

POINTWISE BOUNDS FOR A SOLUTION OF A NONLINEAR BOUNDARY VALUE PROBLEM IN PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Pointwise bounds are obtained for the solution of a Dirichlet problem involving the nonlinear Liouville equation in the plane. Illustrative calculations are performed for a square domain.

1. Introduction

Consider the boundary value problem described by the equations

$$\nabla^2 u = e^u \quad \text{in } V, \tag{1.1}$$

$$u = 1 \quad \text{on } B, \tag{1.2}$$

where V is some closed domain in the plane with B as its boundary. Taken on its own, the differential equation (1.1) possesses an exact solution due to Liouville [3], and for simple domains V this can be used to solve the complete boundary value problem in (1.1) and (1.2). For instance the exact solution of the problem when V is a disc has been obtained by Hart [2]. For other domains it may not be possible to find the exact solution and in such cases approximate solutions must be sought, using numerical or variational approaches for example. The main drawback of these approaches, however, is that extensive computation is often required. Here we shall adopt a very much simpler approach and present analytical pointwise bounds on the exact solution by using the maximum principle for differential equations [1]. Illustrative results are obtained in a simple case.

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2. General pointwise bounds

The boundary value problem can be written as

$$-\nabla^2 u = f(u) \quad \text{in } V, \quad (2.1)$$

with

$$u = 1 \quad \text{on } B, \quad (2.2)$$

where

$$f(u) = -e^u. \quad (2.3)$$

We note that $f(u)$ is a continuous function of u , and

$$\partial f / \partial u = -e^u < 0. \quad (2.4)$$

At the end of this section we establish the existence and uniqueness of a solution of (2.1) and (2.2). We shall denote the solution by u . Now suppose that two functions u_1 and u_2 can be found so that

$$-\nabla^2 u_1 \leq f(u_1) \quad \text{in } V, \quad u_1 = 1 \quad \text{on } B, \quad (2.5)$$

and

$$-\nabla^2 u_2 \geq f(u_2) \quad \text{in } V, \quad u_2 = 1 \quad \text{on } B. \quad (2.6)$$

Then (2.5) and (2.1) with (2.2) imply

$$-\nabla^2(u_1 - u) \leq f(u_1) - f(u) \leq -K(u_1 - u)$$

for $K \geq 0$, since $\partial f / \partial u < 0$. Hence

$$-\nabla^2(u_1 - u) + K(u_1 - u) \leq 0,$$

implying, by the maximum principle, that

$$u_1 - u \leq 0$$

or

$$u_1 \leq u.$$

In the same way, with (2.6) instead of (2.5), we find that

$$u \leq u_2.$$

Combining these we therefore have shown that the upper and lower bounds

$$u_1(x, y) \leq u(x, y) \leq u_2(x, y) \quad \text{in } V \quad (2.7)$$

are valid.

One way to obtain bounding functions u_1 and u_2 in practice is to choose two functions f_1 and f_2 such that

$$f_1(u) \leq f(u) \quad \text{for all } u, \quad (2.8)$$

and

$$f_2(u) \geq f(u) \quad \text{for all } u, \tag{2.9}$$

and let u_1 and u_2 be solutions of

$$-\nabla^2 u_1 = f_1(u_1) \quad \text{in } V, \tag{2.10}$$

and

$$-\nabla^2 u_2 = f_2(u_2) \quad \text{in } V, \tag{2.11}$$

subject to the boundary conditions (2.2). Because of (2.8) and (2.9) the inequalities in (2.5) and (2.6) are then satisfied. The functions f_1 and f_2 can be taken to be linear functions, and so (2.10) and (2.11) are readily solved analytically for the bounding functions u_1 and u_2 . This procedure for obtaining u_1 and u_2 has been exploited recently in one-dimensional problems by Villadsen and Michelsen [5] and by Varma and Streider [4].

