topology. Clearly the 'if' implication would hold because the extended norm topology is stronger than the topology of a conventional norm on a finite-dimensional space, but the following example shows the 'only if' implication can fail.

Example 2.6. Let X be $\mathbb{R}^2$ equipped with the extended norm $\|\cdot\|$ defined as follows. When $x = (s, 0)$ we define $\|x\| = |s|$, and we let $\|x\| = \infty$ otherwise. Let

$$A = \{(s, 0) : s \in \mathbb{R}\} \quad \text{and} \quad B = \{(s, t) : s \in \mathbb{R}, t \ge 0\}.$$

Then $A = X_{\text{fin}}$ is a norm clopen convex set, as is any translate of it (see [3]), and

$$B = \bigcup_{t \ge 0} (A + (0, t))$$

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is a union of norm open sets, so B is a norm open halfspace (in fact B is norm clopen by local finiteness of the family of translates). Clearly $A \cap B = A$, and the linear functional ϕ defined by $\phi(s,t) = t$ satisfies $\sup_A \phi = 0 = \inf_B \phi < \sup_B \phi$ and so ϕ properly separates A and B.

The next observation clarifies relations between intrinsic interior and interior in extended normed spaces.

Fact 2.7. *Let X be a finite-dimensional extended normed space, and suppose that A is a nonempty convex subset of X.*

(a) *If* $\operatorname{int}(A) \neq \emptyset$, *then* $\text{i-int}(A) \subseteq \text{i-int}(\bar{A}) \subseteq \operatorname{int}(A)$.
(b) *There are examples where* $\operatorname{int}(A)$ *properly contains* $\text{i-int}(A)$.
(c) *It is possible to have* $\operatorname{int}(A) = \emptyset$, *while necessarily* $\text{i-int}(A) \neq \emptyset$.

Proof. (a) The first inclusion is obvious. For the second, because interiors, intrinsic interiors and closures are preserved by translations, we suppose that $0_X \in \operatorname{int}(A)$. If $\text{i-int}(\bar{A}) = \{0_X\}$, there is nothing further to do. So suppose that $x \in \text{i-int}(\bar{A})$, where $x \neq 0_X$. Then $\operatorname{span}(x)$ is in the affine hull of $\bar{A}$, and so we deduce that $(1+\alpha)x \in \bar{A}$ for some $\alpha > 0$. Because $0_X \in \operatorname{int}(A)$, we choose $\delta > 0$ so that $\delta B_X \subset A$. Now choose $y \in A$ such that

$$\|y - (1+\alpha)x\| < \alpha\delta \quad \text{and so} \quad \left\| \frac{1}{1+\alpha} y - x \right\| < \frac{\alpha\delta}{1+\alpha}. \tag{2.1}$$

Because A is convex, we have that

$$\frac{1}{1+\alpha} y + \frac{\delta\alpha}{1+\alpha} B_X = \frac{1}{1+\alpha} y + \frac{\alpha}{1+\alpha}(\delta B_X) \subset A,$$

and this with (2.1) implies $x \in \operatorname{int}(A)$ as desired.

(b) For example, let X be as in Example 2.6, and let $A = \{(s,t) : t \geq 0\}$; then A is an open subset of X, but $\text{i-int}(A) = \{(s,t) : t > 0\}$; one can create this sort of example whenever X_{fin} is a proper subspace of X. Indeed, let $h \in X \setminus X_{\text{fin}}$ and let $A = \bigcup_{t \geq 0}(th + X_{\text{fin}})$. Then $\operatorname{int}(A) = A$ while $\text{i-int}(A) = \bigcup_{t>0}(th + X_{\text{fin}})$.

(c) Whenever $X_{\text{fin}} \neq \{0_X\}$, one can let A be a nonempty convex subset of X_{fin} whose interior is empty. Because A is a nonempty convex subset of a finite-dimensional vector space, [9, Theorem 6.2] ensures that $\text{i-int}(A) \neq \emptyset$. □

The following separation theorem is a consequence of Facts 2.5 and 2.7.

Corollary 2.8. *Suppose that X is a finite-dimensional extended normed space. Let A and B be nonempty convex subsets of X such that* $\operatorname{int}(A) \neq \emptyset$ *and* $\operatorname{int}(A) \cap B = \emptyset$. *Then A and B can be properly separated by some $\phi \in X^*$.*

Proof. By Fact 2.7(a) we have $\text{i-int}(A) \subseteq \operatorname{int}(A)$ and so $\text{i-int}(A) \cap B = \emptyset$. Then, obviously, $\text{i-int}(A) \cap \text{i-int}(B) = \emptyset$ and so proper separation follows from Fact 2.5. □

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The previous corollary may not be as immediate as one might have expected. The condition given is natural for conventional normed spaces whether they are finite-dimensional or infinite-dimensional. However, the subtlety in the extended normed spaces is that $\operatorname{int}(A) \neq \emptyset$ is a weaker condition than in conventional normed spaces.

3. General separation results

Our first observation shows that the separation theorem as given in Corollary 2.8, while valid in general conventional normed spaces, is not valid in infinite-dimensional extended normed spaces.

Proposition 3.1. *Let X be a real vector space. Then the following are equivalent:*

(a) *X is infinite-dimensional;*

(b) *X can be endowed with a nontrivial extended norm such that there exists a nonempty convex cone $C \subset X$ and $x_0 \notin C$ with $d(x_0, C) = \infty$, but $\{x_0\}$ and C cannot be properly separated by* any *linear functional on X;*

(c) *X can be endowed with an extended norm such that there exist disjoint nonempty convex sets A and B with* $\operatorname{int}(A) \neq \emptyset$*, but A and B cannot be properly separated by* any *linear functional on X.*

Proof. (a) $\Rightarrow$ (b). Let $\{e_n : n \in \mathbb{N}\}$ be a countably infinite linearly independent subset of X, and let $Y = \operatorname{span}(\{e_n : n \in \mathbb{N}\})$. Then we equip X with the extended norm

$$\|x\| = \begin{cases} |\alpha| & \text{if } x = \alpha e_1, \\ \infty & \text{otherwise.} \end{cases}$$

Now let C be the cone generated by $\{me_n + e_{n+1} : m \in \mathbb{Z}, n \in \mathbb{N}\} \cup \{0_X\}$ and let $x_0 = -e_2$. Suppose that ϕ is any linear functional on X that is not identically zero on Y. Then $\phi(e_{n_0}) \neq 0$ for some $n_0 \in \mathbb{N}$. Now choose integers m_1 and m_2 with sufficiently large magnitude and appropriate signs so that

$$m_1\phi(e_{n_0}) + \phi(e_{n_0+1}) > \phi(x_0) \quad \text{and} \quad m_2\phi(e_{n_0}) + \phi(e_{n_0+1}) < \phi(x_0).$$

Because $m_1 e_{n_0} + e_{n_0+1}$ and $m_2 e_{n_0} + e_{n_0+1}$ are both in C, this shows that C and $\{x_0\}$ cannot be separated properly by the arbitrarily chosen linear functional ϕ.

Now let $v \in C$. Then v is a linear combination, where all coefficients are positive, of finitely many of the elements from the generating set for C as describe above. Then either $v = 0_X$ or we can write $v = \sum_{j=1}^{k} \alpha_j e_j$, where necessarily $k \geq 2$ and $\alpha_k > 0$. Thus for each $v \in C$, $v - x_0$ fails to be a multiple of e_1, and so $\|v - x_0\| = \infty$; thus, $d(x_0, C) = \infty$ as desired.

