Proceedings of the Edinburgh Mathematical Society (1990) 33, 71–78 ©

COMPLETENESS OF THE L^1-SPACE OF CLOSED VECTOR MEASURES

by WERNER J. RICKER

(Received 4th April 1988)

The notion of a closed vector measure m, due to I. Kluvánek, is by now well established. Its importance stems from the fact that if the locally convex space X in which m assumes its values is sequentially complete, then m is closed if and only if its L^1-space is complete for the topology of uniform convergence of indefinite integrals. However, there are important examples of X-valued measures where X is not sequentially complete. Sufficient conditions guaranteeing the completeness of $L^1(m)$ for closed X-valued measures m are presented without the requirement that X be sequentially complete.

1980 *Math. subject classification* (1985 Revision): Primary 28B05, 46G10.

1. Introduction

The notion of a closed vector measure is due to I. Kluvánek [**7**]. This concept has been intensively studied by various authors [**1, 8, 11**] and has turned out to be fundamental in the study of certain operator algebras [**2, 3, 4, 10, 12**]. One of the reasons why closed measures are important is that, modulo certain completeness hypotheses on the underlying space, their L^1-spaces are complete. More precisely, if X is a locally convex Hausdorff space and $m:\Sigma\to X$ is a closed vector measure, Kluvánek showed that $L^1(m)$ is complete for the topology of uniform convergence of indefinite integrals whenever X is complete or quasicomplete (cf. [**8**, Theorem IV 4.1] and [**10**, Proposition 1]). This is also known to be true if X is merely sequentially complete [**2**, p. 139]. However, there are important examples of measures which assume their values in spaces which are not sequentially complete. For instance, spectral measures in dual Banach spaces X' equipped with their weak-star topology fall into this scheme if the predual X is not weakly sequentially complete (cf. Section 3). Accordingly, it seems useful to have available sufficient conditions which guarantee the completeness of $L^1(m)$ for a given closed measure m. The aim of this note is to present such criteria.

More precisely, if $m:\Sigma\to X$ is a vector measure for which the sequential closure $X[m]$ of the linear span of $R(m)=\{m(E); E\in\Sigma\}$ is sequentially complete, then $L^1(m)$ is complete if and only if m is a closed measure. This is a substantial improvement in practice since $X[m]$ may be a proper subspace of X, not necessarily closed or dense. Some relevant examples are discussed in Section 3. The sequential completeness of $X[m]$ is not a necessary condition for $L^1(m)$ to be complete; see Example 1 in Section 3.

https://doi.org/10.1017/S0013091500028893 Published online by Cambridge University Press

2. Preliminaries and the main result

In this section we establish the notation to be used and summarize those aspects of the theory of integration with respect to vector measures that are needed in the sequel; see [8] for a more comprehensive treatment. The main result is also established.

Let X be a locally convex Hausdorff space and X' be its (continuous) dual space. An X-valued measure is a σ-additive map $m:\Sigma\to X$ whose domain Σ is a σ-algebra of subsets of some set Ω. For each $x'\in X'$, the complex-valued measure $E\to\langle m(E),x'\rangle$, $E\in\Sigma$, is denoted by $\langle m,x'\rangle$. Its variation measure is denoted by $|\langle m,x'\rangle|$.

If q is a continuous seminorm on X, let U_q^0 denote the polar of the closed unit ball of q. Then the q-semivariation of m is the set function $q(m)$ defined by

$$q(m)(E)=\sup\{|\langle m,x'\rangle|(E);x'\in U_q^0\},\qquad E\in\Sigma.$$

For each $E\in\Sigma$, the inequalities

$$\sup\{q(m(F));F\in\Sigma,F\subseteq E\}\leqq q(m)(E)\leqq 4\sup\{q(m(F));F\in\Sigma,F\subseteq E\},\tag{1}$$

hold [8, Lemma II, 1.1].