The existence of a solution to the problem in (2.1) and (2.2) can be established by using a result due to Amman [1]. To apply this result we rewrite (2.1) and (2.2) in the form

$$-\nabla^2 v = g(v) \quad \text{in } V, \quad v = 0 \quad \text{on } B, \tag{2.12}$$

where

$$g(v) = e^{1-v}, \quad v = 1 - u. \tag{2.13}$$

Since $g(0) = e > 0$, and the boundary condition satisfies $v \geq 0$ trivially, Amman's theorem A enables us to state that a necessary and sufficient condition for the existence of a non-negative solution v of the boundary value problem (2.12) is the existence of a non-negative C^2 function w satisfying

$$-\nabla^2 w \geq g(w) \quad \text{in } V, \quad w \geq 0 \quad \text{on } B. \tag{2.14}$$

To construct such a function w we consider the problem

$$-\nabla^2 w = g_2(w) = e \quad \text{in } V, \quad w = 0 \quad \text{on } B. \tag{2.15}$$

Since $-\nabla^2 w > 0$ here, it follows that w , if it exists, cannot have a minimum in V , and therefore

$$w \geq 0 \quad \text{in } V. \tag{2.16}$$

For such values of w ,

$$g_2(w) \geq g(w), \tag{2.17}$$

and so (2.14) is satisfied. In addition, (2.15) has the explicit solution

$$w(x, y) = \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{16e}{n^3\pi^3} (-1)^{(n+1)/2} \cos \frac{n\pi x}{2} \left(\frac{\cosh n\pi y/2}{\cosh n\pi/2} - 1 \right). \tag{2.18}$$

This function w satisfies (2.16) and (2.14), and hence by Amman's theorem there exists a non-negative solution v of (2.12). Since $v = 1 - u$, we have therefore established the existence of a solution u to the problem in (2.1) and (2.2).

To show that the solution u of

$$\nabla^2 u = e^u \text{ in } V, \quad u = 1 \text{ on } B, \tag{2.19}$$

is unique we suppose the contrary and let u_α and u_β denote two distinct solutions of the boundary value problem. If we define the quantity

$$P_{\alpha\beta} = \int_V \left[\{\text{grad}(u_\alpha - u_\beta)\}^2 + (u_\alpha - u_\beta)(e^{u_\alpha} - e^{u_\beta}) \right] dV, \tag{2.20}$$

we see that for any distinct functions u_α and u_β

$$P_{\alpha\beta} > 0. \tag{2.21}$$

But, using the divergence theorem in (2.20), we have

$$\begin{aligned} P_{\alpha\beta} &= \int_V (u_\alpha - u_\beta) \{ -\nabla^2(u_\alpha - u_\beta) + e^{u_\alpha} - e^{u_\beta} \} dV \\ &\quad + \int_B (u_\alpha - u_\beta) \frac{\partial}{\partial n} (u_\alpha - u_\beta) dB, \end{aligned} \tag{2.22}$$

and if u_α and u_β are distinct solutions of (2.19) then

$$P_{\alpha\beta} = 0, \tag{2.23}$$

which contradicts (2.21). Hence the solution u of (2.19) is unique.

3. Pointwise bounds

To illustrate the procedure for finding bounding functions we shall consider the problem when V is the region

$$V = \{ -1 \leq x \leq 1, -1 \leq y \leq 1 \} \tag{3.1}$$

in cartesian coordinates, with boundary B . Since by (1.1) $\nabla^2 u > 0$, the exact solution u cannot have a maximum in V , and it therefore follows that

$$u \leq 1 \text{ in } V. \tag{3.2}$$

Figure 1 shows the relevant part of the $u, f(u)$ diagram for $f(u) = -e^u$, where the parameter $a < 1$ will be determined later.