(b) $\Rightarrow$ (c). Given that (b) holds, we set $A = B_X + x_0$ and $B = C$ where C and x_0 are from (b). Thus (c) holds.

(c) $\Rightarrow$ (a). If such sets exist, then X must be infinite-dimensional according to Theorem 2.3. □

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At this point we introduce some terminology for linear spaces X not assumed to be equipped with a topology. By a *spur* we will mean an absorbing subset E that is star-shaped with respect to the origin, that is, whenever $x \neq 0_X$, there exists $\alpha > 0$ such that $\operatorname{conv}\{0_X, \alpha x\} \subseteq E$. A point x_0 of a convex subset C is called a *core point* of C if there exists a spur E with $x_0 + E \subseteq C$. We denote the (convex) set of core points of C by $\operatorname{core}(C)$. Finally, by $\operatorname{lin}(C)$ we mean the set of all $x \in X$ such that for some $c_x \in C$ and for all $\alpha \in (0, 1]$ we have $\alpha c_x + (1 - \alpha)x \in C$. Note that $C \subseteq \operatorname{lin}(C)$.

A natural reason for the failure of separation in Proposition 3.1 is that, unlike the conventional setting, a nonempty interior does not ensure a nonempty core in extended normed spaces. The next result, like the strict and strong separation results that follow, characterizes the ability to continuously separate in terms of the existence of a convex superset of A that somehow 'shields' A from B (see [2, 5]). Indeed, as we will see, proper separation means that there is a point of A lying simultaneously in both the interior and in the core of the shield C; strict separation means that A and B separately have shields C and D that contain A and B respectively in the intersection of their core and their interior; finally, strong separation means that the shield C for A contains A 'uniformly' in the intersection of its core and its interior.

Theorem 3.2. *Let A and B be nonempty convex subsets of an extended normed space X. Then following conditions are equivalent:*

(a) *there exist $\phi \in X^*$ and $\alpha \in \mathbb{R}$ with $A \subseteq \{x : \phi(x) \leq \alpha\}$, $B \subseteq \{x : \phi(x) \geq \alpha\}$, and $A \cap \{x : \phi(x) < \alpha\} \neq \emptyset$;*

(b) *there exists a convex superset C of A such that $\operatorname{int}(C) \cap \operatorname{core}(C) \cap A \neq \emptyset$ and $B \cap \operatorname{core}(C) = \emptyset$;*

(c) *there exists a convex superset C of A such that $\operatorname{int}(C) \neq \emptyset$, $\operatorname{core}(C) \cap A \neq \emptyset$, and $B \cap \operatorname{core}(C) = \emptyset$.*

Proof. Let us show that (a) $\Rightarrow$ (b). Pick $a_0 \in A \cap \{x : \phi(x) < \alpha\}$. We will show that $C := \{x : \phi(x) \leq \alpha\}$ does the job. Obviously, C is a convex superset of A. By continuity, $\{x : \phi(x) < \alpha\}$ is an open neighborhood of a_0 contained in C, so $a_0 \in \operatorname{int}(C)$. Obviously, no core point of C can lie on the bounding hyperplane, that is, $\operatorname{core}(C) \subseteq \{x : \phi(x) < \alpha\}$ from which $\operatorname{core}(C) \cap B = \emptyset$. That $a_0 \in \operatorname{core}(C)$ only uses linearity of ϕ is left as an exercise to the reader.

As the implication (b) $\Rightarrow$ (c) is trivial, it remains to show that (c) $\Rightarrow$ (a). Pick $a_0 \in \operatorname{core}(C) \cap A$. We now follow the standard geometric proof of the algebraic proper separation theorem given in [8] or [10]. First, by the Stone lemma [8, page 7], there exist complementary convex supersets B_0 and C_0 of B and C, respectively. Since $a_0 \notin \operatorname{lin}(B_0)$, the set $\operatorname{lin}(B_0) \cap \operatorname{lin}(C_0)$ must be a hyperplane (see [8, page 14] or [10, pages 20–21]). We can choose a nonconstant linear functional ϕ and $\alpha \in \mathbb{R}$ with $C \subseteq \{x : \phi(x) \leq \alpha\}$ and $B \subseteq \{x : \phi(x) \geq \alpha\}$, and we must have $\phi(a_0) < \alpha$ because $a_0 \in \operatorname{core}(C)$. Finally, ϕ is bounded above on the nonempty interior of C, from which continuity follows. □

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Because core and interior play such vital roles in separation, we list some elementary observations concerning their relations.

PROPOSITION 3.3. *Let X be an extended normed space and suppose that A is a nonempty convex subset of X.*

(a) *It need not be true that* $\operatorname{int}(A) \subseteq \operatorname{core}(A)$. *Moreover, even when X is finite-dimensional, it may occur that* $0_X \in \operatorname{int}(A)$, $\operatorname{span}(A) = X$ *and yet* $0_X \notin \operatorname{core}(A)$.

(b) *It need not be true that* $\operatorname{core}(A) \subseteq \operatorname{int}(A)$.

(c) *Suppose that* $\operatorname{int}(A) \neq \emptyset$. *Then* $\operatorname{core}(A) \subseteq \operatorname{core}(\bar{A}) \subseteq \operatorname{int}(A)$ *(where* $\operatorname{core}(\bar{A})$ *is possibly empty).*

PROOF. (a) Suppose that X_{fin} is a proper subset of X, and let $A = X_{\text{fin}}$. Then A is open, but $\operatorname{core}(A) = \emptyset$. Moreover, when X is a finite-dimensional space where X_{fin} is a proper subspace of X, we write $X = X_{\text{fin}} \oplus \operatorname{span}(\{e_1, e_2, \ldots, e_n\})$ where $\{e_1, e_2, \ldots, e_n\}$ is a linearly independent set. Let

$$A = \{z \in X : z = x + \alpha_1 e_1 + \alpha_2 e_2 + \cdots \alpha_n e_n, \text{ where } x \in X_{\text{fin}}, \alpha_k \geq 0 \text{ for } 1 \leq k \leq n\}.$$

Then A has the claimed properties.

(b) A standard example is as follows. Let c_{00} denote the vector space of finitely supported vectors in c_0. That is, $x \in c_{00}$ if $x = \sum_{k=1}^{\infty} \alpha_k e_k$ where there are only finitely many $\alpha_k \neq 0$ and $\{e_k : k \in \mathbb{N}\}$ is the standard coordinate basis of c_0. Let X denote c_{00} endowed with the inherited standard c_0 norm, and let

$$A = \left\{x \in X : x = \sum_{k=1}^{\infty} \alpha_k e_k,\ |\alpha_k| \leq \frac{1}{k} \forall k \in \mathbb{N}\right\}.$$

Then A is norm closed and $0_X \in \operatorname{core}(A)$ but $\operatorname{int}(A) = \emptyset$. See also Remark 3(b) for examples where the sets are not closed.

