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Some Applications of Critical Point Theory of Distance Functions on Riemannian Manifolds*

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Abstract. In this paper, we use the theory of critical points of distance functions to study the rigidity and topology of Riemannian manifolds with sectional curvature bounded below. We prove that an n-dimensional complete connected Riemannian manifold M with sectional curvature $K_M \geqslant 1$ is isometric to an n-dimensional Euclidean unit sphere if M has conjugate radius bigger than $\pi/2$ and contains a geodesic loop of length 2π. We also prove that if M is an $n(\geqslant 3)$-dimensional complete connected Riemannian manifold with $K_M \geqslant 1$ and radius bigger than $\pi/2$, then any closed connected totally geodesic submanifold of dimension not less than two of M is homeomorphic to a sphere.

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1. Introduction

Let M be a complete Riemannian manifold. For a point $p \in M$, we denote the distance from p to x by $d(p, x)$ and set $d_p(x) = d(p, x)$. Notice that the distance function d_p is not a smooth function (on the cut locus of p). Hence, the critical points of d_p are not defined in a usual sense. The notion of critical points of d_p was introduced by Grove and Shiohama in [GS].

A point $q(\neq p) \in M$ is called a critical point of d_p if there is, for any non-zero vector $v \in T_qM$, a minimal geodesic γ from q to p making an angle $\angle(v, \gamma'(0)) \leqslant \pi/2$ with v. We simply say that q is a critical point of p.

Grove and Shiohama established the theory of critical points to prove their diameter sphere theorem which states that an n-dimensional complete connected Riemannian manifold M with $K_M \geqslant 1$ and diameter bigger than $\pi/2$ is homeomorphic to an n-sphere.

Critical points of distance function is an important tool in global Riemannian geometry. Many interesting results have been proven by using this tool. One can find some of them, e.g., in [A], [AG], [CX], [G], [GP1], [GP2], [P1], [Pe1], [Pe2], [SS], [S1], [S2], [X].

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The purpose of this paper is to study some metric and topological rigidities of Riemannian manifolds with sectional curvature bounded below by using the theory of critical points.

A well-known theorem of Toponogov [T] states that a two-dimensional complete, connected Riemannian manifold with sectional curvature $K_M \geqslant 1$ is isometric to a unit 2-sphere if it has a closed geodesic without self-intersections of length 2π. Our first application of the critical point theory is to prove a similar result for the higher-dimensional case. Before mentioning the result, we fix the following definition for the conjugate radius.

DEFINITION 1.1. Let M be a Riemannian manifold and p be a fixed point of M. Denote by $C(p)$ the conjugate locus of p, that is, the set of the first conjugate points to p, for all the geodesics that start at p. We define the conjugate radius of M at p to be

$$\rho(p) = \begin{cases} +\infty, & \text{if} \quad C(p) = \emptyset, \\ \mathrm{dist}(p, C(p)), & \text{if} \quad C(p) \neq \emptyset. \end{cases}$$

The conjugate radius of M is given by $\rho(M) = \inf_{p \in M} \rho(p)$.

Now we can state our first theorem as follows:

THEOREM 1.2. *Let M be an n-dimensional complete connected Riemannian manifold with sectional curvature $K_M \geqslant 1$ and conjugate radius $\rho(M) > \pi/2$. If M contains a geodesic loop of length 2π, then M is isometeric to an n-dimensional unit sphere $S^n(1)$.*

DEFINITION 1.3. Let (X, d) be a compact metric space and $x \in X$. The radius of X at x is defined as $\mathrm{rad}\ x = \max_{y \in X} d(x, y)$. The radius of X is given by $\mathrm{rad}\ X = \min_{x \in X} \mathrm{rad}\ x$.

The concept of radius was invented in [SY]. As a second application of the critical points theory, we have the following sphere theorem for totally geodesic submanifolds in a manifold of positive sectional curvature.

THEOREM 1.4. *Let M be an $n(\geqslant 3)$-dimensional complete Riemannian manifold with sectional curvature $K_M \geqslant 1$ and radius $\mathrm{rad}\ M > \pi/2$. Suppose that N is a $k(\geqslant 2)$-dimensional closed connected totally geodesic submanifold. Then N is homeomorphic to a k-dimensional Euclidean sphere S^k.*

One can take the real projective space and consider the totally geodesic submanifolds of it to understand that our condition '$\mathrm{rad}\ \mathrm{M} > \pi/2$' in

Theorem 1.4 is essential. We don't know if it can be weakened so that 'the diameter of M is bigger than $\pi/2$'.