A complex-valued, Σ-measurable function f on Ω is said to be m-integrable if it is integrable with respect to each measure $\langle m,x'\rangle$, $x'\in X'$, and if, for every $E\in\Sigma$, there exists an element $\int_E f\,dm$ of X such that

$$\left\langle \int_E f\,dm,x'\right\rangle=\int_E f\,d\langle m,x'\rangle,$$

for each $x'\in X'$. The map $fm:\Sigma\to X$ specified by

$$(fm)(E)=\int_E f\,dm,\qquad E\in\Sigma,$$

is called the indefinite integral of f with respect to m. The Orlicz–Pettis lemma implies that it is a vector measure. The element $(fm)(\Omega)=\int_\Omega f\,dm$ is denoted simply by $m(f)$. The set of all m-integrable functions is denoted by $L(m)$. An m-integrable function is said to be m-null if its indefinite integral is the zero measure. Two m-integrable functions f and g are equal m-almost everywhere (m-a.e.) if $|f-g|$ is m-null.

If f is an m-integrable function, then for each continuous seminorm q on X we define $q(m)(f)=q(fm)(\Omega)$. The function

$$f\to q(m)(f),\qquad f\in L(m),\tag{2}$$

is then a seminorm on $L(m)$. It is clear from (1) that an equivalent seminorm is given by

$$f\to\sup\left\{q\left(\int_E f\,dm\right),E\in\Sigma\right\},\qquad f\in L(m).\tag{3}$$

Denote by $\tau(m)$ the topology on $\mathcal{L}(m)$ specified by the family of seminorms (2) (or (3)) for every continuous seminorm q on X or, at least for enough continuous seminorms q determining the topology of X. The resulting locally convex space is not necessarily Hausdorff. The quotient space of $\mathcal{L}(m)$ with respect to the subspace of all m-null functions is denoted by $L^1(m)$. The resulting Hausdorff topology on $L^1(m)$ is again denoted by τ(m). It is clear from (3) that $\tau(m)$ is the topology of uniform convergence on Σ of indefinite integrals.

A set $E \in \Sigma$ is said to be m-null if χ_E is m-null. Two sets $E, F \in \Sigma$ are m-equivalent if $|\chi_E - \chi_F|$ is m-null. The set of all equivalence classes of Σ with respect to m-equivalence is denoted by $\Sigma(m)$. The set $\Sigma(m)$ can be identified with the subset $\{[\chi_E]_m; E \in \Sigma\}$ of $L^1(m)$, where the square brackets $[\cdot]_m$ signify cosets with respect to m-equivalence. Since

$$q(m)(E) = q(m)(\chi_E), \qquad E \in \Sigma,$$

for each continuous seminorm q on X, the topology and uniform structure $\tau(m)$ has a natural restriction to $\Sigma(m)$ which is again denoted by $\tau(m)$. The vector measure m is said to be closed [7] if $\Sigma(m)$ is a complete space with respect to the uniform structure $\tau(m)$.

Let M be a subset of a locally convex Hausdorff space X. The sequential closure of M is the smallest set in X which contains M and is sequentially closed. The sequential closure of M is a vector subspace of X whenever M is a vector subspace.

Let $m:\Sigma \to X$ be a vector measure and let $X[m]$ denote the sequential closure of the linear span of $R(m) = \{m(E); E \in \Sigma\}$. Equip $X[m]$ with the relative topology from X. If X is sequentially complete, then so is $X[m]$.

Theorem 1. *Let $m:\Sigma \to X$ be a closed vector measure and $\tilde{X}$ be a sequentially complete space containing X as a dense subspace (for the relative topology). Let $\tilde{m}:\Sigma \to \tilde{X}$ denote the measure m considered as taking its values in $\tilde{X}$. If $\int_E f\, d\tilde{m}$ belongs to X, for every $E \in \Sigma$ and every $f \in L^1(\tilde{m})$, then $L^1(m) = L^1(\tilde{m})$ and the space $L^1(m)$ is complete.*

Proof. It is clear that $\tilde{m}$ is a closed measure. Since the continuous seminorms determining the topology of $\tilde{X}$ are just extensions of the continuous seminorms determining the topology of X, it is also clear that $L^1(m) \subseteq L^1(\tilde{m})$ and that $\tau(\tilde{m})$ induces the topology $\tau(m)$ on $L^1(m)$. But, $L^1(\tilde{m})$ we know to be complete and so it suffices to show that $L^1(\tilde{m}) = L^1(m)$. Since $(\tilde{X})' = X'$, it follows from the definition of integrability that any $\tilde{m}$-integrable function f satisfying $\int_E f\, d\tilde{m} \in X$, for each $E \in \Sigma$, is m-integrable. $\square$

Theorem 2. *Let X be a locally convex Hausdorff space and $m:\Sigma \to X$ be a vector measure. If $X[m]$ is sequentially complete, then m is a closed measure if and only if $L^1(m)$ is complete.*

Proof. The completeness of $L^1(m)$ always implies that m is a closed measure (even if $X[m]$ is not sequentially complete). This is because $\Sigma(m)$ is a closed subset of $L^1(m)$.