(c) If $\operatorname{core}(\bar{A}) = \emptyset$ there is nothing further to do. Otherwise, by translation, we may and do assume that $0_X \in \operatorname{int}(A)$. Suppose that $x \in \operatorname{core}(\bar{A})$. If $x = 0_X$, we are done. If $x \neq 0_X$, it follows that $x + \alpha x = (1 + \alpha)x \in \bar{A}$ for some $\alpha > 0$. Following the last part of proof of Fact 2.7(a) word for word, one can conclude that $x \in \operatorname{int}(A)$. □

Theorem 3.2 captures a standard version of a proper separation theorem for conventional normed linear spaces (see [7, Corollary 2.13]). Indeed, if A and B are nonempty convex subsets of a conventional normed space where $\operatorname{int}(A) \neq \emptyset$ and $\operatorname{int}(A) \cap B = \emptyset$, then $\operatorname{core}(A) = \operatorname{int}(A)$ and so we can apply Theorem 3.2 to properly separate A and B where we applied part (b), with the set C chosen as A itself. Some additional observations concerning the conditions in Theorem 3.2 are as follows.

REMARK. (a) Proposition 3.1 shows that the conditions $\operatorname{int}(A) \neq \emptyset$ and $\operatorname{int}(A) \cap B = \emptyset$ are not sufficient for proper separation of convex sets A and B in extended normed spaces. Thus the condition $\operatorname{core}(C) \cap A \neq \emptyset$ is not redundant in (b) of Theorem 3.2.

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(b) Let X be an infinite-dimensional normed space, and let ϕ be a discontinuous linear functional on X. Consider the convex sets

$$A = \{x \in X : \phi(x) \leq 1\} \quad \text{and} \quad B = \{x \in X : \phi(x) > 1\}.$$

Then A and B are disjoint sets, both of which have nonempty core, but no *continuous* linear functional can separate A and B. This shows that the condition $\operatorname{int}(C) \neq \emptyset$ in Theorem 3.2 is not redundant.

(c) When X is an *extended Banach space*, that is, the extended metric determined by the extended norm is Cauchy complete (equivalently, X_{fin} is a Banach space [3]) and C is closed, the condition $\operatorname{int}(C) \neq \emptyset$ is redundant in Theorem 3.2. Indeed, by translation we may assume that $0_X \in \operatorname{core}(C)$. It then follows from Baire's theorem applied to $\{nC : n \in \mathbb{N}\}$ that $\operatorname{int}(C) \neq \emptyset$. The example just given in (b) shows the assumption that C is closed is needed here.

We now turn to stronger forms of separation. As is customary, we say that ϕ *strongly separates* A and B if $\sup_A \phi < \inf_B \phi$. Choosing $\alpha \in (\sup_A \phi, \inf_B \phi)$, we say that the hyperplane $\{x : \phi(x) = \alpha\}$ *strongly separates* A and B. Between proper separation and strong separation is *strict separation* [8, page 16]: there exists a nonconstant linear functional ϕ and $\alpha \in \mathbb{R}$ such that for each $a \in A$ and $b \in B$, $\phi(a) < \alpha < \phi(b)$, so that the hyperplane $\{x : \phi(x) = \alpha\}$ lies strictly between the two sets.

With this terminology in hand, we are ready for a companion strong separation theorem which must invoke some kind of uniform coreness condition. There is more than one way to do this. From this, we get necessary and sufficient conditions for a closed convex set to be the intersection of the weakly closed halfspaces that contain it.

Theorem 3.4. *Let A and B be nonempty convex subsets of an extended normed space X. Then the following conditions are equivalent:*

(a) *there exists $\phi \in X^*$ that strongly separates A and B;*

(b) *there exist a convex superset C of A disjoint from B, a spur E and $\mu > 0$ such that both $A + E \subseteq C$ and $A + \mu B_X \subseteq C$;*

(c) *there exists a convex superset C of A such that* $\operatorname{int}(C) \neq \emptyset$ *and* $\operatorname{core}(C) \cap B = \emptyset$ *and a spur E such that $A + E \subseteq C$.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\phi \in X^*$ with $\sup_A \phi = \alpha < \beta = \inf_B \phi$. Put $C = \{x : \phi(x) < \beta\}$; we claim that C satisfies condition (b) with respect to A and B. Clearly, $A \subseteq C$ and $B \cap C = \emptyset$. By continuity, there exists $\mu > 0$ such that $\phi(\mu B_X) \subseteq (\alpha - \beta, \beta - \alpha)$ and as a result for all $x \in A + \mu B_X$, $\phi(x) < \beta$. This yields $A + \mu B_X \subseteq C$. Addressing the uniform coreness condition, let Δ consist of a representative taken from each equivalence class of the equivalence relation on $X \backslash \{0_X\}$ defined by $x_1 \sim x_2$ provided x_1 is a positive multiple of x_2. If $x \in \Delta$ and $\phi(x) \leq 0$, let $y_x = x$. If $\phi(x) > 0$, put $y_x = (1/2\phi(x))(\beta - \alpha)x$. Then with $E = \bigcup_{x \in \Delta} \operatorname{conv}\{0_X, y_x\}$ we have $a + E \subseteq C$ for each $a \in A$.

(b) $\Rightarrow$ (c). This is trivial.

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(c) $\Rightarrow$ (a). By Theorem 3.2 we can find a nonconstant $\phi \in X^*$ where $\sup_C \phi \le \inf_B \phi$. Choose $x \in X$ with $\phi(x) > 0$ and then $\lambda > 0$ such that $\lambda x \in E$. As $a + \lambda x \in C$ for each $a \in A$,

$$\sup_A \phi + \lambda\phi(x) \le \sup_C \phi \le \inf_B \phi,$$

and the separation is strong. □

Corollary 3.5. *Let X be an extended normed space. Then a proper nonempty subset B of X is convex and weakly closed if and only if it is the intersection of the weakly closed halfspaces that contain it.*

Proof. Sufficiency is immediate. While necessity follows from the general theory of locally convex spaces, we can give a self-contained justification. Indeed, suppose that B is weakly closed and convex where $\emptyset \ne B \ne X$. If $a_0 \notin B$, then by the definition of the weak topology, we can find $\{\phi_1, \phi_2, \ldots, \phi_n\} \subset X^*$ and $\epsilon > 0$ such that

$$a_0 \in \{x : \forall j \le n,\ \phi_j(a_0) - \epsilon < \phi(x) < \phi_j(a_0) + \epsilon\} \subseteq X \backslash B.$$

With $A = \{a_0\}$ and $C = \{x : \text{for all } j \le n,\ \phi_j(a_0) - \epsilon < \phi(x) < \phi_j(a_0) + \epsilon\}$, condition (c) of the last theorem is satisfied with respect to A, B and C, so we can strongly separate $\{a_0\}$ from B, that is, there exist $\psi \in X^*$ and $\alpha \in \mathbb{R}$ with $B \subset \{x : \psi(x) \ge \alpha\}$ while $a_0 \notin \{x : \psi(x) \ge \alpha\}$. □

Continuity aside, it is well known that convex sets A and B can be strongly separated by a nonconstant linear functional if and only if there exists an absorbing convex set V with $(A + V) \cap B = \emptyset$ [8, page 16]. This condition is a consequence of condition (c) of Theorem 3.4. We intend to show that $V = \operatorname{conv}(\frac{1}{2}E)$ is adequate to the task. To see this, note that for each $a \in A$, $a + \frac{1}{2}E \subseteq \operatorname{core}(C)$ so that by the convexity of the core,

$$A + V = \bigcup_{a \in A} (a + V) = \bigcup_{a \in A} \operatorname{conv}\left(a + \frac{1}{2}E\right) \subseteq \operatorname{core}(C).$$

The classical condition follows from our assumption that $\operatorname{core}(C) \cap B = \emptyset$.