2. Proof of the Results

Throughout this paper, all geodesics are assumed to have unit speed.

Proof of Theorem 1.2. Since $K_M \geqslant 1$, M is compact by the Bonnet-Myers Theorem [CE]. We denote by $i(M)$ the injectivity radius of M and for any $x \in M$, let $C_m(x)$ be the cut locus of x. It is well known that the function $f: M \to R^+$ defined by $f(x) = d(x, C_m(x))$ is continuous and that $i(M) = \inf_{x \in M} f(x)$, where d denotes the distance function on M. Thus there exists a point $p \in M$ such that $i(M) = d(p, C_m(p))$. Since $C_m(p)$ is closed and so is compact, there exists $q \in C_m(p)$ such that q assumes the distance from p to $C_m(p)$. By Proposition 2.12 in [C, p.274], we conclude

(a) either that there exists a minimizing geodesic σ from p to q along which q is conjugate to p.
(b) or that there exists exactly two minimizing geodesics σ_1 and σ_2 from p to q with $\sigma_1'(l) = -\sigma_2'(l)$, $l = d(p, q)$.

If (a) holds, then the assumption on the conjugate radius implies that $i(M) = d(p, q) > \pi/2$. Now we assume that (b) holds. Since $q \in C_m(p)$, we have that $p \in C_m(q)$ and, by its very definition, p realizes the distance from q to $C_m(q)$. It follows that $\sigma_1'(0) = -\sigma_2'(0)$. Since $K_M \geqslant 1$, the Rauch comparison theorem implies that $C(x) \neq \emptyset$, $\forall x \in M$ and, consequently, for any $x \in M$, we have

$$\operatorname{rad} x = \max_{y \in M} d(x, y) \geqslant \max_{y \in C(x)} d(x, y) \geqslant \rho(x) \geqslant \rho(M) > \frac{\pi}{2}. \tag{2.1}$$

For any $x \in M$, let $y \in M$ with $d(x, y) = \operatorname{rad} x$; then, by using the Toponogov comparison theorem, one can prove that x has only y as a critical point (cf. [GS], [P1]). Since any local maximal point of d_x is a critical point of x according to Berger's lemma (Cf. [CE]), one concludes therefore that for any $x \in M$, there exists a unique point $A(x)$ which is at maximal distance from x. We claim that the map $A: M \to M$ is continuous. In fact, let $\{x_n\} \subset M$, $x_n \to x_0$, be a convergent sequence in M, then $d(x_n, A(x_n)) \to d(x_0, Ax_0)$, since the map $x \to \max_{y \in M} d(x, y) = d(x, A(x))$ is obviously continuous. For any convergent subsequence $\{Ax_{n_k}\} \subset \{Ax_n\}$ with $Ax_{n_k} \to x_0'$, we conclude from

$$|d(x_{n_k}, Ax_{n_k}) - d(x_0, x_0')| \leqslant d(x_{n_k}, x_0) + d(Ax_{n_k}, x_0')$$

that $d(x_{n_k}, Ax_{n_k}) \to d(x_0, x_0')$. Thus we have $d(x_0, x_0') = d(x_0, Ax_0)$ and so $x_0' = Ax_0$ because Ax_0 is the unique point which is at maximal distance from x_0. The continuity of A follows. Since M is homeomorphic to S^n and $Ax \neq x$, for any $x \in M$, the Brouwer fixed point theorem implies that $A: M \to M$ is surjective. Assume now that

$p = A(r)$ is the unique point which is at maximal distance from some $r \in M$; then $d(p, r) > \pi/2$. If $r = q$, then $d(p, q) > \pi/2$. Consider now the case that $r \neq q$ and take a minimal geodesic σ_3 from q to r; then either

$$\angle(\sigma_3'(0), -\sigma_1'(l)) \leqslant \frac{\pi}{2}, \quad \text{or} \quad \angle(\sigma_3'(0), -\sigma_2'(l)) \leqslant \frac{\pi}{2}.$$

We assume without loss of generality that $\angle(\sigma_3'(0), -\sigma_1'(l)) \leqslant \pi/2$.