The converse proceeds along the lines of [**8**, Proposition 1]. So, suppose that m is closed. Let $\tilde{X}$ denote the completion of X and $\tilde{m}$ be the vector measure defined as in Theorem 1. By Theorem 1, the proof is complete if $L^1(m) = L^1(\tilde{m})$. Let $f \geqq 0$ be

$\tilde{m}$-integrable. Choose Σ-simple functions $s_n \geqq 0$, $n=1,2,\ldots$, such that $s_n \uparrow f$ pointwise. The Dominated Convergence Theorem for $\tilde{m}$ implies $\int_E s_n\, d\tilde{m} \to \int_E f\, d\tilde{m}$ in $\tilde{X}$, for each $E \in \Sigma$. However, $\int_E s_n\, d\tilde{m} = \int_E s_n\, dm$ belongs to $X[m]$, for every $E \in \Sigma$ and every $n=1,2,\ldots$. Since $X[m]$ is sequentially complete $\int_E f\, d\tilde{m}$ belongs to $X[m] \subseteq X$, for each $E \in \Sigma$. Hence f is m-integrable. □

3. Some examples

It is straightforward to produce examples of closed measures $m:\Sigma \to X$, with X not sequentially complete, such that $L^1(m)$ is complete. Indeed, let $X = U \times V$ where U is a Banach space and V is a normed space (but, not complete). Define a vector measure $m:\Sigma \to X$ by

$$m(E) = (n(E), 0), \qquad E \in \Sigma,$$

where $n:\Sigma \to U$ is any vector measure. Then the L^1-spaces of m and n are isomorphic and so $L^1(m)$ is complete. However, such examples are somewhat artificial since the measure assumes its values in a sequentially complete, complemented subspace of X.

The following example, suggested by I. Kluvánek, is more illuminating.

Example 1. Let $\Pi = \{z \in \mathbb{C}; |z| = 1\}$ be the circle group and $\mathbb{Z}$ denote the additive group of integers. Let X denote the subspace of $c_0(\mathbb{Z})$ consisting of the Fourier transforms of elements from $L^1(\Pi)$. Then X is a dense, proper subspace of $c_0(\mathbb{Z})$ for the relative topology. In particular, X is not sequentially complete. Let Σ denote the Borel subsets of $\Omega = \Pi$. Then the set function $m:\Sigma \to X$ defined by

$$m(X) = \hat{\chi}_E, \qquad E \in \Sigma,$$

where $\hat{}$ denotes the Fourier transform, is σ-additive. Since X is a normed space m is necessarily closed, [**8**, Theorem IV 7.1]. For each $\xi \in l^1(\mathbb{Z}) = X'$, let ψ_ξ denote the continuous function

$$\psi_\xi : z \to \sum_{n \in \mathbb{Z}} \xi_n z^{-n}, \qquad z \in \Pi.$$

Then the complex measure $\langle m, \xi \rangle$ is equal to $\psi_\xi(z)\, dz$, for every $\xi \in X'$. In particular, if ξ^0 is the element of X' with 1 in the co-ordinate $n=0$ and zero elsewhere, then $\langle m, \xi^0 \rangle = dz$. Using these observations it can be shown that $L^1(m) = L^1(\Pi)$ as linear spaces and, for $f \in L^1(m)$, the indefinite integral of f with respect to m is given by

$$\int_E f\, dm = (f\chi_E)\hat{}, \qquad E \in \Sigma.$$

It is then clear that $X[m] = X$ and so $X[m]$ is not sequentially complete. Accordingly, Theorem 2 does not apply. Nevertheless, $L^1(m)$ is still complete. This follows from