Note that when the convex set A is a singleton, condition (c) of Theorem 3.4 reduces to condition (b) of the preceding proper separation result. Upon reflection, this should not be surprising, because proper separation as described in condition (a) of Theorem 3.2 becomes strong separation when A is a singleton. For this and related results, we will use $d(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}$ to denote the *gap* between two nonempty sets A and B in $\langle X, \|\cdot\|\rangle$.

When A is a singleton, we have a more attractive formulation of the condition.

Proposition 3.6. *Let B be nonempty convex subset of an extended normed space X and suppose that $a_0 \notin B$. Then $\{a_0\}$ and B can be strongly separated by a continuous linear functional if and only if there exists a convex set D with $a_0 \in \operatorname{core}(D)$ and $d(B, D) > 0$.*

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Proof. If strong separation is possible, by Theorem 3.4, we can find a convex set C with these properties: (i) for some spur E, $a_0 + E \subseteq C$; (ii) for some $\varepsilon > 0, a_0 + \varepsilon B_X \subseteq C$; (iii) $B \cap \operatorname{core}(C) = \emptyset$. Then it is not hard to see that $a_0 + \operatorname{conv}(\frac{1}{3}E) + \frac{1}{2}\epsilon B_X \subseteq \operatorname{core}(C)$ and is thus disjoint from B. Putting $D := a_0 + \operatorname{conv}(\frac{1}{3}E)$, we have $d(B, D) > \varepsilon/2$.

Conversely, if such a D exists, take $\delta > 0$ with $d(B, D) > \delta$. Then $C = D + \delta B_X$ satisfies condition (c) of Theorem 3.4 with respect to $A = \{a_0\}$ and B. □

If the set B in the last proposition were closed, one might attempt to remove the positive gap condition. However, this is not possible in general, but it is possible when X/X_{fin} is finite-dimensional; see Example 4.3 and Theorem 4.4, respectively. That said, the result we give remains a characterization and yields the following satisfactory corollary.

Corollary 3.7. *Let B be a (closed) convex subset of an extended normed space. Then B is the intersection of the weakly closed halfspaces that contain it if and only if for each $a \in X \backslash B$ there exists a convex subset D_a with $a \in \operatorname{core}(D_a)$ and $d(B, D_a) > 0$.*

For this corollary to have value, one must demonstrate that a norm closed convex set in an extended normed space need not be an intersection of weakly closed halfspaces. The next result in this direction addresses this requirement.

Theorem 3.8. *Let X be an extended normed space. The following conditions are equivalent:*

(a) *X is a conventional normed linear space;*

(b) *each nonempty norm closed convex subset is the intersection of weakly closed halfspaces that contain it.*

Proof. Only (b) $\Rightarrow$ (a) requires proof. To this end, suppose that X contains vectors of infinite norm, that is, $X_{\text{fin}} \neq X$. Let $\{b_i : i \in I\}$ be a distance basis for X as described in [3], that is, a linearly independent set of vectors such that $X_{\text{fin}} \oplus \operatorname{span}(\{b_i : i \in I\}) = X$. Any two distinct vectors in $\operatorname{span}(\{b_i : i \in I\})$ lie an infinite distance apart, and as a result, each nonempty subset of $\operatorname{span}(\{b_i : i \in I\})$ has no limit points and is thus norm closed in X. Let B be the set of all positive linear combinations of elements of $\{b_i : i \in I\}$. Evidently this is a closed convex subset of X. While $0_X \notin B$, the origin cannot be in the core of any convex set disjoint from B as $0_X \in \operatorname{lin}(B)$. Thus $\{0_X\}$ cannot be strongly separated from B by a continuous linear functional, that is, each weakly closed halfspace containing B must contain 0_X as well. □

Actually a norm closed convex set in an extended normed space need not even be the intersection of halfspaces of any kind.

Proposition 3.9. *Let X be an extended normed space with* $\dim(X) \geq 2$. *Then the following are equivalent:*

(a) *X is a conventional space;*

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(b) *each nonempty norm closed convex proper subset of X is the intersection of halfspaces that contain it;*

(c) *each nonempty norm closed convex cone properly contained in X is the intersection of halfspaces that contain it.*

Proof. (a) $\Rightarrow$ (b) follows from conventional theory, and (b) $\Rightarrow$ (c) is trivial, so we show (c) $\Rightarrow$ (a). Suppose that $\|\cdot\|$ is not a conventional norm. Fix $z \in X$ such that $\|z\| = \infty$. By standard linear algebra we can find a subspace H of codimension one containing X_{fin} such that $X = H \oplus \{\alpha z : \alpha \in \mathbb{R}\}$; note that $\dim(H) \geq 1$. Define $\phi \in X'$ by $\phi(x) = \alpha$ where $x = \alpha z + h$ with $h \in H$. Since $\phi|X_{\text{fin}}$ is continuous, $\phi \in X^*$. Let $C = \{x : \phi(x) > 0\} \cup \{0_X\}$. As $X_{\text{fin}} \subseteq H = \operatorname{Ker}(\phi)$, it follows from Proposition 2.2 that C is the union of two norm closed sets and is thus norm closed. Clearly, C is a convex cone.

Now if $\{x : \psi(x) > \beta\}$ or $\{x : \psi(x) \geq \beta\}$ were a halfspace containing C, then it is a standard exercise in linear algebra to show that $\psi = \mu\phi$ for some $\mu > 0$. There is therefore no halfspace of any kind containing C not containing H as well. □

The last result fails if $\dim(X) = 1$, because if X is not a conventional normed space, then the topology of X is discrete, so that each ray in X, whether or not it contains its endpoint, is norm closed. These are the halfspaces of X, and any proper convex subset is the intersection of at most two of them. It is also notable that the set C constructed in the proof of Proposition 3.9 is a norm closed cone that is not weakly closed. The next result shows that such an example is never possible for closed flats.

Theorem 3.10. *Let X be an extended normed space and let $N \subset X$ be a norm closed flat. Then N is weakly closed.*

Proof. We may assume that N is a subspace that is neither $\{0_X\}$ nor X. Since the kernel of each continuous linear functional is weakly closed, it suffices to show that whenever $x \notin N$, there exists $\phi \in X^*$ such that $N \subseteq \operatorname{Ker}(\phi)$ and $x \notin \operatorname{Ker}(\phi)$.

Let $M = \operatorname{span}(\{x\} \cup N)$, and define $\psi : M \to \mathbb{R}$ by $\psi(\alpha x + n) = \alpha$ where $n \in N$ and $\alpha \in \mathbb{R}$. We claim that ψ is continuous on M; once this is established, applying Hahn–Banach as in [3, Theorem 5.7] produces a continuous linear extension ϕ of ψ with the desired properties.

Suppose that $\langle \alpha_j x + n_j \rangle$ is norm convergent to $\alpha x + n$ where $n, n_1, n_2, \ldots,$ lie in N. Put

$$\epsilon_j = \|(\alpha - \alpha_j)x - (n - n_j)\| \quad (j \in \mathbb{N}).$$

Continuity is established if we can show that $\lim_{j\to\infty} \alpha_j = \alpha$.