Applying the Toponogov comparison theorem to the hinge (σ_1, σ_3), we obtain

$$\begin{aligned} \cos d(p, r) \geqslant\ & \cos d(p, q) \cos d(q, r) + \\ & + \sin d(p, q) \sin d(q, r) \cos \angle(\sigma_3'(0), -\sigma_1'(l)) \\ \geqslant\ & \cos d(p, q) \cos d(q, r). \end{aligned} \tag{2.2}$$

Using $d(p, r) > d(q, r)$ and $d(p, r) > \pi/2$, we deduce from (2.2) that

$$d(p, q) > \frac{\pi}{2}. \tag{2.3}$$

Summarizing the above discussions, we know that the injectivity radius of our M satisfies $i(M) > \pi/2$ and so we can find a sufficiently small $\delta > 0$ such that $i(M) > \pi/2 + \delta$.

Let $\gamma\colon [0, 2\pi] \to M$ be a geodesic loop of length 2π with base point $x = \gamma(0) = \gamma(2\pi)$. Since $i(M) > \pi/2 + \delta$, γ has no self-intersections. Let

$$y = \gamma\left(\frac{\pi}{2} + \delta\right), \qquad m = \gamma(\pi) \quad \text{and} \quad z = \gamma\left(\frac{3\pi}{2} - \delta\right)$$

and set

$$\gamma_1 = \gamma|_{[0, \frac{\pi}{2}+\delta]}, \qquad \gamma_2 = \gamma|_{[\frac{\pi}{2}+\delta, \pi]}, \qquad \gamma_3 = \gamma|_{[\pi, \frac{3\pi}{2}-\delta]} \quad \text{and} \quad \gamma_4 = \gamma|_{[\frac{3\pi}{2}-\delta, 2\pi]};$$

then $\gamma_i (i = 1, \ldots, 4)$ are minimal geodesics. Take a minimal geodesic τ from m to x. We claim that the length of τ satisfies $L(\tau) = \pi$ and therefore, M is isometric to $S^n(1)$ by Cheng's maximal diameter theorem [Ch]. Assume, on the contrary, that $L(\tau) < \pi$ and set

$$\alpha = \angle(\tau'(0), -\gamma'(\pi)) \quad \text{and} \quad \beta = \angle(\tau'(0), \gamma'(\pi)).$$

Applying the Toponogov comparison theorem to the geodesic triangles $(\gamma_1, \gamma_2, \tau)$ and $(\gamma_3, \gamma_4, \tau)$, respectively, we can construct two geodesic triangles $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\tau})$ and $(\overline{\gamma}_3, \overline{\gamma}_4, \overline{\tau})$ in $S^2(1)$ with vertices $\overline{x}, \overline{y}, \overline{m}$ and $\overline{x}, \overline{z}, \overline{m}$, respectively, and satisfying

$$L(\overline{\gamma}_i) = L(\gamma_i), \quad i = 1, 2, 3, 4; \qquad L(\overline{\tau}) = L(\tau), \tag{2.4}$$

and

$$\overline{\alpha} \leqslant \alpha, \ \overline{\beta} \leqslant \beta, \tag{2.5}$$

where $\overline{\alpha}$ and $\overline{\beta}$ are the inner angles of $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\tau})$ and $(\overline{\gamma}_3, \overline{\gamma}_4, \overline{\tau})$ at $\overline{m}$, respectively.

Let $\overline{x}'$ be the antipodal point of $\overline{x}$ in $S^2(1)$ and let $\overline{\tau}_1$ be the minimal geodesic from $\overline{m}$ to $\overline{x}'$. Denote by $\overline{\alpha}'$ and $\overline{\beta}'$ the inner angles of the triangles $\Delta_{\overline{y},\overline{m},\overline{x}'}$ and $\Delta_{\overline{z},\overline{m},\overline{x}'}$ at $\overline{m}$, respectively. Let $\overline{d}$ be the distance function on $S^2(1)$. From

$$\overline{d}(\overline{x}', \overline{y}) = \pi - \overline{d}(\overline{x}, \overline{y}) = \pi - d(x, y) = \frac{\pi}{2} - \delta(2.14) = \overline{d}(\overline{m}, \overline{y}), \tag{2.6}$$

one obtains by using the cosine law to the triangle $\Delta_{\overline{y},\overline{m},\overline{x}'}$ that

$$\begin{aligned} &\sin\left(\frac{\pi}{2} - \delta\right) \sin d(\overline{m}, \overline{x}') \cos \overline{\alpha}' \\ &\quad = \cos\left(\frac{\pi}{2} - \delta\right) - \cos\left(\frac{\pi}{2} - \delta\right) \cos d(\overline{m}, \overline{x}') \\ &\quad = \sin \delta \ (1 - \cos d(\overline{m}, \overline{x}')) \\ &\quad > 0, \end{aligned}$$