Theorem 1, for example, by choosing $\tilde{X}=c_0(\mathbb{Z})$. The completeness of $L^1(m)$ can also be seen directly. Indeed, the semivariation norm is given by

$$\|fm\|(\Pi)=\sup\left\{\int_\Omega |f|d|\langle m,\xi\rangle|;\,\xi\in X',\|\xi\|\leqq 1\right\},$$

for every $f\in L^1(m)=L^1(\Pi)$, from which it follows (cf. (1)) that

$$\|f\|_{L^1(\Pi)}\leqq\|fm\|(\Pi)\leqq 4\|f\|_{L^1(\Pi)},\qquad f\in L^1(m).$$

Example 2. Let Y be a locally convex space and $L(Y)$ denote the space of all continuous linear operators of Y into itself. Then $L_s(Y)$ denotes $L(Y)$ equipped with the topology of pointwise convergence on Y. Let $P:\Sigma\to L_s(Y)$ be a measure with domain Σ a σ-algebra of subsets of some set Ω. Then P is σ-additive if and only if the $\mathbb{C}$-valued set function

$$\langle Py,y'\rangle:E\to\langle P(E)y,y'\rangle,\qquad E\in\Sigma,$$

is σ-additive for each $y\in Y$ and $y'\in Y'$. A measure $P:\Sigma\to L_s(Y)$ is called a spectral measure if $P(\Omega)=I$ (the identity operator on Y) and $P(E\cap F)=P(E)P(F)$, for every $E,F\in\Sigma$.

The determination of integrability with respect to a spectral measure $P:\Sigma\to L_s(Y)$ is somewhat simpler than for arbitrary vector measures. Namely, a Σ-measurable function f is P-integrable if and only if it is $\langle Py,y'\rangle$-integrable, for each $y\in Y$ and $y'\in Y'$, and there exists an element $P(f)$ in $L(Y)$, also denoted by $\int_\Omega f\,dP$, such that

$$\langle P(f)y,y'\rangle=\int_\Omega f\,d\langle Py,y'\rangle,$$

for all $y\in Y$ and $y'\in Y'$. The indefinite integral of f is given by

$$\int_E f\,dP=P(E)P(f)=P(f)P(E),\qquad E\in\Sigma.$$

Suppose that Z is a Banach space and $P:\Sigma\to L_s(Z)$ is a spectral measure. Then the range $R(P)$, of P, is a uniformly bounded subset of $L(Z)$ and the P-integrable functions are precisely the P-essentially bounded functions. The measure P is a closed measure if and only if $R(P)$ is a closed subset of $L_s(Z)$. Separability of Z is a sufficient condition for P to be a closed measure. All of the above statements concerning spectral measures can be found in [2].

Let Z' be equipped with its (dual) norm topology. Let $L_u(Z')$ denote $L(Z')$ equipped with the uniform operator topology and let $L_{w*}(Z')$ denote $L(Z')$ equipped with the weak-star operator topology. Furthermore, let $L_w(Z)$ denote $L(Z)$ with its weak operator topology. The adjoint operation renders $L_w(Z)$ a dense subspace of $L_{w*}(Z')$. Although $L_{w*}(Z')$ is always quasicomplete, $L_w(Z)$ is sequentially complete if and only if Z is weakly sequentially complete.

We can now formulate our class of examples. An additional feature will be that the only integrable functions are the bounded ones. For the remainder of this section Z is a Banach space which is not weakly sequentially complete and which admits a closed spectral measure $P:\Sigma \to L_s(Z)$. As noted earlier if Z is separable, then any such P is necessarily closed. Define

$$m(E) = P(E), \qquad E \in \Sigma, \tag{4}$$

in which case $m:\Sigma \to L_w(Z)$ is also a spectral measure. For example, we can take $Z = c_0$ with Σ the σ-algebra of all subsets of $\Omega = \{1, 2, \ldots\}$ and $P(E)$ the operator of coordinatewise multiplication by χ_E, for each $E \in \Sigma$.