If this fails, then by passing to a subsequence, we can assume that, for all j, $|\alpha_j - \alpha| \geq \delta > 0$. Then

$$\left\| x - \frac{n - n_j}{\alpha - \alpha_j} \right\| \leq \frac{\epsilon_j}{\delta} \quad (j \in \mathbb{N}).$$

This puts $x \in \overline{N}$, a contradiction, and the proof is complete. □

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It is also worth noting that any norm compact convex set can be represented as an intersection of norm closed halfspaces because norm compact sets are weakly closed. We next characterize convex sets that are intersections of norm closed halfspaces.

THEOREM 3.11. *Let B be a nonempty proper convex subset of an extended normed space X. Then B is the intersection of the norm closed halfspaces that contain it if and only if at each $x \in X \backslash B$, at least one of the following two conditions holds:*

(a) *there exists a convex set C such that $x \in \operatorname{int}(C) \cap \operatorname{core}(C)$ and $\operatorname{core}(C) \cap B = \emptyset$;*
(b) *there exists $\phi \in X^*$ such that for all $w \in x + X_{\text{fin}}$, for all $b \in B$, $\phi(w) < \phi(b)$.*

PROOF. If B is the intersection of norm closed halfspaces, then for each $x \in X \backslash B$, there exist $\phi \in X^*$ and $\alpha \in \mathbb{R}$ such that either (i) $B \subset \{w : \phi(w) \geq \alpha\}$ and $\phi(x) < \alpha$, or (ii) $B \subset \{w : \phi(w) > \alpha\}$, $X_{\text{fin}} \subseteq \operatorname{Ker}(\phi)$ and $\phi(x) \leq \alpha$ (see Proposition 2.2). Clearly, condition (a) holds if condition (i) holds (take $C = \{w : \phi(w) < \alpha\}$), and condition (b) holds if (ii) holds.

Conversely, if (a) holds, by Theorem 3.4 there is a weakly closed halfspace that contains B but not x. If (b) holds, then $X_{\text{fin}} \subseteq \operatorname{Ker}(\phi)$, for otherwise $\phi(x + X_{\text{fin}}) = \mathbb{R}$. By Proposition 2.2, B is contained in the norm closed halfspace $\{w : \phi(w) > \phi(x)\}$. □

We now turn to strict separation. The following proposition is the key to our strict separation result.

PROPOSITION 3.12. *Let A and B be nonempty convex subsets of an extended normed space X. Then following conditions are equivalent:*

(a) *there exist $\phi \in X^*$ and $\alpha \in \mathbb{R}$ with $A \subseteq \{x : \phi(x) < \alpha\}$ and $B \subseteq \{x : \phi(x) \geq \alpha\}$;*
(b) *there exists a convex superset C of A such that $A \subseteq \operatorname{int}(C) \cap \operatorname{core}(C)$ and $B \cap \operatorname{core}(C) = \emptyset$;*
(c) *there exists a convex superset C of A such that $\operatorname{int}(C) \neq \emptyset$, $A \subseteq \operatorname{core}(C)$, and $B \cap \operatorname{core}(C) = \emptyset$.*

PROOF. (a) $\Rightarrow$ (b). If such ϕ and α exist, take $C = \{x : \phi(x) < \alpha\}$. Clearly (b) $\Rightarrow$ (c), so we suppose that (c) holds. Then apply Theorem 3.2 to C and B to find a nonconstant $\phi \in X^*$ and $\alpha \in \mathbb{R}$ with $C \subseteq \{x : \phi(x) \leq \alpha\}$ and $B \subseteq \{x : \phi(x) \geq \alpha\}$. Evidently, $A \subseteq \operatorname{core}(C) \Rightarrow$ for all $a \in A, \phi(a) < \alpha$. □

Before moving to strict separation, we state a corollary that is interesting in its own right.

COROLLARY 3.13. *Let A and B be nonempty convex subsets of an extended normed space X. Then the following conditions are equivalent:*

(a) *there exists $\phi \in X^*$ such that for all $(a, b) \in A \times B$, $\phi(a) < \phi(b)$;*
(b) *either there exists a convex superset C of A such that $\operatorname{int}(C) \neq \emptyset, A \subseteq \operatorname{core}(C)$, and $B \cap \operatorname{core}(C) = \emptyset$, or there exists a convex superset D of B such that $\operatorname{int}(D) \neq \emptyset$, $B \subseteq \operatorname{core}(D)$, and $A \cap \operatorname{core}(D) = \emptyset$;*

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(c) *there exists a convex superset C of $A - B$ such that* $\operatorname{int}(C) \neq \emptyset$, $A - B \subseteq \operatorname{core}(C)$, *and* $0_X \notin \operatorname{core}(C)$.

Proof. We just look at the equivalence of conditions (a) and (c). If (a) holds, take for C the open halfspace $\{x : \phi(x) < 0\}$. Conversely, if (c) holds, apply the previous proposition to the convex sets $A - B$ and $\{0_X\}$. □

Even in conventional linear analysis, the separation statement in the corollary is weaker than strict separation: in the Euclidean plane, take $A = \{(s, t) : t \leq 0\}$ and for B take the epigraph of the natural exponential function. Amazingly, if we replace 'or' by 'and' in condition (b) of Corollary 3.13, we obtain what we are after.

Theorem 3.14. *Let A and B be nonempty convex subsets of an extended normed space X. Then the following conditions are equivalent:*

(a) *there exists $\phi \in X^*$ that strictly separates A and B;*

(b) *there exist convex C and D such that $A \subseteq \operatorname{core}(C) \cap \operatorname{int}(C)$, $B \subseteq \operatorname{core}(D) \cap \operatorname{int}(D)$, and $\operatorname{core}(C) \cap \operatorname{core}(D) = \emptyset$;*

(c) *there exists a convex superset C of A such that $\operatorname{int}(C) \neq \emptyset$, $A \subseteq \operatorname{core}(C)$, and $B \cap \operatorname{core}(C) = \emptyset$, and there exists a convex superset D of B such that $\operatorname{int}(D) \neq \emptyset$, $B \subseteq \operatorname{core}(D)$, and $A \cap \operatorname{core}(D) = \emptyset$.*

Proof. (a) $\Rightarrow$ (b). If $\{x : \phi(x) = \alpha\}$ strictly separates, put $C = \{x : \phi(x) < \alpha\}$ and $D = \{x : \phi(x) > \alpha\}$. Clearly (b) $\Rightarrow$ (c), and so we suppose that (c) holds. Then by Proposition 3.12 we can find $\phi \in X^*, \psi \in X^*, \alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}$ such that whenever $a \in A$ and $b \in B$, both

$$\phi(a) < \alpha \leq \phi(b) \text{ and } \psi(b) < \beta \leq \psi(a).$$

It follows that for each $a \in A$ and $b \in B$, $(\phi - \psi)(a) < \alpha - \beta < (\phi - \psi)(b)$, and we have strict separation of A and B by $\phi - \psi \in X^*$. □

Alternate approaches to separation involve examining the core and interior of sets with respect to their spans. While we do not list natural formulations in this direction, we make the following observation.

Proposition 3.15. *Suppose that X is an extended normed space, and suppose that A is a convex subset of X such that $0_X \in \operatorname{int}(A)$. Then* $\operatorname{span}(\operatorname{int}(A)) = \operatorname{span}(A) = \operatorname{span}(\bar{A})$.