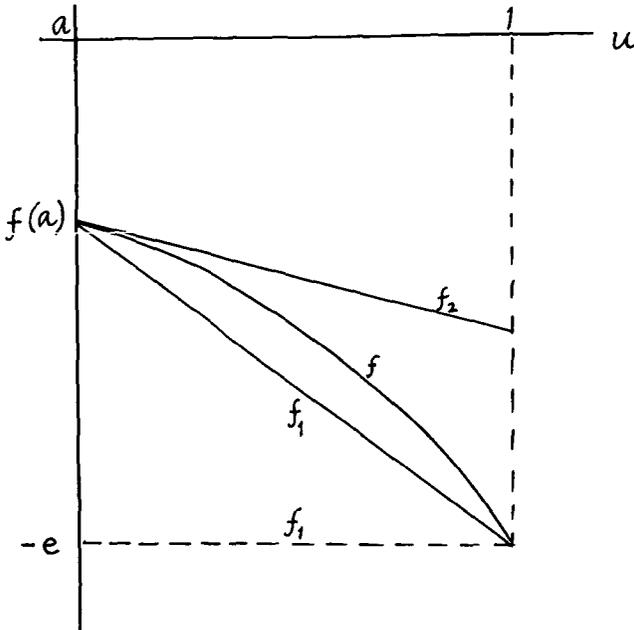


Figure 1. The relevant part of the $u, f(u)$ diagram for $f(u) = e^{-u}$.

3.1 Lower bounding function

To obtain a pointwise lower bound u_1 for u we first choose the function $f_1(u)$ as the horizontal line in Figure 1 through the point $(1, -e)$. This line lies below the function $f(u) = -e^u$ in the region specified by (3.2) and is given by

$$f_1(u) = -e. \tag{3.3}$$

For this choice of $f_1(u)$ the inequality (2.8) holds and by (2.10) the function u_1 satisfies

$$\nabla^2 u_1 = e \text{ in } V, \tag{3.4}$$

$$u_1 = 1 \text{ on } B. \tag{3.5}$$

Equations (3.4) and (3.5) have the solution

$$u_1(x, y) = 1 + \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{16e}{n^3\pi^3} (-1)^{(n-1)/2} \cos \frac{n\pi x}{2} \left(\frac{\cosh(n\pi y/2)}{\cosh(n\pi/2)} - 1 \right), \tag{3.6}$$

which from the theory of Section 2 is a lower bounding function for the exact solution u of (1.1) and (1.2). The smallest value of $u_1(x, y)$ in (3.6) occurs at $x = 0, y = 0$, and is given by

$$\begin{aligned} u_1(0, 0) &= 0.198966 \\ &= a_0 \text{ say,} \end{aligned} \tag{3.7}$$

and this therefore provides a global lower bound on u . Thus we now know that

$$a_0 \leq u \leq 1 \quad \text{in } V. \tag{3.8}$$

Referring to Figure 1 we see that we can confine u to the region (3.8) by choosing the value of the parameter a appropriately. If we do this we can improve on the bounding function in (3.6) by replacing the function $f_1(u)$ in (3.3) by the chord joining the points $(a, f(a))$ and $(1, -e)$. This chord lies below the function $f(u) = -e^u$ for the range (3.8) and is given by

$$f_1(u) = -k^2(u - a) + f(a), \tag{3.9}$$

where

$$k^2 = -\frac{f(1) - f(a)}{1 - a} = \frac{e - e^a}{1 - a}. \tag{3.10}$$

For this choice of $f_1(u)$ the inequality (2.8) holds and by (2.10) the function u_1 satisfies

$$\nabla^2 u_1 = k^2(u_1 - a) - f(a) \quad \text{in } V, \tag{3.11}$$

$$u_1 = 1 \quad \text{on } B. \tag{3.12}$$

The solution of (3.11) and (3.12) is

$$u_1(x, y) = a + k^{-2}f(a) + (1 - a - k^{-2}f(a))w_1(x, y), \tag{3.13}$$

where

$$f(u) = -e^u, \quad k^2 = \frac{e - e^a}{1 - a}, \tag{3.14}$$

and

$$w_1(x, y) = 1 + \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{4k^2}{n\pi k_n^2} (-1)^{(n-1)/2} \cos \frac{n\pi x}{2} \left(\frac{\cosh k_n y}{\cosh k_n} - 1 \right), \tag{3.15}$$

with

$$k_n^2 = k^2 + \frac{n^2\pi^2}{4}. \tag{3.16}$$

With $a = a_0$ given by (3.7), we find from (3.13) that the exact function u is bounded below by

$$\begin{aligned} u_1(0, 0; a = a_0) &= 0.436983 \\ &= a_1 \quad \text{say.} \end{aligned} \tag{3.17}$$