and so

$$\overline{\alpha}' < \frac{\pi}{2}. \tag{2.7}$$

Similarly, one deduces that

$$\overline{\beta}' < \frac{\pi}{2}. \tag{2.8}$$

Combining (2.5), (2.7) and (2.8), we find

$$\overline{\alpha}' + \overline{\beta}' + \overline{\alpha} + \overline{\beta} < \pi + \overline{\alpha} + \overline{\beta} \leqslant \pi + \alpha + \beta = 2\pi, \tag{2.9}$$

which is a contradiction. Thus $d(x, m) = \pi$ and so M is isometric to $S^n(1)$. □

Proof of Theorem 1.2. We denote by d and d^N the distance functions on M and N, respectively. Let p_1 and p_2 be in N to realize the diameter of N, say $s := d^N(p_1, p_2) = \operatorname{diam} N$. From Berger's Lemma ([CE]), we know that p and q are mutually critical points in N. That is, if we denote by $\Gamma_{p_1p_2}$ (resp., $\Gamma_{p_2p_1}$) the set of unit vectors in $T_{p_1}N$ (resp., $T_{p_2}N$) corresponding to the set of normal minimal geodesics of N from p_1 to p_2 (resp., p_2 to p_1), then $\Gamma_{p_1p_2}$ (resp., $\Gamma_{p_2p_1}$) is $\pi/2$-dense in $S_{p_1}N$ (resp., $S_{p_2}N$), here S_xN denotes the unit tangent sphere of N at x. Since a $\pi/2$-dense subset of a great sphere S^l in a unit sphere S^m, $l < m$, is also $\pi/2$-dense in S^m, we know that $\Gamma_{p_1p_2}$ is $\pi/2$-dense in $S_{p_1}M$. Similarly, $\Gamma_{p_2p_1}$ is $\pi/2$-dense in $S_{p_2}M$.

Since N is totally geodesic, it has sectional curvature $K_N \geqslant 1$. Thus, in order to prove that N is homeomorphic to S^k, it suffices to show that $s > \pi/2$ by the Grove and Shiohama diameter sphere theorem [GS]. We assume, on the contrary, that $s \leqslant \pi/2$. Take a point $q \in M$ such that $l := d(p_1, q) = \max_{x \in M} d(p_1, x)$; then $l \geqslant \operatorname{rad} M > \pi/2$. Let $\gamma: [0, l] \to M$ be a minimal geodesic from p_1 to q. Since $\Gamma_{p_1p_2}$ is $\pi/2$-dense in $T_{p_1}M$, there is a $v \in \Gamma_{p_1p_2}$ such that $\angle(v, \gamma_1'(0)) \leqslant \pi/2$. By the definition of $\Gamma_{p_1p_2}$, we can find a minimal geodesic γ_1 of N from p_1 to p_2 such that $\gamma_1'(0) = v$. Note that γ_1 is also a geodesic of M since N is totally geodesic. Set

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$t = d(p_2, q)$. Applying the Toponogov comparison theorem to the hinge (γ, γ_1), we get

$$\cos t \geqslant \cos s \cos l + \sin s \sin l \cos \angle(\gamma'(0), \gamma_1'(0)) \geqslant \cos s \cos l. \tag{2.10}$$

Similarly, let $\sigma\colon [0, t] \to M$ be a minimal geodesic from p_2 to q and since $\Gamma_{p_2 p_1}$ is $\pi/2$-dense in $T_{p_2}M$, we can take a geodesic σ_1 of length s of M from p_2 to p_1 such that $\angle(\sigma'(0), \sigma_1'(0)) \leqslant \pi/2$. Then one can apply the Toponogov inequality to the hinge (σ, σ_1), and obtain

$$\cos l \geqslant \cos s \cos t + \sin s \sin t \cos \angle(\sigma'(0), \sigma_1'(0)) \geqslant \cos s \cos t. \tag{2.11}$$

Since $s \leqslant \pi/2$, we find from (2.10) and (2.11) that

$$\cos l \sin^2 s \geqslant 0, \tag{2.12}$$

which contradicts to the fact that $l > \pi/2$. Thus $s > \pi/2$. This completes the proof of Theorem 1.4. $\square$

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