Proof. It suffices to show that $\bar{A} \subset \operatorname{span}(\operatorname{int}(A))$. So let $x \in \bar{A}$; because $0_X \in \operatorname{int}(A)$ we choose $\delta > 0$ so that $\delta B_X \subset A$. Now choose $y \in A$ such that $\|y - x\| < \delta$. Then

$$\tfrac{1}{2}y + \tfrac{1}{2}(\delta B_X) \subset A$$

because A is convex. Moreover, $\|\frac{1}{2}y - \frac{1}{2}x\| < \frac{\delta}{2}$. Therefore, $\frac{1}{2}x \in \operatorname{int}(A)$ and so $x \in \operatorname{span}(\operatorname{int}(A))$ as desired. □

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Many of the results in this section seem to suggest that things are business as usual once one has included a core condition along with the natural interior condition. The following example is yet another a reminder that additional care is needed in the extended norm setting.

Example 3.16. In a conventional normed space, whenever A is a closed convex set with $\operatorname{int} A \neq \emptyset$ one has $A = \overline{\operatorname{int}(A)}$; however, this need not be true in an extended normed space. To see this, let $X = \mathbb{R}^2$ with $\|x\| = |s|$ when $x = (s, 0)$ and $\|x\| = \infty$ otherwise. Put $A = \{(s,t) : 0 \leq s \leq 1, s \leq t \leq 1\}$. Let $x_0 = (0,0)$ be the origin. Then $x_0 \in A$, but $x_0 \notin \operatorname{int}(A)$ since $\{x : \|x - x_0\| < \delta\} = \{(s,t) : |s| < \delta, t = 0\} \not\subset A$ for each $\delta > 0$. Thus if $x \in \operatorname{int}(A)$ where $x = (s,t)$, we know that $t > 0$. Consequently, $\|x - x_0\| = \infty$ when $x \in \operatorname{int}(A)$ and so $x_0 \notin \overline{\operatorname{int}(A)}$. Thus, the norm closed convex set A satisfies $\operatorname{int}(A) \neq \emptyset$ and $\operatorname{core}(A) \neq \emptyset$, yet $\overline{\operatorname{int}(A)}$ is a proper subset of A.

4. Finer results concerning separating points and convex sets

Although the conditions in Proposition 3.6, are, of course, necessary and sufficient for strong separation of points and convex sets, there are natural questions concerning the possible tightening of the conditions given therein. We state three such natural questions, all of which will be addressed below, and in doing so some subtle distinctions between extended and conventional normed spaces will be exposed.

Question 4.1. *Let $\langle X, \|\cdot\|\rangle$ be an extended normed space. Suppose that A is a nonempty convex set such that $0_X \notin \overline{A}$ and suppose that B is a nonempty norm closed convex set such that $0_X \in \operatorname{core}(B)$ and $A \cap B = \emptyset$.*

(a) *Is there a continuous linear functional that strongly separates $\{0_X\}$ and A? Stated another way, given convex sets B and C that both miss A with $0_X \in \operatorname{core}(B)$ and $0_X \in \operatorname{int}(C)$, is there a third convex set D that misses A such that $0_X \in \operatorname{core}(D)$ and $0_X \in \operatorname{int}(D)$?*

(b) *Is it possible to find a spur K with $0_X \in K$ and $K \cap \overline{A} = \emptyset$?*

(c) *If, additionally, we assume that A is norm closed, is there a continuous linear functional that strongly separates $\{0_X\}$ and A?*

The following example provides negative answers to Questions 4.1(a) and (b).

Example 4.2. Let X be the vector space c_{00} endowed with the extended norm $\|\cdot\|$ defined for $x = \sum_{k=1}^{\infty} \alpha_k e_k$ by

$$\|x\| = \begin{cases} \max\limits_{k \in \mathbb{N}} |\alpha_k| & \text{if } \alpha_1 = 0, \\ \infty & \text{if } \alpha_1 \neq 0. \end{cases}$$

Then there exist a convex subset $A \subseteq X$ with $d(A, 0_X) = \infty$ and a closed convex set B with $0_X \in \operatorname{core}(B)$ and $A \cap B = \emptyset$ and yet $\{0_X\}$ and A cannot be strongly separated by any linear functional in X^*. Further, $0_X \in \operatorname{lin}(\overline{A})$, so there is no spur containing 0_X that misses $\bar{A}$.

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Proof. For each $n \in \mathbb{N}$, we define $x_n \in X$ by $x_n = \sum_{k=1}^{n}(1/n)e_k$, and we let $A = \operatorname{conv}(\{x_1, x_2, x_3, \ldots\})$. Then for any $x \in A$, we observe

$$x = \sum_{k=1}^{\infty} \alpha_k e_k \quad \text{where } \alpha_1 > 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha_k = 1. \tag{4.1}$$

Because $\alpha_1 > 0$, we know that $\|x\| = \infty$, so $d(0_X, A) = \infty$ as desired.

Next, define the (closed) convex set B by

$$B = \Big\{x \in X : x = \sum_{k=1}^{\infty} \alpha_k e_k \text{ where } |\alpha_k| \le \frac{1}{4^k} \text{ for } k \in \mathbb{N}\Big\}.$$

It is easy to see that $0_X \in \operatorname{core}(B)$. Indeed, given $h \in X$, we write $h = \sum_{k=1}^{N} \alpha_k e_k$. Now choose $\delta > 0$ sufficiently small so that

$$\delta \max\{|\alpha_k| : 1 \le k \le N\} < 4^{-N}.$$

Then $0_X + th \in B$ for $0 \le t \le \delta$. It follows from (4.1) and the definition of B that $A \cap B = \emptyset$.

Next let $\phi \in X^*$ with $\|\phi\|_{\mathrm{op}} = M$. Then

$$\begin{aligned} |\phi(x_n)| &\le \left|\frac{1}{n}\phi(e_1)\right| + \left|\phi\Big(\sum_{j=1}^{n-1} \frac{1}{n} e_j\Big)\right| \\ &\le \frac{1}{n}|\phi(e_1)| + M \cdot \frac{1}{n} \to 0. \end{aligned}$$

Thus, ϕ does not strongly separate $\{0_X\}$ and A.

We fix $0 < \beta < 1$, and we will show that $\beta e_1 \in \overline{A}$. For $n > 1/\beta$, we choose $0 < \lambda_n < 1$ so

$$\lambda_n + (1 - \lambda_n)\frac{1}{n} = \beta, \quad \text{that is, } \lambda_n = \frac{n\beta - 1}{n - 1}.$$

Next put $u_n = \lambda_n x_1 + (1 - \lambda_n)x_n$ for x_n as defined above; then $u_n \in A$ for $n > 1/\beta$. By definition,

$$\begin{aligned} u_n &= \lambda_n e_1 + (1 - \lambda_n)\Big(\frac{1}{n}e_1 + \sum_{k=2}^{n} \frac{1}{n} e_k\Big) \\ &= \Big(\lambda_n + (1 - \lambda_n)\frac{1}{n}\Big)e_1 + (1 - \lambda_n)\sum_{k=2}^{n} \frac{1}{n} e_k = \beta e_1 + (1 - \lambda_n)\sum_{k=2}^{n} \frac{1}{n} e_k. \end{aligned}$$

Then the sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ converges to βe_1 and so $\beta e_1 \in \overline{A}$. Consequently, $0_X \in \operatorname{lin}(\overline{A})$ as desired. This, of course, provides a second rationale for the failure of strong separation of $\{0_X\}$ from A. □

Thus we have shown that, for a nonempty convex set A in an extended normed space, unlike the conventional setting, it is more restrictive to require that there is a

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convex set B with $0_X \in \operatorname{core}(B)$ such that $B \cap \overline{A} = \emptyset$ than to require that $d(0_X, A) > 0$ and $B \cap A = \emptyset$. Thus, the negative answer to Question 4.1(a) given above does not ensure a negative answer to Question 4.1(c). Nevertheless, the following theorem answers Question 4.1(c), also in the negative, but at the cost of using an infinite-dimensional quotient X/X_{fin}. However, Theorem 4.4 shows that the infinite-dimensional quotient is unavoidable.