The example. *Let $P:\Sigma \to L_s(Z)$ be a closed spectral measure, $X = L_w(Z)$ and $m:\Sigma \to X$ be the (spectral) measure defined by* (4). *Then,*

(i) *X is not sequentially complete,*

(ii) *m is a closed measure,*

(iii) *$L^1(m)$ is a complete space, and*

(iv) *the m-integrable functions are the m-essentially bounded functions.*

That X is not sequentially complete has already been noted.

Since the strong operatopr and weak operator topologies on $L(Z)$ are compatible, by [**11**, Proposition 2] m is a closed measure. This establishes property (ii).

Let $f:\Omega \to \mathbb{C}$ be an m-integrable function. Then f is also integrable with respect to the spectral measure $P:\Sigma \to L_s(Z)$, with the same indefinite integral as for m, and hence f is P-essentially bounded. Since the P-null sets and m-null sets coincide it follows that f is m-essentially bounded. Since P-essentially bounded functions are certainly P-integrable it follows that m-essentially bounded functions are necessarily m-integrable. This establishes (iv).

To establish (iii), let Z'_σ denote Z' equipped with the weak-star topology. Then the adjoint map $S \to S'$ is an isometry of $L_u(Z)$ onto the closed subspace $L(Z'_\sigma)$ of $L_u(Z')$ and a bicontinuous isomorphism of $L_w(Z)$ onto $L_s(Z'_\sigma)$. Let $Y = L_s(Z'_\sigma)$ and $m':\Sigma \to Y$ denote the measure $m'(E) = P(E)'$, $E \in \Sigma$. Via the adjoint map we can identify $X[m]$ with $Y[m']$. Since $Y[m']$ is a part of the quasicomplete space $L_{w*}(Z')$ it follows from the Dominated Convergence Theorem for m' that $m'(L^1(m')) \subseteq Y[m']$. Since $L^1(m) \simeq L^1(m')$ we conclude that $m(L^1(m)) \subseteq X[m]$. On the other hand the range $R(m) = R(P)$ is a complete Boolean algebra of projections on Z. Let $\langle R(P) \rangle_u$ denote the closed linear span of $R(P)$ in $L_u(Z)$. By [**9**, Theorem 2], $\langle R(P) \rangle_u \simeq \langle R(P') \rangle_u \subseteq Y$ is a closed subspace of $L_{w*}(Z')$ and hence $\langle R(P) \rangle_u$ is sequentially complete in X. By property (iv) and [**5**, XVII Theorem 2.10] we have $m(L^1(m)) = \langle R(P) \rangle_u$. Since $X[m] \subseteq \langle R(P) \rangle_u$ (e.g. [**5**, XVII Corollary 3.17]) it follows that $X[m] = m(L^1(m)) = \langle R(P) \rangle_u$ and hence $X[m]$ is sequentially complete. By Theorem 2, $L^1(m)$ is complete which is (iii).

It was noted above that $L_s(Z'_\sigma) \simeq L_w(Z)$. Combined with Example 2 this leads to the scheme mentioned in the introduction.