This allows us to make an immediate improvement in the lower bound function by working in the region

$$a_1 \leq u \leq 1 \tag{3.18}$$

instead of that in (3.8). With $a = a_1$ we find from (3.13) that the exact function u is bounded below by

$$u_1(0, 0; a_1) = 0 \cdot 455458 = a_2 \quad \text{say.} \tag{3.19}$$

Continuing in this way we generate a sequence $\{a_i\}$ of global lower bounds on u by

$$a_{i+1} = u_1(0, 0; a_i), \quad i = 0, 1, 2, \dots \tag{3.20}$$

The numbers a_i form a monotonic increasing sequence that is bounded above. This sequence therefore has a limit which after generating a_3, a_4, \dots, a_9 we find to be

$$a = 0 \cdot 457094 \tag{3.21}$$

to six places of decimals. This therefore is the best value of a_i to take in the lower bound calculation based on a single chord. Some values of the corresponding lower bounding function are given in Table 1.

TABLE 1. Lower bounding function $u_1(x, y) = u_1(y, x)$.

$\pm x$	0·0	0·2	0·4	0·6	0·8	1·0
$\pm y$						
0·0	0·457	0·473	0·524	0·618	0·769	1·0
0·2		0·488	0·537	0·628	0·774	1·0
0·4			0·580	0·660	0·792	1·0
0·6				0·721	0·826	1·0
0·8					0·886	1·0
1·0						1·0

3.2 Upper bounding function

To obtain a pointwise upper bound u_2 for u we choose the function $f_2(u)$ as the tangent to $f(u)$ at the point $(a, f(a))$ where a is given by (3.21). From Figure 1 we see that this tangent lies above $f(u)$ and is given by

$$f_2(u) = -K^2(u - a) + f(a), \tag{3.22}$$

where

$$K^2 = -f'(a) = e^a. \tag{3.23}$$

For this choice of $f_2(u)$ the inequality (2.9) holds and by (2.11) the function u_2 satisfies

$$\nabla^2 u_2 = K^2(u_2 - a) - f(a) \quad \text{in } V, \tag{3.24}$$

$$u_2 = 1 \quad \text{on } B. \tag{3.25}$$

As for (3.11) and (3.12) this boundary value problem has the solution

$$u_2(x, y) = a + K^{-2}f(a) + (1 - a - K^{-2}f(a))w_2(x, y), \tag{3.26}$$

where

$$f(u) = -e^u, \quad K^2 = e^a, \tag{3.27}$$

and

$$w_2(x, y) = 1 + \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{4K^2}{n\pi K_n^2} (-1)^{(n-1)/2} \cos \frac{n\pi x}{2} \left(\frac{\cosh K_n y}{\cosh K_n} - 1 \right), \tag{3.28}$$

with

$$K_n^2 = K^2 + \frac{n^2\pi^2}{4}. \tag{3.29}$$

Some values of the function u_2 are given in Table 2, the parameter a being given by (3.21).

TABLE 2. Upper bounding function $u_2(x, y) = u_2(y, x)$.

c	$\pm x$	0.0	0.2	0.4	0.6	0.8	1.0
$\pm y$							
0.0		0.470	0.486	0.537	0.631	0.779	1.0
0.2			0.502	0.551	0.641	0.784	1.0
0.4				0.594	0.673	0.801	1.0
0.6					0.733	0.835	1.0
0.8						0.893	1.0
1.0							1.0

From Tables 1 and 2 we see that the exact solution u has been determined to within a few per cent throughout the domain V . These are the best bounds that can be obtained by taking a single chord and single tangent in the $u, f(u)$ diagram. To improve on these it is necessary to introduce more elaborate functions f_1 and f_2 , but, since the main ideas have been illustrated, we shall not pursue this further here.

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