Example 4.3. Let $X = c_{00} \oplus c_{00}$ endowed with the norm

$$\|x\| = \begin{cases} \|u\|_{c_0} & \text{if } x = (u, 0_X) \in c_{00} \oplus c_{00}, \\ \infty & \text{otherwise.} \end{cases}$$

Then there are nonempty norm closed convex sets A and B in X such that $A \cap B = \emptyset$, $0_X \in \operatorname{core}(B)$ and yet $\{0_X\}$ and A cannot be strongly separated by any $\phi \in X^*$.

Proof. The proof will have many similarities with Example 4.2, but the construction will be more delicate in order to control the closure. Notationally, we will write $x \in X$ as

$$x = \sum_{n=1}^{\infty} \alpha_n e_n + \sum_{n=1}^{\infty} \beta_n z_n$$

where $\alpha_n \neq 0$ for only finitely many n, $\beta_n \neq 0$ for only finitely many n, and for x so expressed,

$$\|x\| = \begin{cases} \|x\|_{c_0} = \max\limits_{1 \le n < \infty} |\alpha_n| & \text{when } \beta_n = 0 \text{ for all } n, \\ \infty & \text{otherwise.} \end{cases}$$

So $\{e_n : n \in \mathbb{N}\}$ and $\{z_n : n \in \mathbb{N}\}$ are the standard coordinate bases of $c_{00} \oplus \{0_X\}$ and $\{0_X\} \oplus c_{00}$, respectively ($\{z_n : n \in \mathbb{N}\}$ is a *distance basis* as introduced in [3]).

Next, for $k \ge 1$, define

$$x_{3,k} = \frac{1}{3}e_1 + \frac{1}{3}e_2 + \left(\frac{1}{3} - \frac{1}{3k}\right)e_3 + \frac{1}{k}\left(\frac{1}{9}z_1 + \frac{1}{9}z_2 + \frac{1}{9}z_3\right)$$

Generally, for $n \ge 3$ and $k \in \mathbb{N}$, define

$$x_{n,k} = \frac{1}{n}e_1 + \frac{1}{n}e_2 + \cdots + \frac{1}{n}e_{n-1} + \left(\frac{1}{n} - \frac{1}{nk}\right)e_n + \frac{1}{k}\left(\frac{1}{n^2}z_1 + \frac{1}{n^2}z_2 + \cdots + \frac{1}{n^2}z_n\right). \quad (4.2)$$

Notice that the basis coefficients in the definition of the $x_{n,k}$ are all nonnegative and sum to one. Now let $C = \operatorname{conv}(\{x_{j,k} : 3 \le j < \infty, k \in \mathbb{N}\})$ and let $A = \overline{C}$.

We define the closed convex set B as

$$B = \left\{x \in X : x = \sum_{i=1}^{\infty} \alpha_i e_i + \sum_{j=1}^{\infty} \beta_i z_i,\ |\alpha_i| \le \frac{1}{4^i},\ |\beta_i| \le \frac{1}{4^i}\right\}.$$

As in the proof of Example 4.2, it is not hard to see that $0_X \in \operatorname{core}(B)$. It is also important to note that if $x \in C$, where $x = \sum_{i=1}^{\infty} \alpha_i e_i + \sum_{j=1}^{\infty} \beta_i z_i$, then

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{j=1}^{\infty} \beta_i = 1 \quad \text{and} \quad \beta_i \neq 0 \text{ for some } i \in \mathbb{N}. \quad (4.3)$$

Consequently, $d(0_X, C) = \infty$ and so $0_X \notin A$.

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Next we show that $A \cap B = \emptyset$. Indeed, suppose that $b \in B \cap A$. Then there is a sequence $\langle u_n \rangle \subseteq C$ such that $\|u_n - b\| \to 0$. Write $b = \sum_{i=1}^m \alpha_i e_i + \sum_{i=1}^m \beta_i z_i$. Then for all sufficiently large n we must have

$$u_n = \sum_{i \in \Delta_n} \alpha_{i,n} e_i + \sum_{i=1}^m \beta_i z_i, \quad \text{where } \Delta_n \text{ is a finite subset of } \mathbb{N},$$

and the β_i are as in the expansion for b, for otherwise $\|u_n - b\| = \infty$. For convenience, we will assume that this is true for all n in the sequence. Then

$$u_n \in \operatorname{conv}(\{x_{j,k} : 3 \le j \le m, k \in \mathbb{N}\}), \tag{4.4}$$

for if not, then $\beta_i > 0$ for some $i > m$. Now consider $\phi \in \ell_1 \oplus \ell_1$ defined by

$$\phi\Big(\sum_{i=1}^\infty \alpha_i e_i + \sum_{i=1}^\infty \beta_i z_i\Big) = \sum_{i=1}^m \alpha_i.$$

Then ϕ is continuous on $c_{00} \oplus c_{00}$, and observe that

$$\phi(x_{j,k}) \ge 1 - \frac{1}{j} \ge \frac{2}{3} \quad \text{whenever } 3 \le j \le m, \tag{4.5}$$

and then by (4.4) and (4.5), $\phi(u_n) \ge 2/3$ for each n. However,

$$\phi(b) \le \sum_{j=1}^m |\alpha_i| \le \sum_{j=1}^m \frac{1}{4^i} < \frac{1}{3}.$$

Because ϕ is continuous, we conclude that the sequence $\langle u_n \rangle$ cannot converge in norm to b. This contradiction shows that $A \cap B = \emptyset$.

Finally, we show for any $\phi \in X^*$ that $\inf_A \phi = 0$. So let $\epsilon > 0$ and fix $\phi \in X^*$. Then choose and fix $n_0 \in \mathbb{N}$, $n_0 \ge 3$, so large that

$$\frac{\|\phi\|_{\mathrm{op}}}{n_0} < \frac{\epsilon}{2}. \tag{4.6}$$

Now we choose $m_0 \in \mathbb{N}$ such that

$$\frac{1}{m_0} \sum_{j=1}^{n_0} |\phi(z_j)| < \frac{\epsilon}{2}. \tag{4.7}$$

For n_0 and m_0 as given above, let $x_{n_0,m_0} \in A$ be as defined in (4.2). Since $\sum_{j=1}^{n_0-1} e_j + (1 - (1/m_0))e_{n_0}$ lies on the surface of the unit ball,

$$\begin{aligned} |\phi(x_{n_0,m_0})| &\le \Big|\phi\Big(\sum_{j=1}^{n_0-1} \frac{1}{n_0} e_j + \Big(\frac{1}{n_0} - \frac{1}{n_0 m_0}\Big) e_{n_0}\Big)\Big| + \frac{1}{n_0^2 m_0} \sum_{j=1}^{n_0} |\phi(z_j)| \\ &\le \frac{1}{n_0} \|\phi\|_{\mathrm{op}} + \frac{1}{n_0^2 m_0} \sum_{j=1}^{n_0} |\phi(z_j)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

by (4.6) and (4.7). This shows that $\inf_A \phi = 0$, and so A and $\{0_X\}$ cannot be strongly separated by any $\phi \in X^*$. □

We conclude by showing that the infinite-dimensional quotient X/X_{fin} was necessary in Example 4.3.