Remark. Let $P:\Sigma\to L_s(Z)$ be a closed spectral measure. Then for some hyperstonian space S, some normal (order continuous) strictly positive regular Borel measure μ on S and an equivalent norm on Z, one has that $L^\infty(S,\mu)\simeq\overline{\langle R(P)\rangle}_u$. Moreover for each $z\in Z$ and $z'\in Z'$, the linear functional $T\to\langle Tz,z'\rangle$ is an order continuous linear functional on $L^\infty(S,\mu)\simeq\overline{\langle R(P)\rangle}_u$ (e.g. [**9**, Theorem 1]) and hence belongs to $L^1(S,\mu)$. Conversely, suppose that $f\in L^1(S,\mu)$. Now $R(P)$ is a complete Boolean algebra of projections on Z. Also if $\{E_\alpha\}$ is a family in Σ consisting of sets whose pairwise intersections are μ-null, then $\mu(E_\alpha\cap\operatorname{supp}(f))$ is non-zero for at most countably many α (supp(f) is the support of f). Therefore by [**5**, XVII Lemma 3.5] there is a vector in Z whose carrier projection [**5**, p. 2266] is the characteristic function of supp(f). Hence by [**6**, Theorem 4.2] there exist $z\in Z$ and $z'\in Z'$ such that the linear functional $T\to\langle Tz,z'\rangle$ on $L^\infty(S,\mu)\simeq\overline{\langle R(P)\rangle}_u$ may be identified with $f\in L^1(S,\mu)$ acting on $L^\infty(S,\mu)$ as a linear functional. Also, if $z_1,\dots,z_n\in Z$ and $z'_1,\dots,z'_n\in Z'$, then there exist elements $z\in Z$ and $z'\in Z'$ such that $\langle Tz,z'\rangle=\sum_{i=1}^n\langle Tz_i,z'_i\rangle$ for all $T\in\overline{\langle R(P)\rangle}_u\simeq L^\infty(S,\mu)$ [**6**, Corollary 4.3]. Let $m:\Sigma\to L_w(Z)$ be defined as in (4). By the above discussion and arguments similar to those needed to establish the inequality (1) one may show that the topology on $L^1(m)$ is identifiable with the $0(L^\infty(S,\mu),L^1(S,\mu))$-topology on $L^\infty(S,\mu)$; here $L^\infty(S,\mu)=m(L^1(m))=\overline{\langle R(P)\rangle}_u$ is regarded as a complex Banach lattice. That is, it is the topology on $L^\infty(S,\mu)$ generated by the seminorms $\psi\to\langle|\psi|,|f|\rangle$ for each $\psi\in L^\infty(S,\mu)$ and fixed $f\in L^1(S,\mu)$ [**13**, II Exercise 28]. Since $L^\infty(S,\mu)$ is complete for the $0(L^\infty(S,\mu),L^1(S,\mu))$-topology [**13**, II Exercise 28(b)], this gives an alternative method for describing the topology on $L^1(m)$ and establising the completeness of $L^1(m)$. However, since [**6**, Theorem 3.5] is needed to prove [**6**, Theorem 4.2], it is not necessarily a simpler approach.

Acknowledgement. The author wishes to thank the referee for valuable suggestions. The support of an Alexander von Humboldt Fellowship is gratefully acknowledged.

REFERENCES

1. C. H. Brook and W. H. Graves, Closed measures (Proc. Conf. on Integration, Topology and Geometry in Linear Spaces), *Contemp. Math.* 2 (1980), 145–160.

2. P. G. Dodds and W. J. Ricker, Spectral measures and the Bade reflexivity theorem, *J. Funct. Anal.* **61** (1985), 136–163.

3. P. G. Dodds, B. dePagter and W. J. Ricker, Reflexivity and order properties of scalar-type spectral operators in locally convex spaces. *Trans. Amer. Math. Soc.* **293** (1986). 355–380.

4. P. G. Dodds and B. dePagter, Algebras of unbounded scalar-type spectral operators, *Pacific J. Math.* **130** (1987), 41–74.

5. N. Dunford and J. T. Schwartz, *Linear Operators III: Spectral Operators* (Wiley-Interscience, New York, 1971).

6. T. A. Gillespie, Boolean algebras of projections and reflexive algebras of operators, *Proc. London Math. Soc.* **37** (1978), 56–74.

7. I. Kluvánek, The range of a vector-valued measure, *Math. Systems Theory* **7** (1973), 44–54.

8. I. Kluvánek and G. Knowles, *Vector Measures and Control Systems* (North-Holland Math. Stud. **20,** Amsterdam, 1976).

9. M. Orhon, Boolean algebras of commuting projections, *Math. Z.* **183** (1983), 531–537.

10. W. J. Ricker, On Boolean algebras of projections and scalar-type spectral operators, *Proc. Amer. Math. Soc.* **87** (1983), 73–77.

11. W. J. Ricker, Criteria for closedness of vector measures, *Proc. Amer. Math. Soc.* **91** (1984), 75–80.

12. W. J. Ricker, Spectral measures, boundedly σ-complete Boolean algebras and applications to operator theory, *Trans. Amer. Math. Soc.* **304** (1987), 819–838.

13. H. H. Schaefer, *Banach Lattices and Positive Operators* (Springer, Berlin, 1974).

Fachbereich Mathematik
Universität des Saarlandes
D-6600 Saarbrücken
Federal Republic of Germany

Current Address:
School of Mathematics
University of New South Wales
Kensington, NSW, 2033
Australia