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THEOREM 4.4. *Let $\langle X, \|\cdot\|\rangle$ be an extended normed space such that X/X_{fin} is finite-dimensional. Suppose that A is a closed convex subset of X, $b \notin A$, and there is a convex set B with $b \in \text{core}(B)$ and $A \cap \text{core}(B) = \emptyset$. Then b and A can be strongly separated by some $\phi \in X^*$.*

PROOF. Write $X = X_{\text{fin}} \oplus Z$ where Z is a finite-dimensional vector space. By translation we may assume that $b = 0_X$ and so $0_X \in \text{core}(B)$. Because $0_X \in \text{core}(B)$, there exists

$$\{b_1, b_2, \ldots, b_N\} \text{ a basis of } Z, \text{ such that } \pm b_i \in \text{core}(B) \text{ for } 1 \le i \le N. \tag{4.8}$$

Next we let $P = \text{conv}(\{\pm b_i\}_{i=1}^n)$. Then P is a polytope contained in $\text{core}(B)$.

Now suppose that $d(P, A) > 0$. Then we choose $0 < \delta < d(P, A)$ and we let $U = P + \delta B_X$. Then U is a convex set such that

$$U \cap A = \emptyset \quad \text{and} \quad 0_X \in \text{core}(U) \cap \text{int}(U).$$

By Theorem 3.4, $\{0_X\}$ and A can be strongly separated by some $\phi \in X^*$. So it remains to address the case when $d(P, A) = 0$.

We now suppose that $d(P, A) = 0$. Then we choose sequences $\langle a_n\rangle \subset A$ and $\langle p_n\rangle \subset P$ such that $\|a_n - p_n\| \to 0$. We write $a_n = x_n + z_n$ where $x_n \in X_{\text{fin}}$ and $z_n \in Z$. Because $\|z_n - p_n\| = \infty$ as soon as $p_n \neq z_n$, the only possibility is $p_n = z_n$ for all large n. Consequently, $\|x_n\| \to 0$ and we assume that $p_n = z_n$ for all $n \in \mathbb{N}$.

Let $|||\cdot|||$ be a conventional norm on the finite-dimensional vector space Z. By passing to a subsequence if necessary, we assume that $\|x_n\| \le 1$ for all $n \in \mathbb{N}$ and $\langle z_n\rangle$ converges to some $\bar{z} \in P$ in the $|||\cdot|||$-topology on Z. Let $C = \text{conv}(\{z_n : n \in \mathbb{N}\})$. Then $\text{i-int}(C) \neq \emptyset$ (where $\text{i-int}(C)$ agrees with the relative interior of C as determined by Z equipped the conventional norm as in [9]) and so we choose $w \in \text{i-int}(C)$. Write $w = \sum_{i\in\Delta} \lambda_i z_i$ as a convex combination where Δ is a finite subset of $\mathbb{N}$ and put $\tilde{w} = \sum_{i\in\Delta} \lambda_i a_i$ using the same Δ and λ_i as used in w. Then

$$\tilde{w} \in A \quad \text{and} \quad \tilde{w} = w + u \text{ for some } u \in X_{\text{fin}}. \tag{4.9}$$

Because $0_X \in \text{core}(B)$, we fix $0 < \lambda < 1$ so that $\lambda\tilde{w} \in \text{core}(B)$.

Now put $\hat{z} = (1-\lambda)\bar{z} + \lambda w$ and $\hat{z}_n = (1-\lambda)z_n + \lambda w$. Then $|||\hat{z}_n - \hat{z}||| \to 0$. Moreover, $\hat{z} \in \text{i-int}(C)$ because $w \in \text{i-int}(C)$ and $\bar{z}$ is in the $|||\cdot|||$-closure of C; see [9]. Since $\hat{z} \in \text{i-int}(C)$ we can find $\alpha_n \to 0^+$ and $v_n \in C$ so that

$$(1-\alpha_n)\hat{z}_n + \alpha_n v_n = \hat{z} = (1-\lambda)\bar{z} + \lambda w \quad \text{for all } n \in \mathbb{N}. \tag{4.10}$$

Next choose $\tilde{v}_n$ in a fashion analogous to the choice of $\tilde{w}$, that is, given n, express v_n as some convex combination of the z_i, say $v_n = \sum_{i\in\Delta_n} \lambda_i z_i$ where Δ_n is a finite subset of $\mathbb{N}$; then put $\tilde{v}_n = \sum_{i\in\Delta_n} \lambda_i a_i$. Thus

$$\tilde{v}_n \in A \text{ and we can write } \tilde{v}_n = v_n + u_n \text{ where } \|u_n\| \le 1; \tag{4.11}$$

the fact that $\|u_n\| \le 1$ follows because u_n is a convex combination of elements in B_X. Now let

$$y_n = (1-\alpha_n)[(1-\lambda)a_n + \lambda\tilde{w}] + \alpha_n\tilde{v}_n.$$

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Then $y_n \in A$ for each n because each y_n is a convex combination of elements in A. Moreover, for each $n \in \mathbb{N}$,

$$\begin{aligned} y_n &= (1-\alpha_n)[(1-\lambda)(x_n+z_n)+\lambda(w+u)]+\alpha_n(v_n+u_n) \quad [\text{see } (4.9), (4.11)] \\ &= (1-\alpha_n)[(1-\lambda)z_n+\lambda w+(1-\lambda)x_n+\lambda u]+\alpha_n v_n+\alpha_n u_n \\ &= (1-\alpha_n)\hat{z}_n+\alpha_n v_n+(1-\alpha_n)[(1-\lambda)x_n+\lambda u]+\alpha_n u_n \quad [\text{by definition of } \hat{z}_n] \\ &= (1-\lambda)\bar{z}+\lambda w+(1-\alpha_n)[(1-\lambda)x_n+\lambda u]+\alpha_n u_n \quad [\text{see } (4.10)]. \end{aligned} \tag{4.12}$$

Because $\|x_n\| \to 0$ and $\|\alpha_n u_n\| \le \alpha_n \to 0$, it follows from (4.12) that $\langle y_n \rangle$ converges in norm to

$$(1-\lambda)\bar{z}+\lambda w+\lambda u = (1-\lambda)\bar{z}+\lambda\tilde{w} \in \operatorname{core}(B).$$

This is a contradiction because A is norm closed and $A \cap \operatorname{core}(B) = \emptyset$. □

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G. BEER, Department of Mathematics,
California State University Los Angeles,
5151 State University Drive, Los Angeles,
CA 90032, USA
e-mail: gbeer@cslanet.calstatela.edu

J. VANDERWERFF, Department of Mathematics,
La Sierra University, 4500 Riverwalk Parkway,
Riverside, CA 92515, USA
e-mail: jvanderw@lasierra.edu

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